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Coefficient Estimates and the Fekete–Szegö Problem for New Classes of *m*-Fold Symmetric Bi-Univalent Functions

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Abstract: The results presented in this paper deal with the classical but still prevalent problem of introducing new classes of m-fold symmetric bi-univalent functions and studying properties related to coefficient estimates. Quantum calculus aspects are also considered in this study in order to enhance its novelty and to obtain more interesting results. We present three new classes of bi-univalent functions, generalizing certain previously studied classes. The relation between the known results and the new ones presented here is highlighted. Estimates on the Taylor–Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ are obtained and, furthermore, the much investigated aspect of Fekete–Szegő functional is also considered for each of the new classes.

Keywords: m-fold symmetric; bi-univalent functions; analytic functions; Fekete–Szegö functional; coefficient bounds; coefficient estimates

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1. Introduction and Preliminary Results

The study of bi-univalent functions has its origins in a 1967 paper published by Lewin [1], where he introduced and first investigated the class of bi-univalent functions. It was then proved that $|a_2| < 1.51$, with the estimation being further investigated only a few years later [2,3]. The definition of this class involves the well-known class of functions A consisting of the functions having the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0, f'(0) = 1$. The subclass $S \subset A$ is formed of functions in class A which are univalent in U.

In [4], the Koebe One-Quarter Theorem stated guarantees that a disk of radius $1/4$ is contained in $f(U)$ for every univalent function f. Hence, every function $f \in S$ admits an inverse function f^{-1}, defined as follows:

$$f^{-1}(f(z)) = z, z \in U$$

and

$$f(f^{-1}(w)) = w, |w| < r_0(f), r_0(f) \geq 1/4,$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U.

Let Σ denote the class of all bi-univalent functions in U given by (1).

The interest in bi-univalent functions resurfaced in 2010 when a paper authored by H. M. Srivastava et al. in [5] was published. It opened the door for many interesting developments on the topic. Soon, other new subclasses of bi-univalent functions were introduced [6,7], and special classes of bi-univalent functions were investigated such as Ma-Minda starlike and convex functions [8], analytic bi-Bazilevič functions [9] and recently a family of bi-univalent functions associated with Bazilevič functions and the $\lambda-$ pseudo-starlike functions [10]. Brannan and Clunie's conjecture [3] was further investigated [11] and subordination properties were also obtained for certain subclasses of bi-univalent functions [12]. New results continued to emerge in the recent years, such as coefficient estimates for some general subclasses of analytic and bi-univalent functions [13–15]. Horadam polynomials were used for applications on Bazilevič bi-univalent functions satisfying subordination conditions [16] and for introducing certain classes of bi-univalent functions [17]. Operators were also included in the study as it can be seen in earlier publications [18] and in very recent ones [19]. In 2014, Srivastava et al. [20] defined m-fold symmetric bi-univalent functions following the concept of m-fold symmetric univalent functions. In this paper, some important results were proved, such as the fact that each bi-univalent function generates an m-fold symmetric bi univalent function for each $m \in \mathbb{N}$.

A domain D is said to be $m-$ fold symmetric if a rotation of D about the origin through an angle $2\pi/m$ carries D on itself.

A function f holomorphic in D is said to be $m-$ fold symmetric if $f(e^{\frac{2\pi i}{m}}z) = e^{\frac{2\pi i}{m}}f(z)$.

A function is said to be $m-$ fold symmetric if it has the following normalized form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1}, z \in U, m \in \mathbb{N} \cup \{0\}. \tag{3}$$

The normalized form of f is given as in (3) and the series expansion for $f^{-1}(z)$ is given below [20]:

$$\begin{aligned} g(w) = f^{-1}(w) = w - a_{m+1}w^{m+1} + \\ +[(m+1)a_{m-1}^2 - a_{2m+1}]w^{2m+1} - \\ -[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}]w^{3m+1} + \ldots \end{aligned} \tag{4}$$

Let ξ_m the class of $m-$fold symmetric univalent functions in U that are normalized by (3).

The functions in the class ξ are one-fold symmetric.

Examples of $m-$ fold symmetric bi-univalent functions are

$\{\frac{z^m}{1-z^m}\}^{\frac{1}{m}}; \quad [-log(1-z^m)]^{\frac{1}{m}}; \quad \frac{1}{2}log(\frac{1+z^m}{1-z^m})^{\frac{1}{m}}$.

Interesting results regarding $m-$fold symmetric bi-univalent functions were published in the same year when this notion was introduced [21]; this continued to appear in the following years [22–25] and is still researched today [26,27], proving that the topic remains in development.

The Fekete–Szegö problem is the problem of maximizing the absolute value of the functional $|a_3 - \mu a_2^2|$.

The Fekete–Szegö inequalities introduced in 1933, see [28], preoccupied researchers regarding different classes of univalent functions [29,30]; hence, it is obvious that such inequalities were obtained regarding bi-univalent functions too and very recently published papers can be cited to support the assertion that the topic still provides interesting results [31–33]. Inspiring new results emerged when quantum calculus was involved in the studies, as can be seen in many papers [34,35] and in studies published very recently [36–40]. Some elements of the (p; q)-calculus must be used for obtaining the original results contained in this paper. Further information can be found in [34,35].

Definition 1 ([34]). *Let $f \in \mathcal{A}$ given by (1) and $0 < q < p \leq 1$. Then, the (p, q)-derivative operator or p, q-difference operator for the function f of the form (1) is defined by*

$$D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p - q)z}, z \in U^* = U - \{0\} \tag{5}$$

and

$$(D_{p,q}f)(0) = f'(0) \tag{6}$$

provided that the function f is differentiable at 0.

From relation (2), we deduce that

$$D_{p,q}f(z) = 1 + \sum_{k=2}^{\infty} [k]_{p,q} a_k z^{k-1} \tag{7}$$

where the (p, q)-bracket number or twin-basic is given by

$$[k]_{p,q} = \frac{p^k - q^k}{p - q} = p^{k-1} + p^{k-2}q + p^{k-3}q^2 + \ldots + pq^{k-2} + q^{k-1}, p \neq q$$

which is a natural generalization of the q-number .

Additionally, $\lim_{p \to 1^-} [k]_{p,q} = [k]_q = \frac{1-q^k}{1-q}$.

Definition 2 ([41]). *Let $f \in \mathcal{A}, 0 \leq d < 1$ and $s \geq 1$ is real. Then, $f \in L_s(d)$ of s-pseudo-starlike function of order d in U if and only if*

$$Re(\frac{z[f'(z)]^s}{f(z)}) > d.$$

Lemma 1 ([4,42]). *Let the function $w \in \mathcal{P}$ be given by the following series $w(z) = 1 + w_1 z + w_2 z^2 + \ldots, z \in U$, where we denote by $\mathcal{P}$ the class of Carathéodory functions analytic in the open disk U,*

$$\mathcal{P} = \{w \in \mathcal{A} | w(0) = 1, Re(w(z)) > 0, z \in U\}.$$

The sharp estimate given by $|w_n| \leq 2, n \in \mathbb{N}^*$ holds true.

The tremendous impact quantum calculus has had when associated to univalent functions theory is nicely highlighted in the recent review paper [43].

In the next section of the paper, the original results obtained by the authors are presented in three definitions of new subclasses of bi-univalent functions and theorems concerning coefficient estimates and Fekete–Szegő functional for the newly defined classes defined by (p, q)-derivative operator given in relations (5)–(7). The connection with previously known results is revealed in some remarks following each result presented.

2. Main Results

Definition 3. *The class $\mathcal{N} - FS_{\Sigma,m}^{p,q,s,n}(d)(m \in \mathbb{N}, 0 < q < p \leq 1, s \geq 1, 0 < d \leq 1, (z, w) \in U, 0 \leq n \leq 1)$ contains all the functions f given by (3) if the following conditions are satisfied:*

$$\begin{cases} f \in \Sigma_m \\ |arg\{(1-n)(D_{p,q}f(z))^s + n[z(D_{p,q}f(z))' + D_{p,q}f(z)](D_{p,q}f(z))^{s-1}\}| < \frac{d\pi}{2}, (z \in U) \end{cases} \tag{8}$$

and

$$|arg\{(1-n)(D_{p,q}g(w))^s + n[w(D_{p,q}g(w))' + D_{p,q}g(w)](D_{p,q}g(w))^{s-1}\}| < \frac{d\pi}{2} \tag{9}$$

where the function g is given by (4).

Remark 1. *When $n = 0$ and $s = 1$, we obtain the class $H_{\Sigma}^{p,q,\alpha}$ introduced in [15].*

Remark 2. *In the case when $p = 1, n = 0, s = 1, m = 1 (onefold - case)$ we have $lim_{q\to 1^-}\mathcal{N} - FS_{\Sigma,1}^{1}(d) = FS_{\Sigma}(d)$ and we obtain the class which was introduced by Srivastava et al. in [5].*

The next theorem gives coefficient bounds for the functions belonging to the class $\mathcal{N} - FS_{\Sigma,m}^{p,q,s,n}(d)$.

Theorem 1. *Let f be a function in the class $\mathcal{N} - FS_{\Sigma,m}^{p,q,s,n}(d), (m \in \mathbb{N}, 0 < q < p \le 1, s \ge 1, 0 < d \le 1, (z,w) \in U, 0 \le n \le 1)$, which has the form (3). Then,*

$$|a_{m+1}| \le \tag{10}$$

$$\le \frac{2d}{\sqrt{(m+1)d[2m+1]_{p,q}(s+2mn) + (s+2mn)(s-1)[m+1]_{p,q}^2 d - (d-1)(s+mn)^2[m+1]_{p,q}^2}}$$

and

$$|a_{2m+1}| \le \frac{2d}{(s+2mn)[2m+1]_{p,q}} + \frac{2(m+1)d^2}{(s+mn)^2[m+1]_{p,q}^2}. \tag{11}$$

Proof. If we use relations (8) and (9), we obtain

$$(1-n)(D_{p,q}f(z))^s + n[z(D_{p,q}f(z))' + D_{p,q}f(z)](D_{p,q}f(z))^{s-1} = [\alpha(z)]^d, z \in U \tag{12}$$

and

$$(1-n)(D_{p,q}g(w))^s + n[w(D_{p,q}g(w))' + D_{p,q}g(w)](D_{p,q}g(w))^{s-1} = [\beta(w)]^d, w \in U \tag{13}$$

where $\alpha(z)$ and $\beta(w)$ in $\mathcal{P}$ are given by

$$\alpha(z) = 1 + \alpha_m z^m + \alpha_{2m} z^{2m} + \alpha_{3m} z^{3m} + \dots \tag{14}$$

and

$$\beta(w) = 1 + \beta_m w^m + \beta_{2m} w^{2m} + \beta_{3m} w^{3m} + \dots. \tag{15}$$

If we compare the coefficients in relations (12) and (13), we have

$$(s+mn)[m+1]_{p,q}a_{m+1} = d\alpha_m, \tag{16}$$

$$(s+2mn)[2m+1]_{p,q}a_{2m+1} + \frac{(s+2mn)(s-1)[m+1]_{p,q}^2}{2}a_{m+1}^2 =$$

$$= d\alpha_{2m} + \frac{d(d-1)}{2}\alpha_m^2, \tag{17}$$

$$-(s+mn)[m+1]_{p,q}a_{m+1} = d\beta_m, \tag{18}$$

$$(s+2mn)[2m+1]_{p,q}([m+1]a_{m+1}^2 - a_{2m+1}) + \frac{(s+2mn)(s-1)[m+1]_{p,q}^2}{2}a_{m+1}^2 =$$

$$= d\beta_{2m} + \frac{d(d-1)}{2}\beta_m^2. \tag{19}$$

From relations (16) and (18), we obtain

$$\alpha_m = -\beta_m \tag{20}$$

and
$$2(s+mn)^2[m+1]_{p,q}^2 a_{m+1}^2 = d^2(\alpha_m^2+\beta_m^2) \tag{21}$$

Now, from relations (17), (19) and (21), we obtain the next relation
$$(s+2mn)(s-1)[m+1]_{p,q}^2 a_{m+1}^2 + (m+1)(s+2mn)[2m+1]_{p,q} a_{m+1}^2$$
$$= d(\alpha_{2m}+\beta_{2m}) + \frac{d(d-1)}{2}\left[\frac{2(s+mn)^2[m+1]_{p,q}^2}{d^2}\right] a_{m+1}^2.$$

Therefore, we obtain
$$a_{m+1}^2 =$$
$$= \frac{d^2(\alpha_{2m}+\beta_{2m})}{(m+1)(s+2mn)[2m+1]_{p,q}d + (s+2mn)(s-1)[m+1]_{p,q}^2 d - (d-1)(s+mn)^2[m+1]_{p,q}^2}.$$

Now, for the coefficients α_{2m} and β_{2m}, if we apply Lemma 1, we obtain relation (10):
$$|a_{m+1}| \le$$
$$\le \frac{2d}{\sqrt{d(m+1)(s+2mn)[2m+1]_{p,q} + (s+2mn)(s-1)[m+1]_{p,q}^2 d - (d-1)(s+mn)^2[m+1]_{p,q}^2}}.$$

If we use relations (17) and (19), we obtain
$$2(s+2mn)[2m+1]_{p,q}(m+1)a_{2m+1} - (m+1)(s+2mn)[2m+1]_{p,q} a_{m+1}^2 =$$
$$= d(\alpha_{2m}-\beta_{2m}) + \frac{d(d-1)}{2}(\alpha_m^2-\beta_m^2). \tag{22}$$

From relations (20)–(22), we obtain
$$a_{2m+1} = \frac{d(\alpha_{2m}-\beta_{2m})}{2(s+2mn)[2m+1]_{p,q}} + \frac{d^2(\alpha_m^2+\beta_{2m}^2)(m+1)}{4(s+mn)^2[2m+1]_{p,q}^2}. \tag{23}$$

If we apply Lemma 1 for the coefficients $\alpha_m, \alpha_{2m}, \beta_m, \beta_{2m}$ and relation (23), we obtain relation (11):
$$|a_{2m+1}| \le \frac{2d}{(s+2mn)[2m+1]_{p,q}} + \frac{2(m+1)d^2}{(s+mn)^2[m+1]_{p,q}^2}.$$

□

Definition 4. *The class $\mathcal{N}-FS_{\Sigma,m}^{p,q,s,n}(e)$ $(0<q<p\le 1, 0\le e<1, m\in\mathbb{N}, s\ge 1, (z,w)\in U, 0\le n\le 1)$ contains all the functions f given by (3) if the following conditions are satisfied:*

$$\begin{cases} f\in\Sigma_m \\ Re\{(1-n)(D_{p,q}f(z))^s + n[z(D_{p,q}f(z))' + D_{p,q}f(z)](D_{p,q}f(z))^{s-1}\} > e, z\in U \end{cases} \tag{24}$$

$$Re\{(1-n)(D_{p,q}g(w))^s + n[w(D_{p,q}g(w))' + D_{p,q}g(w)](D_{p,q}g(w))^{s-1}\} > e, w\in U, \tag{25}$$

where the function g is defined by relation (4).

Remark 3. *(a). When $n=0$ and $s=1$, we obtain the class $H_\Sigma^{p,q,\beta}$, which was introduced in [15].*
(b). When $p=1, m=1(onefold-case), n=0, s=1$ and $lim_{q\to 1^-}\mathcal{N}-FS_{\Sigma,m}^{p,q,s,n}(e) = \mathcal{N}-FS_{\Sigma,m}^{s}(e)$, we obtain the class which was introduced by Srivastava et al. in [5].

The next theorem gives the coefficient bounds for the functions class $\mathcal{N}-FS_{\Sigma,m}^{p,q,s,n}(e)$.

Theorem 2. *Let f be a function in the class $\mathcal{N} - FS^{p,q,s,n}_{\Sigma,m}(e)$, $(m \in \mathbb{N}, 0 < q < p \leq 1, s \geq 1, 0 \leq e < 1, (z,w) \in U, 0 \leq n \leq 1)$, which has the form (3). Then,*

$$|a_{m+1}| \leq min\{\frac{2(1-e)}{(s+mn)^2[m+1]_{p,q}}, 2\sqrt{\frac{(1-e)}{(s+2mn)(s-1)[m+1]^2_{p,q} + (m+1)(s+2mn)[2m+1]_{p,q}}}\} \tag{26}$$

and

$$|a_{2m+1}| \leq \frac{2(1-e)}{(s+2mn)[2m+1]_{p,q}}. \tag{27}$$

Proof. We use relations (24) and (25) to obtain

$$(1-n)(D_{p,q}f(z))^s + n[z(D_{p,q}f(z))' + D_{p,q}f(z)](D_{p,q}f(z))^{s-1} = e + (1-e)\alpha(z), \quad z \in U \tag{28}$$

and

$$(1-n)(D_{p,q}g(w))^s + n[w(D_{p,q}g(w))' + D_{p,q}g(w)](D_{p,q}g(w))^{s-1} = e + (1-e)\beta(w), \quad w \in U \tag{29}$$

respectively, where $\alpha(z)$ and $\beta(w)$ in $\mathcal{P}$ are given by relations (14) and (15).

We compare the coefficients from (28) and (29) and we obtain the following relations:

$$(s+mn)[m+1]_{p,q}a_{m+1} = (1-e)\alpha_m, \tag{30}$$

$$(s+2mn)[2m+1]_{p,q}a_{2m+1} + \frac{(s+2mn)(s-1)[m+1]^2_{p,q}}{2}a^2_{m+1} = (1-e)\alpha_{2m}, \tag{31}$$

$$-(s+mn)[m+1]_{p,q}a_{m+1} = (1-e)\beta_m, \tag{32}$$

$$(s+2mn)[2m+1]_{p,q}[(m+1)a^2_{m+1} - a_{2m+1}] + \\ + \frac{(s+2mn)(s-1)[m+1]^2_{p,q}}{2}a^2_{m+1} = (1-e)\beta_{2m}. \tag{33}$$

Now, we obtain, from relations (32) and (30),

$$\alpha_m = -\beta_m \tag{34}$$

and

$$2(s+mn)^2[m+1]^2_{p,q}a^2_{m+1} = (1-e)^2(\alpha^2_m + \beta^2_m). \tag{35}$$

We obtain, from relations (33) and (31), the next relation

$$(s+2mn)[2m+1]_{p,q}(m+1)a^2_{m+1} + (s+2mn)(s-1)[m+1]^2_{p,q}a^2_{m+1} = \\ = (1-e)(\alpha_{2m} + \beta_{2m}). \tag{36}$$

We apply Lemma 1 for the coefficients $\alpha_m, \alpha_{2m}, \beta_m, \beta_{2m}$ and obtain

$$|a_{m+1}| \leq 2\sqrt{\frac{1-e}{[2m+1]_{p,q}(m+1)(s+2mn) + (s-1)(s+2mn)[m+1]^2_{p,q}}}$$

and then relation (26) hold. We use relations (33) and (31) to find the bound on $|a_{2m+1}|$, we obtain

$$-(m+1)(s+2mn)[2m+1]_{p,q}a^2_{m+1} + 2(s+2mn)[2m+1]_{p,q}(m+1)a_{2m+1} = (1-e)(\alpha_{2m} - \beta_{2m}). \tag{37}$$

From relation (37), we obtain

$$a_{2m+1} = \frac{(1-e)(\alpha_{2m} - \beta_{2m})}{2(s+2mn)[2m+1]_{p,q}} + \frac{(m+1)}{2}a^2_{m+1}. \tag{38}$$

From relation (35), if we substitute the value of a_{m+1}^2, we obtain

$$a_{2m+1} = \frac{(1-e)(\alpha_{2m} - \beta_{2m})}{2(s+2mn)[2m+1]_{p,q}} + \frac{(m+1)(1-e)^2(\alpha_m^2 + \beta_m^2)}{4(s+mn)^2[m+1]_{p,q}^2}. \tag{39}$$

Now, we apply Lemma 1 for the coefficients $\alpha_m, \alpha_{2m}, \beta_m, \beta_{2m}$ and relation (39), and we obtain

$$|a_{2m+1}| \le \frac{2(1-e)}{(s+2mn)[2m+1]_{p,q}} + \frac{2(m+1)(1-e)^2}{(s+mn)^2[m+1]_{p,q}^2}.$$

From relations (36) and (38), if we apply Lemma 1, we find that relation (27) holds:

$$|a_{2m+1}| \le \frac{2(1-e)}{(s+2mn)[2m+1]_{p,q}}.$$

□

In the next theorem, we compute the Fekete–Szegö functional for the class $\mathcal{N} - FS_{\Sigma,m}^{p,q,s,n}(d)$.

Theorem 3. *Let f be a function of the form (3) be in the class $\mathcal{N} - FS_{\Sigma,m}^{p,q,s,n}(d)$. Then,*

$$|a_{2m+1} - \rho a_{m+1}^2| \le \begin{cases} \frac{2d}{(s+2mn)[2m+1]_{p,q}}, |r(\rho)| \le \frac{1}{(s+2mn)[2m+1]_{p,q}} \\ 4d(s+2mn)[2m+1]_{p,q}^2|r(\rho)|, |r(\rho)| \ge \frac{1}{(s+2mn)[2m+1]_{p,q}}, \end{cases} \tag{40}$$

where we denote by

$$r(\rho) = \frac{d\{(m+1)[m+1]_{p,q}^2 - 2\rho[2m+1]_{p,q}^2\}}{2[2m+1]_{p,q}^2(s+mn)\{(m+1)d[2m+1]_{p,q} + [m+1]_{p,q}^2(d(s-2)+1)\}}.$$

Proof. From the proof of Theorem 1, we know the values of the coefficients a_{m+1}^2 and a_{2m+1} :

$$a_{2m+1} = \frac{d(\alpha_{2m} - \beta_{2m})}{2(s+2mn)[2m+1]_{p,q}} + \frac{(m+1)d^2(\alpha_m^2 + \beta_m^2)}{4(s+mn)^2[2m+1]_{p,q}^2}$$

$$a_{m+1}^2 = \frac{d^2(\alpha_{2m} + \beta_{2m})}{(m+1)d(s+2mn)[2m+1]_{p,q} + d(s+2mn)(s-1)[m+1]_{p,q}^2 - (d-1)(s+mn)^2[m+1]_{p,q}^2}.$$

We will start to compute $a_{2m+1} - \rho a_{m+1}^2$.
It follows that

$$a_{2m+1} - \rho a_{m+1}^2 =$$

$$= d\{\alpha_{2m}[\frac{1}{2(s+2mn)[2m+1]_{p,q}} +$$

$$+ \frac{d\{(m+1)[m+1]_{p,q}^2 - 2\rho[2m+1]_{p,q}^2\}}{2[2m+1]_{p,q}^2(s+2mn)\{(m+1)d[2m+1]_{p,q} + [m+1]_{p,q}^2(d(s-2)+1)\}}]$$

$$+\beta_{2m}[\frac{d((m+1)[m+1]_{p,q}^2 - 2\rho[2m+1]_{p,q}^2)}{2[2m+1]_{p,q}^2(s+2mn)\{(m+1)d[2m+1]_{p,q} + [m+1]_{p,q}^2(d(s-2)+1)\}} -$$

$$- \frac{1}{2(s+2mn)[2m+1]_{p,q}}]\}$$

According to Lemma 1 and after some computations, we obtain relation (40):

$$|a_{2m+1} - \rho a_{m+1}^2| \leq \begin{cases} \frac{2d}{(s+2mn)[2m+1]_{p,q}}, |r(\rho)| \leq \frac{1}{(s+2mn)[2m+1]_{p,q}} \\ 4d(s+2mn)[2m+1]_{p,q}^2 |r(\rho)|, |r(\rho)| \geq \frac{1}{(s+2mn)[2m+1]_{p,q}} \end{cases}.$$

□

In the next theorem, we compute the Fekete–Szegö functional for the class $\mathcal{N} - FS_{\Sigma,m}^{p,q,s,n}(e)$.

Theorem 4. *Let f be a function in the class $\mathcal{N} - FS_{\Sigma,m}^{p,q,s,n}(e)$ which has the form (3). Then*

$$|a_{2m+1} - \rho a_{m+1}^2| \leq \begin{cases} \frac{2(1-e)}{(s+2mn)[2m+1]_{p,q}}, |r(\rho)| \leq \frac{1}{2(s+2mn)[2m+1]_{p,q}} \\ 4(s+2mn)(1-e)[2m+1]_{p,q}^2 |r(\rho)|, |r(\rho)| \geq \frac{1}{2(s+2mn)[2m+1]_{p,q}} \end{cases}, \tag{41}$$

where we denote by

$$r(\rho) = \frac{m+1-2\rho}{2(s+2mn)[(2m+1)_{p,q}(m+1) + (s-1)[m+1]_{p,q}^2]}.$$

Proof. We will compute $a_{2m+1} - \rho a_{m+1}^2$, using the values of the coefficients a_{m+1}^2 and a_{2m+1} from the proof of Theorem 2:

$$a_{2m+1} = \frac{(1-e)(\alpha_{2m} - \beta_{2m})}{2[1+2m]_{p,q}(s+2mn)} +$$

$$+ \frac{(m+1)(1-e)(\alpha_{2m} + \beta_{2m})}{2(s+2mn)\{[2m+1]_{p,q}(m+1) + (s-1)[m+1]_{p,q}^2\}},$$

$$a_{m+1}^2 = \frac{(1-e)(\alpha_{2m} + \beta_{2m})}{(s+2mn)\{[2m+1]_{p,q}(m+1) + (s-1)[m+1]_{p,q}^2\}}.$$

It follows that

$$a_{2m+1} - \rho a_{m+1}^2 =$$

$$= (1-e)\{\alpha_{2m}[\frac{1}{2[1+2m]_{p,q}(s+2mn)} +$$

$$+ \frac{m+1-2\rho}{2(s+2mn)[(m+1)[2m+1]_{p,q} + (s-1)[m+1]_{p,q}^2]}] +$$

$$+ \beta_{2m}[\frac{m+1-2\rho}{2(s+2mn)[[2m+1]_{p,q}(m+1) + (s-1)[m+1]_{p,q}^2]} -$$

$$- \frac{1}{2[1+2m]_{p,q}(s+2mn)}]\}.$$

According to Lemma 1 and after some computations, we obtain relation (41):

$$|a_{2m+1} - \rho a_{m+1}^2| \leq \begin{cases} \frac{2(1-e)}{(s+2mn)[2m+1]_{p,q}}, |r(\rho)| \leq \frac{1}{2(s+2mn)[2m+1]_{p,q}} \\ 4(s+2mn)(1-e)[2m+1]_{p,q}^2 |r(\rho)|, |r(\rho)| \geq \frac{1}{2(s+2mn)[2m+1]_{p,q}} \end{cases}.$$

□

Definition 5. *Let $l, t : U \to \mathbb{C}$ be analytic functions with the properties* $\min\{Re(l(z)), Re(t(z))\} > 0$, *where* $z \in U, l(0) = t(0) = 1$.

The class $\mathcal{N}-FS^{l,t}_{\Sigma,m}$ contains all the functions f given by (3) *if the following conditions are satisfied:*

$$\left((1-n)(D_{p,q}f(z))^s+n[z(D_{p,q}f(z))'+D_{p,q}f(z)](D_{p,q}f(z))^{s-1}\right)\in l(U), z\in U \tag{42}$$

and

$$\left((1-n)(D_{p,q}g(w))^s+n[w(D_{p,q}g(w))'+D_{p,q}g(w)](D_{p,q}g(w))^{s-1}\right)\in t(U), w\in U, \tag{43}$$

where the function g is given by (4).

In the next theorem, we obtain coefficient bounds for the functions class $\mathcal{N}-FS^{l,t}_{\Sigma,m}$.

Theorem 5. *Given a function f in the class $\mathcal{N}-FS^{l,t}_{\Sigma,m}$, we have*

$$|a_{m+1}|\le \min\{\sqrt{\frac{|l_1'(0)|^2+|t_1'(0)|^2}{2(s+mn)^2[m+1]^2_{p,q}}},\sqrt{\frac{|l_2''(0)|+|t_2''(0)|}{(s+2mn)(s-1)[m+1]^2_{p,q}+(s+2mn)(m+1)[2m+1]_{p,q}}}\}; \tag{44}$$

$$|a_{2m+1}|\le \min\{\frac{(|l'(0)|^2+|t'(0)|^2)[(m+1)[2m+1]_{p,q}-(s-1)[m+1]^2_{p,q}]}{8(s+mn)^2[2m+1]_{p,q}[m+1]^2_{p,q}}+\frac{|l''(0)|^2+|t''(0)|^2}{2(s+2mn)[2m+1]_{p,q}},$$
$$\frac{|l''(0)|+|t''(0)|}{2(s+2mn)[2m+1]_{p,q}}+\frac{[(m+1)[2m+1]_{p,q}+(1-s)[m+1]^2_{p,q}](|l''(0)|+|t''(0)|)}{4(s+2mn)[2m+1]_{p,q}[(s-1)[m+1]^2_{p,q}+(m+1)[2m+1]_{p,q}]}\}. \tag{45}$$

Proof. Using relations (42) and (43), we obtain the following relations:

$$(1-n)(D_{p,q}f(z))^s+n[z(D_{p,q}f(z))'+D_{p,q}f(z)](D_{p,q}f(z))^{s-1}=l(z), \tag{46}$$

and

$$(1-n)(D_{p,q}g(w))^s+n[w(D_{p,q}g(w))'+D_{p,q}g(w)](D_{p,q}g(w))^{s-1}=t(w), \tag{47}$$

where functions $l(z)$ and $t(w)$ having the following forms and satisfy the conditions from Definition 5:

$$l(z)=1+l_1z+l_2z^2+\ldots \tag{48}$$

$$t(w)=1+t_1w+t_2w^2+\ldots \tag{49}$$

Substituting relations (48) and (49) into (46) and (47), respectively, and equating the coefficients, we obtain

$$(s+mn)[m+1]_{p,q}a_{m+1}=l_1; \tag{50}$$

$$(s+2mn)[2m+1]_{p,q}a_{2m+1}+\frac{(s+2mn)(s-1)[m+1]^2_{p,q}}{2}a^2_{m+1}=l_2; \tag{51}$$

$$-(s+mn)[m+1]_{p,q}a_{m+1}=t_1; \tag{52}$$

$$(s+2mn)[2m+1]_{p,q}((m+1)a^2_{m+1}-a_{2m+1})+\frac{(s+2mn)(s-1)[m+1]^2_{p,q}}{2}a^2_{m+1}=t_2. \tag{53}$$

We obtain

$$l_1=-t_1 \tag{54}$$

and

$$l_1^2+t_1^2=2(s+mn)^2[m+1]^2_{p,q}a^2_{m+1} \tag{55}$$

from relations (50) and (52).

Adding relations (51) and (53), we obtain

$$a^2_{m+1}\{(s+2mn)(s-1)[m+1]^2_{p,q}+(s+2mn)(m+1)[2m+1]_{p,q}\}=l_2+t_2. \tag{56}$$

Now, from (55) and (56), we obtain

$$a_{m+1}^2 = \frac{l_1^2 + t_1^2}{2(s+mn)^2[m+1]_{p,q}^2} \tag{57}$$

and

$$a_{m+1}^2 = \frac{l_2 + t_2}{(s+2mn)(s-1)[m+1]_{p,q}^2 + (s+2mn)(m+1)[2m+1]_{p,q}}. \tag{58}$$

We find, from relations (57) and (58), that

$$|a_{m+1}|^2 \le \frac{|l_1'(0)|^2 + |t_1'(0)|^2}{2(s+mn)^2[m+1]_{p,q}^2}$$

and

$$|a_{m+1}|^2 \le \frac{|l_2''(0)| + |t_2''(0)|}{(s+2mn)(s-1)[m+1]_{p,q}^2 + (s+2mn)(m+1)[2m+1]_{p,q}}.$$

Hence, the coefficient $|a_{m+1}|$ has the form given in (44).
Next, by substracting relation (53) from (51), we obtain

$$\begin{aligned} &2(s+2mn)[2m+1]_{p,q}a_{2m+1}+ \\ &+\frac{(s+2mn)(s-1)[m+1]_{p,q}^2 - (s+2mn)[2m+1]_{p,q}(m+1)}{2}a_{m+1}^2 = \\ &= l_2 - t_2. \end{aligned} \tag{59}$$

Substituting the value of a_{m+1}^2 from (57) into (59), it follows that

$$\begin{aligned} &a_{2m+1} = \\ &\frac{l_2 - t_2}{2(s+2mn)[2m+1]_{p,q}} + \frac{[(m+1)[2m+1]_{p,q} - (s-1)[m+1]_{p,q}^2](l_1^2 + t_1^2)}{8(s+mn)^2[m+1]_{p,q}^2[2m+1]_{p,q}}. \end{aligned}$$

So,

$$|a_{2m+1}| \le \frac{(|l'(0)|^2 + |t'(0)|^2)([2m+1]_{p,q}(m+1) - (s-1)[m+1]_{p,q}^2)}{8(s+mn)^2[m+1]_{p,q}^2[2m+1]_{p,q}} + \frac{|l''(0)|^2 + |t''(0)|^2}{2(s+2mn)[2m+1]_{p,q}}.$$

Using in relation (59), a_{m+1}^2 given by (58), we have

$$\begin{aligned} &a_{2m+1} = \\ &\frac{l_2 - t_2}{2(s+2mn)[2m+1]_{p,q}} + \frac{(l_2 + t_2)([2m+1]_{p,q}(m+1) + (1-s)[m+1]_{p,q}^2)}{4(s+2mn)[2m+1]_{p,q}\{(s-1)[m+1]_{p,q}^2 + [2m+1]_{p,q}[m+1]\}}. \end{aligned}$$

It follows that

$$\begin{aligned} &|a_{2m+1}| \le \\ &\frac{|l''(0)| + |t''(0)|}{2(s+2mn)[2m+1]_{p,q}} + \frac{[(m+1)[2m+1]_{p,q} + (1-s)[m+1]_{p,q}^2](|l''(0)| + |t''(0)|)}{4(s+2mn)[2m+1]_{p,q}\{(s-1)[m+1]_{p,q}^2 + (m+1)[2m+1]_{p,q}\}}. \end{aligned}$$

□

3. Conclusions

Following the line of research initiated by Srivastava et al. [20], three new classes of m-fold bi-univalent functions are introduced in Definitions 3–5. The classes introduced here have previously introduced and studied classes of bi-univalent functions as special cases. For these new classes, coefficient estimates are given regarding the Taylor–Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ and the Fekete–Szegő problem is investigated for each class in the theorems from the Main Results section. The bounds of coefficient estimates obtained here are not sharp, and thus further investigation is required in order to improve these estimates.

The results are particularly interesting as a result of adding quantum calculus aspects in the research, an approach often seen in recent published and cited studies.

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