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The Study of the New Classes of m-Fold Symmetric bi-Univalent Functions

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Abstract: In this paper, we introduce three new subclasses of m-fold symmetric holomorphic functions in the open unit disk U, where the functions f and f^{-1} are m-fold symmetric holomorphic functions in the open unit disk. We denote these classes of functions by $FS^{p,q,s}_{\Sigma,m}(d)$, $FS^{p,q,s}_{\Sigma,m}(e)$ and $FS^{p,q,s,h,r}_{\Sigma,m}$. As the Fekete-Szegö problem for different classes of functions is a topic of great interest, we study the Fekete-Szegö functional and we obtain estimates on coefficients for the new function classes.

Keywords: Fekete-Szegö problem; coeffcient bounds and coeffcient estimates; bi-univalent functions; bi-pseudo-starlike functions; m-fold symmetric; analytic functions

1. Introduction and Preliminary Results

Let $\mathcal{A}$ denote the family of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0, f'(0) = 1$.

Let $S \subset \mathcal{A}$ denote the subclass of all functions in $\mathcal{A}$ which are univalent in U (see [1]).

In [1], the Koebe one-quarter theorem ensures that the image of the unit disk under every $f \in S$ function contains a disk of radius $1/4$.

It is well known that every function $f \in S$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z, z \in U$$

and

$$f(f^{-1}(w)) = w, |w| < r_0(f), r_0(f) \geq 1/4,$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of all bi-univalent functions in U given by (1).

The class of bi-univalent functions was first introduced and studied by Lewin [2] and it was shown that $|a_2| < 1.51$.

The domain D is m-fold symmetric if a rotation of D about the origin through an angle $2\pi/m$ carries D on itself.

We said that the holomorphic function f in the domain D is m-fold symmetric if the following condition is true: $f(e^{\frac{2\pi i}{m}} z) = e^{\frac{2\pi i}{m}} f(z)$.

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A function is said to be m-fold symmetric if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, z \in U, m \in \mathbb{N} \cup \{0\}. \quad (3)$$

The normalized form of f is given as in (3) and the series expansion for $f^{-1}(z)$ is given below (see [3]):

$$\begin{aligned} g(w) = f^{-1}(w) = w - a_{m+1}w^{m+1} \\ +[(m+1)a_{m-1}^2 - a_{2m+1}]w^{2m+1} \\ -[\tfrac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}]w^{3m+1} + \ldots \end{aligned} \quad (4)$$

We can give examples of m-fold symmetric bi-univalent functions: $\{\frac{z^m}{1-z^m}\}^{\frac{1}{m}}$; $[-log(1-z^m)]^{\frac{1}{m}}$; $\frac{1}{2}log(\frac{1+z^m}{1-z^m})^{\frac{1}{m}}$.

The important results about the m-fold symmetric analytic bi-univalent functions are given in [3–7].

The Fekete-Szegö problem is the problem of maximizing the absolute value of the functional $|a_3 - \mu a_2^2|$.

Fekete-Szegö inequalities for different classes of functions are studied in the papers [8–14].

Many authors obtained coefficient estimates of bi-univalent functions in the articles [2,14–25].

Definition 1. *Let $f \in \mathcal{A}$ be given by (1) and $0 < q < p \leq 1$. Then, the (p,q)-derivative operator for the function f of the form (1) is defined by*

$$D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p-q)z}, z \in U^* = U - \{0\} \quad (5)$$

and

$$(D_{p,q}f)(0) = f'(0) \quad (6)$$

and it follows that the function f is differentiable at 0.

We deduce from (2) that

$$D_{p,q}f(z) = 1 + \sum_{k=2}^{\infty} [k]_{p,q} a_k z^{k-1} \quad (7)$$

where the (p,q)-bracket number is given by

$$[k]_{p,q} = \frac{p^k - q^k}{p - q} = p^{k-1} + p^{k-2}q + p^{k-3}q^2 + \ldots + pq^{k-2} + q^{k-1}, p \neq q$$

which is a natural generalization of the q-number.

Too $\lim_{p \to 1^-} [k]_{p,q} = [k]_q = \frac{1-q^k}{1-q}$, see [26,27].

Definition 2 ([28])**.** *Let the function $f \in \mathcal{A}$, where $0 \leq d < 1, s \geq 1$ is real. The function $f \in L_s(d)$ of s-pseudo-starlike function of order d in the unit disk U if and only if*

$$Re(\frac{z[f'(z)]^s}{f(z)}) > d.$$

Lemma 1 ([1], p. 41). *Let the function $w \in \mathcal{P}$ be given by the following series: $w(z) = 1 + w_1 z + w_2 z^2 + \dots, z \in U$, where we denote by $\mathcal{P}$ the class of Carathéodory functions analytic in the open disk U,*

$$\mathcal{P} = \{w \in \mathcal{A} | w(0) = 1, Re(w(z)) > 0, z \in U\}.$$

The sharp estimate given by $|w_n| \leq 2, n \in \mathbb{N}^*$ holds true.

2. Main Results

Definition 3. *The function f given by (3) is in the function class $FS^{p,q,s}_{\Sigma,m}(d)(m \in \mathbb{N}, 0 < q < p \leq 1, s \geq 1, 0 < d \leq 1, (z,w) \in U)$ if:*

$$\begin{cases} f \in \Sigma, \\ |arg(D_{p,q}f(z))^s| < \frac{d\pi}{2}, z \in U \end{cases} \tag{8}$$

and

$$|arg(D_{p,q}g(w))^s| < \frac{d\pi}{2}, w \in U, \tag{9}$$

where g is the function given by (4).

Remark 1. *In the case when $m = 1$ (one-fold case) and $s = 1$, we obtain the class defined in [29].*

Remark 2. *In the case when $p = 1$, we obtain $lim_{q \to 1^-} FS^1_{\Sigma,1}(d) = FS_\Sigma(d)$, the class which was introduced by Srivastava et al. in [24].*

We obtain coefficient bounds for the functions class $FS^{p,q,s}_{\Sigma,m}(d)$ in the next theorem.

Theorem 1. *Let f given by (3) be in the class $FS^{p,q,s}_{\Sigma,m}(d)(m \in \mathbb{N}, 0 < q < p \leq 1, s \geq 1, 0 < d \leq 1, (z,w) \in U)$. Then,*

$$|a_{m+1}| \leq \frac{2d}{\sqrt{sd(m+1)[2m+1]_{p,q} - s(d-s)[m+1]^2_{p,q}}} \tag{10}$$

and

$$|a_{2m+1}| \leq \frac{2d}{s[2m+1]_{p,q}} + \frac{2(m+1)d^2}{s^2[m+1]^2_{p,q}}. \tag{11}$$

Proof. If we use the relations (8) and (9), we obtain

$$(D_{p,q}f(z))^s = [\alpha(z)]^d \tag{12}$$

and

$$(D_{p,q}g(w))^s = [\beta(w)]^d, (z, w \in U) \tag{13}$$

where the functions $\alpha(z)$ and $\beta(w)$ are in $\mathcal{P}$ and are given by

$$\alpha(z) = 1 + \alpha_m z^m + \alpha_{2m} z^{2m} + \alpha_{3m} z^{3m} + \dots \tag{14}$$

and

$$\beta(w) = 1 + \beta_m w^m + \beta_{2m} w^{2m} + \beta_{3m} w^{3m} + \dots. \tag{15}$$

It is obvious that

$$[\alpha(z)]^d = 1 + d\alpha_m z^m + (d\alpha_{2m} + \frac{d(d-1)}{2}\alpha_m^2) z^{2m} + \dots,$$

$$[\beta(w)]^d = 1 + d\beta_m w^m + (d\beta_{2m} + \frac{d(d-1)}{2}\beta_m^2) w^{2m} + \dots,$$

$$(D_{p,q}f(z))^s = 1 + s[m+1]_{p,q}a_{m+1}z^m$$
$$+(s[2m+1]_{p,q}a_{2m+1} + \frac{s(s-1)}{2}[m+1]_{p,q}^2 a_{m+1}^2)z^{2m} + \dots$$

and

$$(D_{p,q}g(w))^s = 1 - s[m+1]_{p,q}a_{m+1}w^m$$
$$-s[2m+1]_{p,q}a_{2m+1}w^{2m} + (s(m+1)[2m+1]_{p,q}a_{m+1}^2 + \tfrac{s(s-1)}{2}[m+1]_{p,q}^2 a_{m+1}^2)w^{2m} + \dots$$

If we compare the coefficients in the relations (12) and (13), we have

$$s[m+1]_{p,q}a_{m+1} = d\alpha_m, \tag{16}$$

$$s[2m+1]_{p,q}a_{2m+1} + \frac{s(s-1)}{2}[m+1]_{p,q}^2 a_{m+1}^2$$
$$= d\alpha_{2m} + \frac{d(d-1)}{2}\alpha_m^2, \tag{17}$$

$$-s[m+1]_{p,q}a_{m+1} = d\beta_m, \tag{18}$$

$$-s[2m+1]_{p,q}a_{2m+1} + (s(m+1)[2m+1]_{p,q} + \frac{s(s-1)}{2}[m+1]_{p,q}^2)a_{m+1}^2$$
$$= d\beta_{2m} + \frac{d(d-1)}{2}\beta_m^2. \tag{19}$$

We obtain from the relations (16) and (18)

$$\alpha_m = -\beta_m \tag{20}$$

and

$$2s^2[m+1]_{p,q}^2 a_{m+1}^2 = d^2(\alpha_m^2 + \beta_m^2) \tag{21}$$

Now, from the relations (17), (19) and (21), we obtain that

$$s(s-1)d[m+1]_{p,q}^2 a_{m+1}^2 + (m+1)sd[2m+1]_{p,q}a_{m+1}^2$$
$$-(d-1)s^2[m+1]_{p,q}^2 a_{m+1}^2 = d^2(\alpha_{2m} + \beta_{2m}).$$

We have

$$a_{m+1}^2 = \frac{d^2(\alpha_{2m} + \beta_{2m})}{s[m+1]_{p,q}^2(s-d) + (m+1)sd[2m+1]_{p,q}}. \tag{22}$$

If we apply Lemma 1 for the coefficients α_{2m} and β_{2m}, we have

$$|a_{m+1}| \leq \frac{2d}{\sqrt{(m+1)sd[2m+1]_{p,q} - (d-s)s[m+1]_{p,q}^2}}.$$

If we use the relations (17) and (19), we obtain the next relation

$$\begin{aligned} 2s[2m+1]_{p,q}a_{2m+1} - s(m+1)[2m+1]_{p,q}a_{m+1}^2 \\ = d(\alpha_{2m} - \beta_{2m}) + \tfrac{d(d-1)}{2}(\alpha_m^2 - \beta_m^2). \end{aligned} \tag{23}$$

It follows from (20), (21) and (23) that

$$a_{2m+1} = \frac{(m+1)d^2(\alpha_m^2 + \beta_m^2)}{4s^2[m+1]_{p,q}^2} + \frac{d(\alpha_{2m} - \beta_{2m})}{2s[2m+1]_{p,q}}. \tag{24}$$

If we apply Lemma 1 for the coefficients $\alpha_m, \alpha_{2m}, \beta_m, \beta_{2m}$, we obtain

$$|a_{2m+1}| \leq \frac{2d}{[2m+1]_{p,q}s} + \frac{2d^2(m+1)}{s^2[m+1]_{p,q}^2}.$$

□

Remark 3. *For one-fold case $m = 1$ and $s = 1$ in Theorem 1, we obtain the results obtained in [29].*

Remark 4. *For a one-fold case and $p = 1$, we have*

$$lim_{q\to 1^-} FS^{q,1}_{\Sigma,1}(d) = FS_{\Sigma}(d),$$

the results of Srivastava et al. [24].

Definition 4. *The function f given by (3) is in the class $FS^{p,q,s}_{\Sigma,m}(e)(0 \leq e < 1, 0 < q < p \leq 1, s \geq 1,$ $(z,w) \in U, m \in \mathbb{N})$ if the following conditions are satisfied:*

$$\begin{cases} f \in \Sigma, \\ \mathcal{R}\{(D_{p,q}f(z))^s\} > e, z \in U \end{cases} \tag{25}$$

$$\mathcal{R}\{(D_{p,q}g(w))^s\} > e, w \in U, \tag{26}$$

where the function g is defined by Relation (4).

Remark 5. *For $m = 1$ (one-fold case) and $s = 1$, we obtain the class of functions obtained in [29].*

Remark 6. *When $p = 1$, we obtain $lim_{q\to 1^-} FS^{1}_{\Sigma,1}(e) = FS_{\Sigma}(d)$, the class which was introduced by Srivastava et al. in [24].*

In the next theorem, we obtain coefficient bounds for the function class $FS^{p,q,s}_{\Sigma,m}(e)$.

Theorem 2. *Let the function f given by (3) be in the function class $FS^{p,q,s}_{\Sigma,m}(e), (m \in \mathbb{N},$ $0 < q < p \leq 1, s \geq 1, 0 \leq e < 1, (z,w) \in U)$. Then,*

$$|a_{m+1}| \leq min\{\frac{2(1-e)}{s[m+1]_{p,q}}, 2\sqrt{\frac{(1-e)}{s(s-1)[m+1]_{p,q}^2 + (m+1)s[2m+1]_{p,q}}}\} \tag{27}$$

$$|a_{2m+1}| \leq \frac{2(1-e)(m+1)}{s(s-1)[m+1]_{p,q}^2 + (m+1)s[2m+1]_{p,q}} + \frac{2(1-e)}{s[2m+1]_{p,q}}. \tag{28}$$

Proof. If we use Relations (25) and (26), we obtain

$$(D_{p,q}f(z))^s = e + (1-e)\alpha(z) \tag{29}$$

and

$$(D_{p,q}g(w))^s = e + (1-e)\beta(w), \quad z, w \in U, \tag{30}$$

respectively, where

$$\alpha(z) = 1 + \alpha_m z^m + \alpha_{2m} z^{2m} + \alpha_{3m} z^{3m} + \ldots$$

and

$$\beta(w) = 1 + \beta_m w^m + \beta_{2m} w^{2m} + \beta_{3m} w^{3m} + \ldots,$$

$\alpha(z)$ and $\beta(w)$ are in $\mathcal{P}$.

It is obvious that

$$e + (1-e)\alpha(z) = 1 + (1-e)\alpha_m z^m + (1-e)\alpha_{2m} z^{2m} + \dots,$$

and

$$e + (1-e)\beta(w) = 1 + (1-e)\beta_m w^m + (1-e)\beta_{2m} w^{2m} + \dots$$

Already,

$$(D_{p,q}f(z))^s = 1 + s[m+1]_{p,q}a_{m+1}z^m + (s[2m+1]_{p,q}a_{2m+1} + \tfrac{s(s-1)}{2}[m+1]^2_{p,q}a^2_{m+1})z^{2m} + \dots$$

and

$$(D_{p,q}g(w))^s = 1 - s[m+1]_{p,q}a_{m+1}w^m - s[2m+1]_{p,q}a_{2m+1}w^{2m}$$
$$+(s(m+1)[2m+1]_{p,q}a^2_{m+1} + \frac{s(s-1)}{2}[m+1]^2_{p,q}a^2_{m+1})w^{2m} + \dots$$

From the relations (29) and (30), if we compare the coefficients, we obtain the following relations:

$$s[m+1]_{p,q}a_{m+1} = (1-e)\alpha_m, \tag{31}$$

$$s[2m+1]_{p,q}a_{2m+1} + \frac{s(s-1)}{2}[m+1]^2_{p,q}a^2_{m+1} = (1-e)\alpha_{2m}, \tag{32}$$

$$-s[m+1]_{p,q}a_{m+1} = (1-e)\beta_m, \tag{33}$$

$$-s[1+2m]_{p,q}a_{2m+1} + (s[2m+1]_{p,q}(m+1)$$
$$+\frac{s(s-1)}{2}[1+m]^2_{p,q})a^2_{m+1} = (1-e)\beta_{2m}. \tag{34}$$

We obtain from Relations (31) and (33)

$$\alpha_m = -\beta_m \tag{35}$$

and

$$2s^2[m+1]^2_{p,q}a^2_{m+1} = (1-e)^2(\alpha^2_m + \beta^2_m). \tag{36}$$

We obtain now from Relations (32) and (34) the following relation:

$$s(s-1)[m+1]^2_{p,q}a^2_{m+1} + (m+1)s[2m+1]_{p,q}a^2_{m+1} = (1-e)(\alpha_{2m} + \beta_{2m}). \tag{37}$$

From Lemma 1 for the coefficients $\alpha_m, \alpha_{2m}, \beta_m, \beta_{2m}$, we obtain that

$$|a_{m+1}| \le 2\sqrt{\frac{1-e}{(m+1)s[2m+1]_{p,q} + s(s-1)[m+1]^2_{p,q}}}.$$

If we use Relations (32) and (34) to find the bound on $|a_{2m+1}|$, we obtain the following relation:

$$-s(1+m)[1+2m]_{p,q}a^2_{m+1} + 2s[1+2m]_{p,q}a_{2m+1} = (1-e)(\alpha_{2m} - \beta_{2m}), \tag{38}$$

or equivalently

$$a_{2m+1} = \frac{(1-e)(\alpha_{2m} - \beta_{2m})}{2s[2m+1]_{p,q}} + \frac{(m+1)}{2}a^2_{m+1}. \tag{39}$$

From Relation (36), if we substitute the value of a^2_{m+1}, we obtain

$$a_{2m+1} = \frac{(1-e)(\alpha_{2m} - \beta_{2m})}{2s[2m+1]_{p,q}} + \frac{(m+1)(1-e)^2(\alpha^2_m + \beta^2_m)}{4s^2[m+1]^2_{p,q}}. \tag{40}$$

Now, if we apply Lemma 1 for the coefficients $\alpha_m, \alpha_{2m}, \beta_m, \beta_{2m}$, we obtain

$$|a_{2m+1}| \leq \frac{2(1-e)}{s[2m+1]_{p,q}} + \frac{2(m+1)(1-e)^2}{s^2[m+1]_{p,q}^2}.$$

From Relations (37) and (39) applying Lemma 1, we obtain

$$|a_{2m+1}| \leq \frac{2(m+1)(1-e)}{s(s-1)[m+1]_{p,q}^2 + (m+1)s[2m+1]_{p,q}} + \frac{2(1-e)}{s[2m+1]_{p,q}}.$$

□

Remark 7. *For one fold case ($m = 1$) and $s = 1$ in Theorem 2, we obtain the results given in [29].*

Remark 8. *For a one-fold case, in Theorem 2, choosing $p = 1, q \to 1^-$, we obtain the following corollary.*

Corollary 1. *[24] Let the function $f \in FS_\Sigma(e), (s = 1, 0 \leq e < 1, (z, w) \in U)$ be given by (1). Then,*

$$|a_2| \leq \sqrt{\frac{2(1-e)}{3}}$$

and

$$|a_3| \leq \frac{(1-e)(5-3e)}{3}.$$

In the following theorems, we provide the Fekete-Szegö type inequalities for the functions of the families $FS_{\Sigma,m}^{p,q,s}(d)$ and $FS_{\Sigma,m}^{p,q,s}(e)$.

Theorem 3. *Let f be a function of the form (3) in the class $FS_{\Sigma,m}^{p,q,s}(d)$. Then,*

$$|a_{2m+1} - \sigma a_{m+1}^2| \leq \begin{cases} \frac{2d}{s[2m+1]_{p,q}}, |t(\sigma)| \leq \frac{1}{s[2m+1]_{p,q}} \\ 4sd|t(\sigma)|, |t(\sigma)| \geq \frac{1}{s[2m+1]_{p,q}}, \end{cases} \tag{41}$$

where

$$t(\sigma) = \frac{d(m+1-2\sigma)}{2s[m+1]_{p,q}^2(s-d) + 2s(m+1)d[2m+1]_{p,q}}.$$

Proof. We want to calculate $a_{2m+1} - \sigma a_{m+1}^2$.

For this, from Relations (22) and (24), where we know the values of the coefficients a_{m+1}^2 and a_{2m+1} :

$$a_{m+1}^2 = \frac{d^2(\alpha_{2m} + \beta_{2m})}{s[m+1]_{p,q}^2(s-d) + (m+1)sd[2m+1]_{p,q}},$$

$$a_{2m+1} = \frac{(m+1)d^2(\alpha_m^2 + \beta_m^2)}{4s^2[m+1]_{p,q}^2} + \frac{d(\alpha_{2m} - \beta_{2m})}{2s[2m+1]_{p,q}},$$

it follows that

$$a_{2m+1} - \sigma a_{m+1}^2 =$$

$$d[\alpha_{2m}(\frac{1}{2s[2m+1]_{p,q}} + \frac{d(m+1-2\sigma)}{2s[m+1]_{p,q}^2(s-d) + 2s(m+1)d[2m+1]_{p,q}})$$

$$+\beta_{2m}(\frac{d(m+1-2\sigma)}{2s[m+1]_{p,q}^2(s-d) + 2sd(m+1)[2m+1]_{p,q}} - \frac{1}{2s[2m+1]_{p,q}})].$$

According to Lemma 1 and after some computations, we obtain

$$|a_{2m+1}-\sigma a_{m+1}^2| \le \begin{cases} \frac{2d}{s[2m+1]_{p,q}}, |t(\sigma)| \le \frac{1}{s[2m+1]_{p,q}} \\ 4sd|t(\sigma)|, |t(\sigma)| \ge \frac{1}{s[2m+1]_{p,q}}. \end{cases}$$

□

Theorem 4. *Let f be a function of the form (3) in the class $FS^{p,q,s}_{\Sigma,m}(e)$. Then,*

$$|a_{2m+1}-\sigma a_{m+1}^2| \le \begin{cases} \frac{2(1-e)}{s[2m+1]_{p,q}}, |t(\sigma)| \le \frac{1}{2s[2m+1]_{p,q}} \\ 4s(1-e)|t(\sigma)|, |t(\sigma)| \ge \frac{1}{2s[2m+1]_{p,q}} \end{cases}, \quad (42)$$

where

$$t(\sigma) = \frac{(1-2\sigma+m)}{2s(s-1)d[m+1]_{p,q}^2 + 2s(m+1)[2m+1]_{p,q}}.$$

Proof. We will compute $a_{2m+1}-\sigma a_{m+1}^2$, using the values of the coefficients a_{m+1}^2 and a_{2m+1} given in Relations (37) and (39).

It follows that

$$a_{2m+1}-\sigma a_{m+1}^2$$
$$= (1-e)[\alpha_{2m}(\frac{1}{2s[2m+1]_{p,q}} + \frac{1-2\sigma+m}{2s(s-1)d[m+1]_{p,q}^2 + 2(m+1)s[2m+1]_{p,q}})$$
$$+\beta_{2m}(\frac{(1+m-2\sigma)}{2s(s-1)d[m+1]_{p,q}^2 + 2s(m+1)[2m+1]_{p,q}} - \frac{1}{2s[2m+1]_{p,q}})].$$

According to Lemma 1 and after some computations, we obtain

$$|a_{2m+1}-\sigma a_{m+1}^2| \le \begin{cases} \frac{2(1-e)}{s[2m+1]_{p,q}}, |t(\sigma)| \le \frac{1}{2s[2m+1]_{p,q}} \\ 4s(1-e)|t(\sigma)|, |t(\sigma)| \ge \frac{1}{2s[2m+1]_{p,q}} \end{cases}.$$

□

Definition 5. *Let $h, r : U \to \mathbb{C}$ be analytic functions and $\min\{Re(h(z)), Re(r(z))\} > 0$, where $z \in U, h(0) = r(0) = 1$.*

A function f given by (3) is said to be in the class $FS^{p,q,s,h,r}_{\Sigma,m}$, where $s \ge 1, 0 < q < p \le 1$, $m \in \mathbb{N}$ if the conditions are satisfied:

$$\left(D_{p,q}f(z)\right)^s \in h(U), z \in U \quad (43)$$

and

$$\left(D_{p,q}g(w)\right)^s \in r(U), w \in U, \quad (44)$$

where the function g is given by (4).

We obtain coefficient bounds for the functions class $FS^{p,q,s,h,r}_{\Sigma,m}$ in the following theorem.

Theorem 5. *Let the function f given by (3) be in the class $FS^{p,q,s,h,r}_{\Sigma,m}$. Then,*

$$|a_{m+1}| \le \min\{\sqrt{\frac{|h_1'(0)|^2+|r_1'(0)|^2}{2s^2[m+1]_{p,q}^2}}, \sqrt{\frac{|h_2''(0)|+|r_2''(0)|}{s(s-1)[m+1]_{p,q}^2 + s(m+1)[2m+1]_{p,q}}}\}; \quad (45)$$

$$|a_{2m+1}| \leq \min\{\frac{(|h'(0)|^2+|r'(0)|^2)(m+1)}{4s^2[m+1]_{p,q}^2}+\frac{|h''(0)|+|t''(0)|}{2s[2m+1]_{p,q}},$$
$$\frac{|h''(0)|+|r''(0)|}{2s[2m+1]_{p,q}}+\frac{(m+1)(|h''(0)|+|r''(0)|)}{2s\{(m+1)[2m+1]_{p,q}+(s-1)[m+1]_{p,q}^2\}}\}. \tag{46}$$

Proof. In Relations (43) and (44), the equivalent forms of the argument inequalities are

$$(D_{p,q}f(z))^s = h(z), \tag{47}$$

and

$$(D_{p,q}g(w))^s = r(w), \tag{48}$$

where $h(z)$ and $r(w)$ satisfy the conditions from Definition 5, and have the following Taylor–Maclaurin series expansions:

$$h(z) = 1 + h_1 z + h_2 z^2 + \dots \tag{49}$$

$$r(w) = 1 + r_1 w + r_2 w^2 + \dots \tag{50}$$

If we substitute (49) and (50) into (47) and (48), respectively, and equate the coefficients, we obtain

$$s[m+1]_{p,q}a_{m+1} = h_1; \tag{51}$$

$$s[2m+1]_{p,q}a_{2m+1} + \frac{s(s-1)}{2}[m+1]_{p,q}^2 a_{m+1}^2 = h_2; \tag{52}$$

$$-s[m+1]_{p,q}a_{m+1} = r_1; \tag{53}$$

$$-s[2m+1]_{p,q}a_{2m+1} + (s(m+1)[2m+1]_{p,q} + \frac{s(s-1)}{2}[m+1]_{p,q}^2)a_{m+1}^2 = r_2. \tag{54}$$

We obtain that

$$h_1 = -r_1 \tag{55}$$

and

$$h_1^2 + r_1^2 = 2s^2[m+1]_{p,q}^2 a_{m+1}^2 \tag{56}$$

from Relations (51) and (53).

Adding Relations (52) and (54), we obtain that

$$a_{m+1}^2\{s(s-1)[m+1]_{p,q}^2 + s(m+1)[2m+1]_{p,q}\} = h_2 + r_2. \tag{57}$$

Now, from (56) and (57), we obtain

$$a_{m+1}^2 = \frac{h_1^2 + r_1^2}{2s^2[m+1]_{p,q}^2} \tag{58}$$

$$a_{m+1}^2 = \frac{h_2 + r_2}{s(s-1)[m+1]_{p,q}^2 + s(m+1)[2m+1]_{p,q}}. \tag{59}$$

We obtain from Relations (58) and (59) that

$$|a_{m+1}|^2 \leq \frac{|h_1'(0)|^2 + |r_1'(0)|^2}{2s^2[m+1]_{p,q}^2}$$

and

$$|a_{m+1}|^2 \leq \frac{|h_2''(0)| + |r_2''(0)|}{s(s-1)[m+1]_{p,q}^2 + s(m+1)[2m+1]_{p,q}}.$$

So, we obtain the estimate on the coefficient $|a_{m+1}|$ as in (45).

Next, substracting (54) from (52), we obtain the following relation:

$$2s[2m+1]_{p,q}a_{2m+1} - s(m+1)[2m+1]_{p,q}a_{m+1}^2 = h_2 - r_2. \tag{60}$$

Substituting the value of a_{m+1}^2 from (58) into (60), it follows that

$$a_{2m+1} = \frac{h_2 - r_2}{2s[2m+1]_{p,q}} + \frac{(m+1)(h_1^2 + r_1^2)}{4s^2[m+1]_{p,q}^2}.$$

Therefore,

$$|a_{2m+1}| \le \frac{(|h'(0)|^2 + |r'(0)|^2)(m+1)}{4s^2[m+1]_{p,q}^2} + \frac{|h''(0)| + |t''(0)|}{2s[2m+1]_{p,q}}.$$

Upon substituting the value of a_{m+1}^2 from (59) into (60), it follows that

$$a_{2m+1} = \frac{h_2 - r_2}{2s[2m+1]_{p,q}} + \frac{(m+1)(h_2 + r_2)}{\{(s-1)[m+1]_{p,q}^2 + (m+1)[2m+1]_{p,q}\}2s}.$$

So, it follows that

$$|a_{2m+1}| \le \frac{|h''(0)| + |r''(0)|}{2s[2m+1]_{p,q}} + \frac{(m+1)(|h''(0)| + |r''(0)|)}{2s\{(m+1)[2m+1]_{p,q} + (s-1)[m+1]_{p,q}^2\}}.$$

□

3. Conclusions

As future research directions, the symmetry properties of this operator, the (p,q)-derivative operator, can be studied.

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