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Article

On the Distribution of the spt-Crank

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Abstract: Andrews, Garvan and Liang introduced the spt-crank for vector partitions. We conjecture that for any n the sequence $\{N_S(m,n)\}_m$ is unimodal, where $N_S(m,n)$ is the number of S-partitions of size n with crank m weight by the spt-crank. We relate this conjecture to a distributional result concerning the usual rank and crank of unrestricted partitions. This leads to a heuristic that suggests the conjecture is true and allows us to asymptotically establish the conjecture. Additionally, we give an asymptotic study for the distribution of the spt-crank statistic. Finally, we give some speculations about a definition for the spt-crank in terms of "marked" partitions. A "marked" partition is an unrestricted integer partition where each part is marked with a multiplicity number. It remains an interesting and apparently challenging problem to interpret the spt-crank in terms of ordinary integer partitions.

Keywords: partitions; partition crank; partition rank; spt-crank; unimodal

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1. Introduction and Statement of Results

The spt-function, introduced by the first author [1], counts the total number of appearances of the smallest parts in the partitions of n. For example, $\mathrm{spt}(4) = 10$ because the partitions of 4 are 4, $3+1, 2+2, 2+1+1, 1+1+1+1$. The first author established the following remarkable congruences

$$\begin{aligned}\mathrm{spt}(5n+4) &\equiv 0 \pmod{5}\\ \mathrm{spt}(7n+5) &\equiv 0 \pmod{7}\\ \mathrm{spt}(13n+6) &\equiv 0 \pmod{13}\end{aligned}$$

These congruences bear a striking resemblance to Ramanujan's congruences for the usual partition counting function $p(n)$, namely

$$\begin{aligned}p(5n+4) &\equiv 0 \pmod{5}\\ p(7n+5) &\equiv 0 \pmod{7}\\ p(11n+6) &\equiv 0 \pmod{11}\end{aligned}$$

The rank statistic of a partition was defined by the second author [2] as the largest part of the partition minus the number of parts. Let $N(m,n)$ denote the number of partitions of n with rank m and $N(m,t,n)$ denote the number of partitions of n with rank congruent to m modulo t. The second author conjectured, and Atkin and Swinnerton-Dyer [3] proved, that

$$\begin{aligned}N(k,5,5n+4) &= \frac{p(5n+4)}{5} &&\text{for } 0 \le k \le 4\\ N(k,7,7n+5) &= \frac{p(7n+5)}{7} &&\text{for } 0 \le k \le 6\end{aligned}$$

Therefore, the rank provides a combinatorial interpretation of Ramanujan's congruences modulo 5 and 7. Moreover, the second author observed that the rank is not sufficient to decompose Ramanujan's congruence modulo 11, and he conjectured the existence of a statistic called the "crank" that would explain all three congruences.

Garvan [4] found the crank statistic for vector partitions and together with the first author [5] presented a definition for the crank of a ordinary partition, namely

$$\mathrm{crank}(\lambda) := \begin{cases}\text{largest part of } \lambda & \text{if } o(\lambda) = 0\\ \mu(\lambda) - o(\lambda) & \text{else}\end{cases}$$

where $o(\lambda)$ is the number of 1s in the partition λ and $\mu(\lambda)$ is the number of parts of λ strictly larger than $o(\lambda)$. Let $M(m,n)$ be the number of partitions of n with crank m and $M(m,t,n)$ be the number of partitions of n with crank congruent to m modulo t. Garvan proved [4]

$$\begin{aligned}M(k,5,5n+4) &= \frac{p(5n+4)}{5} &&\text{for } 0 \le k \le 4\\ M(k,7,7n+5) &= \frac{p(7n+5)}{7} &&\text{for } 0 \le k \le 6\\ M(k,11,11n+6) &= \frac{p(11n+6)}{11} &&\text{for } 0 \le k \le 10\end{aligned}$$

Hence, the crank provides a combinatorial interpretation of all three of Ramanujan's congruences for the partition counting function.

Recently Garvan, Liang and the first author [6] defined the spt-crank and used it to provide a combinatorial interpretation of the spt-congruences modulo 5 and 7. To describe the spt-crank we introduce the set of vector partitions, denoted by V. Then V is the Cartesian product

$$V = \mathcal{D} \times \mathcal{P} \times \mathcal{P}$$

where $\mathcal{D}$ is the set of partitions into distinct parts and $\mathcal{P}$ is the set of all integer partitions. For $\pi = (\pi_1, \pi_2, \pi_3) \in V$, let $|\pi| = |\pi_1| + |\pi_2| + |\pi_3|$, where $|\cdot|$ is the sum of the parts of a partition. If $|\pi| = n$ we say that π is a vector partition of n. The crank of a vector partition is defined as $\#(\pi_2) - \#(\pi_3)$, where $\#(\cdot)$ is the number of parts in an integer partition.

To define the spt-crank we introduce the set of S-partitions. Let

$$S := \{\pi = (\pi_1, \pi_2, \pi_3) \in V : 1 \le s(\pi_1) < \infty \text{ and } s(\pi_1) \le \min(s(\pi_2), s(\pi_3))\}$$

where $s(\pi)$ is the smallest part in the partition. For $\pi \in S$ define a weight ω_1, by $\omega_1(\pi) = (-1)^{\#(\pi_1)-1}$. If $\pi \in S$ has crank m, then we refer to the spt-crank as $\omega_1(\pi)$. Define

$$N_S(m,n) := \sum_{\pi \in S, |\pi|=n, \mathrm{crank}(\pi)=m} \omega_1(\pi)$$

and $N_S(m,t,n) = \sum_{k \equiv m \pmod{t}} N_S(k,n)$. Garvan, Liang and the first author [6] establish the following combinatorial interpretation of the spt-congruences modulo 5 and 7

$$\begin{aligned} N_S(k,5,5n+4) &= \frac{\mathrm{spt}(5n+4)}{5} && \text{for } 0 \le k \le 4 \\ N_S(k,7,7n+5) &= \frac{\mathrm{spt}(7n+5)}{7} && \text{for } 0 \le k \le 6 \end{aligned}$$

In a second paper [7], they prove a number of basic results about these values. For instance,

$$N_S(m,n) = N_S(-m,n) \tag{1.1}$$

and, surprisingly,

$$N_S(m,n) \ge 0$$

Later, a simpler proof of this result was given by the second author [8].

Table 1 suggests that the sequence $\{N_S(m,n)\}_m$ is (weakly) unimodal. Precisely, we give the following conjecture.

Conjecture 1.1. *For each $m \ge 0$ and $n \ge 0$ we have*

$$N_S(m,n) \ge N_S(m+1,n)$$

Remark. *Chen, Ji, and Zang have announced a proof of this conjecture [].*

Table 1. A table of values of $N_S(m,n)$.

$n\backslash m$	0	1	2	3	4	5	6	7	8	9
0	1									
1	1									
2	1	1								
3	1	1	1							
4	2	2	1	1						
5	2	2	2	1	1					
6	4	4	3	2	1	1				
7	5	4	4	3	2	1	1			
8	7	7	6	5	3	2	1	1		
9	10	9	8	6	5	3	2	1	1	
10	13	13	11	10	7	5	3	2	1	1
11	17	16	15	12	10	7	5	3	2	1
12	24	24	21	18	14	11	7	5	3	2
13	31	29	27	23	19	14	11	7	5	3
14	40	40	36	32	26	21	15	11	7	5
15	53	51	48	41	35	27	21	15	11	7
16	69	68	62	56	46	38	29	22	15	11

This property is not true for the ordinary rank or crank statistic. For example,

$$N(n-1,n) = N(n-3,n) = 1 \text{ and } N(n-2,n) = 0$$

for all $n > 2$ and a similar statement holds for the crank. Our first statement reinterprets this conjecture in terms of the rank and crank. Define the cumulative density functions of the rank and crank as follows:

$$N_{\leq m}(n) := \sum_{|r|\leq m} N(r,m) \quad \text{and} \quad M_{\leq m}(n) := \sum_{|r|\leq m} M(r,m) \tag{1.2}$$

Theorem 1.2. *For all $n > 1$ and any $m \geq 0$ we have*

$$N_S(m,n) \geq N_S(m+1,n)$$

if and only if

$$N_{\leq m}(n) \geq M_{\leq m}(n)$$

Remark. *The statement that*

$$N_{\leq m}(n) \geq M_{\leq m}(n)$$

is true for each n was conjectured by Bringmann and Mahlburg [10].

Remark. *Kaavya [11] conjectured that $N_{\leq 0}(n) = N(0,n) \geq M(0,n) = M_{\leq 0}(n)$ for all n.*

This theorem leads to a good heuristic reason to believe that the spt-crank is unimodal. Define the moments of the rank and crank statistic by

$$N_{2\ell}(n) := \sum_{m} m^{2\ell} N(m,n) \quad \text{and} \quad M_{2\ell}(n) := \sum_{m} m^{2\ell} M(m,n) \tag{1.3}$$

Remark. *Since $N(-m,n) = N(m,n)$ and $M(m,n) = M(-m,n)$ the odd moments of these statistics are zero.*

Garvan [12] conjectured that

$$M_{2\ell}(n) > N_{2\ell}(n) \tag{1.4}$$

for each $\ell > 0$ and n, despite the fact that

$$N_{2\ell}(n) \sim M_{2\ell}(n) \quad \text{as} \quad n \to \infty$$

This conjecture says that while the rank and crank are distributed asymptotically the same, the crank distribution is slightly "wider" for any fixed n. The first author [1] proved that $\mathrm{spt}(n) = \frac{1}{2}(M_2(n) - N_2(n))$, which yields the $\ell = 1$ case of Garvan's conjecture. Garvan [13] later proved his own conjecture by introducing higher order spt-functions. As a result, we expect that

$$N_{\leq m}(n) \geq M_{\leq m}(n)$$

which is by Theorem 1.2 is equivalent to Conjecture 1.1.

The next theorem provides an asymptotic result supporting Conjecture 1.1.

Theorem 1.3. *For each $m \geq 0$ we have*

$$N_{\leq m}(n) \sim M_{\leq m}(n) \sim \frac{(2m+1)\pi}{48\sqrt{2}n^{\frac{3}{2}}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \quad \text{as } n \to \infty$$

Moreover, we have

$$(N_{\leq m}(n) - M_{\leq m}(n)) \sim \frac{(2m+1)\pi^2}{192\sqrt{3}n^2} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \quad \text{as } n \to \infty$$

Remark. *This result implies that for and fixed m and sufficiently large n we have*

$$N_S(m,n) > N_S(m+1,n)$$

Remark. *For fixed m one may obtain an expansion $N_{\leq m}(n) \sim \frac{(2m+1)\pi}{48\sqrt{2}n^{\frac{3}{2}}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)\left(1 + \sum_{r\geq 1} \frac{\beta_r}{n^{\frac{r}{2}}}\right)$ as $n \to \infty$ with computable β_r.*

We close this section by giving an asymptotic for the distribution of the numbers $N_S(m,n)$. Let

$$N_{S,k}(n) := \sum_{m} m^k N_S(m,n) \tag{1.5}$$

be the moments of the spt-crank.

Remark. *By* (1.1) *the odd moments will be identically zero.*

To define the asymptotic result we give the following definitions: Define

$$\gamma(a,b,c) := \frac{(2(a+b+c))!}{(a+1)!(2b)!(2c)!} B_{2b}\left(\frac{1}{2}\right) B_{2c}\left(\frac{1}{2}\right) (-1)^{a+c} 4^{-a-c} \pi^{-a} (3^{a+1}-1) \tag{1.6}$$

where $B_n(x)$ is the nth Bernoulli polynomial. Define the Kloosterman sum

$$K_k(n) := \sum_{\substack{0 \le h < k \\ (h,k)=1}} \omega_{h,k} e^{-\frac{2\pi i h n}{k}} \tag{1.7}$$

where $\omega_{h,k} := \exp\left(\pi i \, s(h,k)\right)$ with

$$s(h,k) := \sum_{\mu \pmod k} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right)$$

the Dedekind sum, and

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

is the sawtooth function.

Theorem 1.4. *As* $n \to \infty$ *we have*

$$N_{S,2\ell}(n) = \frac{1}{2} \sum_{k < \sqrt{n}} \frac{K_k(n)}{k} \sum_{a+b+c=\ell} k^{a+1} \gamma(a,b,c)(24n-1)^{c+\frac{a}{2}-\frac{1}{4}} I_{\frac{1}{2}-a-2c}\left(\frac{\pi\sqrt{24n-1}}{6k}\right) + O\left(n^{2\ell+\epsilon}\right)$$

where I_ν *denotes the modified Bessel function of order* ν.

Remark. *Using the asymptotic* $I_\nu(x) \sim \frac{1}{\sqrt{2\pi x}} e^x$ *as* $x \to \infty$ *we have*

$$N_{S,2\ell}(n) = \frac{\sqrt{3}}{\pi}(-1)^\ell B_{2\ell}\left(\frac{1}{2}\right)(24n)^{\ell-\frac{1}{2}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$$

Since $\mathrm{spt}(n) = N_{S,0}(n) \sim \frac{1}{2\pi\sqrt{2n}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$ *and* $B_{2\ell}\left(\frac{1}{2}\right) = -B_{2\ell}\frac{2^{2\ell-1}-1}{2^{2\ell-1}}$*, we have*

$$\frac{N_{S,2\ell}(n)}{\mathrm{spt}(n)(6n)^\ell} \sim (2^{2\ell}-2)\,|B_{2\ell}|$$

The results of Bringmann, Mahlburg, and the third author [14] show that

$$\frac{N_{2\ell}(n)}{p(n)(6n)^\ell} \sim \frac{M_{2\ell}(n)}{p(n)(6n)^\ell} \sim (2^{2\ell}-2)\,|B_{2\ell}|$$

Therefore, the spt*-crank (after normalization) has the same distribution as the rank and crank of a partition. This distribution is known to be the same as the distribution of difference of two independent extreme value distributions. See the results of Diaconis, Janson, and the third author [15] for details.*

In Section 2 we prove Theorem 1.2. In Section 3 we use the results of Bringmann, Mahlburg, and the third author [16] on the moments of the rank and crank statistics to establish Theorem 1.4. In Section 4 we use the circle method to calculate the asymptotics of Theorem 1.3. Finally, in Section 5 we discuss the spt-crank in terms of ordinary integer partitions. It seems a challenging and interesting problem to find an interpretation of the spt-crank in terms of ordinary integer partitions.

2. Generating Functions for $N_S(m, n)$

In this section we prove Theorem 1.2. Garvan, Liang and the first author (Corollary 2.5 of [6]) give

$$\sum_{n\geq 1, m\in\mathbb{Z}} N_S(m,n) z^m q^n = \frac{z^{-1}}{(1-z^{-1})^2} \sum_{n=0}^{\infty} \sum_{m\in\mathbb{Z}} \left(N_V(m,n) - N(m,n)\right) z^m q^n$$

where $N_V(m, n)$ is the number of vector partitions with crank m. Note that $N_V(m, n) = M(m, n)$ for $n > 1$. Formal q-series manipulations lead to the following: for any $m \geq 0$ we have

$$\sum_{n=1}^{\infty} N_S(m,n) q^n = \sum_{n>1}^{\infty} \sum_{\ell\geq m} (\ell - m) \left(N_V(\ell,n) - N(\ell,n)\right) q^n \tag{2.1}$$

For example, when $m = 0$ we obtain

$$\sum_{n=1}^{\infty} N_S(0,n) q^n = \sum_{n=1}^{\infty} \operatorname{ospt}(n) q^n$$

where

$$\operatorname{ospt}(n) = \sum_{\ell\geq 0} \ell \left(M(\ell,n) - N(\ell,n)\right)$$

The ospt function is the difference of "first" moments of the crank and rank distributions, see [17]. From (2.1) we have

$$\sum_{n=1}^{\infty} (N_S(m,n) - N_S(m+1,n)) q^n = \sum_{n>1} \sum_{\ell>m} (N_V(\ell,n) - N(\ell,n)) q^n \tag{2.2}$$

Remark. *This also follows from (37) of [8] and Equations* (4.1) *and* (4.2) *below.*

Using the symmetry of the rank and crank statistics, and the fact that $\sum_m N(m, n) = \sum_m M(m, n) = p(n)$ we have

$$\begin{aligned}
\sum_{n>0} (N_S(m,n) - N_S(m+1,n)) q^n &= \sum_{n>0} \sum_{\ell>m} (N_V(\ell,n) - N(\ell,n)) q^n \\
&= \sum_{n=1}^{\infty} \frac{1}{2} \left(\sum_{\ell} N_V(\ell,n) - N(\ell,n)) + \sum_{-m\leq \ell\leq m} (N(m,n) - N_V(m,n)) \right) q^n \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \sum_{-m\leq\ell\leq m} (N(m,n) - N_V(m,n)) q^n
\end{aligned} \tag{2.3}$$

This establishes Theorem 1.2

3. Asymptotics for the Moments of the spt-Crank Statistic

In this section we will calculate the asymptotic for the moments of the spt-crank statistic. This calculation uses the results of [16] and establishes Theorem 1.4. For details see [16].

Let

$$S(x;q) = \sum_{n=1}^{\infty} \sum_{m} N_S(m,n) x^m q^n = -\frac{1}{(1-x)(1-x^{-1})} \left(C(x;q) - R(x;q)\right) \tag{3.1}$$

where $C(x;q)$ is the crank generating function and $R(x;q)$ is the rank generating function. Notice that

$$S(e^{2\pi iu};q) = \sum_{k\geq 0}\frac{(2\pi iu)^k}{k!}\sum_{n=1}^{\infty}\left(\sum_m m^k N_S(m,n)\right)q^n \tag{3.2}$$

By the symmetry of the statistic we have $\sum_m m^k N_S(m,n) = 0$ for all n when k is odd. We define $S_k(q) = \sum_{n=1}^{\infty}\left(\sum_m m^k N_S(m,n)\right)q^n$ to be the S-crank moment generating functions and $N_{S,k}(n) = \sum_m m^k N_S(m,n)$ to be S-crank moments weighted by ω_1.

The proof of Theorem 1.4 follows in a straightforward way from the results of [16] and a simple modification of some of the lemmas therein.

Throughout the remainder of this section let $z \in$ with $\mathrm{Re}\,(z) > 0$ and $0 \leq h < k$ with $(h,k) = 1$. We define $[a]_b$ the inverse of a modulo b. Moreover, for fixed h and k we let $q = e^{\frac{2\pi i}{k}(h+iz)}$. Define $\chi(h,[-h]_k,k)$ to be the multiplier of the Dedekind eta-function. In particular,

$$\chi(h,[-h]_k,k) = i^{-\frac{1}{2}}\omega_{h,k}^{-1}e^{-\frac{\pi i}{12k}([-h]_k-h)}$$

Finally, we define

$$f_\nu(u;z) := e^{\frac{\nu\pi u^2}{z}}\frac{\sin(\pi u)}{\sinh\left(\frac{\pi u}{z}\right)}$$

Proposition 3.1 (Section 3.2 of [16])**.** *In the notation above*

$$C(e^{2\pi iu};q) = -i^{\frac{3}{2}}e^{\frac{\pi i}{12k}(h-[-h]_k)}\chi^{-1}(h,[-h]_k,k)e^{\frac{\pi}{12k}\left(\frac{1}{z}-z\right)}z^{-\frac{1}{2}}f_k(u;z) + \sum_{r=0}^{\infty}\frac{a_r(z)u^r}{r!}$$

where $|a_r(z)| \ll |z|^{\frac{1}{2}-r}\,e^{-\frac{\alpha}{k}Re\left(\frac{1}{z}\right)}$ *for some* $\alpha > 0$ *independent of* k*.*

Proposition 3.2 (Proof of Proposition 3.5 of [16])**.** *In the notation above*

$$R(e^{2\pi iu};q) = -i^{\frac{3}{2}}e^{\frac{\pi i}{12k}(h-[-h]_k)}\chi^{-1}(h,[-h]_k,k)e^{\frac{\pi}{12k}\left(\frac{1}{z}-z\right)}z^{-\frac{1}{2}}f_{3k}(u;z) + \sum_{r=0}^{\infty}\frac{a_r(z)u^r}{r!}$$

where $|a_r(z)| \ll k^{\frac{1}{2}}\,|z|^{\frac{1}{2}-r}$.

Combining Propositions 3.1 and 3.2 and (3.1) we have the following lemma.

Lemma 3.3. *In the notation above,*

$$S(e^{2\pi iu};q) = -\frac{1}{4}i^{\frac{3}{2}}e^{\frac{\pi i}{12k}(h-[-h]_k)}\chi^{-1}(h,[-h]_k,k)e^{\frac{\pi}{12k}\left(\frac{1}{z}-z\right)}z^{-\frac{1}{2}}\frac{e^{\frac{3\pi ku^2}{z}}-e^{\frac{\pi ku^2}{z}}}{\sin(\pi u)\sinh\left(\frac{\pi u}{z}\right)} + \sum_{r=0}^{\infty}\frac{a_r(z)u^r}{r!}$$

where $|a_r(z)| \ll k^{\frac{1}{2}}\,|z|^{\frac{1}{2}-r}$.

Taylor expanding the expression in Lemma 3.3 with respect to u and using (3.2) give asymptotics for $S_k(q)$. The circle method can now be used to turn those asymptotics for the generating functions into asymptotics for the coefficients. Applying the following theorem gives Theorem 1.4. The theorem is a general circle method result, which is a slight modification of Theorem 4.1 of [16].

Theorem 3.4. *Assume that*

$$F_{r,\ell}\left(e^{\frac{2\pi i}{k}(h+iz)}\right)=\sum_{n}c_{r,\ell}(n)e^{\frac{2\pi i}{k}(h+iz)}$$

is holomorphic function of z satisfying

$$F_{r,\ell}\left(e^{\frac{2\pi i}{k}(h+iz)}\right)=-i^{\frac{3}{2}}e^{\frac{\pi i}{12k}(h-[-h]_k)}\chi^{-1}(h,[-h]_k,k)e^{\frac{\pi}{12k}\left(\frac{1}{z}-z\right)}\sum_{a+b+c=\ell}C(a,b,c)z^{-p(a,b,c)}+E_{r,\ell,k}(z)$$

with $E_{r,\ell,k}(z)\ll_{r,\ell}k^{\frac{1}{2}}|z|^{-\frac{1}{2}-2\ell}$, $C(a,b,c)$ are some constants and $p(a,b,c)$ is a polynomial in a, b, and c. Then

$$c_{r,\ell}(n)=2\pi\sum_{k\leq\sqrt{n}}\frac{K_k(n)}{k}\sum_{a+b+c=\ell}C(a,b,c)(24n-1)^{\frac{p(a,b,c)}{2}-\frac{1}{2}}I_{\frac{1}{4}-p(a,b,c)}\left(\frac{\pi\sqrt{24n-1}}{6k}\right)+O\left(n^{2\ell+\epsilon}\right)$$

4. The Circle Method and False Theta Functions

In this section we consider the cumulative density functions of the rank and crank. We show that these generating functions are partial theta functions times the partition generating function. Obtaining an asymptotic expansion for the coefficients of such a generating function via the circle method is classical (see [18], for example). We have the following well known generating functions for $N(m,n)$ and $M(m,n)$

$$\sum_{n\geq 0}N(m,n)q^n=\frac{1}{(q)_\infty}\sum_{n>0}(-1)^{n+1}q^{\frac{n(3n-1)}{2}+|m|n}(1-q^n) \tag{4.1}$$

and

$$\sum_{n\geq 0}M(m,n)q^n=\frac{1}{(q)_\infty}\sum_{n>0}(-1)^{n+1}q^{\frac{n(n-1)}{2}+|m|n}(1-q^n) \tag{4.2}$$

Fine [19] showed that

$$R_{\leq m}(q)=\sum_{n}N_{\leq m}(n)q^n=\frac{1}{(q)_\infty}\left(2\sum_{n=0}^{\infty}(-1)^n q^{\frac{3n^2+n}{2}+mn}-1\right)$$

Similarly, we have from (4.2)

$$C_{\leq m}(q)=\sum_{n}N_{\leq m}(n)q^n=\frac{1}{(q)_\infty}\left(2\sum_{n=0}^{\infty}(-1)^n q^{\frac{n^2+n}{2}+mn}-1\right)$$

Remark. *This shows that the generating function for each cumulative density function is a partial theta functions times the partition generating function.*

Note that

$$q^{\frac{n^2+n}{2}+mn}=q^{\frac{1}{2}(n^2+2mn+n)}=q^{\frac{1}{2}(n^2+2mn+n+m+\frac{1}{4})-\frac{1}{2}\left(m^2+m+\frac{1}{4}\right)}=q^{\frac{1}{2}\left(m+n+\frac{1}{2}\right)^2-\frac{1}{2}\left(m+\frac{1}{2}\right)^2}$$

So we have

$$C_{\le m}(q) = \frac{1}{(q)_\infty}\left(2(-1)^m q^{-\frac{1}{8}(2m+1)^2}\sum_{n>2m}^{\infty}\left(\frac{-4}{n}\right)q^{\frac{n^2}{8}} - 1\right) \tag{4.3}$$

where $\left(\frac{-4}{\cdot}\right)$ is the Kronecker symbol. Similarly, we have

$$R_{\le m}(q) = \frac{1}{(q)_\infty}\left(2q^{-\frac{1}{24}(2m+1)^2}\sum_{n>2m}^{\infty}\chi_m(n)q^{\frac{n^2}{24}} - 1\right) \tag{4.4}$$

where

$$\chi_m(n) = \begin{cases} +1 & n \equiv 2m+1 \pmod{12} \\ -1 & n \equiv 2m+7 \pmod{12} \\ 0 & \text{else} \end{cases} \tag{4.5}$$

We set $q = e^{-s}$ and consider the asymptotic as $s \to 0^+$.

The following proposition is a slight variation of a proposition of Lawrence and Zagier [20]. Since the proof is analogous and standard, we do not include it here.

Proposition 4.1 (p. 98 of [20])**.** *Let $C : \mathbb{Z} \to$ be a periodic function with mean value 0. Then for each $m \ge 0$ the L-series $L_m(s, C) = \sum_{n>m}^{\infty} C(n)n^{-s}$ (Re $(s) > 1$) extends holomorphically to all of and the function $\sum_{n>m}^{\infty} C(n)e^{-n^2 t}$ $(t > 0)$ has the asymptotic expansion*

$$\sum_{n>m}^{\infty} C(n)e^{-n^2 t} \sim \sum_{r=0}^{\infty} L_m(-2r, C)\cdot\frac{(-t)^r}{r!}$$

as $t \to 0^+$. The numbers $L_m(-r, C)$ are given explicitly by

$$L_m(-r, C) = -\frac{M^r}{r+1}\sum_{n=(m+1)}^{m+M} C(n)B_{r+1}\left(\frac{n}{M}\right) \quad (r = 0, 1, \ldots)$$

where $B_k(x)$ denotes the kth Bernoulli polynomial and M is any period of the function $C(n)$. Moreover, these expansions are valid in the region $|t| < \frac{2\pi}{M}$.

This proposition readily yields an asymptotic for the infinite series in (4.3) and (4.4).

Proposition 4.2. *With $q = e^{-s}$ we have the following asymptotic expansions valid in the region $|s| < \frac{\pi}{6}$.*

$$\sum_{n>2m}^{\infty}\left(\frac{-4}{n}\right)q^{\frac{n^2}{8}} \sim (-1)^m\left(\frac{1}{2} + \left(2m^2 - \frac{1}{2}\right)\frac{(-s)}{8} + \left(8m^4 - 12m^2 + \frac{5}{2}\right)\frac{(-s)^2}{8^2\cdot 2} + \cdots\right)$$

$$\sum_{n>2m}^{\infty}\chi_m(n)q^{\frac{n^2}{24}} \sim \frac{1}{2} + \left(2m^2 - 4m - \frac{5}{2}\right)(-s) + \frac{1}{2}(2m-5)(2m+1)(4m^2 - 8m - 41)\frac{s^2}{2} + \cdots$$

Using $(q)_\infty^{-1} = \sqrt{\frac{s}{2\pi}}e^{\frac{\pi^2}{6s} - \frac{s}{24}}(1+O(s^N))$ for any $N > 0$ (this follows from Euler–Maclaurin summation formula or the modularity of the Dedekind eta-function, see [21] page 53), we see that

$$C_{\le m}(e^{-s}) \sim \sqrt{\frac{s}{2\pi}}e^{\frac{\pi^2}{6s} - \frac{s}{24}}\left(\frac{(2m+1)}{4}s + \frac{(2m+1)}{16}s^2 + \cdots\right)$$

and

$$R_{\leq m}(e^{-s}) \sim \sqrt{\frac{s}{2\pi}} e^{\frac{\pi^2}{6s} - \frac{s}{24}} \left(\frac{(2m+1)}{4} s + 3 \frac{(2m+1)}{16} s^2 + \cdots \right)$$

A standard application of the circle method (see, for instance, Wright [18] for a similar situation) gives the theorem.

5. Some Guesses for the spt-Crank

This section collects some observations concerning the values of $N_S(m, n)$. In particular, we are concerned with defining the spt-crank in terms of partitions (perhaps with their parts marked by the multiplicity).

A *marked* partition means a pair (λ, k) where λ is a partition and k is an integer identifying one of its smallest parts. If there are s smallest parts then the $k = 1, 2, \cdots, s$. Evidently, a good first approximation for the spt-crank is

$$F(\lambda, k) := \begin{cases} p - k & \text{if } p > 0 \\ 1 - k & \text{if } p = 0 \end{cases}$$

where p is the number of parts in λ greater than or equal to k. If $T(n, m)$ is the number of marked partitions of n with $F(\lambda, k) = m$, then the difference

$$D(n, m) := T(n, m) - N_S(n, m)$$

is zero for most of the possible values of n and m. Tables 2 and 3 give the values of $T(n, m)$ and $D(n, m)$ for small $n \leq 12$.

Table 2. A table of values of $T(m, n)$.

$n\backslash m$	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
1										1									
2									1	1	1								
3								1	1	1	1	1							
4							1	1	2	2	2	1	1						
5						1	1	2	3	1	2	2	1	1					
6					1	1	2	4	4	3	4	3	2	1	1				
7				1	1	2	3	5	5	3	4	4	3	2	1	1			
8			1	1	2	3	6	8	6	6	6	6	5	3	2	1	1		
9		1	1	2	3	5	7	11	8	8	8	8	6	5	3	2	1	1	
10	1	1	2	3	5	8	12	15	10	11	11	11	10	7	5	3	2	1	1

Table 3. A table of values of $D(m, n)$.

$n\backslash m$	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	10	11
1												0											
2											0	0	0										
3										0	0	0	0	0									
4									0	0	0	0	0	0	0								
5								0	0	0	1	−1	0	0	0	0							
6							0	0	0	1	0	−1	0	0	0	0	0						
7						0	0	0	0	1	1	−2	0	0	0	0	0	0					
8					0	0	0	0	1	2	−1	−1	−1	0	0	0	0	0	0				
9				0	0	0	0	0	1	3	−1	−2	−1	0	0	0	0	0	0	0			
10			0	0	0	0	0	1	2	4	−3	−2	−2	0	0	0	0	0	0	0	0		
11		0	0	0	0	0	0	1	4	4	−2	−4	−2	−1	0	0	0	0	0	0	0	0	
12	0	0	0	0	0	0	1	2	6	4	−5	−4	−3	−1	0	0	0	0	0	0	0	0	0

The differences $D(n, m)$ are non-zero and have a simple regular behavior in the central angle between the two lines

$$n = 3m + 5 \text{ and } n = -2m + 2$$

and zero everywhere else. Additionally, the numbers become periodic on the boundaries. (This is hard to tell from the Table 2, but easy to see from a larger table.) The left side boundary has period 2 and the right hand boundary has period 3. After removing those periodic parts, there are two more boundary lines

$$n = -3m + 3 \text{ and } n = 4m + 11$$

which separate the regions where the numbers are periodic from the regions where they are not. So it is easy to conjecture that there is a series of boundaries $N = km + c$, $N = -km + c$, for each integer k, separating regions with period $k - 1$ from regions with period k.

Finally, we speculate that a definition of the spt-crank may be different depending on the size of the smallest part. It remains a challenge to find a definition of the spt-crank for ordinary partitions.

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