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How Learning to Speak the Language of a Computer-Based Digital Environment Can Plant Seeds of Algebraic Generalisation: The Case of a 12-Year-Old Student and eXpresser

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Abstract: When learning in a digital interactive mathematics learning environment (DIMLE) designed to foster the development of specific mathematics content, students come to express their ideas through different languages and representations. We devise a method based on the Theory of Instrumental Genesis (TIG) to analyse aspects of a middle school student's learning about algebraic generalisation in a DIMLE called "eXpresser". Our analytic scheme allows us to capture changes in her instrumented schemes when accomplishing a certain task repeatedly, gradually modifying her interactions with the system. The results concern both insights into a specific mathematics learning journey in a DIMLE, and methodological progress at a more general level. Indeed, the method we devised and explored in this specific case can be applied to infer students' schemes from their actions as they interact with other DIMLEs. This possibility yields great potential because more and more actions can now be recognized directly by software. This has important implications for computer-supported personalised learning, and AI in general.

Keywords: algebraic generalisation; digital interactive mathematics learning environment (DIMLE); instrumentation scheme; theory of instrumental genesis

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1. Introduction: Students' Learning in Digital Interactive Mathematics Learning Environments

The branch of research in mathematics education concerned with studying students' learning in computer-based learning environments was inaugurated decades ago by Papert's work with Logo [1]. He coined the notion of "microworld" to describe an environment designed to foster specific mathematics learning: indeed, such an environment needs to be designed to "incorporate" certain mathematical ideas that are informally encountered by students as they interact with the microworld. Over the decades new terminology has been proposed to refer to digital environments with similar characteristics; it is beyond the scope of this paper to review or discuss such terminology, but in order for our work to resonate with current research, instead of "microworld", we will speak of Digital Interactive Mathematics Learning Environment (DIMLE) (e.g., [2,3]). The DIMLE we will introduce was designed with the explicit goal of promoting certain concepts and forms of algebraic reasoning that we will illustrate below. By interacting with digital objects in the DIMLE, students need to discover and express relationships between them to accomplish certain tasks (that make sense within the DIMLE); during such a process, the student is supposed to encounter fundamental algebraic ideas that can be metaphorically seen as seeds of the algebraic concepts and forms of reasoning that are "planted" by the experience with the DIMLE. Such a "planting of seeds", in a carefully designed DIMLE such as the one we will introduce, should occur as a result of the student becoming successful in interacting with the system: to do so, she needs to become fluent in expressing her thoughts and strategies in the "language of the DIMLE", which in our case is a sort of graphic programming environment (e.g., [4]). Becoming fluent in the language of a digital computer-based environment can be seen as a skill of techno-mathematical literacy [5], which is fundamental in order to create "constructive interference" between the 4th Industrial Revolution and the teaching and learning of mathematics modernisation of our educational systems, as highlighted by Lew and Baccaglini-Frank [6] at the plenary panel of the 44th Conference of the International Group for the Psychology of Mathematics Education.

Moreover, as we will discuss in Section 2.1, becoming fluent in the language of the system can make use of and foster the development of forms of computational thinking (CT) (e.g., [4,7]) that are "the thought processes involved in formulating problems and their solutions, so that the solutions are represented in a form that can be effectively carried out by an information-processing agent" ([8], p. 20). A recent systematic review of such literature reveals that most studies, in particular those conducted to improve the conceptual basis and practice of computationally enhanced mathematics learning (e.g., [9]), suggest that CT's integration in mathematics instruction is associated to gains in learning outcomes, especially in the domains of algebra and geometry [4,9]. Moreover, among the computational concepts and practices associated with CT-based mathematical activities for which learning gains have been found, the following are particularly relevant to our work in this paper: the notion of variable, abstraction—especially in the form of pattern recognition, modelling, and testing and debugging, which are notions described by many authors (e.g., [10]).

Generally speaking, in this paper, we will elaborate on a method to help researchers answer questions such as the following that have been addressed over the years within the general literature in mathematics education (e.g., [3,11–13]). When a student achieves successful outcomes following interactions with a DIMLE designed to embed aspects of a mathematical domain, what learning occurs? How can we make sense of the interaction between the student, the computer and nearby educators (visually, acoustically or from responses to computer-generated benchmarks, etc.) so as to trace students' learning (and eventually design the system to better support students' learning)?

More specifically, our goal here is to qualitatively capture in fine-grained detail a student's learning within a DIMLE, associating sets of (visible) actions accomplished by a student to solve a specific task with mathematical ways of thinking that are fundamental for conquering algebraic generalisation. Although we will be focusing on a specific DIMLE and showcasing the analyses in the case of a specific student, we argue that our method is generalisable to other learning journeys, and to different DIMLEs (with the necessary changes in the intended learning and a priori analysis of the DIMLE's design).

2. Our Perspective on Algebraic Generalisation, Elaboration of the Theoretical Lens and Introduction of the DIMLE

We start by making more explicit the content of the student's learning that we prepare to observe, within the broader perspective of digital technology as a gateway to algebraic generalisation. We then introduce how we will use the Theory of Instrumental Genesis [14–17] to analyse the interactions between a student and the DIMLE to untangle how becoming fluent in the language of the DIMLE can be linked to the planting of seeds of algebraic generalisation. We finally introduce the main features of the DIMLE in focus, eXpresser, explaining its potential for promoting the learning intended.

2.1. Digital Technology as a Gateway to Overcome Difficulties with Algebraic Generalisation

What is algebra? Most secondary students, if they can provide any answer at all, would answer that it is a set of rules of transformation, a way of symbolically manipulating expressions. Mathematicians similarly recognise the importance of the mathematical "machinery" of algebra; but they would also acknowledge that algebra is a language through which structure can be expressed in a general way. Most students, it seems, even those who can carry out the requisite manipulations successfully, cannot use algebra as a tool for thinking mathematically and do not appreciate that algebra is about expressing generality, and that symbolic rules are the language of this expression (e.g., [18–20]).

The literature is replete with examples of students' difficulties in algebra, first documented in large-scale studies such as the Concepts in Secondary Mathematics and Science [CSMS] programme (see [21]). A summary is provided by Küchemann and Hoyles [22] and good arguments and examples are shared by Arcavi, Drijvers and Stacey [20]. Students often struggle to understand the idea behind using letters to represent any value and are inexperienced with using mathematical vocabulary to express generality (e.g., [23–25]). Even students who are capable of expressing a general rule using words, like "always" or "every", struggle to build their arguments using letters and symbols.

Moving flexibly between different forms of representations has been identified as an invaluable mathematical orientation and one that is feasible for students if they are given appropriate support [26]. However, even though students are often provided with various representations and models to assist them in their efforts to understand, justify and communicate structural arguments, they tend not to make expressive links between these different forms of representations (symbolic, iconic, numeric) or to perceive their equivalence [22,27,28].

Such difficulties are not alleviated by the role played by algebra in the traditional mathematics curriculum: the purpose of algebra tends to get lost, the end (to express generality) confused with the means (to learn the language of algebra). As Kaput [29] put it, students are routinely asked to "learn representation systems without anything to represent" (p. 546). Yet the need to express and justify generality can be considered "the heart, root and purpose of algebra" ([30], p. 2).

The challenge, therefore, is to design tasks and technologies that together have the potential to help students appreciate the power of different representations. Such a design challenge was faced by the project that led to the development of the MiGen system, a suite of software components, and in particular a DIMLE, eXpresser, to help 11- to 14-year-old students appreciate algebra as a language for generalisation [31]. The project ended a decade ago, but the results reported in this paper are new and we believe they should inform both current research on the teaching and learning of algebra, and current research on student learning in DIMLEs with AI support.

The MiGen system aimed at fostering the development of a key competence, that of expressing ideas through a different language and representations, the eXpresser language. The students explored eXpresser using its language to express their ideas and communicate with this tool. Such processes (Exploring and Expressing) were evident in other, more recent, projects. For example, the ScratchMaths project that explored students' mathematics learning during structured interactions with the Scratch programming language (e.g., [32]), which also considered CT's influence towards mathematical learning (as suggested in [7]). This project offered a framework of design principles for the actions learners get to use when programming. These are: "Explore (mathematical ideas), Explain (a process in computer language), Envisage (or predict possible outcomes), Exchange (ideas and processes with others), and bridgE (mathematical and computational concepts)" [33]. In both these environments, learners were encouraged to gain new skills to be able to interact with the digital tools in question and learn a new language so as to explore, explain their mathematical ways of thinking and, in fact, communicate with the digital tools. They had to apply their mathematics knowledge and use their mathematical skills in a new context, that of a digital tool. However, of course, they had to gain technical knowledge and develop technical skills relevant to the digital tool they interacted with.

This idea of such (inevitably connected) processes was also reported by Hoyles et al. [5] in their "mathematics-at-work" project, where they proposed the term "Techno-mathematical Literacies". These are literacies regarding the breadth of mathematics knowledge as well as technical knowledge required within contemporary workplaces. Jacinto and Carreira [34] extended the idea of techno-mathematical literacies, and introduced the term techno-mathematical fluency, as they wanted to also capture "the idea of being able to produce mathematical thinking by means of digital tools, to reformulate or generate new knowledge, and to express such thinking technologically" (p. 1122). They added that "techno-mathematical fluency emphasises the need to be fluent in a 'language' that entails both mathematical and technological knowledge, the skilful use of digital tools and the efficient interpretation and communication of the solution to a problem" (p. 1122).

2.2. The Theory of Instrumental Genesis

To study the interplay between students' activity in the microworld and the development of their algebraic thinking, we use the Theory of Instrumental Genesis (e.g., [15–17]). Within this theory, artefacts are distinguished from instruments: an artefact is a physical or digital object [35] that is used by a learner to carry out a given task; the artefact becomes an instrument for solving a certain task when it is endowed with a scheme, which is a psychological construct. The notion of scheme has its roots in Vergnaud's work [36], upon which the Theory of Instrumental Genesis (TIG) was developed [17].

Within the TIG, Drijvers, Godino, Font and Trouche [13] have described three dualities: the artefact–instrument duality, involving the process of how a user transforms an artefact into an instrument; the instrumentation–instrumentalization duality, focusing on how a tool can shape a student's thinking and actions (instrumentation) and how a student's knowledge, thinking and actions can shape the use of an artefact (instrumentalization); and the scheme–technique duality, which refers to "the relationships between thinking and gesture" ([17], p. 26). Within this third duality, some parts of students' work on solving a particular task are "visible"—the techniques, while others are not directly observable—the schemes, "the cognitive foundations of these techniques" ([17], p. 27). We note that an artefact may be a digital tool as a whole or specific symbolic features within it, as discussed by Monaghan, Trouche and Borwein [35].

Various studies grounded within the TIG have described students' learning of certain mathematical concepts through the use of digital artefacts: for example, Roorda, Ros, Drijvers and Goedhart [37] did this for the concept of derivative; Drijvers, Godino, Font and Trouche [17] used both the TIG and the onto-semiotic approach to capture students' learning in the context of quadratic equations; and Gregersen also worked in this direction, focusing on students' justification processes in situations that involve a variable represented as a slider in GeoGebra applets [38,39]. Our study is situated within this line of research.

Specifically, in this paper, we will make use of the distinction between techniques and schemes to identify emergent instrumentation schemes in eXpresser. This will allow us to offer a fine-grained example of how a student's dialogue with the DIMLE leads her to form and express ideas in different languages: verbally and in the language of eXpresser. The techniques we will identify mostly show how the student learns to use the programming language of the DIMLE, which is based on signs of various types: graphical, written words and icons in "property windows" (see Section 2.3), and algebraic to form expressions.

2.3. The Microworld eXpresser and Three Key Ways of Thinking for Algebraic Generalisation

eXpresser is a DIMLE designed to help 11–14 year-old students appreciate algebra as a language for generalisation (Noss et al., 2009 [31]), and to develop, in particular, the following three algebraic ways of thinking (AWOT) [40]:

- AWOT.1—perceiving structure and exploiting its power: recognising the constituent elements of a complex structure, and using them to build the structure—both physically and mentally;
- AWOT.2—seeing the general in the particular: identifying variants and invariants, manipulating a special and familiar case in order to get a sense of what stays the same and what changes;
- AWOT.3—recognising and articulating generalisations, including expressing them symbolically: describing structure by using variables (symbols) and expressing symbolically their relationships to structure and generality; that is, building an algebraic rule.

As discussed earlier in Section 2.1, the main obstacle students are known to face is making the step from using relationships (correctly but) implicitly in calculations to expressing these relationships explicitly. This is specifically what eXpresser was designed to help students overcome. In eXpresser, students are presented with a dynamic (animated) figural coloured pattern and their task is to construct the same pattern on their own in eXpresser, but also find the "general" rule for calculating the number of tiles in any pattern, e.g., the nth term in a figural pattern sequence. Students had to perceive the structure in the given pattern or what was repeated and what remained the same in any instantiation of the pattern (AWOT1); then they had to explore what stayed the same and what varied and consider constants and variables (AWOT2); before taking the final step of articulating the "rule" that gave the total number in any pattern instantiation writing it using the eXpresser language (AWOT3). In other words, students are asked to create a pattern on a grid using a set of coloured tiles and repeat it with a specific regularity. Below, we present an example to further showcase eXpresser's design and the specific tasks it poses to students who interact with it (Figure 1).

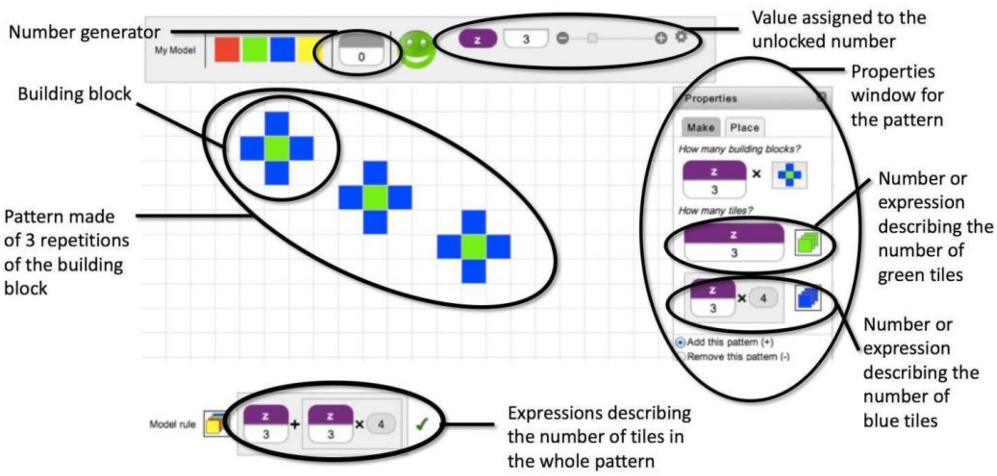


Figure 1. Screenshot of eXpresser, in which a "crosses" pattern has been constructed by repeating a building block made of 4 blue tiles and a green tile. Rules for the numbers of tiles of each colour in the pattern have been constructed using the "unlocked" number (which is 3 in this instance) named "z".

For example, Figure 1 shows a pattern in eXpresser created with a set of four blue tiles and one green tile in the middle, forming a "cross" (referred to as "building block" in the eXpresser language), repeated three times; below the pattern, a model rule has been constructed: it represents the total number of green and blue tiles in the pattern. "z" is chosen as the letter to represent the number of repetitions of each group of tiles making a building block. Any number used to describe a pattern, like "z", can be unlocked; that is, it can be allowed to change. Once a number is unlocked, its variation can be imposed either manually or automatically through animation. The possibility for students to develop a scheme in which they unlock a number and use it to symbolically describe a pattern was designed to foster the development and understanding of algebraic generalisation, and, in particular, the meaning of algebraic variable, a number that can change giving generality to an expression (seen as a symbolic representation of a pattern).

Sets of tiles can become coloured: from being greyed out, they will turn into the colour they were intended to be whenever the student correctly answers the question "How many tiles [of that intended colour]?" within the properties window (the main artefact that students learn to use to express their ideas in the DIMLE's language) relative to a pattern.

Different techniques, the observable instances of a solver's activity carried out for this (or another) task, are related to different schemes that can also be inferred with the help of the solver's verbal communication. For example, to find the number of squares in case 3 (of Figure 1), a student could count up the tiles and type in a numerical answer; or they could use the number found previously for case 2 and add the squares needed to construct case 3, so computing 10 + 5 either orally or by constructing an expression and calculating its value; or they could visualise a "unit" defining the figure's structure (in Figure 1 a "flower" with 4 blue tiles and 1 green tile) and multiply the number of squares that compose it (5) by its number of repetitions in case 3, constructing the expression 5×3, or $3 + 3 \times 4$ if she counts the blue tiles and the green tiles separately.

The answer provided by the student may be correct only in a specific case (e.g., for 3 repetitions of the building block) but not in general; that is, when animation is turned on and the number of repetitions of the building block varies. For example, Figure 1 shows the rules "3" and "3×4" in the properties window built to express the total number of green and blue tiles, respectively, in the pattern. This feature was designed to help students focus their attention on the number of tiles in a pattern with a certain structure (AWOT.1), and on expressing their evolving counting strategies in a general algebraic language (AWOT.2 and AWOT.3).

The possibility offered by eXpresser of working with geometrical structures and linking these to symbolic representations is a key feature with respect to all three algebraic ways of thinking. In eXpresser, the learner is asked to construct a sequence and "make it stay coloured" (we refer to this as the colouring task) as the number of repetitions varies. While solving such a task, a student's emergent instrumented schemes should come to include the unlocking of a number in expressions. Fluency in her techniques for using this artefact can be interpreted as an important step in the student's learning to express her ideas in the language of eXpresser, and, more in general, towards algebraic generalisation.

2.4. Our Specific Research Question and Broader Aim

With this study we specifically aim to investigate the following research question.

What instrumented schemes for the colouring task in eXpresser can be inferred from students' instrumented techniques emerging during their interactions with the DIMLE, and what are the emerging algebraic ways of thinking?

At a more general level, we wish to shed light on the method we used to set up an analytic scheme that allowed us to infer the student's instrumented schemes and sketch out the progress of her mathematics learning.

3. Capturing a Student's Learning Journey in eXpresser

In this section, we present our methodological approach and our focus, for this paper, on the case of one student, Molly.

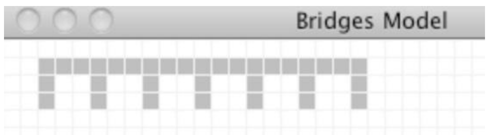
3.1. Participants and Data Collected

We introduced the DIMLE eXpresser to six seventh graders in a school in London, after which we conducted two or three observed-activity sessions. The introductory lesson was planned to prepare and motivate students for the activities of the following sessions, in which observations would be conducted. During the lesson, a model of repeating cross shapes was used to introduce the fundamental features of eXpresser, including the support offered by the system. Students were asked to work in pairs and a researcher visited each pair in turn to answer specific questions. Students were shown how to drag numbers onto the canvas and form expressions with them. They were not, however, shown how to build a general rule: this problem was left to be solved during the sessions. We video-recorded the students' interactions with eXpresser during the sessions using the screen capturing software Camtasia 2 for Mac OS.

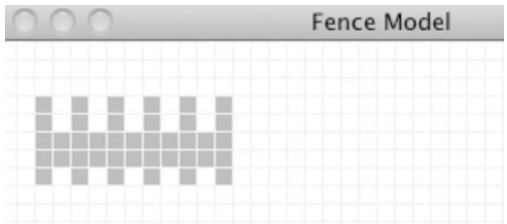
However, such data were collected more than a decade ago and the MiGen system is no longer available in the form in which it was experienced by the participants. We therefore re-enacted [41] a set of vignettes from the experience of one of our participants, who we refer to here with the pseudonym Molly, using a new web-based version of the DIMLE (accessible at web-expresser.appspot.com, accessed on 13 April 2024), and using completely anonymised excerpts we had transcribed from the Thinking Aloud protocol (e.g., [42]) used during the original study. Such a protocol provided us with a "window" into her schemes and emerging algebraic ways of thinking [11]. Molly, like all students who took part in our studies, worked on many different tasks in eXpresser. She participated in three observed-activity sessions, and in the last session, she agreed to explain how eXpresser worked to a classmate who had been absent. In this paper, we present vignettes from Molly's sessions with eXpresser, focussing especially on the one that took place in her last session.

3.2. Re-Enactments of the Observed-Activity Sessions

The vignettes we will present took place around three activities, each of which took up a 40 min session: the first two comprised working individually with eXpresser to build models, and the third one to help another student construct and colour a given model. The activities involved the construction of two models: the bridges model and the fences model (see Figure 2a,b). During these activities, two researchers were present (one was the first author of this paper) and they would stop and talk to students if they appeared to be "stuck".



(**a**)



(**b**)

Figure 2. (**a**,**b**). The bridges model and the fence model as presented on the screen in activities (**a**) 1 and (**b**) 2.

The simplified general rule for the bridges model is "5n + 3" and for the fence model is "7n + 5". In the third activity, Molly was asked to help a fellow student learn to use eXpresser to make a model and find a rule for the number of tiles for any model number. In this activity, students were expected to articulate and share their thinking processes to support a fellow student solve the given task and derive a general rule.

Each vignette depicts a successive instantiation of Molly's of the colouring task, which she faced a total of ten times. We consider her colouring tiles of a single colour an instantiation of the task (or "specific task", that can be seen as a subtask of the general colouring task). In each instantiation, we will identify the following:

- The specific task Molly is solving;
- The artefacts she uses (a visible part of how Molly interacts with the DIMLE);
- Her instrumentation schemes, through the techniques and the technical elements (the visible parts of her becoming fluent in the language of the DIMLE), as well as the conceptual elements (that we will associate to the emergence of the expected algebraic ways of thinking).

While the technical elements are relatively easy to identify objectively through the observations of the student's actions on the screen and her utterances, we can only infer the conceptual elements. Indeed, the conceptual elements constitute the invisible part of the scheme that needs to be inferred. To this aim, we will strengthen our inferences by considering Molly's verbal interactions with the nearby researcher and with her classmate.

To carry out the analyses of Molly's case, we started by transcribing all her sessions in Transana 2.2 (www.transana.com accessed on 13 April 2024), a Computer-Assisted Qualitative Data Analysis Software for qualitative analysis of multifaceted data [43], as was performed for the analyses of all students' interactions with eXpresser [44]. In Transana, we organised Molly's media files into smaller units of analysis, each corresponding to an instantiation of the colouring task; then, we coded them using thematic or conceptual keywords so as to investigate certain phenomena in her interactions, including actions that we associate to her techniques that we report on in this paper.

We provide an analytic account of the visible part of each instantiation, in the form of vignettes, with screenshots and dialogue excerpts when necessary, followed by a table summarising all aspects analysed, and by an analytic comment of the process depicted in each vignette.

4. Results

Molly completes all three activities, constructing two patterns for each model, and twice using multiple colour tiles in the same patterns. She faces the task again twice, after unlocking the number of repetitions of the building block. Therefore, overall, she faces the colouring task a total of 10 times: 5 times in activity 1 (instantiations 1–5), 3 times in activity 2 (instantiations 6–8) and twice in activity 3 (instantiations 9–10). We present her first instantiation and then the four instantiations in which the most significant changes in her instrumented scheme appear.

4.1. Vignette 1

When Molly first attempts this task, she constructs the bridges model using a pattern with tiles that overlap and a pattern of negative tiles. Patterns with negative tiles have the "ability" to erase existing tiles on the eXpresser and they are indicated with an "x" when placed on top of existing tiles, as shown in Figure 3.

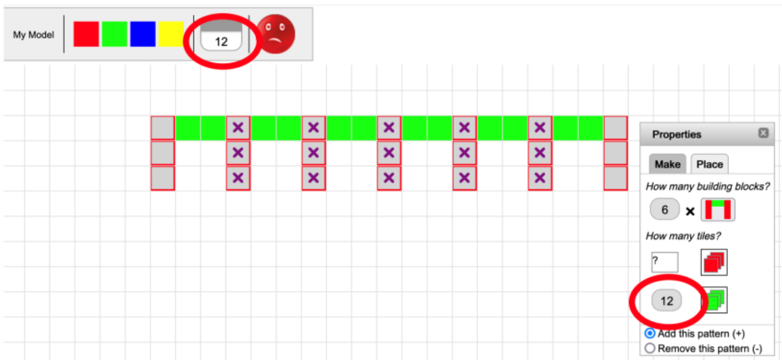


Figure 3. Molly successfully coloured the green tiles for the 6 repetitions using the number generator, as highlighted by the red circles added to the screenshot.

Account of Molly's actions

Molly counts the green tiles as sets of two, saying "two, four, six, eight, ten, twelve". She types 12 into the number generator and drags it into the properties window. Table 1 presents the aspects analysed of Molly's first instantiation.

Table 1. Molly's instantiation 1 of instrumentation scheme for the colouring task.

Specific Task	Artefacts Used	Instrumentation Scheme: Compute and Answer
colour the green tiles (see Figure 3)	number generator; properties window	technique: compute the number of tiles mentally; answer with a number conceptual elements: perceiving structure visually (as per AWOT.1) and describing it arithmetically: seeing 12 as 6 blocks of 2 technical elements: use the number generator to write the number of green tiles; drag this into the "?" under "How many tiles?" in the properties window

Analytic comment

In this instantiation, through the visible technical elements, we identify both processes of instrumentation and of instrumentalization. The former relates to how seeing the 6 sets of 2 green tiles leads Molly to a counting approach, and to how her response to the eXpresser's question "How many tiles?" in the properties window promotes her answer "12" that she creates by using the number generator artefact, which is a tool that enables interaction with eXpresser. Instrumentalization also seems to be occurring because Molly seems to be "imposing onto the DIMLE" her intuitive approach of counting the green tiles and using the number generator to record the exact number of these and answer the question. At this point, while her scheme does suggest that Molly has perceived the relationship between the visual–geometrical layout of the pattern and its numerical description in a particular case, she is still using a counting approach that is very context-specific.

4.2. Vignette 2

Now, Molly seeks to colour the red tiles (see Figure 3).

Account of Molly's actions

Molly initially repeats her previous actions. However, she has trouble finishing the computation. The researcher is watching and they suggest: "How about writing a rule?". Molly hesitates, and the following dialogue takes place.

> **Researcher:** Alright let's write it down here with the expression. So [. . .], over here, you take the 6 and then . . . what are you going to do?
>
> **Molly:** [she drags out the 6] and then I think I have to [she moves the mouse on the screen but does not find what she seems to be expecting.]
>
> **Researcher:** How do you put them together, if you want? You drag it on top of the other one right?
>
> **Molly:** oh yeah. I think it's times. . .
>
> **Researcher:** okay and then what do you need to do?
>
> **Molly:** calculate it and that's 36. So, I think you have to do, drag the 36 there.

Molly successfully colours the red tiles in her static pattern, using the scheme described in Table 2.

Table 2. Molly's instantiation 2 of instrumentation scheme for the colouring task.

Specific Task	Artefacts Used	Instrumentation Scheme: Calculate Value of Expression
colour the red tiles (see Figure 3)	geometric representation of the pattern; expression -blocks; properties window; calculate value	technique: make an expression for the number of tiles and calculate its value; answer with a number conceptual elements: a numerical calculation can be represented symbolically; a number must be used to answer, "How many red tiles?"; perceiving structure, and recognizing and articulating generalisations expressing them symbolically through the expression-blocks (as per AWOT.3) technical elements: drag onto the canvas a number from the building block properties window; drag onto the canvas another number; multiply the two numbers on the canvas; calculate the value; use this number to answer, "How many tiles?"

Analytic comment

In her second instantiation of the colouring task, thanks to her difficulties in counting the tiles which prompted the researcher's intervention, Molly uses a new artefact, the expression-blocks, to interact with eXpresser, communicating to it the calculation she intended to carry out.

We recognize here a process of instrumentation: Molly's actions of using expression blocks by dragging them from the properties window in eXpresser and combining them enable her to create an expression (or a "rule" as she calls it) for the number of red tiles, or in other words, to communicate with eXpresser using its language, a language that shares many symbols with mathematical language. On the other hand, instrumentalization is also occurring, as Molly learns to drop the one expression block (number of repetitions of the building block) onto the other expression block (number of red tiles within the building block) and select the operation to connect them. Moreover, Molly has eXpresser "calculate it"; that is, she uses the newly constructed expression not as an independent algebraic rule providing an algebraic translation of the geometrical structure of the pattern but as a "calculator" to obtain the correct "answer". In doing this, Molly is shaping the tool to make it produce what she wants; this might be the case because she is not sure about the meanings of the numbers that she used to construct the rule and she still needs to compare the result with the number she calculated mentally. We can also interpret this interaction as a development of Molly's techno-mathematical fluency (Jacinto and Carreira, 2017 [34]): she is using four artefacts of the DIMLE to effectively interact with it in a way that helps her solve the mathematical problem she is facing.

4.3. Vignette 3

Account of Molly's actions

Molly is prompted to unlock her model number and check her colouring of the model. She calls the unlocked number "b" for "building blocks". As soon as b changes, Molly's model no longer remains coloured. The system's intelligent support is designed to intervene in moments like this, prompting the student to find a link between the number of building blocks and the number of tiles in each building block, and then prompting for a rule that links these numbers. These prompts appear on the screen and Molly, who is working alone at the computer, opens the properties windows and starts dragging numbers onto the canvas. She constructs a correct rule for the red tiles, multiplying b times 6; then, she calculates its value and replaces her previous answer with the calculated value (Figure 4).

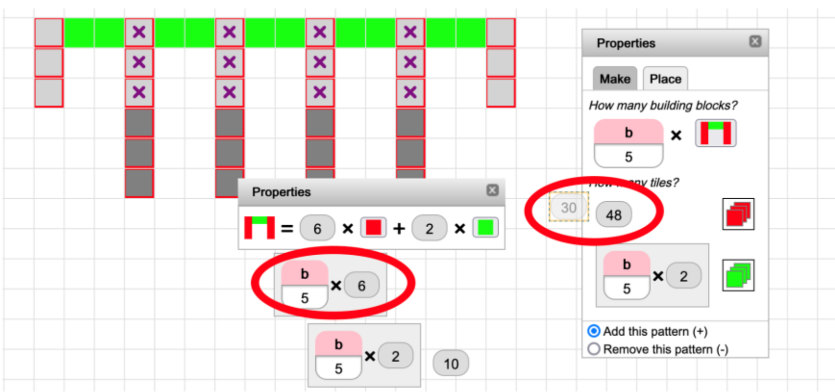


Figure 4. Molly writes a correct rule for the red tiles (5×6), calculates its value (30) and uses this calculated value to replace her previous answer (which she first tried to fix by unlocking it), highlighted by the red circles added to the screenshot.

Molly successfully colours the red tiles (in Figure 4), using the scheme described in Table 3.

Table 3. Molly's instantiation 3 of instrumentation scheme for the colouring task.

Specific Task	Artefacts Used	Instrumentation Scheme: Calculate Value of Expression with Unlocked Number
colour the red tiles (see Figure 4)	geometric representation of the pattern; unlocked number; pattern animation; expression blocks; properties windows; calculate value	technique: make an expression for the number of tiles, using the unlocked number of building blocks, and calculate its value; answer with a number conceptual elements: a numerical calculation can be represented symbolically; a numerical calculation can be related to a geometrical structure (working towards AWOT.1); there are special numbers that "can change" (working towards AWOT.2); a number must be used to answer, "How many red tiles?" (working towards AWOT.3) technical elements: drag out the number of tiles of the desired colour in a building block properties window; drag out the unlocked number of repetitions of the building block; multiply these two numbers; calculate the value; use the calculated value to answer "How many tiles?"

Analytic comment

Here, we see a shift in Molly's instrumented scheme: eXpresser's animation artefact prompts Molly to unlock a number and change its value, which makes Molly's non-general solution evident. We see an important process of instrumentation taking place here. Molly seems to start to realise that it is the unlocked number b that needs to be used in the rules describing the numbers of coloured tiles, not specific numbers typed into the number generator that are good only for a specific case. The system clearly seems to be shaping Molly's actions, and probably her scheme behind such actions, to some extent, while fostering her development of techno-mathematical fluency in the DIMLE.

Although Molly successfully builds a rule containing an unlocked number, she still insists on answering the "How many tiles?" question with a "calculated" number (see her words in instantiation 2). This action suggests that Molly has taken some steps towards seeing the general in the particular (AWOT.2), and towards recognizing and articulating a generalisation (AWOT.3), but that she has not yet completely acquired the sought for algebraic ways of thinking. In terms of instrumentalization, we notice that Molly insists on forcing the tool to "calculate the value" in order to place a single number, not an expression, in response to the "How many tiles?" question. It is possible that Molly's past experience and her lack of experience with algebraic generality prevents her from using her general rule as a number.

We summarise her next four instantiations of the colouring task: during these, Molly comes to use the constructed rule to answer the "How many tiles?" question, increasing her techno-mathematical fluency. However, before inserting such a rule into the properties window, every time, she insists on calculating its value and leaving that number on the screen. We interpret this behaviour as indicative of a persistent difficulty in internalising AWOT.3.

In the fourth and last vignette, we analyse Molly's final instantiation of the colouring task, and then provide a summary of her scheme development.

4.4. Vignette 4

Molly is now working on activity 3, where she has been asked to help a fellow student work in eXpresser.

Account of Molly's actions

Molly swiftly presents to her classmate "what you should do", showing her the sequence of actions to be carried out to colour a pattern. The first time, Molly suggests that her classmate uses the number generator to answer because "here you can type in the answer calculated in your head", but, she continues, "then your pattern doesn't stay coloured". Molly unlocks the number of repetitions of the building block; then, she continues to explain "it's better to use the properties window". Molly creates an expression multiplying the unlocked number times the number of tiles to colour in a single building block; then, she drags the expression into the properties window to answer the "How many tiles?" question.

Table 4 shows Molly's scheme in her last instantiation of the colouring task.

Table 4. Molly's instantiation 10 (the final one) of her instrumentation scheme for the colouring task.

Specific Task	Artefacts Used	Instrumentation Scheme: Answer with General Rule
colour the tiles of a selected colour	geometric representation of the pattern; expression-blocks; number generator; calculate value; properties window; unlocked number	technique: make an expression for the number of tiles; use it to answer conceptual elements: a numerical calculation can be represented symbolically; a numerical calculation can represent properties of a geometrical structure; algebraic rules can answer questions by asking "how many. . .?" (but it is better if you calculate the value in your head ahead of time); perceiving structure, seeing the general in the particular and articulating generalisations (working towards AWOT.1, AWOT.2, AWOT.3) technical elements: calculate the number of tiles and type that number into the number generator; drag out the number of tiles of the desired colour in a building block properties window; unlock the number of repetitions of the building block; drag out the number of repetitions; multiply the two numbers; use the rule in the answer to answer the "How many tiles?" question

Analytic comment

Molly's scheme, which we called "answer with general rule", now includes an explicitation of her mental calculations through a product of two numbers, of which one is unlocked, representing "something" that stands for a number that could be any number. Behind these actions, we infer the development of an instrumentation scheme that has overcome the use of a mental strategy to calculate tiles when the number of building blocks is fixed: Molly's scheme now is based on a general rule that will always represent the correct number of tiles, no matter how many building blocks there are. The process of instrumentation supported by eXpresser and the gradual development of techno-mathematical fluency in eXpresser have led Molly to articulate a generalisation, representing a numerical calculation symbolically and using a whole expression as a single "object". For Molly, this object now has the same function as the number she would calculate and then drag into the properties window as an answer to the "How many tiles?" question. The fact that such a procedure does not maintain the model's colouring in the microworld, seems to be enough to foster the development of Molly's scheme into the one described here. Interestingly, however, when Molly explains to her classmate how to proceed, she still includes the intermediate step of calculating the number of tiles "in your head" and even typing that number into the number generator, as a sort of mental control over the other instrumented actions, or as a remainder of her initial scheme.

4.5. Looking Back at Molly's Scheme Development

We set out to investigate Molly's instrumented schemes for the colouring task in eXpresser, inferring them from the instrumented techniques emerging during her interactions with the DIMLE, and putting them in correspondence with emerging algebraic ways of thinking. Our analyses show how such a development occurred for Molly, who started out using an implicit rule to mentally calculate the total number of tiles (compute and answer scheme), and transitioned to a scheme in which a whole expression, with an unlocked number (a variable) is used in place of that specific single number (answer with the general rule scheme).

A similar evolution was what we hoped for when designing the microworld, since Molly's initial scheme coincided with what students aged 11–14 had been found to typically do in similar situations: they use mental strategies quickly to count up tiles in a pattern, and they calculate the results of simple multiplications each time a new question is asked. However, before conducting this study, we were unsure whether and how students (1) would develop instrumented schemes involving "general" expressions as a way for successfully interacting with the DIMLE, and (2) how such schemes were related to the desired algebraic ways of thinking.

Looking back at Molly's learning journey, and taking into consideration all the data analysed, we can organise her schemes into four phases as follows, in which we can identify an interplay between instrumented techniques and developing algebraic ways of thinking:

- A phase of guided sense-making that coincided with the use of her "compute and answer" scheme, and in which she tries to make sense of the task and of the artefacts in eXpresser she chooses to use, supporting her activity by recalling previous techniques;
- A phase of mental recollection, in which she mentally recalls previous actions associated with her task. The sense-making component is weak in this phase, and the scheme appears to be unstable, as Molly easily falls back to previous schemes (especially to her calculate value of expression scheme or calculate value of expression with unlocked number scheme) and seeks support from eXpresser or a nearby researcher;
- A phase of adjustment and stabilisation of the scheme "answer with general rule", during which the instrumented techniques in eXpresser become condensed and are applied in a more automatic manner. In this phase, Molly's techno-mathematical fluency becomes more apparent as she becomes more and more experienced with how to interact with eXpresser to achieve what she wants;
- A phase of generalisation, though still quite situated, in which Molly explains to her classmate how to "answer with general rule" in a more flexible (possibly more general) way, in the sense that she separates what needs to be done from what "you can do".

When Molly's initial "complete and answer" scheme is destabilised by the loss of colour caused by the animation of the model, after being guided a first time through a solution process for the colouring task, she attempts to recall specific actions (opening properties windows, dragging numbers onto the canvas and combining them through an operation) in (what seems to be) an attempt to recall a technique used before but without the conceptual elements of the instrumented scheme it was part of. The numbers from the two windows are just any numbers in the two windows, while the choice of multiplication to combine them is in line with the mental operation that she has consciously (and correctly) carried out. At this point, Molly seems completely absorbed in trying to express her mental strategy in the language of the DIMLE, in a process of instrumentalization, that fails. For her, the calculation of the result of a multiplication was an important aspect, so she asks the computer to calculate the value of the constructed multiplicative expression. However, she does not react (in any visible way) to the mismatch between the value obtained by the computer and that obtained through a thoughtful calculation carried out with the critical numbers in her head. It is unclear whether she does not realise the mismatch or simply suspends her reasoning because of blind trust in the computer's feedback to a "calculate" prompt.

Throughout her interaction with the DIMLE, there seems to be a tension between different elements of techno-mathematical fluency: learning the language of eXpresser, seen as procedures to be recalled and blindly carried out, and learning to interact with eXpresser in a meaningful (to her) way. When prompted by the system or by the researcher to carry out an action that she had not thought of spontaneously, Molly seems to switch to an "execution mode" and tends to wait to receive further operational prompts. When these do not arrive, she attempts to recall how she had overcome the same impasse in previous iterations of the task, and only as a last resort does she try to make sense of the actions and feedback in order to plan her next move.

An important step in overcoming such a tension is taken when Molly reaches her "calculate value of expression with unlocked number" scheme in her fourth instantiation. This is a key step in the process of instrumentation and in developing meaningful techno-mathematical fluency because, through it, Molly starts to assign meaning to the numbers that appear in the properties windows. She finally expresses in the language of the DIMLE the "rule" that was in her head. Moreover, this scheme marks a fundamental benchmark in "seeing the general in the particular" (AWOT.2): the number of tiles is expressed not as a number but as an open expression, which is an object itself, representing the calculation that the student would have performed in her head every time the model number changed.

Once Molly has become comfortable using "rules" to answer the computer's questions, she communicates how to do this, showing the steps in her instrumented scheme to a classmate. Molly now also explains that "you can type in the answer calculated in your head" but she points out that it is not necessary to accomplish the task, and that "it's better to use the properties window" if you want your model to stay coloured. Thus, we can infer that Molly's scheme has become somewhat general. Moreover, in the end, Molly seems to be comfortable using locked and unlocked numbers in a single expression and then dragging such an expression into the properties window to answer the "How many tiles?" questions. We see these technical elements as indicative of a conceptual component of her emerging scheme: for Molly, expressions can now involve a "general" number as well, which is a "something" that stands for a number that could be any number. Moreover, whole expressions have become objects for Molly that can be used in eXpresser just like single numbers. These are important aspects of the algebraic ways of thinking of seeing the general in the particular and articulating generalisations while achieving techno-mathematical fluency in eXpresser.

5. Discussion and Conclusions

Our aim was to shed light on how the activity—what students actually do and how they express themselves—in a DIMLE can foster the development of instrumented schemes that support the construction of mathematical meanings. We analysed a specific case of a student in eXpresser learning algebraic ways of thinking. Our broader aim was to shed light on aspects of our method that we believe can be applied (after being recontextualised) to study students' mathematics learning in other DIMLEs. In this final discussion, we take a step back and further contextualise and highlight the significance of the results concerning both of our aims.

5.1. A Broader Glance at Planting Seeds of Algebraic Generalisation in eXpresser

In the previous section, we illustrated Molly's interactions with eXpresser, showing how the various artefacts transformed Molly's mathematical knowledge and helped her gain techno-mathematical fluency in eXpresser. From the analyses, we were able to outline shifts in Molly's instrumented schemes for the colouring task that occurred over the three 40 min sessions we analysed, and to make inferences about the algebraic ways of thinking in development.

Overall, this case study shows a tension, common to most of the students who interacted with the system, between different elements of techno-mathematical fluency: one related to rote recall of procedural steps, and the other more related to sense-making. Such a tension between "procedural" and "conceptual" [45] is well known in the broader literary context, as it has been found to characterise many "skill practice" sessions in traditional maths classes. However, Kieran [46] has argued the necessity to overcome such a dichotomy. We find that our results strengthen Kieran's argument because, in the eXpresser scenario, Molly seems to be worried both with recalling/developing a procedure and at the same time making sense of it; that is, constructing situated meanings around a set of actions (both those carried out and those being planned). Thus, dichotomizing "procedural" and "conceptual" in the DIMLE we explored makes little or no sense.

Another aspect of Molly's learning seems to recur in much of the literature on algebraic learning. A sort of "cognitive gap" appears between being able to describe a number of tiles in a pattern as a number which is the final result of a mental calculation, and being able to describe such a number as a "rule". Such difficulties lie at the heart of one of the fundamental aspects of algebra that so many students struggle to master: the conception of a "rule" as a process aimed at obtaining a "result" needs to be overcome (e.g., [18,20]). We see a key role played by the DIMLE in supporting such a transition: in such a context, the student needs to learn to speak the language of the DIMLE, developing what we described as techno-mathematical fluency.

In this section, we refer to the specific case of eXpresser that we analysed, but we believe that our considerations apply to other DIMLEs designed to promote the gradual and constructivist learning of their language. We note that "learning to speak the language of eXpresser", as our analyses suggest, is not a trivial task; it is not a question of simply memorising meaningless procedures. Instead, the DIMLE promotes a gradual learning of the eXpresser language while also fostering the production of mathematical thinking, in this case, particular algebraic ways of thinking. Such ways of thinking may lead to the generation of "new" knowledge and thinking which can be expressed technologically with the DIMLE's language.

5.2. Generalisability of Our Method

Within the scheme–technique duality, the instantiations we presented of the scheme can be seen as "benchmarks" in the learning process: capturing them through their technical elements allowed us to "see" how they change and to infer their relationships with the algebraic ways of thinking that eXpresser was designed to foster. Moreover, our operationalisation of the instrumentation–instrumentalization duality allowed us to identify links between Molly's instrumented activity in the microworld and her mathematical learning.

We see our method as generalisable to mathematics learning in other DIMLEs, with a few caveats. The mathematics knowledge (concepts, ways of thinking, etc.) embedded and how it can be mobilised through interactions with the functionalities of the DIMLE need to be made explicit (as we did in Section 2.3). Moreover, at this stage of a priori analysis, significant recurring tasks that the learner is likely to face need to be identified. Once such an a priori analysis has been carried out, it will be possible to identify units of analysis in the collected data from the learners' interactions with the DIMLE, and an a posteriori analysis can be conducted, using an analytic scheme similar to the one we set up and used to generate Tables 1–4.

We hope that the research presented in this paper may also contribute to the problem of designing DIMLEs that support personalised learning. At a time when the affordances of technology are developing apace, collecting accounts of students' instrumentation schemes seems to be a fundamental step in pursuing insightful design of learning environments and of tasks within them (e.g., Refs. [37,47–49]). In particular, being able to infer students' schemes from their actions—potentially actions that can be recognized by the software itself—can lead to technological environments with intelligent support systems that make informed predictions about the student's learning and support it in personalised ways. Moreover, considering how students can develop skills, such as techno-mathematical fluency, that can be applicable in other digital environments and other contexts, should be a key priority now that digital technologies are everywhere.

Our future work entails a further investigation of how students develop instrumented schemes and skills that allow them to interact, talk and solve mathematical problems in digital environments and enact both their mathematical competencies and their digital competencies. Such processes have been described as mathematical digital competency [50] and are key for students' mathematics learning in the digital era and their digital citizenship.

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