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When Is $\sigma(A(t)) \subset \{z \in \mathbb{C}; \Re z \leq -\alpha < 0\}$ the Sufficient Condition for Uniform Asymptotic Stability of LTV System $\dot{x} = A(t)x$?

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Abstract: In this paper, the class of matrix functions $A(t)$ is determined for which the condition that the pointwise spectrum $\sigma(A(t)) \subset \{z \in \mathbb{C}; \Re z \leq -\alpha\}$ for all $t \geq t_0$ and some $\alpha > 0$ is sufficient for uniform asymptotic stability of the linear time-varying system $\dot{x} = A(t)x$. We prove that this class contains as a proper subset the matrix functions with the values in the special orthogonal group $SO(n)$.

Keywords: linear time-varying system; stability; logarithmic norm

MSC: 34A30; 34D20

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Citation: Vrabel, R. When Is $\sigma(A(t)) \subset \{z \in \mathbb{C}; \Re z \leq -\alpha < 0\}$ the Sufficient Condition for Uniform Asymptotic Stability of LTV System $\dot{x} = A(t)x$? *Mathematics* **2022**, *10*, 141. https://doi.org/10.3390/math10010141

Academic Editor: Ivan Slapničar

Received: 7 December 2021
Accepted: 1 January 2022
Published: 4 January 2022

1. Introduction

Stability analysis for linear time-varying (LTV) systems is of constant interest in the international dynamical systems and control community. One reason is that, for example, the LTV systems naturally arise when one linearizes nonlinear systems about a non-constant nominal trajectory. In contrast to the linear time-invariant (LTI) cases which have been thoroughly understood, many properties of the LTV systems are still not completely resolved in general. In this context the stability analysis is offered as a good example.

The stability characteristics of an LTI system of ordinary differential equations $\dot{x} = Ax$ can be characterized completely by the placement of the eigenvalues of the constant system matrix A in the complex plane. For LTV systems described by

$$(dx/dt \triangleq)\ \dot{x} = A(t)x, \quad t \geq t_0, \tag{1}$$

someone would intuitively expect that if, for each t, the "frozen-time" system is stable of any kind, then the time-varying system should also be stable provided $A(t)$ is bounded. However, these conditions are still not strong enough to guarantee the uniform asymptotic stability. A LTV system can be unstable even if all eigenvalues of its system matrix $A(t)$ are constant and have negative real parts, and the system can also be asymptotically stable even if all eigenvalues of $A(t)$ are constant and some have positive real parts [1,2]. Thus, additional restrictions suitably constraining the rate of variation in $A(t)$ have to be imposed. The best known results were given by C. A. Desoer [3], W. A. Coppel [4] and H. H. Rosenbrock [5] in their studies of slowly varying systems. The results are summarized and slightly strengthened in [6] (Theorem 3.2) (the notations used here will be listed and explained in the following subsection):

Theorem 1. *Suppose that $A(t)$ is (piecewise) continuous matrix function $A(\cdot) : [0, \infty) \to \mathbb{R}^{n\times n}$ which satisfies:*

(i) *there exists $M > 0$ such that the induced operator norm $N(A(t)) < M$ for all $t \geq 0$,*

(ii) *there exists* $\alpha > 0$ *such that the spectrum*

$$\sigma(A(t)) \subset \{z \in \mathbb{C}; \Re z < -\alpha\} \tag{2}$$

for all $t \geq 0$ *(i.e., the real parts of all eigenvalues of* $A(\cdot)$ *are negative and less than* $-\alpha$*).*

Then any of the following conditions guarantees uniform asymptotic stability of (1):

(C1) $\alpha > 4M$ *for all* $t \geq 0$*;*

(C2) $A(\cdot)$ *is piecewise differentiable and* $N(\dot{A}(t)) < \frac{2}{2n-1}\frac{\alpha^{4n-2}}{2M^{4n-4}}$ *for all* $t \geq 0$*;*

(C3) *For some* $k \geq 0$, $\eta \in (0,1)$, $\alpha > 2M\eta + \left(\frac{n-1}{k}\right)\log\eta$ *and*

$$\sup_{0 \leq \tau \leq k} N(A(t+\tau) - A(t)) < \eta^{n-1}(\alpha - 2M\eta + \frac{n-1}{k}\log\eta);$$

(C4) $\alpha > n-1$ *and for some* $\eta \in (0,1)$,

$$\sup_{h>0} N\left(\frac{A(t+h) - A(t)}{h}\right) < 2\eta^{n-1}(\alpha - 2M\eta + (n-1)\log\eta).$$

The symbol "log" in the previous theorem denotes the natural logarithm.

In the present paper, we will proceed in a different way. Despite the fact that eigenvalues of the "frozen-time" LTV systems cannot be used to determine the system's stability in general, as they do not share the same physical meaning as their LTI counterparts any longer, we will try to determine the widest possible class of the matrix functions $A(t)$, $t \geq t_0$, for which the condition (2) will be the sufficient condition for the uniform asymptotic stability of the LTV system (1) without further additional conditions and constrains. Not to mention that calculating norms in the criteria C2-C4 above is not an easy task.

We will gradually expand the classes of the systems (1) for which "LTI system's spectral criterion for asymptotic stability" is sufficient to be the LTV system (1) uniformly asymptotically stable. The findings are formulated as Observations 1–3, where

$$\text{Obs } 1 \subset \text{Obs } 2 \subset \text{Obs } 3\,.$$

This chain of inclusions is not terminated and can be continued.

First, we recall and define the concepts and summarize the results that we will need in our further analyses.

Notations, Assumptions and Preliminary Results

Given a norm $n(\cdot)$ on $\mathbb{C}^n$, let $N(\cdot)$ denote the norm on the vector space of all $n-$by$-n$ complex matrices, induced by $n(\cdot)$, that is, defined by

$$N(A) = \max\{n(Ax) : n(x) = 1\}.$$

Let the matrix function $A(\cdot) : [t_0, \infty) \to \mathbb{R}^{n \times n}$ in (1) is continuous.

The spectrum $\sigma(A)$ of a matrix A is the set of its eigenvalues $\lambda_i(A)$, $i = 1, \ldots, n$. The value $\lambda_{\max}(A)$ denotes the maximum eigenvalue from $\sigma(A)$ if $\sigma(A) \subset \mathbb{R}$.

The various types of stability of the LTV systems can be expressed in the terms of a fundamental matrix [7] (p. 54) [8], and for periodic LTV systems, see [9].

Lemma 1. *Let* $\Phi(t)$ *be a fundamental matrix for* $\dot{x} = A(t)x$, $t \geq t_0$*. Then the system* $\dot{x} = A(t)x$ *is*

(S) *stable if and only if there exists a positive constant* K *such that*

$$N(\Phi(t)) \leq K \text{ for all } t \geq t_0,$$

(US) *uniformly stable if and only if there exists a positive constant K such that*

$$N\Big(\Phi(t)\Phi^{-1}(\tau)\Big) \le K \text{ for all } t_0 \le \tau \le t < \infty,$$

(AS) *asymptotically stable if and only if*

$$N(\Phi(t)) \to 0 \text{ as } t \to \infty,$$

(UAS) *uniformly asymptotically stable (⇔ uniformly exponentially stable) if and only if there exist positive constants K, α such that*

$$N\Big(\Phi(t)\Phi^{-1}(\tau)\Big) \le Ke^{-\alpha(t-\tau)} \text{ for } t_0 \le \tau \le t < \infty.$$

The complexity of the LTV systems and nonavailability of explicit solutions was the primary motivation for the development of the qualitative theory of dynamical systems, which determine the properties of solutions without explicitly solving the equations. One such suitable tool is the concept of a "logarithmic norm", which allows the estimation of the state transition matrix and bounds on the solutions only on the basis of the entries of system matrix $A(t)$.

The logarithmic norm of a matrix function $A(t)$ we denote by $\mu[A(t)]$. The standard definition is

$$\mu[A(t)] = \lim_{h\to 0^+} \frac{N(I_n + hA(t)) - 1}{h},\ t \ge t_0, \tag{3}$$

where I_n denotes the identity matrix on $\mathbb{R}^n$.

Specifically, for the Euclidean vector norm $n_2(x) \triangleq \left(\sum\limits_{i=1}^{n} x_i^2\right)^{1/2}$ in $\mathbb{R}^n$, we have

$$N_2(A) \triangleq \max_{n_2(x)=1} n_2(Ax) = \sqrt{\lambda_{\max}(A^TA)} \text{ and } \mu_2[A] = \frac{1}{2}\lambda_{\max}\Big(A + A^T\Big), \tag{4}$$

see, e.g., [7,10–14]. Here and elsewhere in the paper, the superscript 'T' denotes transposition.

While the matrix norm $N(A)$ is always positive if $A \neq 0$, the logarithmic norm $\mu(A)$ may also take negative values, e.g., for the Euclidean vector norm $n_2(\cdot)$ and when A is negative definite because $\frac{1}{2}(A + A^T)$ is also negative definite, Ref. [15] (p. 215). Therefore, the logarithmic norm does not satisfy the axioms of a norm. On the other hand, the logarithmic norm is useful in the analysis of stability of the systems due to the following estimates:

(E1) Ref. [16] Let $\Phi(t)$ is a fundamental matrix for $\dot{x} = A(t)x$, $t \ge t_0$. Then

$$e^{-\int_\tau^t \mu[-A(s)]ds} \le N\Big(\Phi(t)\Phi^{-1}(\tau)\Big) \le e^{\int_\tau^t \mu[A(s)]ds}$$

for all $t_0 \le \tau \le t < \infty$;

(E2) Ref. [17] (p. 34) The solution $x(t)$ of $\dot{x} = A(t)x$ satisfies for all $t \ge t_0$ the inequalities

$$n(x(t_0))e^{-\int_{t_0}^t \mu[-A(s)]ds} \le n(x(t)) \le n(x(t_0))e^{\int_{t_0}^t \mu[A(s)]ds}.$$

These properties together with Lemma 1 immediately gives the following [7] (p. 59):

Lemma 2 (Stability criteria)**.** *The LTV system $\dot{x} = A(t)x$ is*

(S) *stable if*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \mu[A(s)]ds < \infty$$

(US) *uniformly stable if*

$$\mu[A(t)] \leq 0 \text{ for } t \geq t_0$$

(AS) *asymptotically stable if*

$$\lim_{t\to\infty} \int_{t_0}^{t} \mu[A(s)]ds = -\infty$$

(UAS) *uniformly asymptotically stable if*

$$\mu[A(t)] \leq -\alpha < 0 \text{ for } t \geq t_0$$

(U) *unstable if*

$$\liminf_{t\to\infty} \int_{t_0}^{t} \mu[-A(s)]ds = -\infty.$$

We focus here on the uniform asymptotic stability of the LTV systems, but analogous results we obtain for the weaker types of stability (S, US, AS) without any complications.

2. Results from the Matrix Theory

First, we introduce several known and also new results from the matrix theory that will be needed in our further considerations, see e.g., [18] for more details and also for the interesting theory behind this.

For a matrix $A \in \mathbb{R}^{n\times n}$, $S(A) = \frac{1}{2}(A + A^T)$ denotes the *symmetric part* of A and $\not{S}(A) = \frac{1}{2}(A - A^T)$ denotes the *skew-symmetric part* of A; notice that $A = S(A) + \not{S}(A)$.

In general, all eigenvalues of a symmetric matrix S (i.e., $S^T = S$) lie on the real axis of the complex plane, the eigenvalues of a skew-symmetric matrix $\not{S}$ (i.e., $\not{S}^T = -\not{S}$) lies on the imaginary axis.

For mechanical systems, the equations encountered are typically of the special form of the system (1), namely,

$$M(t)\ddot{q}(t) + G(t)\dot{q}(t) + K(t)q(t) = 0.$$

Here $M(= M^T > 0)$, G and K are $(n \times n)$ real matrices, where the symmetric and skew-symmetric parts of G correspond to damping and gyroscopic forces, and the symmetric and skew-symmetric parts of K correspond to stiffness and circulatory forces [19].

A square matrix $A \in \mathbb{C}^{n\times n}$ is *normal* if it commutes with its complex conjugate transpose; for the real matrices this reduces to $A^T A = AA^T$. All orthogonal (i.e., the columns and rows are orthogonal unit vectors), symmetric, and skew-symmetric matrices are normal. A square matrix R is a *rotation matrix* if and only if $R^T = R^{-1}$ ($\Rightarrow RR^T = R^T R = I_n$) and determinant equals 1 ($\det(R) = 1$). The set of all orthogonal matrices of size n with determinant $+1$ constitutes a group known as the *special orthogonal group* $SO(n)$ and forms group inside the set of normal matrices. Rotation matrices, describing rotations about the origin, provide an algebraic description of such rotations, and are used extensively for computations in mechanics, robotics [20], geometry, physics and computer graphics.

Let $\not{S}, D, N \in \mathbb{R}^{n\times n}$ is a skew-symmetric, diagonal ($d_{ij} = 0$ if $i \neq j$) and normal matrix, respectively. The following properties are proved in Appendix A:

Property 1. *$\not{S} + D$ is normal if and only if $\not{S}$ and D commute (for example, if $D = \beta I_n$, $\beta \in \mathbb{R}$).*

Property 2. $\mu_2[\mathcal{S} + D] = \mu_2[D]$

Property 3. *If $A \in \mathbb{R}^{n\times n}$, then*
$$\mu_2[A] = \lambda_{\max}(S(A))$$

Property 4. *If $A \in \mathbb{R}^{n\times n}$, then*
$$\mu_2[\mathcal{S}(A)] = 0$$

3. Results and Simulation Experiments

Theorem 2. *If $A \in \mathbb{R}^{n\times n}$ is symmetric, then $\mu_2[A] = \lambda_{\max}(A)$.*

Proof. The claim of theorem follows from Property 3 taking into account that $S(A) = A$, if A is symmetric. □

This theorem in combination with Lemma 2 gives the first result.

Observation 1. *If for every $t \geq t_0$ is $A(t)$ a symmetric matrix and*
$$\sigma(A(t)) \subset \{z \in \mathbb{C};\, \Re z \leq -\alpha\} \text{ (in fact, } \sigma(A(t) \subset \mathbb{R}) \tag{5}$$
where $\alpha > 0$ is a constant, then the LTV system $\dot{x} = A(t)x$, $t \geq t_0$ is uniformly asymptotically stable (UAS).

Example 1. *Now we give an example to show that the property (5) is not the necessary condition to be the system with symmetric $A(t)$ UAS. Consider the system (1) with*
$$A(t) = \begin{bmatrix} -\frac{11}{2} + \frac{15}{2}\sin 12t & \frac{15}{2}\cos 12t \\ \frac{15}{2}\cos 12t & -\frac{11}{2} - \frac{15}{2}\sin 12t \end{bmatrix}.$$

The eigenvalues of $A(t)$ are 2 and -13 for all t, but the system is UAS [1].
These eigenvalues are obtained by solving algebraic equation
$$\det(A(t) - \lambda I_2) = \det\left(\begin{bmatrix} -\frac{11}{2} + \frac{15}{2}\sin 12t - \lambda & \frac{15}{2}\cos 12t \\ \frac{15}{2}\cos 12t & -\frac{11}{2} - \frac{15}{2}\sin 12t - \lambda \end{bmatrix}\right)$$
$$= \left(\lambda + \frac{11}{2}\right)^2 - \left(\frac{15}{2}\right)^2 = 0,$$
which is independent of the variable t in this particular case.

With some more effort we can prove analogous result to Observation 1 for the class of normal matrices.

Theorem 3. *If $N \in \mathbb{R}^{n\times n}$ is normal, then*
$$\min_i\{\Re\lambda_i(N)\} = \frac{1}{2}\lambda_{\min}(N + N^T)$$
$$\max_i\{\Re\lambda_i(N)\} = \frac{1}{2}\lambda_{\max}(N + N^T).$$

Proof. For the reader's convenience, let us recall the key concepts and relations on which the proof of Theorem 3 is based; more information on this topic in matrix theory can be found, for example, in [18] (Chapter 1).

The *field of values* of the matrix $A \in \mathbb{C}^{n\times n}$ is
$$F(A) \triangleq \{x^*Ax :\, x \in \mathbb{C}^n, x^*x = 1\}.$$

The superscript $*$ stands for componentwise complex conjugate transpose (sometimes also called a Hermitian transpose).

Property-Projection: For all $A \in \mathbb{C}^{n\times n}$,

$$F\left(\frac{1}{2}(A + A^*)\right) = \Re F(A).$$

Property-Normality: If $A \in \mathbb{C}^{n\times n}$ is normal, then

$$F(A) = \mathrm{Co}(\sigma(A)),$$

where $\mathrm{Co}(\sigma(A))$ denotes the convex hull of $\sigma(A)$, i.e., the smallest closed convex set containing the spectrum of the matrix A.

Now we get the containment of the theorem by combining these two properties of the field of values taking into account the fact that $\min_i\{\Re\lambda_i(N)\}$ and $\max_i\{\Re\lambda_i(N)\}$ are endpoints of the projection of the convex hull (= the polygon whose vertices are the eigenvalues from $\sigma(N)$) of the spectrum of N onto the real axis. The analogous argument holds for $\frac{1}{2}(N + N^T)$, which is a symmetric matrix and, therefore, normal and so $F\left(\frac{1}{2}(N + N^T)\right)$ is a closed real line segment whose endpoints are the largest and smallest eigenvalues of $\frac{1}{2}(N + N^T)$, recall that all eigenvalues of a symmetric matrix are real numbers. □

As an academic example to illustrate Theorem 3, let us consider the normal matrix which is neither orthogonal, symmetric, nor skew-symmetric, namely,

$$N_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Direct calculation gives the spectra

$$\sigma(N_1) = \left\{\frac{1}{2}\left(1 \pm i\sqrt{3}\right), 2\right\}$$

$$\frac{1}{2}\sigma\left(N_1 + N_1^T\right) = \left\{\frac{1}{2}, \frac{1}{2}, 2\right\}$$

and so

$$\min_i\{\Re\lambda_i(N_1)\} = \frac{1}{2} = \frac{1}{2}\lambda_{\min}(N_1 + N_1^T)$$

$$\max_i\{\Re\lambda_i(N_1)\} = 2 = \frac{1}{2}\lambda_{\max}(N_1 + N_1^T).$$

Corollary 1. *If $A \in \mathbb{R}^{n\times n}$ is normal, then $\mu_2[A] = \max_i\{\Re\lambda_i(A)\}$.*

Proof. The containment follows immediately from Theorem 3 and the calculation rule for the logarithmic norm $\mu_2[\cdot]$, (4). □

Observation 2. *If for every $t \geq t_0$ is $A(t)$ a normal matrix and*

$$\sigma(A(t)) \subset \{z \in \mathbb{C};\ \Re z \leq -\alpha\},$$

where $\alpha > 0$ is a constant, then the LTV system $\dot{x} = A(t)x$, $t \geq t_0$ is UAS.

Example 2. *Let us consider the LTV system (1),*

$$\dot{x} = A(t)x,\ t \geq 0,\ x(0) \text{ given},$$

where

$$A(t) = \begin{bmatrix} -2 - t\cos^2 t & 1/2 & -1/2 \\ -1/2 & -2 - t\cos^2 t & 1/2 \\ 1/2 & -1/2 & -2 - t\cos^2 t \end{bmatrix}$$

with the spectrum

$$\sigma(A(t)) = \left\{ -2 - t\cos^2 t, -2 - t\cos^2 t \pm \frac{\sqrt{3}}{2} i \right\}, \; t \geq 0.$$

Because $A(t) = \mathcal{S}(N_1) + (-2 - t\cos^2 t)I_3$, *the matrix* $A(t)$*is normal for all* $t \geq 0$ *(Property 1) and the system is UAS by Observation 2.*

Or alternatively, by employing Property 2,

$$\mu_2[A(t)] = \mu_2[(-2 - t\cos^2 t)I_3] = -2 - t\cos^2 t \leq -2 < 0$$

for all $t \geq 0$. *Thus the estimate (E1) and Lemma 1 also imply uniform asymptotic stability of the system.*

Moreover, in this specific case when $A(t)$ *is of the form* $A(t) = \mathcal{S}(B(t)) + \beta(t)I_n$ *(here,* $B(t) = N_1$ *– in general,* $B(t)$ *could be an arbitrary continuous matrix function), Property 2 implies that* $\mu_2[A(t)] = \beta(t) = -\mu_2[-A(t)]$, *and, by the estimate (E2), the modulus of the solution* $x(t)$,

$$n_2(x(t)) = n_2(x(0))e^{\int_0^t \beta(s)ds} = n_2(x(0))e^{\int_0^t (-2 - s\cos^2 s)ds}.$$

The results of simulation in the MATLAB environment are shown in Figure 1.

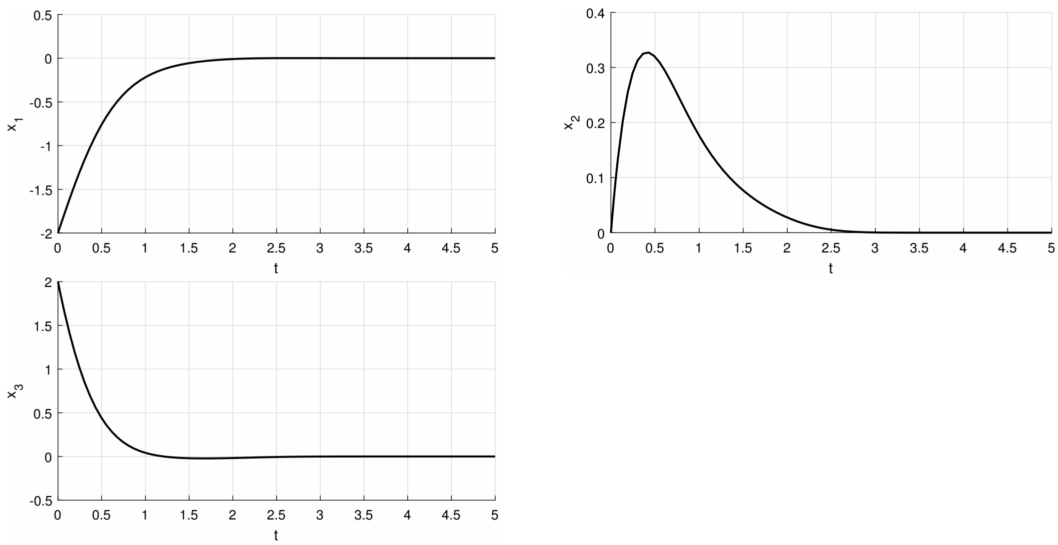

Figure 1. Simulation result for Example 2 with the initial state $x(0) = (-2\ 0\ 2)^T$. Flow of the state−variables x_1, x_2 and x_3.

Example 3. *For the LTV system (1) with*

$$A(t) = \begin{bmatrix} \beta(t) & \gamma(t) \\ -\gamma(t) & \beta(t) \end{bmatrix},$$

where $\beta(\cdot), \gamma(\cdot)$ *are continuous functions on* $[t_0, \infty)$; *the values of the matrix function* $A(t)$ *are in* $SO(2)$ *if* $\beta^2(t) + \gamma^2(t) = 1$.

Proceeding analogously to the previous example, we can observe that

$$n_2(x(t)) = n_2(x(t_0))e^{\int_{t_0}^{t} \beta(s)ds}.$$

For example, if $\beta(t) = \cos t$ and $\gamma(t) = -\sin t$, the system $\dot{x} = A(t)x$, $t \geq 0$ is by Lemma 1 and the estimate (E1) uniformly stable and

$$n_2(x(t)) = n_2(x(0))e^{\sin t}.$$

For illustration purpose, see Figure 2.

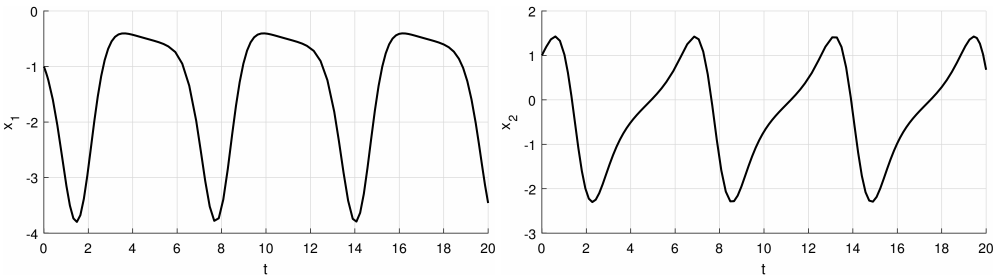

Figure 2. Simulation result for Example 3 with $\beta(t) = \cos t$ and $\gamma = -\sin t$ and the initial state $x(0) = (-1\ 1)^T$. Flow of the state−variables x_1 and x_2.

We will now present the most general statement of this paper, from which Observations 1 and 2 will emerge as special cases. First, let us define some useful concepts and relations between them that will allow us to easily formulate the main result of the paper.

The vector norm $n(\cdot)$ is called *monotonic* if $n(x) \leq n(y)$ for all $x, y \in \mathbb{C}^n$ such that $|x| \leq |y|$, see e.g., [21]. The notations $|\cdot|$ (absolute value), and $\leq$ are to be interpreted componentwise, when applied to vectors. For example, the Euclidean norm $n_2(\cdot)$ is monotone. Further, if the norm $n(\cdot)$ is monotonic, then

$$N(B) = \max_i |b_i| = \rho(B) \tag{6}$$

for all diagonal matrices $B = \mathrm{diag}(b_1, \ldots, b_n)$ [21].

Given $A \in \mathbb{C}^{n\times n}$ *the spectral abscissa* is defined as

$$\alpha(A) \triangleq \max_i \{\Re\lambda_i(A)\}$$

and *the spectral radius* is defined as

$$\rho(A) \triangleq \max_i |\lambda_i(A)|.$$

The *logarithmic inefficiency* [11,22] of a vector norm $n(\cdot)$ with respect to the matrix A is given by

$$q(A) = \mu[A] - \alpha(A)\ (\geq 0).$$

Combining

$$\alpha(A) = \lim_{h\to 0^+} \frac{\rho(I_n + hA) - 1}{h} \quad \text{and} \quad (3)$$

we find that

$$q(A) = \lim_{h\to 0^+} \frac{N(I_n + hA) - \rho(I_n + hA)}{h} \tag{7}$$

and we have the following theorem.

Theorem 4. *Let for every $t \geq t_0$, $A(t)$ is diagonalizable by a nonsingular real matrix $P(t)$ and define $n_{P(t)}(x) \triangleq n(P(t)x)$ for all $t \geq t_0$, where $n(\cdot)$ is a monotonic norm. Then*

$$\mu_{P(t)}[A(t)] = \alpha(A(t)) \text{ for all } t \geq t_0. \tag{8}$$

Proof. From $P(t)A(t)P^{-1}(t) = D(t)$ and using that $\mu_{P(t)}[A(t)] = \mu[D(t)]$ [16] we find that

$$q_{P(t)}(A(t)) = \mu_{P(t)}[A(t)] - \alpha(A(t)) = \mu[D(t)] - \alpha(A(t)).$$

Now, because the similar matrices have the same characteristic polynomial, $\alpha(A(t)) = \alpha(D(t))$, and

$$\mu[D(t)] - \alpha(A(t)) = \mu[D(t)] - \alpha(D(t)) = q(D(t)) \text{ for all } t \geq t_0.$$

The property (6) of a monotonic norm with $B = I_n + hD(t)$ and the equality (7) gives $q(D(t)) = 0$, and so, $q_{P(t)}(A(t)) = 0$ for all $t \geq t_0$, which yields (8). □

Since the estimates (E1) and (E2) hold in general only for the time-invariant vector norm in $\mathbb{R}^n$ (a notable exception are the UAS LTV systems—for details see [23] (Theorem 3, the inequality (7)) and [12]), we must impose an additional assumption:

$$n_{P(t)}(x) \triangleq n(P(t)x) = \hat{n}(\hat{P}x) \triangleq \hat{n}_{\hat{P}}(x), \tag{9}$$

where $\hat{P}$ is a constant nonsingular real matrix. It is worth noting that we do not have to require the norm $\hat{n}(\cdot)$ to be monotonic. The monotonic norms $n(\cdot)$ may also differ depending on t.

The following example documents that this condition may not be omitted.

Example 4. *Consider (1) in the vector space $\mathbb{R}^2$ endowed with the Euclidean norm and*

$$A(t) = \begin{bmatrix} -4 & 3e^{-8t} \\ -e^{8t} & 0 \end{bmatrix}.$$

The pointwise eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -1$ for all $t \geq t_0$, and so the matrix $A(t)$ is diagonalizable with

$$P(t) = \frac{3}{2}\begin{bmatrix} e^{8t} & -1 \\ -\frac{1}{3} & e^{-8t} \end{bmatrix}$$

and $\mu_{P(t)}[A(t)] = \alpha(A(t)) = -1$, by Theorem 4. But on the other hand, the fundamental matrix solution

$$\Phi(t) = \begin{bmatrix} -3e^{-5t} & -e^{-7t} \\ e^{3t} & e^{t} \end{bmatrix}$$

shows that in all balls around the origin $x = 0$ there is solution arbitrarily near the origin being expelled from a neighborhood of the origin, therefore, the system is unstable.

The condition (9) narrows the class of matrix functions to which we can apply Theorem 4 quite substantially, but as we will now see, the class of normal matrices from Observation 2 is one of them. Indeed, the normal matrices are unitarily diagonalizable, that is, $P^T(t)P(t) = I_n$ (for real matrices), for all $t \geq t_0$. Thus, we have for the Euclidean norm in $\mathbb{R}^n$ and for all $t \geq t_0$

$$n_{P(t)}(x) \triangleq n_2(P(t)x) = \sqrt{x^T P^T(t) P(t) x}$$

$$= \sqrt{x^T I_n x} = \sqrt{x^T I_n^T I_n x} = n_2(I_n x) = n_2(x), \text{ i.e., } \hat{P} = I_n \text{ and } \hat{n}(\cdot) = n_2(\cdot)$$

in (9). Theorem 4 then implies $\mu_2[A(t)] = \alpha(A(t))$, $t \geq t_0$, which is in compliance with Corollary 1. For the more general case within the Euclidean norm, when $P^T(t)P(t) = E$ (obviously, $E = E^T > 0$) from the special form of the eigendecomposition for symmetric positive definite matrices, it follows that

$$E = W^T DW = W^T \sqrt{D}^T \sqrt{D}W = \left(\sqrt{D}W\right)^T \left(\sqrt{D}W\right),$$

that is, $\hat{P} = \sqrt{D}W$ in (9) with $n(\cdot)$ and $\hat{n}(\cdot)$ equal to $n_2(\cdot)$.

Observation 3. *Let for every $t \geq t_0$ is $A(t)$ diagonalizable by a nonsingular real matrix $P(t)$ and let for all $t \geq t_0$ is $n_{P(t)}(x) \triangleq n(P(t)x) = \hat{n}(\hat{P}x) \triangleq \hat{n}_{\hat{P}}(x)$, where $n(\cdot)$ are the monotonic norms and $\hat{P}$ is a nonsingular, constant and real matrix.*

If

$$\sigma(A(t)) \subset \{z \in \mathbb{C};\, \Re z \leq -\alpha\},$$

where $\alpha > 0$ is a constant, then the LTV system $\dot{x} = A(t)x$, $t \geq t_0$ is UAS.

4. Conclusions

In this paper we have established the class of matrix functions $A(\cdot) : [t_0, \infty) \to \mathbb{R}^{n\times n}$ for which the condition

$$\sigma(A(t)) \subset \{z \in \mathbb{C};\, \Re z \leq -\alpha < 0\} \text{ for all } t \geq t_0$$

without any additional assumptions constraining the rate of variation in $A(t)$ ensures the uniform asymptotic stability of the LTV system $\dot{x} = A(t)x$, $t \geq t_0$. The main result is formulated in Observation 3. This class consists of the matrix functions with values in the set of normal matrices, the subgroup of which is also an important special orthogonal group $SO(n)$. Moreover, for the LTV systems with $A(t) \in SO(2)$ for all $t \geq t_0$, we have derived the formula for computing the exact value of vector norm of $x(t)$.

The challenge for further research is to find the classes of matrix functions $A(t)$, $t \geq t_0$ for which the assumption (9) applies, that is, for which there is a suitable combination of norms (monotonic norms $n(\cdot)$ and norm $\hat{n}(\cdot)$) and a nonsingular constant matrix $\hat{P}$, similarly to that for matrix functions with values in the set of normal matrices, analyzed just before Observation 3.

The results of this paper could also be an interesting topic for further research regarding control theory as a contribution to the stability theory of linear systems. For example, to design such an adaptive state feedback control law $u = -K(t)x$ for the control system $\dot{x} = A(t,\theta)x + Bu$ so that the closed-loop system matrix function $A(t,\theta) - BK(t)$, $t \geq t_0$ satisfies the conditions established in Observation 1, 2 or 3 stabilizing the control system; θ in the system matrix function represents the parametric uncertainty. For the last achievements and research directions on the field of stabilizability of the linear control systems, see, for example, Refs. [24–27] and the references therein.

Funding: This publication has been published with the support of the Ministry of Education, Science, Research and Sport of the Slovak Republic within project VEGA 1/0193/22 " Návrh identifikácie a systému monitorovania parametrov výrobných zariadení pre potreby prediktívnej údržby v súlade s konceptom Industry 4.0 s využitím technológií Industrial IoT ".

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: Author thanks the editors and the anonymous reviewers for their insightful comments which improved the quality of the paper.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A

For the readers' convenience, in this appendix we present the proofs of Properties 1–4.

Appendix A.1. Property 1

Proof. We need to show that

$$(\mathcal{S}+D)^T(\mathcal{S}+D) = (\mathcal{S}+D)(\mathcal{S}+D)^T.$$

Using that $\mathcal{S}^T = -\mathcal{S}$ and $D = D^T$ we obtain

$$\begin{aligned}
&(\mathcal{S}+D)(\mathcal{S}+D)^T - (\mathcal{S}+D)^T(\mathcal{S}+D) \\
&= \mathcal{S}\mathcal{S}^T + D\mathcal{S}^T + \mathcal{S}D^T + DD^T - \mathcal{S}^T\mathcal{S} - D^T\mathcal{S} - \mathcal{S}^T D - D^T D \\
&= D\mathcal{S}^T + \mathcal{S}D^T - D^T\mathcal{S} - \mathcal{S}^T D = D(\mathcal{S}^T - \mathcal{S}) + (\mathcal{S} - \mathcal{S}^T)D \\
&= -2(D\mathcal{S} - \mathcal{S}D) = 0 \in \mathbb{R}^{n\times n}.
\end{aligned}$$

□

Appendix A.2. Property 2

Proof. Observe that

$$\mu_2[\mathcal{S}+D] = \frac{1}{2}\lambda_{\max}(\underbrace{\mathcal{S}+\mathcal{S}^T}_{=0\in\mathbb{R}^{n\times n}} + \underbrace{D+D^T}_{2D}) = \lambda_{\max}(D) = \mu_2[D].$$

□

Appendix A.3. Property 3

Proof.

$$\mu_2[A] = \mu_2[S(A)+\mathcal{S}(A)] = \frac{1}{2}\lambda_{\max}\Big(S(A)+\mathcal{S}(A)+S^T(A)+\mathcal{S}^T(A)\Big)$$

Because $S^T(A) = S(A)$ and $\mathcal{S}^T(A) = -\mathcal{S}(A)$,

$$\mu_2[A] = \lambda_{\max}(S(A)).$$

□

Appendix A.4. Property 4

Proof. $\mathcal{S}(A)$ is skew-symmetric and so $\mathcal{S}(A)+\mathcal{S}^T(A) = 0$ (null matrix), or in other words, symmetric part of skew-symmetric matrix is the null matrix. □

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