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Development of Definitory and Classificatory Thinking in Geometry through Storytelling and GBL Activities

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Abstract: Little is discussed regarding the meaning of "definition" in primary school, where children often memorize definitions. In geometry, students frequently confuse "definition" with the "description" of a geometric object. The aim of this study is to verify whether a new hybrid methodology between storytelling and game-based learning called Geometrikoland exercises skills that fall within what we understand as "definitory thinking", which the set of competencies and meta-competencies is associated with acquiring a true mastery of definitions in geometry. This goal can be achieved because the dynamics of Geometrikoland alter the approach to definitions. The methodology applied is a hybrid of storytelling and adventure game-based learning applied to a "engine" of quadrilateral theory. The experiment took place in a fifth-grade class in an Italian school. The analysis of some data based on the accuracy of geometry actions during workshops and qualitative analysis based on the analysis of oral arguments produced during the labs lead to the conclusion that, on average, children have gained a greater awareness of what it means to define a geometric object and a better understanding of the relationships between various subsets of the set of quadrilaterals, previously seen as separate sets. Further confirmation has been provided by comparing the results of two mini-tests (a placement test and an exit test), which are useful for assessing the skills associated with the criterion of arbitrariness, the criterion of uniqueness, and classificatory thinking.

Keywords: storytelling; figural concepts; geometry; quadrilaterals; game-based learning; definitory thinking; classificatory thinking; critical attributes

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1. Introduction

This study focuses on the experimentation with an artifact (referred to here as Geometrikoland) designed to develop definitory and classificatory thinking in geometry. The methodology employed involves integrating the educational approach of storytelling with non-elementary game-based learning dynamics. Classificatory skills are crucial when discussing definitory skills. In accordance with Poincaré's writings, to truly understand a definition, a mere description of the object to be defined is not enough: one must be able to observe differences from similar objects and comprehend the meaning and necessity of each individual word contained in the definitions [1]. The difficulties students face in understanding definitions have been confirmed by various studies [2–4]. Other studies also confirm the existence of difficulties in classifying mathematical objects [5–7]. The significance of our study lies in the fact that, at the time of the writing of this article, there is no similar experiment involving geometry, storytelling, and adventure game-based learning in the literature.

Storytelling is an effective form of communication as it creates a symbiosis between the art of writing or narrating stories and the needs of teaching and learning. This methodology was chosen because it promotes greater student engagement compared to traditional methods. Using storytelling, emotionally, helps overcome barriers between mathematics and students with low self-esteem. In this experiment, students undergo an immersive experience, requiring them to rework what they have learned during traditional lessons and view it from a different perspective.

With the term *definitory thinking in geometry*, we aim to encompass the set of skills and meta-skills associated with the activity of defining, which are useful for acquiring a true mastery of definitions in geometry. These include understanding and applying the *criteria of uniqueness*, *criteria of arbitrariness*, and *criteria of minimality* of definitions, as well as the skills of *classificatory thinking* associated with definitions and the recognition of *critical and non-critical attributes* [8]. Additionally, we add a specific skill for geometry, which is the construction in the individual's mind of *Fischbein's figural concepts*, rather than the separate observation of conceptual characters and figural characters that do not interact with each other, hindering the understanding of geometric definitions [9].

During the use of the Geometrikoland artifact, we obtained feedback on the presence of figural concepts in the students' minds and feedback on the consistency of their classification of quadrilaterals.

The chosen disciplinary framework to achieve this goal is the *Inclusive Model of the Quadrilateral Theory* (a version where parallelograms are considered a subset of trapezia). This choice was deemed more appropriate than the *Exclusive Model of the Quadrilateral Theory* (a version where parallelograms are not considered trapezia) because it allows students, during the activity, to make more nuanced observations due to the greater depth of classification. During the laboratory activity, a girl named Vittoria remarked: 'Rectangles are super isosceles trapezia, so they win'. Therefore, Vittoria understood that the class of rectangles is a subset of the class of isosceles trapezia. Observations of this kind are not possible in the exclusive model due to the shallower hierarchy. In this case, Vittoria could have at most said that "Rectangles are parallelograms"; this statement seems obvious and, therefore, does not have the same impact on students.

The methodology employed is a hybrid approach that combines storytelling and game-based learning. In the conclusions of his article, Fischbein writes: 'The development of figural concepts generally is not a natural process. One of the main reasons that geometry is such a difficult topic in school programs is that figural concepts do not develop naturally towards their ideal form. Consequently, one of the main tasks of mathematics education (in the domain of geometry) is to create types of didactical situations which would systematically ask for a strict cooperation between the two aspects, up to their fusion in unitary mental objects' [9] (p. 161).

The research hypothesis is that Geometrikoland fulfils the main tasks of geometry education outlined by Fischbein because it creates learning opportunities that systematically require close collaboration between figural and conceptual aspects, leading to their fusion and the creation of unitary mental objects. We have verified during the study that children, through this hybrid methodology (storytelling and adventure-game-based learning), enhance their skills in definitory thinking and classificatory thinking within the geometric context.

Therefore, it is hypothesized that, with the instructional mediation of the teacher, the hybrid methodology encourages children to stimulate their thinking using Figural Concepts rather than their Image Component. The hypothesis is tested through the administration and analysis of the results of two brief oral tests on definitory thinking and classificatory thinking: a placement test and an output test, and through the analysis of laboratory activities associated with the use of the Geometrikoland artifact.

2. Theoretical Framework

There is a vast body of literature supporting the pleasure of storytelling, both from the perspective of those who narrate them and those who listen [10]. There is also an abundance of anecdotal evidence suggesting that 'telling a story creates more vivid, powerful and memorable images in a listener's mind than does any other means of delivery of the same material' [11] (p. xvii).

Baker e Greene [12] follow Lewis Carroll, who describes stories as "love gifts", and extend this metaphor to storytelling, which they define as "giving a gift". After interviewing hundreds of storytellers, both occasional and professional, Haven [11] has proposed the following definition:

Storytelling: the art of using language, vocalization, and/or physical movement and gesture to reveal the elements and images of a story to a specific, live audience [11] (p. 215).

Haven has also noted that "storytelling is both the most basic mode of human communication and a powerful performance art for" [11] (p. 216). Fortunately, most teachers are masters of this art. They know how to engage students in a story and how to adapt the narrative to suit the audience.

The choice to use storytelling is linked to both motivational and cognitive efficacy aspects [10,13]. Many studies also highlight the benefits of digital storytelling in educational settings, even in mathematics [14–16].

2.1. Definitions of Quadrilaterals

Every time researchers conduct an experiment involving the theory of quadrilaterals and its definitions, a fundamental choice must be made. There are two antagonistic taxonomies of quadrilaterals, and it is necessary to decide which one to adopt: the Exclusivist taxonomy and the Inclusivist taxonomy. The substantial difference concerns the mutual position between the set of parallelograms and the set of trapezia. In the Exclusivist taxonomy, the trapezium is defined as "a quadrilateral with exactly two sides parallel", and therefore, the set of parallelograms is excluded from the set of trapezia; more precisely, we can say that the intersection between the set of parallelograms is equal to the empty set. In the Inclusivist taxonomy, the trapezium is defined as "a quadrilateral with at least two sides parallel", and therefore, the set of parallelograms is included in the set of trapezia. The Inclusivist taxonomy is consistent with higher-level mathematics (infinitesimal calculus). Our experimentation uses the Inclusivist taxonomy, where all parallelograms are trapeziums. The network of definitions for quadrilaterals used in this experiment is the one adopted for the competitions of the first Italian national geometry tournament [17]. This tournament has been held in Italy since 2016 without interruptions, is called 'Torneo Nazionale di Geometriko', and is open to all primary and secondary schools. The definitions adopted for the experiment are as follows:

Definition 1. *A Quadrilateral is defined as any polygon having exactly four sides.*

Definition 2. *A quadrilateral is defined as a Trapezium if and only if it has (at least) a pair of parallel sides. The sides with the opposite side parallel are called the Bases of the Quadrilateral, while the sides without the opposite side parallel are called Lateral Sides.*

Definition 3. *A quadrilateral is defined as a Right-Angled Trapezium if and only if it is a trapezium with (at least) one right angle.*

Definition 4. *A quadrilateral is defined as an Isosceles Trapezium if and only if it is a trapezium with (at least) one base having congruent adjacent angles.*

Definition 5. *A quadrilateral is defined as a Scalene Trapezium if and only if it is a non-isosceles trapezium.*

Definition 6. *A quadrilateral is defined as a Kite if and only if it has two consecutive sides congruent, and the other two sides are also congruent to each other.*

Definition 7. *A quadrilateral is defined as a Parallelogram if and only if it has opposite sides that are parallel.*

Definition 8. *A quadrilateral is defined as a Rhomboid if and only if it is a parallelogram with consecutive sides not congruent and adjacent angles not congruent.*

Definition 9. *A quadrilateral is defined as a Rectangle if and only if it has at least three right angles.*

Definition 10. *A quadrilateral is defined as a Rhombus if and only if it has all sides congruent.*

Definition 11. *A quadrilateral is defined as a Square if and only if it has all sides congruent and at least three right angles.*

In Figure 1, you can see a Venn diagram drawn up by our research team showing the classification of quadrilaterals in relation to these definitions.

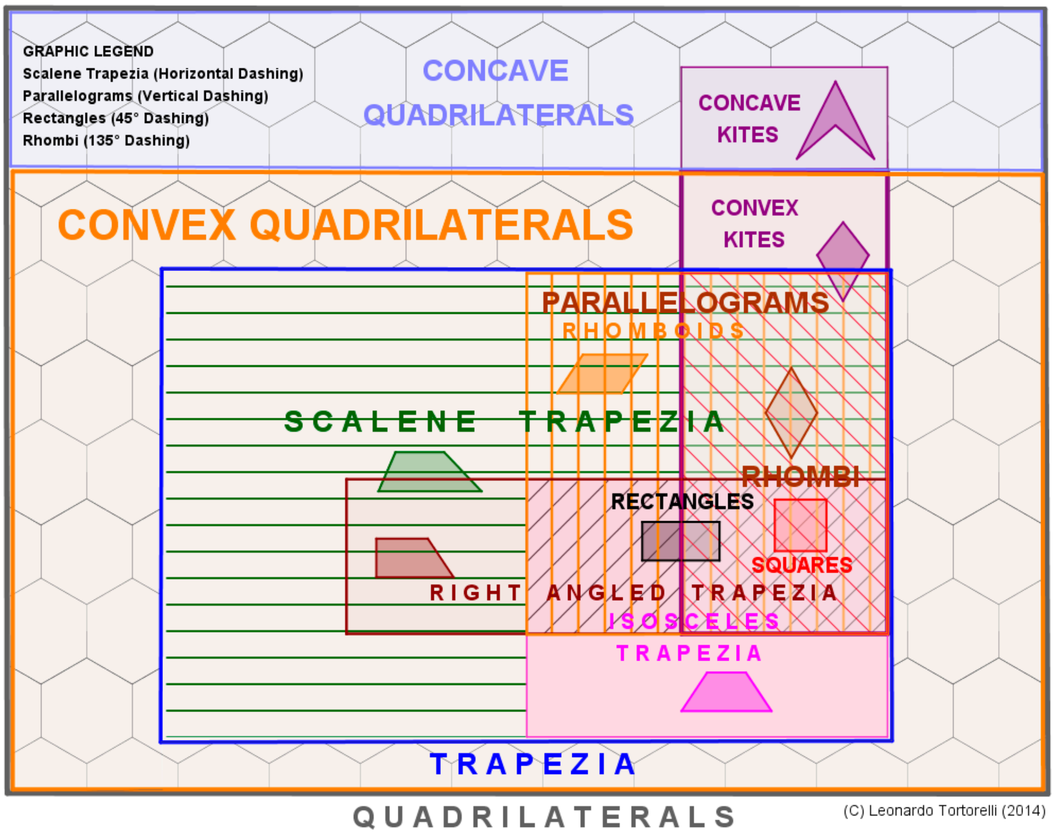

Figure 1. Euler diagram of some types of quadrilaterals.

2.2. The Theoretical Framework for the Methodological Conduct of Classroom Experimentation

The artefact draws inspiration from Vygotsky's socio-historical school and two Vygotskian concepts: the internalization of processes [18] (p. 56) related to the construction of one's competencies through participation and sharing experiences within the classroom group, and the concept of Zone of Proximal Development (ZPD), based on social interaction and intentional asymmetrical cooperation among students, teachers, and peers with different levels of competence [18] (p. 86). According to Vygotsky, when a student needs to grasp a mathematical concept, two subprocesses come into play: a social process and a semiotic process. The social subprocess is a consequence of the fact that to socialize knowledge, communication through signs in interpersonal space is necessary. Geometrikoland serves as a tool for semiotic mediation in the sense of Bartolini Bussi and Mariotti [19]; in Geometrikoland, teachers "orchestrate" the use of the artefact as if they were cultural mediators or experts guiding students in constructing new meanings during its use. The term "orchestration" is used here in the sense of Bartolini Bussi [20].

2.3. Theoretical Framework for Analysing Results

In accordance with Duval, we argue that "il n'ya pas de *noétique* sans *sémiotique*" [21], meaning that when understanding a new mathematical concept (in our case, a geometry concept), conceptual acquisition involves considering one or more semiotic representations of that concept. In the case of our experiment, semiotics comes into play through the representation of geometric concepts via signs. Duval, in one of his articles [22], discusses the importance that Vygotsky also placed on signs. Duval writes that Vygotsky, in the work *Thought and language* [23], already asserted that there is no concept without a sign. In geometry, students often mistakenly believe that the concept and its representation are the same thing.

The chosen theoretical frameworks for the analysis and interpretation of results are the *Theory of Figural Concepts by Fischbein* [9] and the *Theory of Functional Fixedness* [24]. Fischbein argues that geometrical figures are mental entities with a dual nature. Geometrical figures are composed of conceptual characters and figural characters. The quadrilaterals addressed in this experiment are therefore abstract concepts endowed with figural properties (such as shape, parallelism of sides, congruence of angles, and more). Fischbein analyses the continuous tensions to which a student is subjected when learning a figural concept. In many cases, the figure and the concept are in conflict, but weaker students base their reasoning solely on the image because the image component is what has an immediate influence on the direction of thought.

Before Fischbein, the dominant theory was Piéron's [25] (p. 72). Piéron stated that symbolic representations used in the process of abstract thinking correspond to a set of concrete representations of the same concept that share common features. Fischbein's theory is antagonistic to the existing cognitive theories. In previous theories, concepts and mental images were considered as two distinct categories of mental entities. The adoption of Fischbein's theoretical framework appears natural in our experiment. When drawing a specific quadrilateral to verify a property (for example, to verify that opposite sides of a parallelogram are congruent), we should not think about the particular parallelogram drawn but rather an infinite set of objects. All these objects will be united by the fact of having the same critical attributes required in the definition of a parallelogram. Definitions control shapes (generally inspired by real objects) and clarify concepts because they possess the characteristics of ideality, abstractness, absolute perfection, and universality [9].

According to the *Theory of Functional Fixedness* [24], given that an object and a subject who must use it to solve a problem, there is a tendency to see an object only in its traditional functions, ignoring potential alternative uses. In this case, Functional Fixedness can be an obstacle to creativity, as it limits the vision of possibilities beyond the predefined functions of an object. Duncker investigated this phenomenon in geometric contexts (*Geometrical functional fixedness*, GFF), demonstrating that it can be useful for analysing errors and difficulties shown by students in visual perceptual functioning [Ibid]. According to Hoz, "[...] GFF explains why the subject cannot find and/or identify an element in the diagram of a geometrical problem: One function of this element prevents the discovery or the use of other functions of this element, thus fixating the element itself to a certain function" [26].

3. Materials and Methods

3.1. The Experimental Context

The study described in this article is part of vertical research project (still on-going) involving students from 3rd grade to 10th grade. This experiment was conducted during mathematics lessons, by the curricular teacher, supported by our research team. The sample consists of a 5th grade class of a primary school in Southern Italy. The class examined consists of 16 pupils from middle-class families. The class has already studied the Theory of Quadrilaterals with their teacher. Therefore, it is logical to conduct a placement test before the start of the experiment.

This sample was selected for our study because it mirrors the average levels of classrooms in Southern Italy. Indeed, the class comprises a couple of children with mediumto-high ability, a third with low performance in mathematics, and the remaining portion with more or less sufficient proficiency. The teacher's attitude was also of interest to us. Before the experiment, the teacher informed us that her students exhibited little interest and, in some cases, hostility towards geometry because they perceived it as a tedious and challenging discipline. The teacher held the belief that in primary school, it was not feasible to progress beyond a mnemonic approach to definitions. For our research team, this initial situation appeared challenging, prompting us to opt to conduct our research within this context. This marked our inaugural encounter with Geometrikoland. At this juncture, no control group was envisaged, as, among other considerations, it was crucial for us to evaluate the potential of the Geometrikoland tool.

3.2. The Teaching Methodological Cycle

3.2.1. Phase 1: Theoretical Lectures on the Theory of Quadrilaterals

The first phase consists of a series of five one-hour lessons on the Theory of Quadrilaterals. The lessons' topics were: the concept of mathematical definition and quadrilaterals (Lesson 1); trapezia (Lesson 2); kites, parallelograms, and rhomboids (Lesson 3); rhombi, rectangles, and squares (Lesson 4); the classification of quadrilaterals (Lesson 5). Each lesson includes extensive moments dedicated to argumentation. During the lessons with the students, we discussed together multiple times how geometrical figures represent mental constructs that are special mathematical concepts because they have a mental representation of the property in space; this occurs because a geometrical figure is a symbiosis of *conceptual properties and figural properties* [9]. Emphasizing this aspect to the students multiple times was important. Many students became aware of the reasons for their difficulties in understanding geometry definitions and therefore found the courage to try to overcome their limitations.

3.2.2. Phase 2: Storytelling Workshop

Narrative Framework

The narrative falls within the fantasy genre. Our adventure takes place in the Land of Hilbert (see Figure 2), where the kingdom of Geometrikoland exists, a happy and prosperous city ruled by the generous King Euclid. All citizens of the kingdom study geometry with love and use it in their everyday activities. This makes them happy because studying geometry simplifies everyday life in work, arts, and sciences. For this reason, the citizens of Geometrikoland are very intelligent, and their civilization is highly advanced. At the borders of the kingdom lies the Skull Forest, where orcs live, ruled by a sect of evil wizards and witches known as the Coven of Misconception (CoM for short). The Skull Forest is swampy, humid, dark, and inhabited by snakes and fierce beasts, while Geometrikoland is a flat land rich in cultivated fields and wildlife. The CoM does not allow its people to study geometry because citizens who study geometry become more intelligent, and intelligent citizens are more challenging to subjugate. Witches and warlocks do not like Geometry; however, they use it as a powerful weapon to cast their evil spells. Over the centuries, the CoM has attempted several times to conquer Geometrikoland, but the kingdom, protected by the Mage of Varignon and the Knights of Venn, has always repelled these attacks. The Knights of Venn use magical weapons forged by the mage Varignon. The mage Varignon lives outside the kingdom of Geometrikoland. The mage's laboratory is on the peak of Convex Mountain, and he never leaves his home because he must defend and guard powerful magic potions and amulets. The happiness of the kingdom is in danger because the sorcerer Hugo (leader of the CoM, called the goat wizard because he is half man and half goat) has discovered an ancient spell that petrifies people who listen to it. This dangerous spell is particularly insidious because it is recited disguised as definitions and theorems of Euclidean geometry. The sect of warlocks has infiltrated the kingdom of Geometrikoland. The citizens are defenceless because they love geometry and listen carefully to its statements. It has been effortless for the CoM to petrify all citizens and the army of Geometrikoland. Fortunately, the Knights of Venn and King Euclid escaped the petrifying magic because they are protected by magical quadrilateral shields. The shields were built by the mage Varignon and are the only way to escape the petrifying magic. Each quadrilateral shield can repel certain types of magic but not all (remember that magic is disguised as definitions and theorems). Only the King and the knights of the kingdom can use the quadrilateral shields. To save the kingdom from the black magic of the CoM, it is necessary to go to the mage Varignon to give him an important relic: the Chalice of Pythagoras. This ancient artifact will allow the mage to build a powerful amulet that will forever protect Geometrikoland from evil and nullify the petrification spell. In this case, all citizens of the kingdom will come back to life. The company of Geometrikoland, consisting of King Euclid and the Knights of Venn, leaves the kingdom to bring the Chalice of Pythagoras to Mage Varignon. Some spies of the CoM have discovered King Euclid's intentions and reported them to the sorcerer Hugo. The CoM must hinder the journey of Geometrikoland's heroes to Convex Mountain using the petrification magic.

Figure 2. The map of Hilbert's Earth.

The Roles of Students during the Workshop

The students' roles during the workshop are crucial. They are divided into two opposing groups: the witches and warlocks of the Coven of Misconception (CoM) with their leader (the sorcerer Hugo) and King Euclid with his Knights of Venn (Geometrikoland Company). They are brought into a very large room, with each character represented by a pair of students. The sorcerer Hugo and King Euclid are portrayed by the students with the highest geometry proficiency in the class. The two groups must be balanced in terms of knowledge of geometry. In the game, each witch and warlock has six spells to attack the Company (definitions and theorems of quadrilateral theory), while the Sorcerer Hugo has eight spells. Each knight has four quadrilateral shields to defend against malevolent spells, and King Euclid has six quadrilateral shields. In our experiment, the 16 children (8 pairs) are divided into 4 members of the CoM (one of whom is the sorcerer Hugo) and 4 members of the Geometrikoland Company (King Euclid and 3 knights). The teacher assumes the role of the deity Efraim, the protector of all inhabitants of the Land of Hilbert, ready to intervene during the game to assist both the witches and warlocks of the CoM and the inhabitants of Geometrikoland. Specifically, Efraim intervenes to stimulate mathematical discussions whenever a player struggles to understand the definitions and properties in play at that moment. The name Efraim is not coincidental; it refers to Fischbein, to whom the teacher makes frequent references, and whom the researchers discussed during frontal lessons on quadrilaterals.

The Rules of the Game

The goal of the Geometrikoland Company is to successfully bring the Chalice of Pythagoras to the wizard Varignon (the Chalice can only be carried by the King); the goal of the Coven of Misconception (CoM) is to petrify all the knights and their king before they reach the laboratory of Wizard Varignon. Each spell cast by the magicians of the CoM petrifies one shield; when a knight runs out of shields, if attacked again, they are petrified and excluded from the game. Knights need to be hit by five properties to be petrified (as they have four shields), while the King needs to be hit by seven properties (as he has six shields). Be warned! The King carries the Chalice of Pythagoras and can only be petrified when all his knights have been petrified.

The Dynamics of the Game-Based Learning Model

The Geometrikoland Company moves towards the Convex Mountain and can only do along a suspended bridge over a canyon. Their path is generally protected by the magic of Wizard Varignon. However, there are five dark points along the route where there is no protection from Wizard Varignon's magic. In these points, the CoM can attack King Euclid and his knights with petrifying magic. At each of the five dark points, each witch and warlock can attack a chosen knight or King Euclid. In Figure 3 is a diagram summarising the objectives of the two opposing teams.

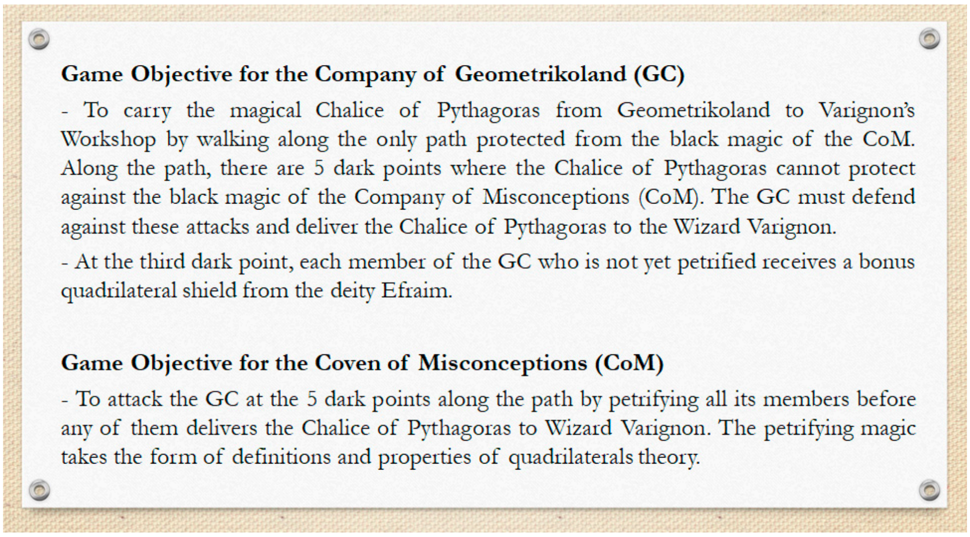

Figure 3. Objectives of the two opposing teams. Each time a hero is attacked by the CoM, they lose a quadrilateral shield. A magical attack is formulated with a definition or theorem printed on a playing card, which, after use, is no longer usable. The one who receives the attack must use an appropriate quadrilateral shield to defend themselves (the shield petrifies and is then no longer usable). The dynamics of the game are now described in more detail. The Geometrikoland Company reaches the first dark point.

Each member of the Company, after suffering the magical attack with a property p, must say 'I won!' (if the set of quadrilaterals X represented on the shield used for defense satisfies property p) or must say 'I lost!' (if the set of quadrilaterals X represented on the shield used for defense does not satisfy property p). The answer (must be correct, otherwise another shield of the defender of Geometrikoland will be petrified (the attacked player loses two shields instead of one in this case). When the set of quadrilaterals X represented by the shield satisfies property p, the knight or the King who used it to defend advances directly to the second dark point, and this game mechanism repeats. At each dark point: each witch or warlock can make only one attack. At each dark point, the same element of the Company cannot be attacked twice (Fair-Play Rule). Since the goal of the CoM is to petrify all fighters of Geometrikoland before they reach Wizard Varignon, each pair of students managing the CoM avatars must intelligently decide which element of the Company to attack based on the shields they still have available (because the quadrilateral shields are clearly visible to all players, but the attack cards with the spells that each member of the CoM can perform are secret and cannot be communicated). At first this was not the case, even the spells were public for everyone, but this created too great an advantage for the CoM because the most skilled of this team played for all the sorcerers. In this way the consultation on the spells to use is limited to the children of the couple who represent the same sorcerer.

3.3. The Teaching Project

The students are divided into two groups. The dynamics related to the geometry skills of each group are analyzed. The witches and warlocks hold some properties among the theorems and definitions. When one of them has to choose which member of the Company to attack, the student is forced to reflect on the definition of each set of quadrilaterals semiotically represented by all the shields still in play. So, each pair of children representing the members of the CoM must compare the quadrilateral shields that the opponents have available and then build a game strategy. The strategy indicates who to attack and with which property. From the experiment, it is observed that children are forced to delve deeper into the understanding of each definition of individual quadrilaterals that could be the target of their attack. Those who have memorized the definitions cannot decide who to attack or attack a Company element with the mistaken belief of winning. If a pair of children is struggling during the game, they can ask for the help of the deity Efraim (teacher). All of this means that strategic thinking is not only useful for winning a didactic game but also useful for guiding children towards a greater understanding of geometric definitions. To build a winning strategy, each child is forced to deepen the meaning of each individual definition of quadrilaterals and their hierarchy. When the game ends, a new game is played by reversing the roles: the members of the CoM become the Company of Geometrikoland and vice versa.

The quadrilateral shields in Figure 4 are repeated, but in each shield of the same type, the represented quadrilateral always has a different position relative to the inscriptions. In this way, students deal with different semiotic representations (even though of the same figural register). It may also happen that a Parallelogram is depicted on a Trapezium shield: the principle is that the shield is the representation of a set, and any of its elements can be taken as the representative. We refer to the meaning that Duval assigns to the expressions: transformation of a representation of a semiotic system, treatment of a representation, and conversion of a representation [21]. During the Geometrikoland activity, students continuously operate transformations of semiotic representations (treatments and conversions of semiotic representations). Students must make a conversion every time they need to translate the multifunctional discursive representation register (represented in this case by the natural language used in geometric statements representing spells) into the non-discursive representation register of geometric configurations (represented in this case by the drawings of quadrilaterals represented on the available shields). Students must make a treatment every time they need to establish within the non-discursive representation register of geometric configurations whether two different shields can constitute a figural representation. Students perform treatments (because they thus move from one representation to another of the same semiotic register of the same mathematical concept) and conversions (because they move from one representation to another of different semiotic registers of the same mathematical concept). It is through the use of signs that each individual develops knowledge. The interaction with the children was conducted in the form of a mathematical discussion in the sense of Bartolini Bussi [27].

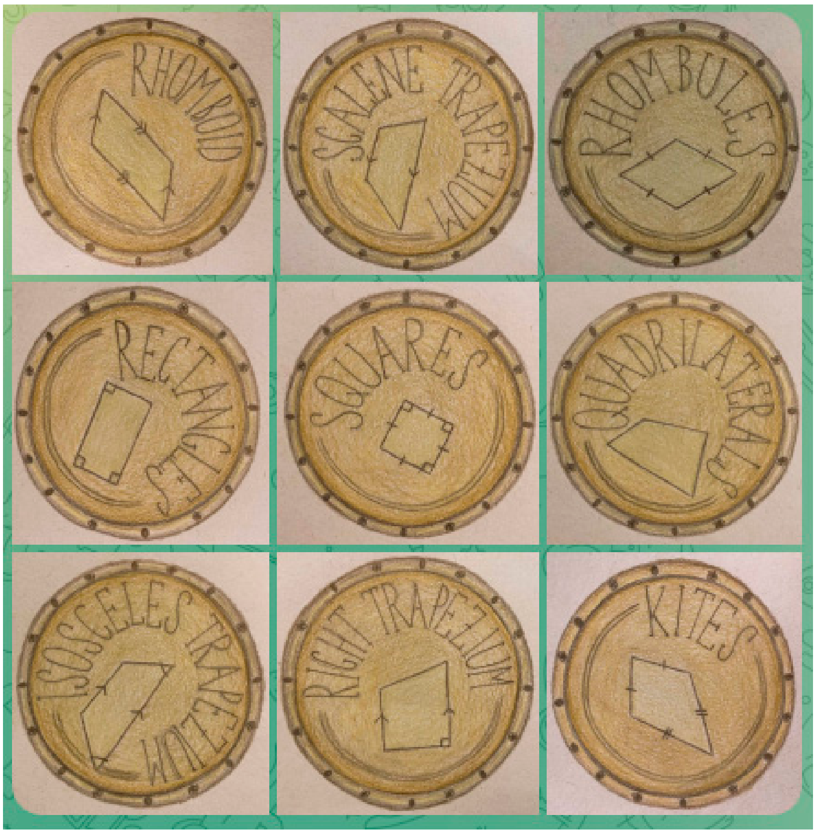

Figure 4. Quadrilateral shields.

4. Results

In this paragraph we analyse some interesting qualitative results. Five workshops were conducted over two months (December 2023 and January 2024). Each workshop lasted two hours and was preceded by an hour of theoretical discussion on the theory of quadrilaterals.

The first workshop confirmed what emerged from the placement test: the level of understanding of the definitions is rather low. This section will narrate some dialogue among children that confirm this observation. The wizard Kepler, represented by the *xy* pair, attacks the knight Emma, represented by the *ab* pair, with a petrifying spell. Knight Emma has only one shield for defence (Figure 5), and therefore, she must use it at the next attack.

(**a**) (**b**)

Figure 5. This figure outlines the attack that the wizard Kepler of the CoM launches against the knight Emma. (**a**) The card with the magic used by Kepler to attack the knight, Emma; (**b**) The shield used by Emma to defend herself from Kepler's spell.

Transcript of the dialogue among the students:

Kepler uses the spell shown in Figure 5a and reads it aloud:

x: 'If you do want to win this battle, you have to use a set of squares!'

Emma uses a winning shield, but the figure misleads them, and the team's speaker says:

a: 'I use the rhombus shield and say I lost!'

The rules of this game require that the knights must justify their statement ("I won!" or "I lost!").

Efraim: 'Why did you lose?'

a: 'Because it's a rotated square, so it's a rhombus, not a square!'

Efraim: 'Did you read the writing on the shield? It says squares'.

The second player of the pair managing the knight Emma also intervenes.

b: 'Yes, but it's a rotated square... It's a rhombus!'

At this point, the teacher reads some selected sentences from the article by the real Fischbein.

Efraim: 'Mental entities (so-called geometric figures) possess both conceptual and figural characters. [9] (p. 139) (...) A square is not an image drawn on a sheet of paper. It is a shape controlled by its definition (though it may be inspired by a real object). [9] (p. 141) (...) Ideally, it is the conceptual system that should absolutely control' [9] (p. 160).

Efraim: 'Unfortunately, in this play, this did not happen; the figural instinct prevailed over conceptual reasoning'.

Efraim stops the game and explains, in simpler terms, the sentences by Fischbein read earlier. He engages the children in a mathematical discussion about what they have just heard, addressing the misconception that none of the children in the class noticed. The episode can be interpreted through the lens of the *Theory of Figural Concepts* [9]. It is evident that the pair *ab* (and all the other children) was misled by the figural character, namely the image drawn on the shield used for defence (Figure 5b). The shield represents a square in the prototypical position of a rhombus, and this led the children into error [28]. On other occasions, *a* had demonstrated knowledge of the definition of a square. This episode highlights the absence of symbiosis between figural characters and conceptual characters and the prevalence of figural characters over conceptual characters.

The game's dynamics involve a geometric penalty when a player says 'I lost!' but should say 'I won!' (and vice versa). The geometric penalty is the loss of another shield (chosen at random), and if the knight has no more shields to lose, they are petrified and thus excluded from the game. The class teacher declared using only prototypical figures during her course on quadrilaterals; this aspect could have influenced student learning. The confusion between geometric shapes (in our example, a rotated square was considered a rhombus but not a square) might be due to a combination of factors involving cognitive development and each student's individual school experience. In particular, this type of misconception is caused, for example, by choices of didactic transposition made by the teacher or textbooks (more generally, by the *noosphere*). Among these didactic choices is the provision of only unambiguous and stereotyped representations that lead students to attribute incorrect properties to geometric figures. This issue often arises from the teacher's lack of consideration for the importance of semiotic representations and a deficiency in the teacher's conceptual knowledge. In these cases, the teacher demonstrates knowledge tied to the didactic transposition done by textbooks (more generally, by the noosphere). For these reasons, students of such teachers often confuse mathematical concepts with their semiotic representations.

Figure 6 illustrates a second interesting dialogue that we observed during a workshop.

The witch Hypatia (represented by the pair *uv*) attacks the knight Thales (represented by the pair *cd*) with a petrifying spell. The knight Thales defends himself from the witch's attack with the right-angled trapezium shield (Figure 6).

Transcribing the dialogue between the students. The witch Hypatia uses the spell shown in Figure 6a and reads it aloud:

u: "If you want to win this battle, you have to use a set of quadrilaterals with at least one side that is also a height".

Thales uses a winning shield, but the image deceives them, and the team's speaker says:

c: "I use the right-angled trapezium shield and say that I lost!"

The rules of this game require that the knights justify their declaration ("I won!" or "I lost!"). If the children don't justify their response, the teacher stimulates it.

Efraim: "Why did you lose?"

c: "Because the sides are not heights. . ."

The second player of Team Thales joins the discussion to justify their response.

d: 'This is a side and therefore cannot be a height. . . they are two different things. . . this is a height. . ." (the child draws with their finger on the shield the height drawn from the vertex of the obtuse angle relative to the major base)".

(**a**) (**b**)

Figure 6. This figure outlines the attack that the witch Hypatia from the CoM launches against the knight Thales. (**a**) The card with the magic used by Hypatia to attack Thales; (**b**) the shield used by Thales to defend himself from Hypatia's spell.

We interpret the episode through the lens of *Geometrical Functional Fixedness* [24]. It is evident that the pair embodying the knight Thales—and many other children who did not challenge their response—exhibited a difficulty in visual perception. Thales' children were unable to identify a height in the right-angled trapezium because the function of being a side of the trapezium hindered the recognition of the function of being a height of the trapezium, thereby fixing the element itself to a certain function.

Five workshops, each lasting two hours, were conducted. During each session, the CoM used 20 petrifying spells against the knights, and the researchers listened to the corresponding 20 responses from the knights. The most interesting responses from an educational standpoint were noted.

The number of errors made by the knights was also recorded during each workshop. Not all errors made by the knights were counted; only those resulting from a poor understanding of definitions and classification were considered. Errors due to distractions, for example, were ignored as they are not the focus of our study. The graph in Figure 7 represents the frequency of errors attributable to a lack of understanding of definitions and their classification recorded during each workshop.

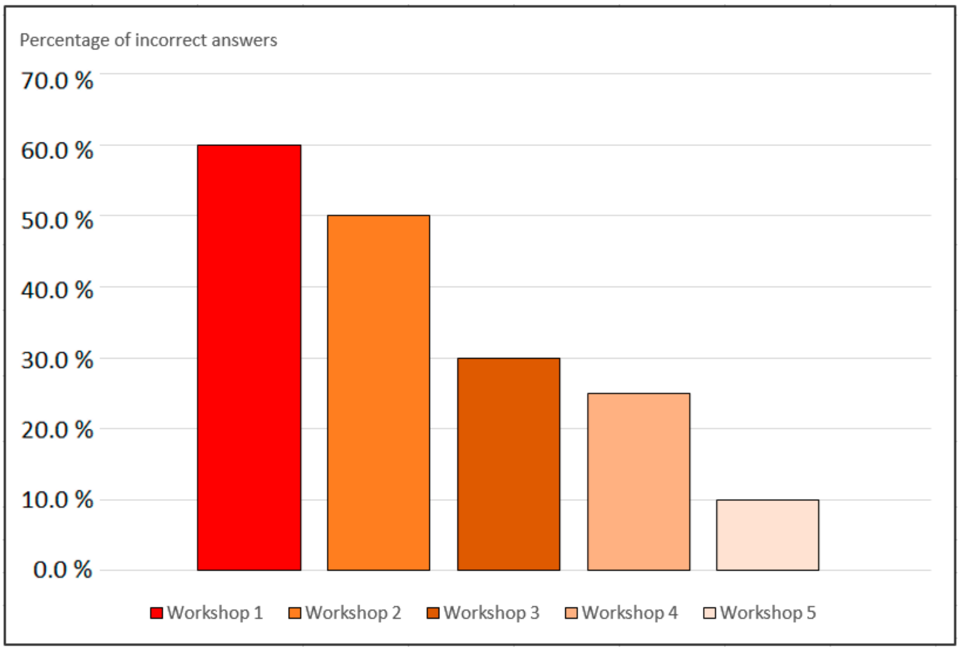

Figure 7. The percentage of incorrect answers attributable to errors in understanding definitions.

5. Experimental Conclusions

In this final part of the results section, the outcomes of both the placement test and the exit test are presented. Each of the 16 children underwent both tests through a conversational interview format conducted by the teacher, with the researcher noting down the responses. This approach was necessary to make the children feel comfortable, especially since at the beginning of the research process, they showed reluctance towards external evaluation. The placement test consists of three items probing different facets of the same macro-competence we have termed *definitory thinking*. The first item focused on the *criterion of uniqueness* in defining, the second on the *criterion of arbitrariness* in defining, and the third on *classificatory thinking*. Figure 8 illustrates the three questions of the placement test alongside the percentages of correct responses.

"Geometrikoland" Experimentation (2023/2024) - PLACEMENT TEST Source of questions: National Geometriko Tournament Archive (Italy) - Quadrilateral Theory with Inclusivist Taxonomy	Placement Test	
(1) Definitory Thinking in geometry: Uniqueness Criterion A proposition can be a definition of the geometrical object X if and only if it describes it uniquely.	**Percentage of responses**	
Read very carefully the following proposed definitions of some quadrilaterals that you have already studied. Do these proposals meet the **Uniqueness Criterion** of definition? If the answer is no, justify it with a counterexample.	YES	NO
[Y] [N] (a) A quadrilateral is a **Rhombus** if and only if it has perpendicular diagonals.	81.3%	18.7%
[Y] [N] (b) A quadrilateral is an **Isosceles Trapezoid** if and only if it is a trapezium with congruent lateral side.	100.0%	0.0%
[Y] [N] (c) A polygon is a **Quadrilateral** if and only if it has four sides.	100.0%	0.0%
[Y] [N] (d) A quadrilateral is a **Rectangle** if and only if it has at least three right angles.	50.0%	50.0 %
(2) Definitory Thinking in geometry: Arbitrariness Criterion Every proposition that satisfies the uniqueness criterion of the geometric object X can be chosen as the definition of X.	**Percentage of Correct**	
Your teacher no longer likes the definition of **Parallelogram** that you have studied: "A quadrilateral is called a parallelogram if and only if it has opposite sides parallel." The teacher asks you to propose a new definition according to the Criterion of Arbitrariness.	6.25%	
(3) Definitory Thinking in geometry: Classificatory Thinking in Geometry In geometry, classificatory thinking refers to the process of categorizing geometric shapes or figures based on their properties, characteristics, or relationships. This involves identifying similarities and differences among shapes and organizing them into distinct groups or classes.	**Percentage of responses**	
Identify whether each of the following propositions of Classificatory Thinking is TRUE or FALSE. Justify your answer.	TRUE	FALSE
A Parallelogram is a Trapezium.	12.5%	87.5%
A Square is a Rhombus.	18.8%	18.2%
A Rectangle is a Right Trapezium.	6.3%	93.7%
The Kite is a Rhombus.	25.0%	75.0%

Figure 8. Placement test: percentages of correct answers highlighted in green cells.

A benefit of interviews compared to written tests is that it is easier to filter and thus exclude false positives from the count. In the two tests, correct answers with inaccurate reasoning were considered incorrect.

In Figure 9, the three questions of the exit test and the percentages of correct answers are reported. In this case, incorrect answers with flawed reasoning were also considered wrong. The first three questions are identical to those of the placement test (administered three months prior). The children were not aware of the answers to the placement test. The opening question of the exit test was: “What do you remember from the placement test? Do you want to renegotiate any answers?” All the children said that too much time had passed since the first test and that they did not remember the questions. The choice to repeat the same questions has the advantage of being able to make a comparison with identical rather than similar questions. We are aware of the pros and cons of this choice, but we trust that the children were sincere in saying that they did not remember anything about the test.

"Geometrikoland" Experimentation (2023/2024) - EXIT TEST Source of questions: National Geometriko Tournament Archive (Italy) - Quadrilateral Theory with Inclusivist Taxonomy	**Exit Test**	
(1) Definitory Thinking in geometry: Uniqueness Criterion A proposition can be a definition of the geometrical object X if and only if it describes it uniquely.	**Percentage of responses**	
Read very carefully the following proposed definitions of some quadrilaterals that you have already studied. Do these proposals meet the **Uniqueness Criterion** of definition? If the answer is no, justify it with a counterexample.	YES	NO
[Y] [N] (a) A quadrilateral is a **Rhombus** if and only if it has perpendicular diagonals.	12.5%	87.5%
[Y] [N] (b) A quadrilateral is an **Isosceles Trapezoid** if and only if it is a trapezium with congruent lateral side.	25.0%	75.0%
[Y] [N] (c) A polygon is a **Quadrilateral** if and only if it has four sides.	18.7%	81.3%
[Y] [N] (d) A quadrilateral is a **Rectangle** if and only if it has at least three right angles.	93.8%	6.2%
(2) Definitory Thinking in geometry: Arbitrariness Criterion Every proposition that satisfies the uniqueness criterion of the geometric object X can be chosen as the definition of X.	**Percentage of Correct Answers**	
Your teacher no longer likes the definition of **Parallelogram** that you have studied: "A quadrilateral is called a parallelogram if and only if it has opposite sides parallel." The teacher asks you to propose a new definition according to the Criterion of Arbitrariness.	87.5%	
(3) Definitory Thinking in geometry: Classificatory Thinking in Geometry In geometry, classificatory thinking refers to the process of categorizing geometric shapes or figures based on their properties, characteristics, or relationships. This involves identifying similarities and differences among shapes and organizing them into distinct groups or classes.	**Percentage of responses**	
Identify whether each of the following propositions of Classificatory Thinking is TRUE or FALSE. Justify your answer.	TRUE	FALSE
A Parallelogram is a Trapezium.	100.0%	0.0%
A Square is a Rhombus.	100.0%	0.0%
A Rectangle is a Right Trapezium.	87.5%	0.0%
The Kite is a Rhombus.	6.3%	93.7%

Figure 9. Exit test: percentages of correct answers highlighted in green cells.

The exit test contains one additional question compared to the placement test (Figure 10). Its purpose is to verify the transfer of skills to theories other than the theory of quadrilaterals. All 7 items used were sourced from the database of the National Geometriko Tournament, which has been held annually in Italy since 2016.

(4) Definitory Thinking in geometry: Transfer of Skills (Uniqueness Criterion and Arbitrariness Criterion) This question aims to verify whether the skills learned during the experimentation have an impact in the classroom on other theories of geometry, such as the *Theory of triangles*.	**Percentage of responses for each option**
Let's define a Right Triangle as "a triangle in which one of the angles is a right angle." Which of the following sentences could be an alternative definition of a Right Triangle?"	
(a) A triangle with a vertical side.	19%
(b) A triangle with two perpendicular sides.	75%
(c) A triangle with two acute angles.	6%
(d) A triangle with one side longer than the other two.	0%

Figure 10. Exit test: a question to assess the transfer of skills in definitory thinking.

6. Conclusions

Unfortunately, there are no similar crossover experiences documented in the literature between the two adopted methodologies. However, this aspect is likely to be considered a strength rather than a flaw of this study, which experiments with an integrated use of the two methodologies. The narrative context during the pure gaming phases is ensured by Efraim (the teacher), who thus has a dual role: as a mediator of learning in Vygotskian style and as a narrator and actor who constructs and maintains a narrative atmosphere in fantasy style around the children throughout the Geometrikoland activity.

The aim of this research Is to ascertain whether the hybrid methodology between storytelling and game-based learning, named Geometrikoland, exercises competencies that fall within what we understand as definitive thinking: understanding of definitions, classificatory thinking, critical attributes, and construction of figural concepts. The two short questionnaires (placement and exit tests) also provided indications of progress regarding the application of the *criterion of uniqueness* and the *criterion of arbitrariness* of definitions, as well as further results on *classificatory thinking*.

From the collected data, a decreasing trend in the number of incorrect responses due to a lack of understanding of the definitions and classification of quadrilaterals is evident. This trend in the error curve can be interpreted as the understanding of definitions improving with increasing experience in Geometrikoland, thus positively influencing this aspect. The analysis of dialogue and arguments produced by students is still under study.

The analysis of the dialogue and arguments produced by the students is still under study; however, from some transcriptions (including those reported in this article), it is evident that the mental activities of the children confirm the research hypotheses. In particular, Geometrikoland fulfills the task assigned by Fischbein [9], as the narrative structure and dynamics of the educational game create continuous learning opportunities that systematically require close collaboration between figural and conceptual aspects, leading to their fusion and the creation of unitary mental objects. Some dialogue among the children during the Geometrikoland workshops (including those transcribed in this paper) also confirm the hypothesis that Geometrikoland is a tool that allows the teacher to detect any misconceptions held by the children in terms of definitory thinking, such as the presence of non-critical attributes in the definitions possessed by the children). For example, the pair of children managing the knight Guy, during the second workshop, were attacked with the spell "congruent diagonals", and after defending themselves with the "Kites" shield with a square drawn on it, they gave the wrong answer, saying, "We have won!". In this case, a cognitive conflict arises due to the lack of collaboration between figural and conceptual aspects. Repair occurred through peer mediation (knights and sorcerers) and the teacher (Efraim), leading to the expansion of the zone of proximal development [18] of *a* and *b*, bringing about their fusion and the creation of unitary mental objects.

The adopted mixed methodology (storytelling and game-based learning) has proven to be effective from both an emotional and motivational perspective for children and in terms of enhancing definitional and classificatory skills. The positive results confirm the hypothesis that *Geometrikoland* is a useful artifact for the development of definitional skills. In fact, full engagement in the *Geometrikoland* story has, as a necessary condition, the consideration of each quadrilateral as a synthesis of its conceptual and figural characters, thus overcoming the barriers raised by their cognitive dissociation.

Additionally, based on observations during the workshops, an important contribution to the obtained results comes from the circular shape of the shields. Having geometric figures drawn within a circumference eliminated any Cartesian reference system (this could have also occurred unconsciously if, for example, the shields had a rectangular shape). This choice ensured that children had to reason solely about the definitions of various quadrilaterals and the properties of non-stereotypical figures without any external reference. Further confirmation of this reflection comes from numerous observations of the following kind. During the fourth workshop, the wizard Hugo attacked the knight Paul with the spell that required him to defend himself with a "Set of quadrilaterals having at least one pair of parallel sides and at least one pair of perpendicular sides". In the first two workshops, children often responded to this attack with the shield "right-angled trapezium" (Figure 6b), only to then say, "We lost!" It was evident from the children's comments that they did not recognize within a circular reference (given by the edge of the shield) the parallelism and perpendicularity of the sides because they were not in prototypical positions. As the workshops progressed, the children made fewer and fewer errors of this type because they learned to make observations of parallelism and perpendicularity independent of the graphic context and the need for a fixed and implicit Cartesian reference such as the one they are accustomed to (for example, the classroom floor and walls or the edges of the notebook or blackboard). Most likely, this situation quickly removed *critical attributes* [8] associated with the orientation of quadrilaterals in relation to a reference system in the majority of children.

From a preliminary comparative analysis of the results of the placement test and the exit test, the improvement in children's performance in all three investigated competencies (application of the uniqueness criterion, application of the arbitrariness criterion, and verification of classificatory thinking) is evident.

Initially, the research team did not expect a positive outcome. However, confidence increased over time as it was observed that during the game, players were not only required to examine the cards and decide which one to respond with but also had to justify their answer. We believe that the constant interaction in their minds between the conceptual definitions of quadrilaterals and their corresponding visual representations is the reason for their better-than-expected scores in the exit test. This point is crucial: requesting justification prompts players to pause and search for it; it is expected that this interruption will block the immediate, intuitive response and make the subject realize the need to resort to a theoretical approach. Despite following their intuition, players are expected to seek a theoretical relationship between the properties in play. The better performance on the arbitrariness criterion can also be explained by the fact that a deeply rooted belief in the children had to be overcome. Before the experiment, they reasoned with "THE definition" rather than "A definition". During the Geometrikoland activities, this obstacle was repeatedly overcome, for example, when the knights, to defend themselves from the magic of "Parallelogram with congruent diagonals", used the shield of "Rectangle" to ward off the spell and advance towards the goal.

The good results of the fourth question are very important because they demonstrate that there has been only a positive effect not only for the theory of quadrilaterals with which the children worked but also in a theory that was not the subject of the experiment (the theory of triangles).

In the spell cards, there are already multiple definitions for the same mathematical object (consider, for example, that for the parallelogram, there are already five spell cards consisting of necessary and sufficient conditions of the parallelogram, thus we have five potential definitions equivalent to the commonly used one where the parallelism of opposite sides is considered as critical attributes). A development of the game involves enhancing this aspect through the design of new cards representing further diverse possible definitions of the same quadrilateral and then studying the behaviour of the students in using this specific type of cards.

The future direction of this research involves confirming some empirical observations that suggest a correlation between the strategic thinking employed to win at Geometrikoland and improvements in terms of definitory thinking.

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References

1. Poincaré, H. Mathematical Definitions and Education. In *Science and Method (Trans. F. Maitland)*; Poincaré, H., Ed.; Thomas Nelson: London, UK, 1914.
2. Vinner, S. The role of definitions in the teaching and learning of mathematics. In *Advanced Mathematical Thinking*; Springer: Dordrecht, The Netherlands, 1991; pp. 65–81.
3. De Villiers, M. To teach definitions in geometry or teach to define? In Proceedings of the Twenty-Second International Conference for the Psychology of Mathematics Education, Stellenbosch, South Africa, 12–17 July 1998.
4. Zaslavsky, O.; Shir, K. Students' conceptions of a mathematical definition. *J. Res. Math. Educ.* **2005**, *36*, 317–346.
5. De Villiers, M. The role and function of a hierarchical classification of quadrilaterals. *Learn. Math.* **1994**, *14*, 11–18.
6. Currie, P.; Pegg, J. Investigating students understanding of the relationships among quadrilaterals. In Proceedings of the Teaching Mathematics in New Times: Conference Proceedings. Melbourne: Mathematics Education Research Group of Australasia Incorporated, Gold Coast, Australia, 5–8 July 1998.
7. Frank, M. What difference does it make? Children's views of the differences between some quadrilaterals. *Educ. Stud. Math.* **2000**, *42*, 179–196.
8. Hershkowitz, R.; Vinner, S. The role of critical and non-critical attributes in the concept image of geometrical concepts. In Proceedings of the Seventh International Conference for the Psychology of Mathematics Education, Jerusalem, Israel, 24–29 July 1983; pp. 223–228.
9. Fischbein, E. The theory of figural concepts. *Educ. Stud. Math.* **1993**, *24*, 139–162. [CrossRef]
10. Zazkis, R.; Liljedahl, P. *Teaching Mathematics as Storytelling*; Sense Publishers: Dordrecht, The Netherlands, 2009.
11. Haven, K.F. *Super Simple Storytelling: A Can-Do Guide for Every Classroom, Every Day*; Libraries Unlimited: Devon, UK, 2000.
12. Baker, A.; Greene, E. *Storytelling: Art and Technique*, 2nd ed.; R. R. Bowker: New York, NY, USA, 1987.
13. Zan, R. The crucial role of narrative thought in understanding story problems. *Curr. State Res. Math. Beliefs* **2011**, *16*, 287–305. [CrossRef]
14. Gould, D.; Schmidt, D.A. Trigonometry comes alive through digital storytelling. *Math. Teach.* **2010**, *104*, 296–301. [CrossRef]
15. Ravanelli, F. Un'Esperienza di Matematica e Digital Storytelling nella Scuola Primaria. 2012. Available online: https://www.rivistabricks.it/wp-content/uploads/2017/08/5_Ravanelli.pdf (accessed on 23 April 2024).
16. Starcic, A.I.; Cotic, M.; Solomonides, I.; Volk, M. Engaging pre-service primary and preprimary school 756 teachers in digital storytelling for the teaching and learning of mathematics. *Br. J. Educ. Technol.* **2015**, *47*, 29–50. [CrossRef]
17. Tortorelli, L. *Quaderni di Geometria Verticale. Vol. 3—Quadrilateri*; Edizioni Dedalo: Bari, Italy, 2021.
18. Vygotsky, L. *Mind in Society: Development of Higher Psychological Processes*; Harvard University Press: Cambridge, MA, USA, 1978.
19. Bartolini Bussi, M.G.; Mariotti, M.A. Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In *Handbook of International Research in Mathematics Education*; Routledge: London, UK, 2008; p. 746.
20. Bartolini Bussi, M.G. Verbal interaction in mathematics classroom: A Vygotskian analysis. In *Language and Communication in Mathematics Classroom*; Steinbring, H., Bartolini Bussi, M.G., Sierpinska, A., Eds.; NCTM: Reston, VA, USA, 1998; pp. 65–84.
21. Duval, R. Registres de représentation sémiotique. *Ann. Didact. Sci. Cognit.* **1993**, *5*, 37–65.
22. Duval, R. Quel cognitif retenir en didactique des mathématiques? *Recherch. Didact. Math.* **1996**, *16*, 349–382.
23. Vygotsky, L. *Thought and Language*; MIT Press: Cambridge, MA, USA, 1962.
24. Duncker, K.; Lees, L.S. On problem-solving. *Psychol. Monogr.* **1945**, *58*, 1–133. [CrossRef]
25. Piéron, H. *Vocabulaire de la Psychologic*; PUE: Paris, France, 1957.
26. Hoz, R. The Effects of Rigidity on School Geometry Learning. *Educ. Stud. Math.* **1981**, *12*, 171–190. [CrossRef]

27. Bartolini Bussi, M.G. Mathematical discussion and perspective drawing in primary school: To Giovanni Prodi on occasion of his 70th birthday. *Educ. Stud. Math.* **1996**, *31*, 11–41. [CrossRef]
28. Rosh, E. Principles of categorization. In *Cognition and Categorization*; Rosh, E., Loyd, B., Eds.; Lawrence Erlbaum Associates Inc.: Hillsdale, NJ, USA, 1978; pp. 27–48.