

Odd-Graceful Total Colorings for Constructing Graphic Lattice

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Abstract: The security of passwords generated by the graphic lattices is based on the difficulty of the graph isomorphism, graceful tree conjecture, and total coloring conjecture. A graphic lattice is generated by a graphic base and graphical operations, where a graphic base is a group of disjointed, connected graphs holding linearly independent properties. We study the existence of graphic bases with odd-graceful total colorings and show graphic lattices by vertex-overlapping and edge-joining operations; we prove that these graphic lattices are closed to the odd-graceful total coloring.

Keywords: lattice-based cryptology; graphic lattice; total coloring; graph labeling; topological authentication

1. Introduction and Preliminary

1.1. Introduction

Considering current text passwords, once the password file is leaked, most of the plaintext passwords can be quickly deciphered using current cracking techniques, and personal privacy and property will face serious losses. Topological authentication is a new technique based on topological coding, a mixed branch of discrete mathematics, number theory, algebraic group, graph theory, and so on. This technology is more and more widely used in real life; for example, the user identity authentication of the smart-phone adopts a graphic password [1,2]. Johnson et al. proposed a picture signature password to determine access to computing devices or services [3]. Wang et al. [4] designed a topological code consisting of a topological structure and graph colorings. According to the order of vertex-edge-vertex, Yao et al. [5] arranged the labels or colors of each edge and two end-vertices of a graph G , with p vertices, and q edges in a topological matrix of order $3 \times q$; that is

$$T_{code} = \begin{pmatrix} f(u_1) & f(u_2) & \cdots & f(u_q) \\ f(e_1) & f(e_2) & \cdots & f(e_q) \\ f(v_1) & f(v_2) & \cdots & f(v_q) \end{pmatrix}_{3 \times q} \quad (1)$$

where f represents a function to obtain the colors of the vertices and edges. The topological matrix of G can derive $(3q)!$ number strings that can be used as digital-based passwords, and the preset coloring f can be needed during authentication, called topological authentication. It is feasible to generate passwords by topological coding, because its deciphering process involves NP-hard problems, for instance, the graph isomorphism problem and the graceful tree conjecture. Tian et al. [6] proved that the passwords generated by the labeling of a graph can resist brute force attacks, dictionary attacks, guessing attacks, reproduction attack, and denial of service attacks.



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The lattice-based cryptosystem is based on NP-Hard or NP-C problems [7,8], which makes lattice-based cryptosystem a core research field in cryptosystems against quantum attacks. A lattice $L(\mathbf{B})$ is defined as the set of all integer combinations

$$L(\mathbf{B}) = \left\{ \sum_{i=1}^n x_i \mathbf{b}_i : x_i \in Z, 1 \leq i \leq n \right\} \tag{2}$$

of n linearly independent vectors $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$ in R^m with $n \leq m$, where Z is the integer set, m is the dimension, and n is the rank of the lattice, and the vector group $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$ is called a lattice base. Lattice theory involves the Shortest Vector Problem [9], Shortest Independent Vector Problem, Successive Minimum Problem [10], Closet Vector Problem, and other difficult problems. The security of the lattice-based cryptology depends on the difficulty of the difficult problems in classical number theory and computational complexity theory [11], many difficult problems in the lattice have been proven to be NP-hard, so this mechanism is generally considered to be resistant to the attacks from quantum computers when it is used in cryptanalysis and design.

Inspired by the ability of lattice-based cryptology to resist supercomputers and quantum computers, we propose graphic lattices for the encryption of network models or complex systems. According to the method of topological coding, large-scale graphs are more reliable than small-scale graphs. Therefore, it is necessary to generate large-scale graphs from existing small-scale graphs for identification. The graphic lattice combines operations and topological coding of the graphic base to generate the topological coding of a large-scale graph [12,13]. In this paper, we extend the theoretical tool of topological coding from graph labeling to graph coloring and define a new type of coloring for building graphic lattices and providing topological coding based on new coloring.

1.2. Preliminary

The graphs described in this paper are graphs that do not contain double edges. The labeling of a graph G refers to $f : V(G) \rightarrow [a, b]$, there is always $f(x) \neq f(y)$ for any pair of vertices $x, y \in V(G)$. The coloring of the graph G means that a function $g : V(G) \rightarrow [a, b]$, such that $g(u) = g(v)$ for some two vertices $u, v \in V(G)$ [14]. The degree of a vertex u of a (p, q) -graph G is denoted as $|N(u)|$, where $N(u)$ represents the neighbor set of the vertex u , and a vertex having degree one is called a leaf. For simplicity of description, we write $f(P) = \{f(w) : w \in P\}$, where $f : P \subset V(G) \cup E(G) \rightarrow [1, M]$, $M > 1$ is a positive integer.

A graph G admits a proper total coloring $f : V(G) \cup E(G) \rightarrow [1, M]$ if it satisfies the following conditions [15,16]. For $u, v, w \in V(G)$, then

- (1) $f(u) \neq f(v)$ for each edge $uv \in E(G)$;
- (2) $f(uv) \neq f(uw)$ for $v, w \in N(u)$;
- (3) $f(uv) \neq f(u)$ for $v \in N(u)$.

The symbols used in this paper are shown in Table 1, and the other notations and terminologies not mentioned here can be found in [15,17].

Table 1. Abbreviations.

Symbol	Meaning
(p, q) -graph	A undirected graph with p vertices and q edges
$[a, b]$	A set $\{x \in Z : a \leq x \leq b\}$ with $a, b \in N, a < b$
$[m, n]^o$	An odd integer set $\{m, m + 2, \dots, n\}$ for two odd integers $1 \leq m < n$
$ X $	The number of elements of a set X
$ N(u) $	The degree of a vertex u
$f(P)$	The label/color set $f(P) = \{f(w) : w \in P\}$, $P \subset V(G) \cup E(G)$ for the labeling or coloring of a graph G

Definition 1. A (p, q) -graph G admits a total coloring $f : V(G) \cup E(G) \rightarrow [1, M]$, if it satisfies (1) Each edge $uv \in E(G)$ is colored as $f(uv) = |f(u) - f(v)|$;

- (2) The edge color set is $f(E(G)) = [1, 2q - 1]^0$;
- (3) The vertex color set is $f(V(G)) \subseteq [1, 2q]$.

Then f is called a quasi-odd-graceful total coloring if (1) and (2) hold, and f is called an odd-graceful total coloring if the above three conditions are met. Furthermore, if the vertex set of the (p, q) -graph G can be divided into $V(G) = X \cup Y$ with $X \cap Y = \emptyset$, so that each edge $uv \in E(G)$ satisfies $u \in X$ and $v \in Y$, besides, $\max f(X) < \min f(Y)$ is true, then we call f a set-ordered odd-graceful total coloring.

Several examples are shown in Figure 1 for understanding Definition 1. Similar to the definition of a lattice, we give the definition of a graphic lattice.

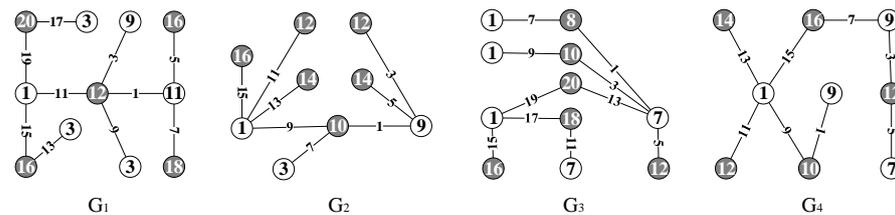


Figure 1. Four graphs G_1, G_2, G_3, G_4 admit set-ordered odd-graceful total colorings.

Definition 2. Let $G = (G_1, G_2, \dots, G_n) = (G_i)_1^n$ be a graphic base composed of n linearly independent graphic vectors under a graphical operation “ \diamond ”, where each G_i is a colored/uncolored graph, and $F_{p,q}$ is a set of colored/uncolored graphs of λ vertices and μ edges with respect to $\lambda \leq p$, $\mu \leq q$ and $2n - 2 \leq p$. We write the resulting graph obtained by performing a graphical operation \diamond on a graph H and the graphic base G with $a_i \in Z^0$, denoted as $G \diamond H = H \diamond_{i=1}^n a_i G_i$. In general, the graph set

$$L(G \diamond F_{p,q}) = \{H \diamond_{i=1}^n a_i G_i : a_i \in Z^0, H \in F_{p,q}\} \tag{3}$$

with $\sum_{i=1}^n a_i \geq 1$ is called a graphic lattice, G a graphic lattice base, p is the dimension and n is the rank of $L(G \diamond F_{p,q})$. Moreover, $L(G \diamond F_{p,q})$ is called a linear graphic lattice if every $H \in F_{p,q}$, each G_i of base G and $H \diamond_{i=1}^n a_i G_i$ are forests or trees. A colored tree-graph lattice is full-rank $p = n$ in Equation (3).

Based on the following two graphical operations, some large-scale graphs containing odd-graceful total colorings can be obtained, which increases the cryptographic space.

Definition 3. For two graphs, M_1 and M_2 , the two types of graphical operations related to the vertices and edges are as follows.

(1) Vertex-overlapping operation refers to overlapping a vertex u in graph M_1 with a vertex v in graph M_2 to form a new vertex w , then the neighbors of w are the union of the neighbors of u and the neighbors of v ; the operation is denoted as $u \odot v$.

(2) Edge-joining operation means adding a new edge between a vertex u in the graph M_1 and a vertex v in the graph M_2 , the operation is expressed as $u \oplus v$.

2. Graphic Lattices with Odd-Graceful Total Colorings

Lemma 1. Each tree admits a quasi-odd-graceful total coloring.

Proof. Let $|E(T)| = q$ be the number of edges of a tree T , we proceed with the proof by induction on the number of edges of tree T . When $q = 1$, T has one edge $e = uv$, there is a quasi-odd graceful total coloring f' defined as: $f'(uv) = 1$, $f'(u) = 2$ and $f'(v) = 3$. Suppose that this lemma is true for the tree with $q - 1$ edges, then, when $|E(T)| = q$, we suppose that the vertex u is a leaf of tree T and its neighbor is vertex v . Because the tree $T - u$ admits a quasi-odd-graceful total coloring f' so that the edge color set of tree $T - u$ is $f'(E(T - u)) = [1, 2(q - 1) - 1]^0$, based on this fact, we define a coloring f of tree T as follows: $f(w) = f'(w)$ for $w \in V(T) \cup E(T) \setminus \{u, uv\}$, $f(u) = f'(v) + 2q - 1$ and

$f(uv) = 2q - 1$, obviously, $f(u) - f(v) = f(uv)$. Therefore, the edge color set of tree T is just $[1, 2q - 1]^0$, which means that f is a quasi-odd-graceful total coloring of tree T . The lemma follows induction. \square

2.1. Results Based on the Vertex-Overlapping Operation

Given a graphic base $\mathbf{G} = (G_i)_1^n$ with disjoint graphs G_1, G_2, \dots, G_n , and a connected graph H with n vertices, we overlap a vertex u of H with a vertex v_i of graph G_i into a new vertex for $i \in [1, n]$, the resulting connected graph is called a, H -graph, denoted as $H \odot_{i=1}^n G_i$.

Theorem 1. Each G_i of a graphic base $\mathbf{G} = (G_i)_1^n$ and another connected bipartite graph H with n vertices admits a set-ordered odd-graceful total coloring for $i \in [1, n]$, then the H -graph is a connected bipartite graph and admits a set-ordered odd-graceful total coloring.

Proof. The numbers of vertices and edges of the connected bipartite graph G_i in the graphic base \mathbf{G} are p_i and q_i . Suppose that (X_i, Y_i) is the bipartition of vertex set of G_i , where $X_i = \{x_{i,1}, x_{i,2}, \dots, x_{i,a_i}\}$, $Y_i = \{y_{i,1}, y_{i,2}, \dots, y_{i,b_i}\}$ and $a_i + b_i = |V(G_i)| = p_i$. According to the assumption in the theorem, each connected bipartite graph G_i admits a set-ordered odd-graceful total coloring f_i , so $\max f_i(X_i) < \min f_i(Y_i)$ holds true, without a loss of generality, we have $f_i(x_{i,k}) \leq f_i(x_{i,k+1})$ for $k \in [1, a_i - 1]$, $f_i(x_{i,a_i}) < f_i(y_{i,1})$ and $f_i(y_{i,l}) \leq f_i(y_{i,l+1})$ for $l \in [1, b_i - 1]$.

Let w_1, w_2, \dots, w_n be the vertices of the connected bipartite graph H , $|V(H)| = p(H)$ and $|E(H)| = q(H)$, the vertex bipartition of the graph H is denoted as (X, Y) , where $X = \{w_1, w_2, \dots, w_s\}$ and $Y = \{w_{s+1}, w_{s+2}, \dots, w_n\}$. Because H admits a set-ordered odd-graceful total coloring α , we can get $\max \alpha(X) < \min \alpha(Y)$, $\alpha(w_n) - \alpha(w_1) = 2q(H) - 1$ and $\alpha(w_{s+1}) - \alpha(w_s) = 1$.

We overlap a vertex of graph H with a vertex of each graph G_i into a new vertex to obtain a connected H -graph, and let $A(n) = \sum_{k=1}^n q_k$, $A^*(m, n) = \sum_{k=m}^n q_k$ for $m \leq n$, $B(k) = \sum_{j=1}^k q_{s+j}$. We first define another total coloring g of the graph H as: $g(w_i) = \alpha(w_i)$ for $i \in [1, s]$, $g(w_j) = \alpha(w_j) + 2A(n)$ for $j \in [s + 1, n]$, then each edge $w_i w_j$ is colored as

$$g(w_i w_j) = g(w_j) - g(w_i) = \alpha(w_j) + 2A(n) - \alpha(w_i) = 2A(n) + \alpha(w_i w_j). \tag{4}$$

So the edge color set of graph H is $g(E(H)) = [2A(n) + 1, 2A(n) + 2q(H) - 1]^0$. Next, we give each graph G_i of the base \mathbf{G} a new total coloring.

Overlap the vertex $y_{s+1, b_{s+1}}$ of graph G_{s+1} and the vertex w_{s+1} of graph H into a vertex $y_{s+1, b_{s+1}} \odot w_{s+1}$, and the vertex $y_{s+1, b_{s+1}}$ is recolored as

$$\begin{aligned} g(y_{s+1, b_{s+1}}) &= g(w_{s+1}) = \alpha(w_{s+1}) + 2A(n) \\ &= f_{s+1}(y_{s+1, b_{s+1}}) + \alpha(w_{s+1}) + 2A(n) - 2q_{s+1}. \end{aligned} \tag{5}$$

The other vertices of graph G_{s+1} can be colored as

$$g(x_{s+1, k}) = f_{s+1}(x_{s+1, k}) + \alpha(w_{s+1}) + 2A(n) - 2q_{s+1}, \quad k \in [1, a_{s+1}]; \tag{6}$$

$$g(y_{s+1, l}) = f_{s+1}(y_{s+1, l}) + \alpha(w_{s+1}) + 2A(n) - 2q_{s+1}, \quad l \in [1, b_{s+1}]. \tag{7}$$

Thereby, the edge color set of graph G_{s+1} is just $g(E(G_{s+1})) = [1, 2q_{s+1} - 1]^0$.

For graph G_{s+2} , overlapping the vertex $y_{s+2, b_{s+2}}$ of G_{s+2} and the vertex w_{s+2} of graph H into a vertex $y_{s+2, b_{s+2}} \odot w_{s+2}$, the new color of vertex $y_{s+2, b_{s+2}}$ is

$$\begin{aligned} g(y_{s+2, b_{s+2}}) &= g(w_{s+2}) = \alpha(w_{s+2}) + 2A(n) \\ &= f_{s+2}(y_{s+2, b_{s+2}}) + \alpha(w_{s+2}) + 2A(n) - 2q_{s+2}, \end{aligned} \tag{8}$$

we also have

$$g(x_{s+2,k}) = f_{s+2}(x_{s+2,k}) + \alpha(w_{s+2}) + 2A(n) - 2(q_{s+2} + q_{s+1}), k \in [1, a_{s+2}]; \tag{9}$$

$$g(y_{s+2,l}) = f_{s+2}(y_{s+2,l}) + \alpha(w_{s+2}) + 2A(n) - 2q_{s+2}, l \in [1, b_{s+2}]; \tag{10}$$

hence, the edge $x_{s+2,k}y_{s+2,l}$ can be colored as

$$g(x_{s+2,k}y_{s+2,l}) = g(y_{s+2,l}) - g(x_{s+2,k}) = f_{s+2}(x_{s+2,k}y_{s+2,l}) + 2q_{s+1}. \tag{11}$$

We get the edge color set of the graph G_{s+2} as $g(E(G_{s+2})) = [2q_{s+1} + 1, 2(q_{s+1} + q_{s+2}) - 1]^0$. Generally, we overlap the vertex $y_{s+r,b_{s+r}}$ of graph T and the vertex w_{s+r} of graph H into a vertex $y_{s+r,b_{s+r}} \odot w_{s+r}$, and recolor the vertices and edges of connected bipartite graph G_{s+r} for $r \in [3, t]$, where $s + t = n$, we have

$$\begin{aligned} g(y_{s+r,b_{s+r}}) &= g(w_{s+r}) = \alpha(w_{s+r}) + 2A(n) \\ &= f_{s+r}(y_{s+r,b_{s+r}}) + \alpha(w_{s+r}) + 2A(n) - 2q_{s+r}; \end{aligned} \tag{12}$$

the other vertices in the graph G_{s+r} can be colored as

$$g(x_{s+r,k}) = f_{s+r}(x_{s+r,k}) + \alpha(w_{s+r}) + 2A(n) - 2B(r), k \in [1, a_{s+r}]; \tag{13}$$

$$g(y_{s+r,l}) = f_{s+r}(y_{s+r,l}) + \alpha(w_{s+r}) + 2A(n) - 2q_{s+r}, l \in [1, b_{s+r}]; \tag{14}$$

so the coloring of edge $x_{s+r,k}y_{s+r,l}$ is

$$g(x_{s+r,k}y_{s+r,l}) = g(y_{s+r,l}) - g(x_{s+r,k}) = f_{s+r}(x_{s+r,k}y_{s+r,l}) + 2B(r - 1). \tag{15}$$

Thereby, we get the set G_{s+r} as $g(E(G_{s+r})) = [2B(r - 1) + 1, 2B(r) - 1]^0$ for $r \in [3, t]$. Based on the above coloring process, we also get an edge color set $g(\bigcup_{r=1}^t E(G_{s+r})) = [1, 2\sum_{r=1}^t q_{s+r} - 1]^0$, the maximum value of colors used is

$$\max \left\{ g(v) : v \in \bigcup_{r=1}^t V(G_{s+r}) \right\} = \alpha(w_{s+t}) + 2A(n) = 2A(n) + 2q(H). \tag{16}$$

The above part has given the new total colorings of graphs $G_{s+1}, G_{s+2}, \dots, G_n$. Further, we recolor the graphs G_1, G_2, \dots, G_s in the following. For each graph G_i with $i \in [1, s]$, we perform the vertex-overlapping operation on the vertex $x_{i,1}$ of G_i and the vertex w_i of H , then we determine a new total coloring of $G_i, i = 1, 2, \dots, s$.

Recolor vertex $x_{s,1}$ as

$$g(x_{s,1}) = g(w_s) = \alpha(w_s) = f_s(x_{s,1}) + \alpha(w_s) - 1, \tag{17}$$

and other vertices of graph G_s as

$$g(x_{s,k}) = f_s(x_{s,k}) + \alpha(w_s) - 1, k \in [1, a_s], \tag{18}$$

$$g(y_{s,l}) = f_s(y_{s,l}) + 2[A(s - 1) + A^*(s + 1, n)] + \alpha(w_s) - 1, l \in [1, b_s]. \tag{19}$$

Each edge $x_{s,k}y_{s,l}$ of the graph G_s is recolored as

$$g(x_{s,k}y_{s,l}) = g(y_{s,l}) - g(x_{s,k}) = f_s(x_{s,k}y_{s,l}) + 2[A(s - 1) + A^*(s + 1, n)]. \tag{20}$$

Therefore, the edge color set is

$$g(E(G_s)) = [2A(s - 1) + 2A^*(s + 1, n) + 1, 2A(s - 1) + 2A^*(s + 1, n) + 2q_s - 1]^0.$$

We recolor the vertex $x_{s-1,1}$ as

$$g(x_{s-1,1}) = g(w_{s-1}) = h(w_{s-1}) = f_{s-1}(x_{s-1,1}) + \alpha(w_{s-1}) - 1, \tag{21}$$

and recolor other vertices in the graph G_{s-1} as

$$g(x_{s-1,k}) = f_{s-1}(x_{s-1,k}) + \alpha(w_{s-1}) - 1, k \in [1, a_{s-1}]; \tag{22}$$

$$g(y_{s-1,l}) = f_{s-1}(y_{s-1,l}) + 2[A(n) - q_{s-1} - q_s] + \alpha(w_{s-1}) - 1, l \in [1, b_{s-1}]. \tag{23}$$

$$g(x_{s-1,k}y_{s-1,l}) = f_{s-1}(x_{s-1,k}y_{s-1,l}) - 2(q_{s-1} + q_s) + 2A(n). \tag{24}$$

Obviously, we get the edge color set $g(E(G_{s-1})) = [2A(n) - 2q_{s-1} - 2q_s + 1, 2A(n) - 2q_s - 1]^o$. In general, for $d \in [2, s - 1]$, we have

$$g(x_{s-d,1}) = g(w_{s-d}) = \alpha(w_{s-d}) = f_{s-d}(x_{s-d,1}) + \alpha(w_{s-d}) - 1, \tag{25}$$

$$g(x_{s-d,k}) = f_{s-d}(x_{s-d,k}) + \alpha(w_{s-d}) - 1, k \in [1, a_{s-d}]; \tag{26}$$

$$g(y_{s-d,l}) = f_{s-d}(y_{s-d,l}) + 2[A(n) - A^*(s - d, s)] + \alpha(w_{s-d}) - 1, l \in [1, b_{s-d}]. \tag{27}$$

We color each edge $x_{s-d,k}y_{s-d,l}$ of graph G_{s-d} as

$$\begin{aligned} g(x_{s-d,k}y_{s-d,l}) &= g(y_{s-d,l}) - g(x_{s-d,k}) \\ &= f_{s-d}(x_{s-d,k}y_{s-d,l}) + 2[A(n) - A^*(s - d, s)]. \end{aligned} \tag{28}$$

thus, we get the edge color set of the graph G_{s-d} ,

$$g(E(G_{s-d})) = [2A(n) - 2A^*(s - d, s) + 1, 2A(n) - 2A^*(s - d, s) + 2q_{s-d} - 1]^o.$$

By recoloring the graph H and the graphs G_i in the graphic base \mathbf{G} , we obtain the vertex color set of H -graph $H \odot_{i=1}^n G_i$ to be $g(V(H \odot_{i=1}^n G_i)) \subseteq [1, 2A(n) + 2q(H)]$, the edge color set of the H -graph to be $g(E(H \odot_{i=1}^n G_i)) = [1, 2A(n) + 2q(H) - 1]^o$. Thereby, we claim that g is a set-ordered odd-graceful total coloring of graph $H \odot_{i=1}^n G_i$. An example of the proof process of Theorem 1 is shown in Figure 2.

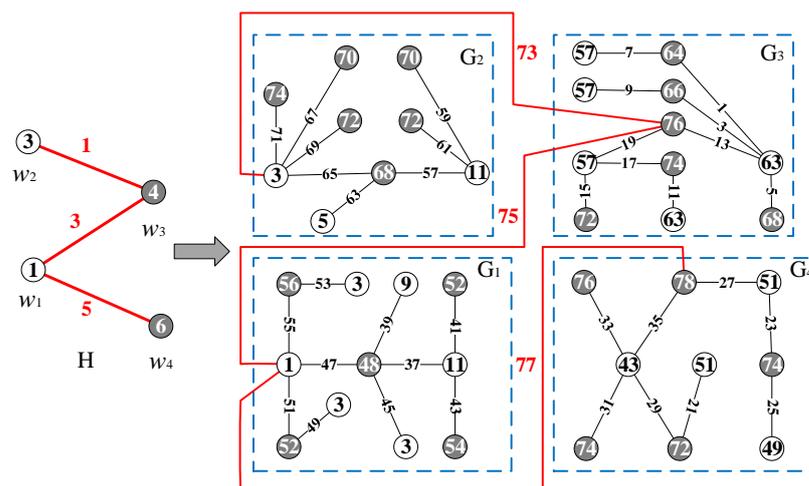


Figure 2. An example for illustrating the proof of Theorem 2.

□

2.2. Results Based on the Edge-Joining Operation

We divide the graphs obtained by edge operation into two categories according to the number of newly added edges between the graphs G_i and G_j in the graphic base \mathbf{G} , where $i, j \in [1, n]$ and $i \neq j$.

2.2.1. Single-Series Graphs

A single-series graph is obtained by connecting a vertex of the graph G_i and a vertex of the graph G_{i+1} with a new edge for $i \in [1, n - 1]$, abbreviated as $E_s \oplus_{i=1}^n G_i$, E_s contains all of the newly added edges.

Theorem 2. *Each G_i in a graphic base \mathbf{G} is a connected bipartite graph and admits a set-ordered odd-graceful total coloring; then a single-series graph admits a set-ordered odd-graceful total coloring.*

Proof. Suppose that (X_i, Y_i) is the bipartition of vertex set of connected bipartite graph G_i in the graphic base \mathbf{G} , where $X_i = \{x_{i,1}, x_{i,2}, \dots, x_{i,s_i}\}$, $Y_i = \{y_{i,1}, y_{i,2}, \dots, y_{i,t_i}\}$ and $s_i + t_i = |V(G_i)| = p_i$, $|E(G_i)| = q_i$. Since graph G_i admits a set-ordered odd-graceful total coloring f_i , the colors of all of the vertices can be arranged as $f_i(x_{i,1}) \leq f_i(x_{i,2}) \leq \dots \leq f_i(x_{i,s_i}) < f_i(y_{i,1}) \leq f_i(y_{i,2}) \leq \dots \leq f_i(y_{i,t_i})$. The edge color set of graph G_i under coloring f_i is

$$\begin{aligned} f_i(E(G_i)) &= \{f_i(x_{i,k}y_{i,l}) = |f_i(y_{i,l}) - f_i(x_{i,k})| : x_{i,k}y_{i,l} \in E(G_i)\} \\ &= [1, 2q_i - 1]^0, \quad i \in [1, n]. \end{aligned} \tag{29}$$

Let $F(r) = \sum_{j=1}^r f_j(x_{j,s_j})$ and $M(r) = \sum_{j=1}^r q_{n-j+1}$, we define another total coloring g of graph G_i as follows: $g(x_{1,k}) = f_1(x_{1,k})$ for $x_{1,k} \in X_1$, $g(x_{2,k}) = f_2(x_{2,k}) + f_1(x_{1,s_1}) + 1$ for $x_{2,k} \in X_2$. Generally, when $3 \leq i \leq n$, we have $g(x_{i,k}) = f_i(x_{i,k}) + F(i - 1) + i - 1$ for $x_{i,k} \in X_i$.

Recolor vertex $y_{n,l}$ in the graph G_n as

$$g(y_{n,l}) = f_n(y_{n,l}) + F(n - 1) + n - 1, \tag{30}$$

then the color of edge $x_{n,k}y_{n,l} \in E(G_n)$ is

$$\begin{aligned} g(x_{n,k}y_{n,l}) &= g(y_{n,l}) - g(x_{n,k}) \\ &= f_n(y_{n,l}) + F(n - 1) + n - 1 - f_n(x_{n,k}) - F(n - 1) - n + 1 \\ &= f_n(x_{n,k}y_{n,l}), \end{aligned} \tag{31}$$

the edge color set of graph G_n is $g(G_n) = [1, 2q_n - 1]^0$. Then, we recolor vertex $y_{n-1,l} \in Y_{n-1}$ in the graph G_{n-1} as

$$g(y_{n-1,l}) = f_{n-1}(y_{n-1,l}) + F(n - 2) + n + 2q_n, \tag{32}$$

then the coloring of edge $x_{n-1,k}y_{n-1,l}$ in the graph G_{n-1} can be expressed as

$$\begin{aligned} g(x_{n-1,k}y_{n-1,l}) &= g(y_{n-1,l}) - g(x_{n-1,k}) \\ &= f_{n-1}(y_{n-1,l}) + F(n - 2) + n + 2q_n - [f_{n-1}(x_{n-1,k}) + F(n - 2) + n - 2] \\ &= f_{n-1}(x_{n-1,k}y_{n-1,l}) + 2q_n + 2, \end{aligned} \tag{33}$$

hence the edge color set of graph G_{n-1} is $g(E(G_{n-1})) = [2q_n + 3, 2(q_n + q_{n-1}) + 1]^0$.

When $2 \leq m \leq n - 1$, we recolor the vertex $y_{n-m,l} \in Y_{n-m}$ as

$$g(y_{n-m,l}) = f_{n-m}(y_{n-m,l}) + F(n - m - 1) + 2M(m) + n + m - 1, \tag{34}$$

and color the edge $x_{n-m,k}y_{n-m,l}$ by

$$\begin{aligned} g(x_{n-m,k}y_{n-m,l}) &= g(y_{n-m,l}) - g(x_{n-m,k}) \\ &= f_{n-m}(y_{n-m,l}) + F(n - m - 1) + 2M(m) + n + m - 1 \\ &\quad - [f_{n-m}(x_{n-m,k}) + F(n - m - 1) + n - m - 1] \\ &= 2M(m) + 2m + f_{n-m}(x_{n-m,k}y_{n-m,l}). \end{aligned} \tag{35}$$

We can calculate that the edge color set of each graph G_{n-m} for $m \in [2, n - 1]$ as

$$g(E(G_{n-m})) = [2M(m) + 2m + 1, 2M(m + 1) + 2m - 1]^o.$$

Combining the following two equations

$$g(x_{n-m,1}) = f_{n-m}(x_{n-m,1}) + F(n - m - 1) + n - m - 1, \tag{36}$$

$$g(y_{n-m-1,1}) = f_{n-m-1}(y_{n-m-1,1}) + F(n - m - 2) + n + 2M(m + 1) + m, \tag{37}$$

we can get $f_{n-m-1}(y_{n-m-1,1}) - f_{n-m}(x_{n-m,1}) = f_{n-m-1}(x_{n-m-1,s_{n-m-1}})$, and the condition $f_{n-m}(x_{n-m,1}) = 1$, then we have

$$\begin{aligned} g(x_{n-m,1}y_{n-m-1,1}) &= g(y_{n-m-1,1}) - g(x_{n-m,1}) \\ &= f_{n-m-1}(y_{n-m-1,1}) + F(n - m - 2) + n + 2M(m + 1) \\ &\quad + m - [f_{n-m}(x_{n-m,1}) + F(n - m - 1) + n - m - 1] \\ &= 2M(m + 1) + f_{n-m-1}(y_{n-m-1,1}) - f_{n-m}(x_{n-m,1}) \\ &\quad - f_{n-m-1}(x_{n-m-1,s_{n-m-1}}) + 2m + 1 \\ &= 2M(m + 1) + 2m + 1. \end{aligned} \tag{38}$$

For $d \in [0, n - 2]$, we connect two vertices $x_{n-d,1}$ and $y_{n-d-1,1}$ with a new edge $x_{n-d,1}y_{n-d-1,1}$ to obtain a connected single-series graph $E_s \oplus_{i=1}^n G_i$, the set E_s means $E_s = \{x_{n-d,1}y_{n-d-1,1} : d \in [0, n - 2]\}$. We can obtain the following color set under total coloring g ,

$$g(E_s) = \{2[M(1) + 1] - 1, 2[M(2) + 2] - 1, 2[M(3) + 3] - 1, \dots, 2[M(n - 1) + n - 1] - 1\}.$$

By the above deduction, the color set of the single-series graph $E_s \oplus_{i=1}^n G_i$ is

$$g(E(E_s \oplus_{i=1}^n G_i)) = g(E_s) \cup \left(\bigcup_{i=1}^n g(E(G_i)) \right) = [1, 2A(n) + 2n - 3]^o,$$

where $A(n) = \sum_{j=1}^n q_j$. We let $X = X_1 \cup X_2 \cup \dots \cup X_n$ and $Y = Y_1 \cup Y_2 \cup \dots \cup Y_n$, then (X, Y) is the vertex set bipartition of $E_s \oplus_{i=1}^n G_i$, in conclusion, g is a set-ordered odd-graceful total coloring of the single-series graph $E_s \oplus_{i=1}^n G_i$. An example of the proof process of Theorem 2 is shown in Figure 3.

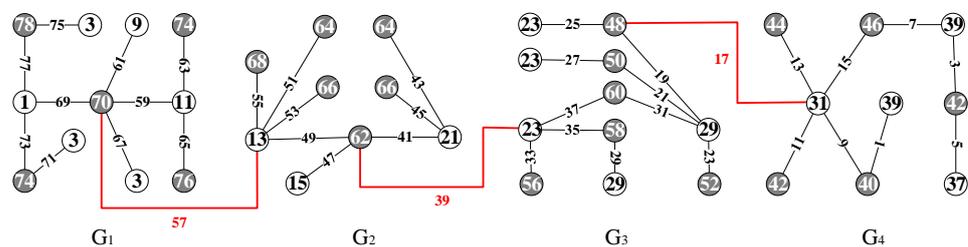


Figure 3. A single-series graph, made by the four graphs shown in Figure 1, admits a set-ordered odd-graceful total coloring.

□

2.2.2. Multiple-Series Graphs

There is only one edge between two adjacent graphs G_{i-1} and G_i in the single-series graph for $i \in [2, n]$. However, when the number of new edges between two graphs G_i and G_j ($i \neq j$) is not equal to 1, but $t \geq 1$ new edges, the resulting graph $E_c \oplus_{i=1}^n G_i$ is called a multiple-series graph, E_c represents the edge set of between G_i and G_j in graphic base \mathbf{G} .

Theorem 3. Each G_i in a graphic base \mathbf{G} admits a set-ordered odd-graceful total coloring, then the multiple-series graph admits a set-ordered odd-graceful total coloring.

Proof. According to the assumptions in the theorem, the connected bipartite graph G_i admits a set-ordered odd-graceful total coloring f_i , we suppose that (X_i, Y_i) is the bipartition of the vertex set of the connected bipartite graph G_i , where $X_i = \{x_{i,1}, x_{i,2}, \dots, x_{i,s_i}\}$, $Y_i = \{y_{i,1}, y_{i,2}, \dots, y_{i,t_i}\}$ and $s_i + t_i = p_i$, $|E(G_i)| = q_i$. Arrange the colors of all vertices as follows

$$f_i(x_{i,1}) \leq f_i(x_{i,2}) \leq \dots \leq f_i(x_{i,s_i}) < f_i(y_{i,1}) \leq f_i(y_{i,2}) \leq \dots \leq f_i(y_{i,t_i}), \tag{39}$$

then the edge color set of graph G_i is

$$\begin{aligned} f_i(E(G_i)) &= \{f_i(x_{i,k}y_{i,l}) = |f_i(y_{i,l}) - f_i(x_{i,k})| : x_{i,k}y_{i,l} \in E(G_i)\} \\ &= [1, 2q_i - 1]^0, \quad i \in [1, n]. \end{aligned} \tag{40}$$

The multiple-series graph $E_c \oplus_{i=1}^n G_i$ is generated by iterations. First of all, we define another total coloring g_1 for the two graphs G_1 and G_2 in graphic base \mathbf{G} . Let t_1 represents the number of new edges added between G_1 and G_2 . We have $g_1(x_{1,k}) = f_1(x_{1,k})$ for $x_{1,k} \in X_1$, $g_1(x_{2,k}) = f_2(x_{2,k}) + f_1(x_{1,s_1}) + 1$ and $g_1(y_{2,l}) = f_2(y_{2,l}) + f_1(x_{1,s_1}) + 1$ for $x_{2,k} \in X_2, y_{2,l} \in Y_2$, also $g_1(y_{1,l}) = f_1(y_{1,l}) + 2(q_2 + t_1)$ for $y_{1,l} \in Y_1$, and $g_1(x_{2,k}y_{2,l}) = f_2(x_{2,k}y_{2,l})$ for $x_{2,k}y_{2,l} \in E(G_2)$; therefore, the edge color set in the connected graph G_2 is $g_1(E(G_2)) = [1, 2q_2 - 1]^0$, the coloring of the edge $x_{1,k}y_{1,l} \in E(G_1)$ can be further calculated as $g_1(x_{1,k}y_{1,l}) = f_1(x_{1,k}y_{1,l}) + 2(q_2 + t_1)$, then $g_1(E(G_1)) = [2(q_2 + t_1) + 1, 2(q_1 + q_2 + t_1) - 1]^0$, we can get $g_1(E(G_1)) \cup g_1(E(G_2)) = [1, 2q_2 - 1]^0 \cup [2(q_2 + t_1) + 1, 2(q_1 + q_2 + t_1) - 1]^0$. Moreover, we have

$$g_1(x_{1,k}y_{2,l}) = g_1(y_{2,l}) - g_1(x_{1,k}) = f_2(y_{2,l}) + f_1(x_{1,s_1}) + 1 - f_1(x_{1,k}), \tag{41}$$

and $f_2(y_{2,t_2}) = 2q_2$. In addition,

$$g_1(y_{1,l}x_{2,k}) = g_1(y_{1,l}) - g_1(x_{2,k}) = f_1(y_{1,l}) + 2(q_2 + t_1) - f_2(x_{2,k}) - f_1(x_{1,s_1}) - 1. \tag{42}$$

For $w \in [1, t_1]$, we get the following inequality from the above derivation,

$$g_1(x_{1,k}y_{2,l}) \geq 2q_2 - 1 + w, \tag{43}$$

$$g_1(y_{1,l}x_{2,k}) \geq 2q_2 - 1 + w. \tag{44}$$

Obviously, (X, Y) is the vertex bipartition of the connected graph $E_{c1} \oplus (G_1 \cup G_2)$, where $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. The edge $x_{1,k}y_{2,l}$ or $y_{1,l}x_{2,k}$ satisfies the above inequality conditions form a set E_{c1} , so that $|E_{c1}| = t_1$, we claim that g_1 is a set-ordered odd-graceful total coloring of $E_{c1} \oplus (G_1 \cup G_2)$. Next, let $O_2 = E_{c1} \oplus (G_1 \cup G_2)$, then consider the total coloring of connected graph O_2 and graph G_3 . According to the above method, the edge set E_{c2} can be obtained, which is the set of all edges between O_2 and G_3 ; thereby, we can obtain a set-ordered odd-graceful total coloring g_2 of $O_3 = E_{c2} \oplus (O_2 \cup G_3)$, in this way, the theorem is proven. An example of the proof process of Theorem 3 is shown in Figure 4.

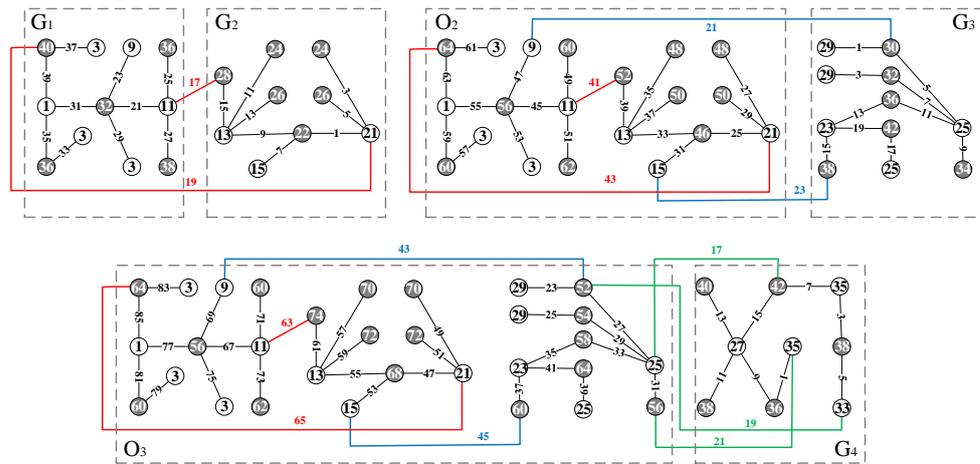


Figure 4. An example for illustrating the proof of Theorem 3. □

3. Conclusions

This paper studies the existence of graphic bases with odd-graceful total colorings and shows graphic lattices by vertex-overlapping operations and edge-joining operations; further, we prove that these graphic lattices are closed to the odd-graceful total coloring. If the password generated by the graphic lattices is used to encrypt important information, then the problem of determining the graph isomorphism and some coloring conjectures will be involved in the process of decryption, so it is not easy to crack it, in addition, it has the characteristics of diverse and simple generation methods.

The theoretical research of graphic lattice and its corresponding key design has great practical significance, the topological authentication generated by combining the topological structure of graph with labeling or a coloring can be used to obtain honeywords [6], and can be used in fields, such as cryptographic design [18], topological coding [19] and network information security [20,21]. Our future work will further promote the concept of the graphic lattices, not limited to odd-graceful total coloring, but combines the topological structure of the graph with different new colorings or new labelings, and uses the graphic lattices to generate more authentication passwords.

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