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Examining Coherence in Preservice Mathematics Teachers' Noticing of Students' Thinking About Classification in Geometry

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Abstract

This study aims to examine the thematic coherence among preservice mathematics teachers' noticing components when analysing students' thinking about classification in geometry, as well as the actions they propose to respond to those students. The research was conducted within an instructional module on the teaching of geometry, embedded in a mathematics methods course of a master's programme. The module was designed to foster preservice secondary mathematics teachers' pedagogical content knowledge alongside their noticing skills. Considering the mathematics education literature about the process of classification in geometry and the components of noticing, an analytical framework was developed to identify the thematic coherence of preservice mathematics teachers' noticing of students' thinking from two fictitious classroom episodes. Data came from individual written responses of 12 preservice mathematics teachers to an instructional task. The results overall patterns reveal strong thematic coherence in attending and interpreting, with responding also showing substantial, though comparatively lower, coherence. The findings also indicate that preservice teachers frequently proposed coherent responses that were both specific and may foster students' conceptual understanding. This study highlights that promoting coherence in professional noticing, particularly within the responding component, is vital for cultivating teaching practices that are both responsive and conceptually grounded.

Keywords: teachers' noticing; noticing of students' thinking; coherence; classification; geometry; preservice secondary mathematics teachers



Academic Editors: Salvador Llinares and Alessandro Jacques Ribeiro

Received: 30 September 2025

Revised: 13 November 2025

Accepted: 14 November 2025

Published: 16 November 2025

Citation: Oliveira, H., Cybulski, F. C., & Cyrino, M. C. d. C. T. (2025).

Examining Coherence in Preservice Mathematics Teachers' Noticing of Students' Thinking About Classification in Geometry. *Education Sciences*, 15(11), 1543. <https://doi.org/10.3390/educsci15111543>

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1. Introduction

Teacher noticing has been increasingly recognized as a central construct in mathematics education (Weyers et al., 2024), as it reflects how teachers attend to, interpret, and respond to students' thinking in ways that inform instructional decisions (Jacobs et al., 2010). Research has highlighted the importance of coherence across these three component processes, since effective noticing depends not only on identifying salient features of students' reasoning but also on aligning interpretations with subsequent instructional responses (Rotem & Ayalon, 2024; Thomas et al., 2023). Decision-making has been identified as the component of noticing that is particularly demanding for preservice teachers (Barnhart & van Es, 2015; Llinares et al., 2025). Despite a recent review indicating that this aspect has received increasing attention in recent studies (Weyers et al., 2024), it remains the least frequently examined facet of noticing (König et al., 2022). Additionally, the extant literature has

predominantly focused on general contexts or broad domains of mathematics. By contrast, relatively few studies have examined teachers' noticing within specific mathematical domains (König et al., 2022).

This gap is particularly salient in geometry (Ulusoy & Çakıroğlu, 2021), namely in what concerns the process of classification, where students' reasoning often entails tensions among prototypical examples, the application of formal definitions, and the hierarchical classification of geometric objects (Fujita, 2012). These complexities present notable challenges for preservice teachers (PTs), who must interpret and respond coherently to such reasoning (Avcu, 2023; Ulusoy & Çakıroğlu, 2021). Accordingly, the present study seeks to address this gap by examining the thematic coherence in PTs' noticing of students' thinking about classification in geometry.

1.1. Preservice Teachers' Professional Noticing and Its Development

Jacobs et al. (2010) considered professional noticing within the specific domain of children's mathematical thinking. According to these authors, teacher noticing comprises three interrelated component skills. The first is attending to the mathematical details in students' strategies. The second is interpreting children's understanding based on their strategies. Finally, the third component consists of deciding how to respond, taking these understandings into account. This latter component is closely connected to the previous two, as it requires teachers to consider both what they have observed and how they have interpreted it when determining their intended responses.

Developing professional noticing effectively is a challenging endeavour, especially for preservice teachers (Barnhart & van Es, 2015). In particular, preservice teachers (PTs) face challenges in the "deciding how to respond" noticing component, as they usually do not know the students or their context, which makes it difficult to choose appropriate responses (Barnhart et al., 2025). Buform et al. (2022) emphasized that participants also struggled more to propose conceptually oriented decisions or task modifications when they perceived that students had understood the concepts involved. Similarly, in the study by Wieman and Webel (2019), participants frequently chose not to respond at all when they interpreted students' mathematical thinking as correct, assuming that such students did not require additional support.

It is essential that preservice teachers are encouraged to focus their attention on how they can establish connections between their interpretations of students' thinking and their subsequent decisions on how to respond (Buform et al., 2022). Therefore, it is necessary to examine the links between professional noticing skills and, more specifically, to investigate the extent to which these skills are coherent with one another (Thomas et al., 2023) to support preservice mathematics teachers' noticing and instructional practices.

1.2. Coherence in Teachers' Professional Noticing

To date, few studies (e.g., Barnhart & van Es, 2015; Rotem & Ayalon, 2024; Sánchez-Matamoros et al., 2019; Thomas et al., 2023) have investigated how, and to what extent, one noticing skill influences and is influenced by another—that is, whether attending, interpreting, and deciding how to respond to students' thinking are interrelated, and the characteristics of such connections. One such study was conducted by Barnhart and van Es (2015), who found a significant relationship between attending and interpreting. According to these authors, high levels of attending to student ideas are necessary for interpreting and deciding in a sophisticated manner, although they are not sufficient conditions, because attending to relevant aspects does not guarantee that these will be used as evidence to interpret and respond. Interpreting and deciding how to respond were also related in the sense that they tended to share the same levels of sophistication. Therefore, the quality

of each skill, among other aspects, depends on the quality of the preceding one, which implies that high levels of attending without effective interpreting are unlikely to result in sophisticated responses. Moreover, preservice teachers were more likely to demonstrate higher levels of sophistication in interpreting than in responding, and higher levels in attending than in interpreting.

These results are consistent with those of [Sánchez-Matamoros et al. \(2019\)](#), who concluded that the more preservice teachers linked what they attended in students' written work with how they interpreted students' understandings, the more appropriate their proposed actions were. Like [Barnhart and van Es \(2015\)](#), they found that attending to the mathematical aspects of students' work was necessary for interpreting their understandings, but not sufficient. It was only when preservice teachers attended to all relevant mathematical elements that they could interpret students' understandings, although they sometimes encountered difficulties in doing so. Furthermore, general interpretations were followed by general instructional actions, whereas more complete interpretations led to conceptual actions that could better support students' learning.

[Rotem and Ayalon \(2024\)](#) suggest that coherence between interpreting and responding does not always occur. In their study, in a few cases, high coherence in proposing alternative teaching strategies was not preceded by high coherence in interpreting students' mathematical statements and teachers' responses. This indicates that high coherence in interpreting is not necessarily required for high coherence in responding. They also concluded that one factor contributing to high coherence levels in responses was the mathematical task itself, which allowed multiple solutions and thus enabled preservice mathematics teachers to reflect on different mathematical elements and students' thinking, making it easier for them to devise coherent teaching strategies.

According to [Thomas et al. \(2023\)](#), one specific type of coherence is *thematic coherence*, which refers to a thematic linkage between noticing skills. Such coherence can be observed when noticing capacities involve a common mathematical or pedagogical theme. In their analytical process, those authors considered only the presence (coherent) or absence of themes. As a result, they found an interplay between themes across the skills and the overall quality of noticing. For example, they reported significant relationships between attending–interpreting coherence and the quality of these skills. In other words, identifying a theme while attending to and maintaining it throughout the other skills was associated with higher professional noticing quality. This continuity of a common theme can indicate more productive and proficient professional noticing. Therefore, thematic coherence is a factor that should be considered in preservice mathematics teacher education and needs to be intentionally developed.

Mathematics education researchers also emphasize the importance of conducting subject-specific studies to investigate potential particularities in professional noticing to inform teaching and curricular decisions ([König et al., 2022](#)). To date, one of the few studies that addressed this purpose in geometry was developed by [Ulusoy and Çakıroğlu \(2021\)](#). Examining professional noticing in the context of trapezoids, they found that PTs most often proposed general instructional actions that were not based on the students' strategies they had attended to and interpreted. However, some issues remain to be investigated, such as those related to noticing and classification in geometry, a topic that poses difficulties for PTs ([Brunheira & Ponte, 2019](#); [Llinares et al., 2016](#)). Therefore, it is important to promote and investigate the PTs' noticing skills in geometry, namely the classification in geometry.

1.3. Classifying in Geometry

The process of classifying involves identifying relationships among objects based on similarities in their attributes ([Mariotti & Fischbein, 1997](#)). In general, as [Villiers \(1994\)](#)

states, classification in geometry can be partitive or hierarchical. Partitive classification considers disjoint sets, that is, sets with no elements in common, and requires exclusive definitions. Hierarchical classification, on the other hand, is based on inclusion relations, where more specific concepts are seen as included within more general ones, or where different sets can share common objects, requiring inclusive definitions. For example, in a hierarchical classification, there is a particular prism that can be classified as a cube and as a parallelepiped and thus referred to by different names. The same does not occur in a partitive organization, in which a geometric object cannot belong to more than one set.

Markman (1989) identified two key characteristics of hierarchical classification, namely asymmetry and transitivity. Inclusion relations are asymmetric because if a subset is included in a more general set, all objects in the subset belong to the more general set, but the reverse is not true. For example, every prism is a polyhedron, but not every polyhedron is a prism. It follows that all critical attributes¹ of polyhedra are also critical attributes of prisms, but not vice versa. Inclusion relations are also transitive: if a subset is included in a general subset that is itself included in another, more general subset, then the first subset is also included in the most general subset. For example, if a cube is a parallelepiped, and a parallelepiped is a prism, then a cube is a prism. Consequently, the critical attributes of a prism are also critical attributes of a cube, but not vice versa. The critical attributes of a geometric object constitute the subset of necessary and sufficient conditions required to define the object without superfluous or redundant information (Villiers et al., 2009). Identifying the critical attributes of a geometric object is a complex task, even for preservice mathematics teachers (Avcu, 2023; Cybulski et al., 2024).

Hierarchical classifications are more commonly used, as they are more functional and enable deductive and global interpretations (Villiers, 1994). At the same time, they can be more challenging, as they require logical deduction and can be influenced by the prototypical phenomenon (Fujita, 2012). In geometry, some representations of concepts tend to be overrated because they include not only the critical attributes but also visually striking characteristics. For example, some prototypical examples in school geometry include the “slanted parallelogram” and the quadrangular pyramid, which is one of the most common examples in the study of pyramids. These are considered prototypes, as they are typically recognized as the main (and sometimes the only) representation of the concept, influenced by visual perception (Hershkowitz, 1990). This prototypical phenomenon can hinder the understanding of inclusion relations, even when definitions are known (Cybulski et al., 2024; Fujita, 2012).

According to Llinares et al. (2016), understanding inclusive relations—for example, that a square can be considered a type of rhombus—is essential for preservice secondary school teachers to learn about the development of students’ notions of quadrilaterals classification, in contrast to a perceptual perspective based on prototypical figures. Brunheira and Ponte (2019) also discussed that prototypical figures can influence preservice primary teachers’ classification of quadrilaterals. Furthermore, these authors highlighted that some difficulties derived from the concept of classification itself, as some participants were unfamiliar with its structure. However, in a second moment in the study, when the PTs had to classify prisms, they could establish hierarchical relations more easily, since prototypical figures did not influence them as much as before. This was mainly because they already knew some characteristics of the classification process and were less familiar with prisms than with quadrilaterals, allowing them to focus more on the prisms’ properties than on their prototypical characteristics.

Considering all this, this study addresses two research questions: (1) to what extent there is coherence among the three noticing component processes of PTs—attending, interpreting,

and responding—when analysing students’ reasoning about classification? and (2) what specific actions are proposed by PTs to respond to students’ thinking about classification?

By situating the analysis in the domain of geometric classification, the study extends research on teacher noticing into a less-examined but conceptually demanding area of mathematics. It also contributes to the literature by foregrounding the responding component, offering insights into how PTs propose instructional actions based on their interpretations. In doing so, the study aims to inform teacher education by identifying strengths and limitations in PTs’ capacity to sustain coherence in their noticing, particularly regarding classification in Geometry, thereby supporting the development of more responsive and conceptually grounded teaching practices.

2. Methods

2.1. The Context of the Study

The study was carried out in the context of an instructional module on the teaching of geometry in a mathematics methods course in the first semester of the second year of a master’s program for teaching mathematics at a Portuguese University. All 12 preservice secondary mathematics teachers (PTs) enrolled in the course had a three-year degree in mathematics, applied mathematics, or statistics.

The instructional module consisted of eight lessons, two hours each, introducing topics central to geometry teaching through practical activities and discussion of theoretical ideas concerning the main reasoning processes in geometry (including defining and classifying), the prototypical phenomenon, characteristics, and properties of some families of polygons and geometric solids, and the use of physical and digital resources for teaching geometry. Several instructional tasks were intended to promote their professional noticing of geometry teaching and learning. Also, previously to this instructional module, the course provided many opportunities to PTs to get involved in approximations of practice (Grossman et al., 2009), for instance, by analysing students’ written work and watching and analysing videos from real classrooms. Overall, the course aims to prepare PTs for student-centred approaches, entitled as “exploratory teaching” (de Jesus et al., 2020), commonly discussed with them in terms of mathematical tasks to foster students’ mathematical understanding and the associated teacher’s role, and classroom dynamics.

The first author assumed simultaneously the role of instructor and researcher, planned the lessons, and designed the instructional tasks in collaboration with the other two authors. Most tasks were carried out in pairs or small groups by the 12 PTs, but the last one, which is used as a resource for this study, was solved individually to better understand the professional noticing of each PT.

2.2. Data Collection

Data for this study were obtained from the second part of the last teacher education task (Figure 1) in the instructional module, which targets PTs’ noticing in geometry. This part consists of two episodes from a fictitious 7th-grade classroom where students are involved in the exploration of how Euler’s law² for polyhedra applies to different families of solids. We asked PTs to address the thinking of the three fictitious students who take part in the two episodes (Cátia, Henrique, and Teresa).

The three questions included in the task were intended to support PTs in showing the three components of noticing concerning the students’ thinking: the first one on attending, the second on interpreting, and the last one on responding. This task’s format is in line with the options assumed in other studies on the noticing of mathematical thinking that proved successful in eliciting the different components of PTs’ noticing (Cabral et al., 2021; Moreno et al., 2025; Sánchez-Matamoros et al., 2019), and also on the coherence between

the interpretation of students' statements and the proposed teaching strategies (Rotem & Ayalon, 2024). Additionally, in the task's first question, we prompt PTs to focus specifically on the processes of defining and classifying in geometry, as in previous instructional tasks, we realised that their understanding of these processes needed further development.

Pyramids and bipyramids

In a 7th grade class, the students had been studying various polyhedra. Since the students were already familiar with the family of pyramids from primary school, the teacher thought it would be interesting to have them explore bipyramids. In accordance with the 7th grade curriculum, she also explored the regular polyhedra.

Episode 1

After verifying that Euler's rule applies to prisms, Henrique became curious to check whether it also holds for bipyramids. He tested a few cases and confirmed that, indeed, Euler's rule still applied. When they went on to verify whether the relation was also valid for regular polyhedra, Henrique immediately stated: "I don't need to check the tetrahedron or the octahedron because I already know it's true!" However, Teresa disagreed with her classmate, arguing: "Bipyramids are two pyramids joined together, so it doesn't make sense that the same relation applies to both types of polyhedra."

Episode 2

After exploring these families of polyhedra, Cátia asked the teacher a question: "Teacher, I have a doubt. The regular tetrahedron and the regular octahedron are regular polyhedra, but the tetrahedron is a pyramid and the octahedron is a bipyramid! So how can they belong to the same family and at the same time to different families?!"

Questions:

1. What can you say about the understanding that each of the students demonstrates regarding the processes of defining and classifying in geometry?
2. What factors might have influenced the way the students thought?
3. What would you do, as a teacher, to support each of these students? (You may include in your answer specific actions, instructions, or questions you would ask them).

Figure 1. The task proposed to preservice mathematics teachers.

The statements of the three students presented in the two episodes intend to illustrate salient aspects of their thinking concerning classification in geometry, in accordance with the literature (Fujita, 2012; Villiers, 1994), and with a focus on topics from the mathematics curriculum for that school year in Portugal, ensuring consistency with those topics. Table 1 presents the main characteristics of students' mathematical thinking on classification that we intended PTs could attend to and interpret.

In a previous session, the PTs had the opportunity to recall Euler's law in the case of regular polyhedra, and they were supposed to know that this law applies to all convex polyhedra. From these sessions, it was our understanding that they were not familiar with the bipyramid's family. Thus, we expected that the situation presented in the task, by integrating both familiar and unfamiliar elements for PTs, would provide an appropriate level of challenge to prompt their noticing of students' thinking regarding the classification process in geometry.

Analysing written documents, such as the episodes incorporated into the proposed task, may foster PTs' attentiveness to detail and their focus on key mathematical aspects, including the strategies employed by students, as this process requires a more deliberate and analytical approach when compared, for instance, with video-based analysis

(Magiera & Zambak, 2021). Accordingly, the examination of students' thinking in written documents, even in the absence of contextual information, has proved to be an effective means of eliciting PTs' noticing (Scheiner, 2023).

Table 1. Main characteristics of students' mathematical thinking on classification in the two episodes.

Student	Episode	Characteristics of Students' Mathematical Thinking on Classification
Henrique	1	- acknowledges a hierarchical classification of solids - applies the inclusion relationship between the octahedron and the set of bipyramids, recognizing that critical attributes of the bipyramids apply to the octahedron - wrongly assumes that a tetrahedron, being a pyramid, is a subset of the set of bipyramids
Teresa	1	- assumes that a bipyramid is the junction of two pyramids, thus classifying the solid by its appearance (prototypical phenomenon) - disregards the critical attributes of each solid (pyramides and bipyramides) in their definition - assumes a partitive classification of the two families of solids, not accepting that the same property (e.g., Euler's law for polyhedra) may apply to both
Cátia	2	- disregards the critical attributes of the tetrahedron and the octahedron in their definition - misses the fact that two solids may belong simultaneously to two different families

2.3. Data Analysis

To analyse the thematic coherence among the three component processes of noticing (Thomas et al., 2023), we examined the responses of 12 PTs to the questions concerning the two episodes used for data collection, in which three hypothetical middle school students participated. Given the content of the episodes in relation to classification in geometry (Table 1) and the literature on this topic (Hershkowitz, 1990; Markman, 1989; Villiers, 1994; Villiers et al., 2009), for data analysis concerning the noticing of students' thinking, we considered the following four dimensions as themes that could emerge from the PTs' responses.

- Classification based on prototypical examples (A): When classification is based on examples that include not only the critical attributes but also visually striking characteristics and are typically recognized as the main representation of the concept (Hershkowitz, 1990).
- Classification based on the mathematical definitions (B): When classification is based on the critical attributes of a geometric object, that is, the subset of necessary and sufficient conditions required to define the object (Villiers et al., 2009).
- Types of classification—Partitive versus Hierarchical (C): When classification considers sets with no elements in common (partitive) or that different sets can share common objects (hierarchical) (Villiers, 1994).
- Inclusion and relations/Properties (D): When classification considers relations or properties such as asymmetry—if a subset is included in a more general set, all objects in the subset belong to the more general set—and transitivity—if a subset is included in a subset that is itself included in a more general subset, then the first subset is also included in the most general subset (Markman, 1989).

As all students' interventions in the two episodes that PTs analysed had some relationship with at least one of the four themes concerning classification in geometry, for the analysis of the thematic coherence between the attending process and the students' thinking, we sought evidence (or absence) of a meaningful connection between what the

PT described from the student's talk as an aspect related to classification and the students' actual thinking. Thus, in addition to identifying the presence or absence of the same theme (Thomas et al., 2023), we have also considered the adequacy of the PTs' description related to the thinking expressed by the students.

The same rationale was applied to analyse the coherence between interpreting and attending. Thus, in each of the PTs' responses that involve interpreting, we looked for a thematic and appropriate connection with what they had referred to in their response regarding attending.

Finally, regarding the responding component of noticing, we also looked for the presence or absence of coherence with what PTs have interpreted. However, considering that a lack of coherence between responding and interpreting does not necessarily imply a weak connection between attending and responding (Rotem & Ayalon, 2024), we assumed a holistic perspective in the analysis of the linkage between responding and interpreting or attending. Thus, we considered that there is thematic coherence between responding and one of the other two components of noticing when a common theme between them emerges in the PT's response, and it appropriately targets the idea that she/he had expressed when attending to and/or interpreting the situation. If the PT responded in a way that did not target the ideas that she/he expressed when interpreting, we classified the response as not coherent.

To illustrate how thematic coherence was analysed in PTs' responses, we provide the examples of PT2 and PT9, focusing on Teresa's intervention. PT2 correctly identifies that Teresa recognizes that pyramids and bipyramids constitute distinct families of solids: "The student appears to know pyramids and bipyramids, since she disagrees with her classmate Henrique when he refers to both solids as bipyramids" (Attending). Additionally, PT2 identifies Teresa's struggle in accepting that these solids may share some properties and interprets this difficulty in light of what she considers to be a partitive view of classification: "the student's difficulties probably stem from having the concepts defined in a partitive way, failing to see relationships between objects belonging to different families" (Interpreting). PT2 then suggests supporting Teresa in identifying the critical attributes of those two families, to counteract the partitive perspective of classification: "it would be important to review the definitions [of pyramids and bipyramids] that the student is considering, so that only critical attributes are used, to avoid partitive definitions and classifications" (Responding). In the PT9 case, which presents similar formulations in his responses regarding the attending and interpretation components, the responding component was classified as incoherent. This PT considers Teresa to be correct when considering pyramids and bipyramids as two distinct families; however, he does not focus on the student's difficulty he identified regarding the partitive classification, stating: "In Teresa's case, I believe she is correct; I have no actions [to propose]".

Additionally, to address the second research question, we looked for actions proposed by the PTs to address students' thinking about classification in the two episodes they had analysed. An inductive thematic analysis (Neuendorf, 2019) allowed us to identify four main themes related to envisioned teacher's actions by the PTs: (i) questioning to prompt students' reflection; (ii) encouraging experimentation, resource exploration, and the use of representations; (iii) connecting with students' prior experiences and using analogies; and (iv) taking general actions.

All PTs' responses were first analysed by one of the researchers (R1) to search for the main themes that emerged from their responses and gain a global perspective on the coherence among the three components of noticing. This researcher also identified the main actions that the PTs were proposing when responding to students. Then a second researcher (R2) went through all the material, codified it, and used a spreadsheet to record the number

of coherent and non-coherent responses for each PT and each noticing component and theme related to classification. After that, the codification and quantification went back to R1, who reviewed them and made some corrections and suggestions. Subsequently, a common understanding was achieved, and the final codification version was produced.

The quantification of coherent and non-coherent responses gives us an indication of the presence of coherence in PTs' responses, which we present in the Section 3. That is complemented by a qualitative analysis of the PTs' responses that illuminates the strengths and difficulties in their noticing concerning classification. Throughout the following section, PTs are mentioned as PT1, PT2, and so forth.

3. Results

To analyse the coherence among the noticing component processes, we considered the responses produced by the 12 PTs about the two episodes included in the task used for data collection, in which three hypothetical middle-school students took part. Given the content of the episodes with respect to classification in geometry, we assessed the presence (or absence) of thematic coherence in PTs' according with the four themes: (A) Classification based on prototypical examples; (B) Classification based on the mathematical definitions; (C) Types of classification—Partitive vs. Hierarchical; and (D) Inclusion and relations/properties (Table 2).

Table 2. Coherence in noticing components and themes in classification in geometry.

Noticing Components	Themes in Classification						
	Coherence	Incidence	(A)	(B)	(C)	(D)	Total
Attending	YES	93%	8	12	23	26	69
	NO	7%	1	0	2	2	5
	Total		9	12	25	28	74
			12%	16%	34%	38%	(38%)
Interpretating	YES	87%	8	9	20	21	58
	NO	13%	0	0	3	6	9
	Total		8	9	23	27	67
			12%	13.4%	34.3%	40.3%	(35%)
Responding	YES	81%	4	7	14	18	43
	NO	19%	0	2	4	4	10
	Total		4	9	18	22	53
			7.5%	17%	34%	41.5%	(27%)

Overall, PTs addressed more than one aspect for each of the three students, with an incidence rate of 2.1 in the case of attending, the highest, with 74 occurrences, and 1.5 in the case of responding, the lowest, with 53 occurrences. It is also worth noting that the frequency for interpreting is quite high, with 67 occurrences. The lower value for responding may be partly associated with the fact that, when PTs judged that the three students displayed a correct notion of classifying the geometric solids referred to in the episode, they no longer felt the need to indicate actions for the teacher to take, sometimes noting that they had nothing to add.

We now turn to the relationship between the three noticing component processes, more specifically, to the thematic coherence that emerges from PTs' noticing as they examine students' thinking about classification (Table 2). In the case of attending, with the highest coherence value (93%), the majority of PTs' responses show the ability to identify aspects related to the classification process that are aligned with the mathematical thinking the

hypothetical students (Cátia, Henrique, and Teresa) display in the episodes. The high level of coherence also recorded for Interpreting (87%) in most PTs' responses shows the ability to make sense of students' ideas, namely, the understanding and the difficulties they display associated with the process of classifying geometric solids. Finally, the incidence of responses that exhibit coherence for Responding is also quite significant (81%). This deserves particular emphasis, since it corresponds to a capacity PTs showed to present proposed actions that are not only thematically coherent with what they had interpreted about students' thinking, but that also proved appropriate in that context. Therefore, in the PTs' responses, there is a strong relationship between Interpreting and Attending, as well as between Responding and Interpreting, along with strong coherence between Attending and the students' thinking revealed in the episodes.

The values in Table 2 also show that PTs' responses focus mostly on the last two classification themes (Types of classification—Partitive vs. Hierarchical and Inclusion and relations/properties), ranging from 34% to 42% approximately and, taken together across the three noticing process components, always amounting to over 70%. Similar values are recorded between the first two and between the last two themes for the three noticing components.

We now examine how these results are distributed among the 12 PTs, to understand to what extent the high incidence of coherent responses reflects the entirety of participants' noticing (Table 3). First, it is noteworthy that all PTs were able to identify a significant set of aspects related to the classification of geometric solids for the three noticing component processes. The number of responses considered coherent for each PT ranges from 11 to 18, with an average of 14.2.

Table 3. Presence of coherence of noticing component processes in PT's responses.

Coherence	PT1	PT2	PT3	PT4	PT5	PT6	PT7	PT8	PT9	PT10	PT11	PT12
Yes	11 (79%)	16 (100%)	12 (80%)	17 (94%)	12 (75%)	15 (79%)	13 (68%)	11 (92%)	16 (89%)	18 (100%)	16 (100%)	12 (92%)
No	3 (21%)	0 (0%)	3 (20%)	1 (6%)	4 (25%)	4 (21%)	6 (32%)	1 (8%)	2 (11%)	0 (0%)	0 (0%)	1 (8%)
Total	14	16	15	18	16	19	19	12	18	18	16	13

Regarding the level of coherence between the three noticing component processes, six PTs (50%) presented only coherent responses or just one non-coherent response. Even so, about 33% of PTs show a percentage over 20% of non-coherent responses, with PT5 and PT7 standing out with 25% and 32%, respectively. Thus, it can be said that most PTs show a high level of coherence in their responses, which is in line with what was observed in Table 2 for coherence levels for each of the three noticing component processes in the four geometry classification themes. In the following section, we aim to identify the component processes in which the greatest difficulties were recorded and examine how these relate to the specific dimensions considered for the classification process.

3.1. Coherence Between Attended Aspects and Students' Thinking

For the attending component, we analysed the coherence between the aspects identified by the PTs in relation to the classification process demonstrated by the students and the content of the episodes, in line with the intentions behind their creation. This noticing component has the largest number of responses (38% of the total) and the highest percentage of coherent responses (Table 2), with quite high values.

The themes (C) Types of classification—partitive versus hierarchical, and (D) Inclusion—relations/properties together received most responses, consistent with the

thinking of the three students in the episodes. The conceptual relationship between these two themes may also explain why they received a similar number of responses (Table 2). Thus, in Henrique's case, most PTs pointed out that he showed an understanding of hierarchical classification between the octahedron and tetrahedron and the bipyramids, even though that was not correct, as PT4 notes: "Henrique shows that he knows the hierarchical relation since he believes that it is possible to apply Euler's law to all bipyramids, assuming that the tetrahedron and octahedron are bipyramids (Classify)." This PT also highlights the inclusion relationship associated with a hierarchical classification—namely, that if a property applies to the class of bipyramids, in this case the Euler's law for polyhedra, it will also apply to the subclass of octahedra and tetrahedra—while pointing out the student's misconception: "After concluding that Euler's relation applied to bipyramids, Henrique assumed that it also applied to the tetrahedron and octahedron because they are bipyramids, which is not true because the tetrahedron does not have the critical attributes to be one; that is, the tetrahedron does not belong to the family of bipyramids".

Regarding Teresa, several PTs attend to the fact that the student's remarks show a partitive perspective on the classification of octahedra and tetrahedra, as expressed by PT6: "Teresa seems to have the definitions of pyramid and bipyramid correct; however, she is having difficulty classifying because she is using a partitive rather than a hierarchical classification".

As for Cátia, the vast majority of PTs recognise that the objection she raises is associated with the notion of inclusion in classifying the solids in question. PT11, for example, mentions that the student seems to have "difficulty understanding that, when classifying mathematical objects, one can consider subsets of families that satisfy specific properties ... she does not understand that, although two objects may belong to 'different families', they can also belong to the 'same, broader family'".

Only four non-coherent responses were recorded, out of 53, for these two themes (Table 2), two of them from PT5 and addressed to student Henrique. This PT claims there is understanding on the student's part regarding hierarchical classification but based on something the student does not say: "The student understands the hierarchical relation between the families of pyramids and bipyramids, since he shows that if a property applies to one regular polyhedron, it also applies to all other regular polyhedra." In fact, the episode states that Henrique said he would not need to check whether Euler's relation applied in the case of the tetrahedron and the octahedron because he had already done so for bipyramids. Therefore, PT5's Attending is not coherent with the student's thinking. Recall that PT5 is one of the PTs who presented a higher percentage of non-coherent responses (Table 3) and, interestingly, she felt the need to make explicit that she had difficulties analysing the episode, stating that she "was not familiar with the family of pyramids, bipyramids, and Euler's relation."

The two themes (A) Classifying based on prototypical examples and (C) Classifying based on the mathematical definitions (critical attributes) had a lower percentage of responses (12% and 16%, respectively—Table 2). In the former case, this appears almost exclusively in association with Teresa's intervention; these PTs managed to identify that the student does not attend to the definitions of pyramid and bipyramid when considering whether Euler's law for polyhedra holds for the two types of solids. PT2 notes in this regard that the student is influenced by the visual characteristics of each solid, asserting that these could not share the same property: "By failing to identify similarities in solids that are visually different, she shows difficulty classifying, since even though they are solids belonging to different families, they can have features in common." For classifying based on the mathematical definitions (critical attributes), PTs mainly note that Henrique does not master the definition of a bipyramid, which leads him to include the tetrahedron in that class. Also in Teresa's case, some PTs attend to the fact that she does not consider all

critical attributes when defining a bipyramid, as PT11 notes: “The student also presents a definition of bipyramid when she says that ‘Bipyramids are two pyramids joined together’, but she does not describe all the critical attributes.”

Although only one non-coherent response was recorded, one would expect PTs to identify more elements in the thinking of students Henrique and Teresa associated with these two themes. However, it was found that PTs’ attention often shifted away from the classification of geometric solids—the focus of the instructional task—to Euler’s law for polyhedra, which meant that some responses were not considered in our analysis for these two themes. For example, in the following excerpt, “Henrique generalised a rule, based on specific cases—he considered that what was valid for one family of solids would also be valid for another,” PT1 attends to the process of generalising Euler’s relation from particular cases but does not mention classification.

3.2. Coherence Between Interpreting and Attended Aspects of Students’ Thinking

Interpreting was the noticing component that received the second highest number of responses from participants (35% of the total), with a high percentage of coherence in the responses (Table 2), which shows a strong relationship between Interpreting and Attending. As with Attending, the themes (C) Types of classification—partitive versus hierarchical and (D) Inclusion—relations/properties had, between them, a similar number of responses and, taken together, concentrated most of the PTs’ responses, in line with the thinking of the three students in the episodes.

In Henrique’s case, for whom most PTs had attended to the hierarchical classification present in his thinking, all—with only one exception—managed to present an interpretation coherent with what they had stated; that is, that he had mistakenly considered pyramids to be a subclass of bipyramids. For example, PT8 states that when Henrique says, “I don’t need to check for the tetrahedron’ whether Euler’s relation holds,” he thinks the tetrahedron is a bipyramid, and adds that this “reveals difficulties with the definition of bipyramid and its properties.” PTs’ interpretation of Teresa’s thinking is also coherent with what they had attended to, namely that a partitive perspective of classification interferes with her perception that solids belonging to different families can share common characteristics. As PT9 notes: “This is an explicit case of partitive classification that greatly limits the relations between categories and their properties.”

The theme Inclusion—relations/properties concentrates the largest number of PTs’ responses, with a strong incidence in the case of student Cátia, which is coherent with the episode’s content and the difficulty they had attending to. PT10 further notes that this difficulty is associated with another episode involving Cátia that links with the classification of another class of geometric solids: “Cátia shows difficulties with the concept of inclusion, being unable to understand that two different families can be contained within a broader family that encompasses both. This difficulty may already be recurrent and was previously noted when she worked with parallelepipeds”.

This theme also recorded the highest proportion of non-coherent responses (22.2%). Of the six responses that did not show coherence between Interpreting and Attending, four refer to Henrique and two to Cátia. In Henrique’s case, the lack of coherence in the interpretation stems from the erroneous assumption that pyramids constitute a subclass of bipyramids and, as PT6 states, that the student would have concluded: “that if Euler’s formula holds for bipyramids, which are the joining of two pyramids, it will also hold in particular for pyramids.” Thus, again, this lack of coherence is associated with PTs’ lack of knowledge about these families of solids.

The two themes (A) Classifying based on prototypical examples and (B) Classifying based on the mathematical definitions (critical attributes) obtained a much lower percentage

of responses than the previous two (12% and 13% of the total, respectively—Table 2). As with Attending, for the former classification theme, interpreting responses appear almost exclusively devoted to student Teresa, which shows coherence between the two noticing components. Even so, in most cases, PTs did not present an interpretation of the objection raised by the student in terms of the prototypical phenomenon, perhaps because the family of solids involved—bipyramids—is one with which these PTs are not very familiar. One of the few cases in which such an interpretation is associated with an inference is, interestingly, PT3, who links the student's objection to transferring previous experiences with solids to this situation: "Teresa may be transferring her experience from other mathematical situations, such as sums of areas or volumes, to the idea that joining two solids means adding all their characteristics (vertices, edges, and faces)."

Also in line with attending, the theme Classifying based on the mathematical definitions (critical attributes) arises essentially in Henrique's case, where again it is highlighted that the student did not consider critical attributes of pyramids and bipyramids, which led him to regard pyramids, in most PTs' view, as a subclass of bipyramids, as PT10 notes: "He seems not to have properly understood the critical attributes of a bipyramid, or at least he did not check whether these applied to the tetrahedron." Also noteworthy is PT9's case, who sought to interpret Henrique's thinking by making inferences about the possible influence of his experience from an episode previously analysed in another task: "Henrique made the mistake of formulating the definition of bipyramid in a way that included pyramids, perhaps because in the previous task Henrique had realised that 'All parallelepipeds are quadrangular prisms!', which implies one definition nested within another. Henrique may have jumped ahead, assuming that in this case, the same would occur with pyramids being bipyramids."

In short, since the number of responses considered under these two themes in Attending was already quite limited, this was also reflected in the number of occurrences for Interpreting, although there was considerable coherence between these two noticing components.

3.3. Coherence Between Responding and Interpreting or Attending to Students' Thinking

The number of responses considered for this noticing component was the lowest among the three (27% of the total), with 14 fewer responses across the four Classification themes compared to Interpreting, which is not substantial. As in the previous dimensions, there is a high percentage of coherence between Responding and Interpreting and Attending (Table 2) and a clear concentration of responses in the last two classification themes (together accounting for 75% of the responding component).

Regarding (C) Types of classification—partitive versus hierarchical, almost all PTs presented one or two responses corresponding to a responding answer that is coherent with what they had interpreted about the type of classification that one or both students expressed in the episode.

Thus, in Teresa's case, many PTs associated the student's objection (coherently in Attending and Interpreting) with a partitive classification perspective; as such, the Responding is associated with teacher actions, more generic or more specific, seeking to support the progression to a hierarchical classification. To this end, several PTs take into account the need for the student to consider, in the definitions, the critical attributes of the geometric solids at issue—in this case, the tetrahedron and the octahedron, and more broadly, bipyramids and pyramids. This is the case with PT10: "It might be necessary to ask about the similarities among bipyramids, pyramids, and the remaining polyhedra, arriving at the critical and non-critical attributes of each, so as to construct a hierarchical classification including the different families."

In Henrique's case, many PTs present a response that points to the need for the student to review the definition of a pyramid, so that he realises he cannot assume that a property that applies to bipyramids will also apply to pyramids and thus becomes aware that the latter do not constitute a subclass of the former. As PT9—who, as we saw for interpreting, infers the origin of the student's error by linking it to an earlier experience—notes that he would “perhaps try to return to the definition of pyramid, hoping that he would identify the fact that it is not compatible with that of bipyramid. Alerting that we are in a different case from the previous one (task—part 1).”

In Cátia's case, responding emerges in a way that is coherent with what PTs had interpreted, aiming to lead the student to understand what is involved in an inclusion relationship. It is interesting in this respect that PT1's interpretation clearly distinguishes the definition of a solid from its classification as the origin of the objection raised by both Cátia and Teresa, noting that they “have some difficulty understanding that several solids, although with different definitions, can be grouped into the same family; that is, that they can have the same classification (for a certain characteristic, or set of characteristics).”

3.4. Proposed Actions Involved in Responding

It is therefore apparent that most PTs' responses are thematically coherent with interpreting and attending and are pertinent to the students' thinking in the two episodes. According to the analysis carried out, these responses are mostly specific and target students' conceptual understanding, as explained below.

One frequent action is questioning to prompt students' reflection that allows them to recognise an error, leads them to make comparisons among families of solids, or to review definitions they are assuming for the families of solids, as PT11 expresses regarding Henrique: “I would ask ‘Why don't you need to check that property for the tetrahedron?’, as I believe that would lead the student to explain that the tetrahedron belongs to the family of bipyramids. . . Then I would ask ‘Why is the tetrahedron a bipyramid?’, with the aim of getting the student to explain the definition of bipyramid and to notice the critical attributes that the tetrahedron lacks.”

Also appearing with some frequency is PTs' encouragement of experimentation, resource exploration, and the use of representations, namely, the construction of Venn diagrams or concept maps to support the classification of solids. For example, for Cátia's case, PT5 suggests building a scheme (concept map), combined with questioning: “I would ask the student ‘What is a regular polyhedron? Can pyramids be regular polyhedra? And bipyramids?’ I would ask the student to make a diagram including pyramids, bipyramids, and regular polyhedra. Then I would ask where she would place the tetrahedron and the octahedron in that diagram.”

Another action that emerged was the attempt to connect with students' prior experiences and to use analogies, either with concepts the students might already know or with their previous experiences. Thus, PT9 draws on a previous episode in which Cátia also takes part, seeking to leverage that experience: “Instead of trying to explain the situation, I could, for example, confront her with the definitions of parallelepiped given by her (Cátia) and by Anabela, in an attempt for her to perceive the possibility of defining the same solid in different ways, and that some are included in others.” In the case of analogies, PT10—for Cátia—would resort to familiar number sets to help her understand that the tetrahedron and octahedron, although having different characteristics, can belong to the same class: “An example that could be given is that of the integers, which are a ‘family’, but within this we can form two families, the positives and the negatives, and both belong to the same large family of integers. Then a Venn diagram could be drawn with the student to make inclusion clearer.”

Finally, some responses refer to general actions that the PTs propose to respond to the students. However, these were considered not thematically coherent, as they did not specifically address the themes that emerged in the PTs' previous interpretation of students' thinking.

4. Discussion and Conclusions

This study set out to examine the extent and nature of coherence among preservice teachers' (PTs) noticing processes—attending, interpreting, and responding—when analysing students' thinking about classification in geometry. Overall results show high thematic coherence in attending (93%) and interpreting (87%), with a still substantial but comparatively lower coherence in responding (81%). The results extend prior research that positions professional noticing as three interrelated component skills (Jacobs et al., 2010) and align with evidence that the quality of later skills depends on the quality of earlier ones, though not deterministically so (Rotem & Ayalon, 2024; Thomas et al., 2023). Our domain-specific findings in geometry confirm that strong attending and interpreting frequently co-occurred (Barnhart & van Es, 2015; Sánchez-Matamoros et al., 2019), while responding—despite its relatively high coherence with the other two components—remained the most fragile link, echoing reviews that also identify responding as the least examined and often most challenging facet of noticing (König et al., 2022).

Two features of PTs' noticing stood out. First, the main theme related to classification in geometry that emerged in their noticing of students' thinking was the pair “partitive vs. hierarchical classification” and “inclusion/relations,” which together accounted for over 70% of coded instances across components, reflecting the centrality of inclusion relations for understanding geometric taxonomies. These results may be related to the pedagogical options in the instructional module, which emphasized those topics as important learning goals to be addressed in the teaching of middle school mathematics. Second, content-knowledge gaps occasionally disrupted coherence between different noticing components, particularly in the case of the definitions and critical attributes of bipyramids and pyramids, leading some PTs to infer incorrect subclass relationships and to overextend properties (e.g., Euler's law for polyhedra) across families. This pattern corroborates reports that domain knowledge shapes the interpretation–response link (Linares et al., 2025).

The findings also indicate that written-work analysis can elicit detailed, theme-sustaining noticing chains—consistent with literature showing that artifact-based tasks reduce in-the-moment pressure and promote fine-grained attention to students' strategies (König et al., 2022; Magiera & Zambak, 2021). However, instances of responding were less frequent than those of attending and interpreting, as some PTs refrained from offering a response when considering the students' reasoning correct, a pattern previously noted in other studies (Wieman & Webel, 2019). Together, these results underscore the need to cultivate principled next moves even in seemingly correct cases.

Our findings also show that PTs often proposed coherent responding actions that were specific and proximal to the attended/interpreted themes. Most proposed actions were of a conceptual nature (Sánchez-Matamoros et al., 2019), including: (a) targeted questioning to surface critical attributes, (b) representation work (e.g., Venn diagrams/concept maps) to externalize inclusion relations, and (c) analogies (e.g., number sets) to bridge from familiar hierarchical structures. These positive results may derive from the fact that the PTs concentrated their analysis on short written episodes focused on a single topic closely that was linked to the work undertaken in the instructional module, with particular emphasis on key aspects of definition and classification in geometry. Additionally, as these PTs were already in their second year of the master's program, they have developed pedagogical

content knowledge that allows them to propose actions that are student-centred and conceptually challenging.

Conceiving noticing as comprising three interrelated components, the desired coherence between the teacher's decisions and the actual student's thinking to whom these decisions are directed requires the development of knowledge across different domains. Thus, this study suggests that analysing classroom situations, even when hypothetical, at more advanced stages of training may enable PTs to integrate diverse domains of knowledge within mathematics education, thereby enhancing their ability to notice students' thinking.

Even so, in some instances, it was found that the PTs did not provide responses that addressed the specific student's needs, particularly in the case of Teresa, one of the students in the first episode, who appeared to be influenced by the prototypical phenomenon in her reasoning (Fujita, 2012). In this instance, some PTs focused on actions directed towards the partitive nature of classification, a conception that the student also seemed to display, perhaps because they regarded it as a conceptually more complex topic. Future research could explore in greater depth the factors that lead PTs to overlook specific students' learning needs, particularly when misconceptions such as the prototypical phenomenon are involved. It would also be valuable to investigate how PTs prioritise or interpret conceptual complexity when deciding which aspects of students' reasoning to address.

This study contributes to understanding the expression of coherence among noticing components and the characteristics of responding within a specific domain that constitutes a recognised source of difficulty for prospective teachers: classification in geometry. Taken together, these findings reinforce recent conceptualizations of thematic coherence—maintaining a common mathematical/pedagogical thread across attending, interpreting, and responding—as a signature of higher-quality noticing (Thomas et al., 2023). They also affirm calls for subject-specific investigations of noticing, because the geometric content here foregrounds challenges and opportunities that differ from, say, early algebra or proportional reasoning (König et al., 2022). The present study may offer insights into designing research in the context of methods courses for preservice mathematics teachers with a focus on the thematic coherence of noticing students' thinking.

Several limitations must be acknowledged in this research. Firstly, the study relied on written responses to hypothetical episodes, which, although valuable for reflection (Scheiner, 2023), lack the immediacy and situational complexity of classroom interactions (Rodrigues et al., 2022), thus may underestimate real-time pressures on responding. The small sample size (12 PTs) restricts generalizability, and the analysis focused exclusively on classification in three-dimensional geometry. Moreover, our coherence coding emphasized thematic alignment rather than fine-grained quality levels within each component, and some PTs' domain-specific unfamiliarity (e.g., with bipyramids) may have confounded interpretation and response. Finally, because the episodes were hypothetical and decontextualized, PTs lacked knowledge about the student or the context that may influence their decision-making (Barnhart et al., 2025). Further research should therefore expand to larger and more diverse samples, include in-service teachers, and explore whether similar coherence patterns emerge in other mathematical domains, such as number, algebra, or functions (Cabral et al., 2021; Rotem & Ayalon, 2024). Future studies could also examine how different instructional designs—such as video-based cases (Ulusoy & Çakiroğlu, 2021), collaborative whole-class discussions (Cybulski et al., 2024; Rodrigues et al., 2022), or scaffolded use of professional materials (Moreno et al., 2025)—influence coherence across noticing processes.

In sum, this study underscores that fostering coherence in professional noticing, especially in the responding component, is essential for developing responsive and conceptually grounded teaching practices. By highlighting both strengths and challenges in PTs' noticing

of classification, it contributes to ongoing efforts to prepare teachers who can attend to, interpret, and respond coherently to students' thinking in ways that support meaningful mathematical learning.

Author Contributions: Conceptualization, H.O., F.C.C. and M.C.d.C.T.C.; Methodology, F.C.C.; Validation, H.O. and F.C.C.; Formal analysis, H.O. and F.C.C.; Investigation, H.O.; Writing—original draft, H.O. and F.C.C.; Writing—review & editing, H.O., F.C.C. and M.C.d.C.T.C. All authors have read and agreed to the published version of the manuscript.

Funding: National Funds through FCT-Portuguese Foundation for Science and Technology, I.P., under the scope of UIDEF—Unidade de Investigação e Desenvolvimento em Educação e Formação (IDB/04107/2025, <https://doi.org/10.54499/UIDB/04107/2020>), to H.O., and through Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES, Brazil, Finance Code 001) by a scholarship to F.C.C., and Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq, Brazil) by a research productivity grant to M.C.d.C.T.C.

Institutional Review Board Statement: Universidade Estadual de Londrina, Brazil 5.001.063 2021-09-27.

Informed Consent Statement: Written informed consent was obtained from all participants in the study.

Data Availability Statement: The data underlying this study are not publicly available, but they can be made available after being de-identified and anonymised in compliance with privacy restrictions, upon request to the corresponding author.

Acknowledgments: During the preparation of this manuscript, the authors used ChatGPT, version 5, for assistance in the discussion of the findings. The authors have reviewed and edited the output and take full responsibility for the content of this publication.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviation

The following abbreviation is used in this manuscript:

PTs Preservice teachers

Notes

- ¹ According to [Hershkowitz \(1990\)](#), critical attributes are those attributes that an instance must have to be a concept example.
- ² Euler's law for polyhedra (or Euler's formula) is a topological result stating that, in any simple polyhedron, the number of vertices plus the number of faces minus the number of edges equals two ([Debnath, 2010](#)).

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