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Conformable Fractional Martingales and Some Convergence Theorems

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Abstract: In this paper, we define conformable Lebesgue measure and conformable fractional countable martingales. Some convergence theorems are proved.

Keywords: conformable fractional; martingales; convergence theorems; fractional conditional expectation

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1. Introduction

Martingales are a main topic in probability theory. They have many applications in our real lives. Fractional martingales have ties and relationships with fractional Brownian motion [1,2]. The main definition of martingales can be written by using the real line as: $E \subseteq \mathbb{R}^+$, where the Lebesgue measurable set is $\mathbb{R}$.

Assume $\mathcal{A}$ to be the σ-algebra of Lebesgue measurable sets in E, and μ is the Lebesgue measure on E, where $(E, \mathcal{A}, \mu)$ is a measure space, $L^1(E, \mu)$ is the space of Lebesgue integrable functions on E, and B_n is a sequence of σ- algebras of the Lebesgue measurable set in $\mathcal{A}$ such that $B_n \subseteq B_{n+1} \subseteq \mathcal{A}, \forall\, n \geq 1$.

Definition 1. *For each n, let $f_n \in L^1(E, B_n, \mu)$. Then, f_n is called a martingale if $\int_D f_n d\mu = \int_D f_m d\mu$, $\forall\ m \geq n$, and $D \in B_n$. The standard notation for f_n is: $E(f_m|B_n) = f_n$, $\forall\, m \geq n$, and is called the conditional expectation of f_m relative to B_n. For more on martingales, we refer to [2–4].*

2. Method and Results

Fractional martingales, as introduced in [1], have a strong relation to fractional Brownian motion. Furthermore, the Riemann–Liouvill fractional integral was used for fractional martingales. Hu, Y. et al pointed out in [1], that fractional martingales are not martingales. Consequently, in this section, we introduce the following: (i) fractional Lebesgue measure, and (ii) fractional martingales. We use conformable fractional integral for the definition of fractional martingales. Furthermore, our definition of fractional martingales ensures that fractional martingales are martingales.

Definition 2. *Let μ be the Lebesgue measure on $E \subseteq \mathbb{R}^+$ and $\mathcal{A}$ be the σ–algebra of Lebesgue measurable sets in E. We define the conformable fractional Lebesgue measure for $\alpha \in (0,1)$* as:

$\mu^\alpha(B) = \int_B \frac{d\mu}{x^{1-\alpha}}$, for any $B \in \mathcal{A}$. One can easily show that μ^α is a measure on E, noting that $B \subseteq [0, \infty)$, so, $x \geq 0$.

Hence,

$$\mu^\alpha([0,1]) = \int_0^1 \frac{d\mu}{x^{1-\alpha}} = 1$$

and,

$$\mu^\alpha([4,9]) = 9^\alpha - 4^\alpha$$

so,

$$\text{if } \alpha = \frac{1}{2}, \text{ then } \mu^{0.5}([4,9]) = 1.$$

One can build a whole theory here using the Lebesgue fractional measure, such as $L^p(E, \mathcal{A}, \mu^\alpha)$, $1 \le p < \infty$. Further, it would be nice to study the relation between $L^p(E, \mathcal{A},\ \mu)$ and $L^p(E, \mathcal{A}, \mu^\alpha)$.

Definition 3. *Let $f \in L^1(E, \mu^\alpha)$, and B be a σ-algebra of Lebesgue measurable sets. Then, a function $g \in L^1(E, B, \mu^\alpha)$ is called the fractional conditional expectation of f relative to B if $\int_A g d\mu^\alpha = \int_A f d\mu^\alpha$, $\forall\ A \in B$.*

We remark that $\int_A f d\mu^\alpha$ is just the fractional integral introduced in [5]. We denote g by $E(f|B)$. Conditional expectation is an important concept in probability theory.

A nice example of fractional conditional expectation is:

Example 1. *Let $A_n = (n, n+1)$. Consider the σ–algebra $\mathcal{A}$ generated by (A_n). Now it is easy to check that $E_\alpha(f|\mathcal{A}) = \sum_{n=1}^{\infty} \frac{\int_{A_n} f d\mu^\alpha}{\mu^\alpha(A_n)} 1_{A_n}$, where 1_{A_n} is the characteristic function of the set A_n [6]. Conditional expectation is the cornerstone of the definition of martingales.*

Note that a fractional martingale is associated with the fractional Lebesgue measure. However, martingales are associated with the usual Lebesgue measure. Therefore, a function could be integrable with respect to Lebesgue measure but not integrable with respect to fractional Lebesgue measure.

Theorem 1. *Let $f \epsilon L^1(E, \mu^\alpha)$. Then $E_\alpha(f|B)$ exists for every σ-algebra B of Lebesgue measurable sets of E. Further,*

$$\| E_\alpha(f|B) \|_1 \le \| f \|_1$$

Proof of Theorem 1. For $A \epsilon B$, define $\gamma(A) = \int_A f d\mu^\alpha$

Clearly,

$$\operatorname*{Lim}_{\mu^\alpha(A)\to 0} \gamma(A) = 0$$

Hence, γ is μ^α-continuous. Then, by the Radon–Nikodym theorem [3], there exists $g \epsilon L^1(E, B, \mu)\ \ni\ \gamma(A) = \int_A g d\mu^\alpha$, for every $A \epsilon B$.

Thus,

$$g = E_\alpha(f|B).$$

The use of Jensen's inequality completes the proof, noting that $x^{1-\alpha} > 0$, on $E \subseteq (0, \infty)$. □

Theorem 2. *Remains true for $f \epsilon L^p(E, \mu^\alpha)$, for $1 < p < \infty$.*

Now, we present the main definition.

Definition 4. *Let (B_n) be a sequence of σ–algebras of Lebesgue measurable sets, such that $B_n \subseteq B_{n+1} \subset \mathcal{A}$, $\forall\, n$. A sequence of functions (f_n) whereby $f_n \epsilon L^1(E, \mathcal{A}, \mu^\alpha)$ and $E_\alpha(f_k|B_n) = f_n\ \ \forall\ k \ge n$, is called a fractional martingale. We will write (f_n, B_n) for such a martingale.*

A nice example of a martingale is:

Example 2. *Let $f \epsilon L^1(E, \mu^\alpha)$ and (B_n) be a sequence of σ–algebras of Lebesgue measurable sets in E. Let $f_n = E_\alpha(f|B_n)$. Then, clearly (f_n) is a fractional martingale.*

Let $\mathcal{A}$ be the $\sigma-$ algebra of all Lebesgue measurable sets in E. So,

$$L^1(E, \mu^\alpha) = L^1(E, \mathcal{A}, E_\alpha)$$

Now, let (f_n, B_n) be a martingale in $L^1(E, \mu)$. So (f_n, B_n) is a fractional martingale if μ is replaced by μ^α.

Now, we prove:

Theorem 3. *A martingale (f_n, B_n) in $L^1(E, \mu^\alpha)$ converges in $L^1(E, \mu^\alpha)$ if, and only if, there exists $f \epsilon L^1(E, \mu^\alpha)$, such that for each $A \epsilon \bigcup_{n=1}^{\infty} B_n$ we have*

$$\lim_{n\to\infty} \int_A f_n d\mu^\alpha = \int_A f d\mu^\alpha$$

Proof of Theorem 3. With no loss of generality, we assume that the σ-algebra generated by $\bigcup_{n=1}^{\infty} B_n = \mathcal{A}$.

Now,

assume that $f_n \overset{n\to\infty}{\to} f$ in $L^1(E, \mu^\alpha)$,

so,

$$\int_E |f_n - f| \, d\mu^\alpha \overset{n\to\infty}{\to} 0 \tag{1}$$

However, for any $A \epsilon \bigcup_{n=1}^{\infty} B_n$, we have

$$\left| \int_A f_n \, d\mu^\alpha - \int_A f \, d\mu^\alpha \right|$$

noting that μ^α is a measure.

$$\leq \int_A |f_n - f| \, d\mu^\alpha$$

By (1) we get

$$\lim_{n\to\infty} \int_A f_n d\mu^\alpha = \int_A f d\mu^\alpha$$

For the converse:

Assume there exists $f \epsilon L^1(E, \mu^\alpha)$ such that

$$\lim_{n\to\infty} \int_A f_n d\mu^\alpha = \int_A f d\mu^\alpha \text{ for all } A\epsilon \bigcup_{n=1}^{\infty} B_n.$$

Since we assume that $\mathcal{A} =$ the σ-algebra generated by $\bigcup_{n=1}^{\infty} B_n$, then we get

$$E_\alpha(f|B_n) = f_n \; \forall \; n \geq 1$$

Now, we claim that $\underset{n\to\infty}{lim} \parallel f_n - f \parallel_1 = 0$.

By assumption, on $\mathcal{A}$ and $\bigcup_{n=1}^{\infty} B_n$, it follows that simple functions of the form $\sum_{i=1}^{n} a_i 1_{A_i}$, $A_i \epsilon \bigcup_{n=1}^{\infty} B_n$ are dense in $L^1(E, \mu^\alpha)$.

Hence, for every $\varepsilon > 0$ there exists $g_\varepsilon = \sum_{i=1}^{m} b_i 1_{E_i}$, such that $\parallel f - g_\varepsilon \parallel_1 < \frac{\varepsilon}{2}$.

Since $B_n \subseteq B_{n+1}$, then there exists m_0, such that

$$E_i \; \epsilon \; B_{m_0}, \;\; \forall \; 1 \leq i \leq m$$

Hence, g_ε is B_n-measurable $\forall \; m \geq m_0$, and

$$E_\alpha(g_\varepsilon|B_m) = g_\varepsilon \; \forall \; m \geq m_0 \tag{2}$$

Now, for $m \geq m_0$, we have:

$$\| f_m - f \|_1 \leq \| f_m - g_\varepsilon \|_1 + \| g_\varepsilon - f \|_1$$

Using (2), we get

$$f_m - g_\varepsilon = E_\alpha((f - g_\varepsilon | B_m)),\ \forall\, m \geq m_0\ .$$

Hence,

$$\begin{aligned}
\| f_m - f \|_1 &\leq \| f_m - g_\varepsilon \|_1 + \| g_\varepsilon - f \|_1 \\
&= \| E_\alpha(f - g_\varepsilon | B_m) \|_1 + g_\varepsilon - f_1 \\
&\leq \| f - g_\varepsilon \|_1 + \| g_\varepsilon - f \|_1 \\
&= 2\, g_\varepsilon - f_1 \leq 2\,\frac{\varepsilon}{2} = \varepsilon \quad \forall\, m \geq m_0
\end{aligned}$$

Thus,

$$f_m \overset{n \to \infty}{\to} f \text{ in } L^1(E, \mu^\alpha).$$

This completes the proof. □

A nice consequence of Theorem 4 which is easy to prove is:

Theorem 4. *A fractional martingale* (f_n, B_n) *in* $L^1(E, \mu^\alpha)$ *is convergent in* $L^1(E, \mu^\alpha)$ *if, and only if, there exists* $f \in L^1(E, \mu^\alpha)$*, such that* $E_\alpha(f | B_n) = f_n\ \forall\, n \geq 1$.

3. Discussion

Conformable fractional martingales have similar properties to the usual martingales.

4. Conclusions

We proved convergence theorems for the conformable fractional martingales similar to the usual martingales.

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