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The Complexities of Mathematical Knowledge and Beliefs within Initial Teacher Education: An Analysis of Three Cases

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Abstract: This study arose from a desire to investigate initial teacher education for primary school mathematics teachers in Hong Kong. A conceptual framework for analyzing pre-service teachers' mathematical knowledge and beliefs informed this qualitative investigation. The purpose of the study was to scrutinize both the cognitive and affective domains in mathematics teaching of pre-service primary teachers within the analytical framework. This paper seeks to capture three cases of Hong Kong pre-service primary teachers holding different views about mathematics teaching and learning as revealed by a beliefs survey. Their mathematical knowledge was examined along with a questionnaire, individual interviews, and an analysis of the lesson plans they provided. Through triangulation, it was found that these pre-service teachers' beliefs about mathematics teaching and learning correspond to their pedagogical content knowledge to some extent. The results of lesson plan analysis were mostly consistent with those in the questionnaire and interviews although some of the interview data were inconsistent with the questionnaire results. Such inconsistency seemed to suggest that pre-service teachers' reform-oriented mathematical beliefs were impacted by some traditional values such as Confucianism. Implications are discussed regarding the relationship between knowledge and beliefs in mathematics education as well as the theory of teacher competencies.

Keywords: case analysis; initial teacher education; lesson plan analysis; primary mathematics; pre-service teachers; teacher knowledge; teacher beliefs

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1. Introduction

Learning to teach mathematics in innovative ways is a highly complex endeavor. Pre-service teachers' learning experiences prior to involvement in teacher education programs and the long-standing education traditions are possible external factors influencing their learning outcomes [1,2]. Factors such as early family experiences, past school experiences, and teacher education programs may influence the beliefs of pre-service teachers in particular [3]. This research investigated the mathematical knowledge and beliefs of pre-service primary teachers in Hong Kong. An integrative analytical framework that incorporates a range of knowledge and belief constructs was used in the investigation. The study aimed to ascertain the nature of pre-service teachers' mathematical knowledge and the characteristics of their espoused mathematical beliefs.

In consideration of the fact that studying pre-service teachers' cognitive abilities and affective traits can foster a comprehensive understanding of their mathematical competencies [4], this paper outlines the theoretical framework as a blueprint for analyzing pre-service teachers' mathematical knowledge and beliefs. Information from the within-case analysis can be used to inform mathematics teacher educators and researchers on enhancing the quality of teacher education programs to prepare subject specialists of mathematics in primary education.

1.1. Cognitive and Affective Domains in Mathematics Teaching

Over the last decade, there has been enormous interest in the cognitive and affective domains of mathematics education [5]. A growing body of research has provided evidence

for the effects of teachers' knowledge and beliefs on their instructional practice in mathematics classrooms [6,7]. As "much of what is true for professional development is also true for teacher preparation" [8] (p. 158), both cognitive and affective aspects are likely to have a significant influence on the preparation of mathematics teachers.

Knowledge and beliefs are intertwined. Human choices and actions are based on what people know and what they believe [9]. Concerning pre-service teachers without much classroom experience, their pre-existing mathematical knowledge and beliefs will make a positive or negative impact on their lesson planning in teaching mathematics [10,11]. In this sense, it can be argued that both knowledge and beliefs of pre-service teachers play a crucial role in the processes of participation and reification in learning to teach [12].

The concept of teacher knowledge has evolved over time. It involves knowledge of the subject matter, curriculum, and pedagogy, as well as global issues and other general themes [13]. Certainly subject-specific knowledge is an essential prerequisite for a competent teacher [14]. However, it is inadequate to equate teaching competence with teachers' knowledge alone. The affectivity relating to teachers' beliefs, values, attitudes, and emotions also needs to be taken into account in considering teacher competencies [15]. In reviewing the literature on the affective domain in mathematics education, subject-oriented beliefs are deemed to be more stable but less intensive than other affections due to their cognitive nature [16,17]. In order to provide a better understanding of pre-service teachers' mathematical knowledge and beliefs, it is necessary to adopt a conceptual framework for studying the essential knowledge and beliefs constructs in the field of mathematics teacher education.

1.2. Frameworks for Analyzing Mathematical Knowledge and Beliefs

A variety of frameworks have been developed for analyzing teachers' mathematical knowledge and beliefs. Early in the 1980s, Shulman [18] had already raised the issue of a lack of attention to the subject matter in teaching and advocated seven categories of the knowledge base of teachers. He classified content-specific knowledge of teachers as content knowledge, curriculum knowledge, and pedagogical content knowledge (PCK), with the other four categories of generic knowledge including general pedagogical knowledge, knowledge of learners, knowledge of educational contexts, and knowledge of educational ends, purposes, and values.

Based on the work of Shulman [18], some influential theories and models of mathematics teachers' knowledge have been generated. For example, Rowland and Turner's [19] "knowledge quartet" framework reflects Shulman's idea of PCK. Ball and her colleagues produced a "practice-based theory of mathematical knowledge for teaching" [20] (p. 389) to clearly differentiate subject matter knowledge from PCK. According to Ball et al., subject matter knowledge contains common content knowledge, specialized content knowledge, and knowledge at the mathematical horizon while PCK encompasses knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum. With respect to the PCK of mathematics teachers, Chick and her research team [21] created a three-category framework with 20 sub-categories for analysis and interpretation. They classified the evidence in accordance with the three categories: clearly PCK, content knowledge in a pedagogical context, and pedagogical knowledge in a content context. These aforementioned theoretical frameworks have a profound impact on research into teacher knowledge in mathematics education [22].

In the sphere of teacher education, some researchers have strived to distinguish beliefs from knowledge in order to delineate the meaning of beliefs. They defend that beliefs constitute a central part of the professional competencies of mathematics teachers. For instance, Ernest [23] proposed a model of cognitive structures connecting mathematics teachers' knowledge, beliefs, and attitudes. He divided mathematics teachers' beliefs into four types: conception of the nature of mathematics, models of teaching mathematics, models of learning mathematics, and principles of education. Since then, more and more researchers categorized or conceptualized beliefs into certain types in order to investigate

teachers' mathematical beliefs [17,24]. A review of research on mathematical beliefs always reveals the existence of dichotomous philosophical perspectives, such as conventional beliefs versus reform-oriented beliefs [25,26]. Both quantitative and qualitative analysis methods were used for analyzing beliefs in the past.

Frameworks that consist of the knowledge and beliefs constructs have shaped the direction of research on mathematics teacher education. Internationally, the *Teacher Education and Development Study in Mathematics* (TEDS-M) was the first large-scale comparative research into the preparation of pre-service primary and lower-secondary mathematics teachers [27]. It was intended to investigate the characteristics of pre-service teachers, teacher educators, and teacher education programs. TEDS-M measured pre-service teachers' learning outcomes in different teacher education systems. At the primary level, a total of 13,871 pre-service primary teachers recruited from 15 participating countries and regions were studied extensively. In the TEDS-M study, a conceptual framework concerning knowledge and beliefs was used for evaluating pre-service teachers' learning outcomes in particular. The TEDS-M framework was derived from Shulman's [18] theory of teacher knowledge, and Richardson and Thompson's [28,29] concepts about affective-motivational characteristics such as professional beliefs, motivation, and self-regulation. Three domains of teacher knowledge including mathematical content knowledge (MCK), mathematics pedagogical knowledge (MPK), and knowledge of teaching (pedagogy) were assessed [27]. MCK covered four knowledge dimensions—Number and Operations, Algebra and Functions, Geometry and Measurement, as well as Data and Chance; MPK comprised curricular knowledge, knowledge of planning for teaching, and knowledge of enacting teaching [30]. Besides, the TEDS-M study also examined pre-service teachers' beliefs about: the nature of mathematics, the nature of teaching mathematics, and the nature of learning mathematics, as well as their self-efficacy and preparedness to teach.

More recently, Lo [31] has established a conceptual framework for the study of mathematical knowledge and beliefs in various educational settings based on the notion of TEDS-M. She synthesized the frameworks used in previous studies conducted by Ernest [23], McLeod [24] and Philipp [17] to identify four belief constructs in mathematics teacher education (see Table 1). According to the four types of mathematical beliefs shown in Table 1, Lo [32] further developed a 25-item mathematical beliefs survey (see Appendix A) to explore pre-service primary teachers' beliefs about: mathematics, mathematics teaching, mathematics learning, as well as beliefs about the social context in relation to mathematics education. Apart from the mathematical beliefs, three facets of mathematical knowledge (including curriculum knowledge, MCK, and PCK) were also highlighted in her conceptual framework.

Table 1. Synthesis of frameworks for researching mathematical beliefs [31] (p. 79).

Type of Beliefs	Ernest [23]	McLeod [24]	Philipp [17]
Beliefs about mathematics	Conception of the nature of mathematics	Beliefs about mathematics	
Beliefs about mathematics teaching	Model of teaching mathematics	Beliefs about mathematics teaching	Beliefs about curriculum Beliefs about technology
Beliefs about mathematics learning	Model of learning mathematics	Beliefs about the self	Beliefs about students' mathematical thinking Beliefs about gender
Beliefs about the social context in relation to mathematics education	Principles of education	Beliefs about the social context	

Each education system has its own cultural tradition in mathematics teaching and learning [33]. In Hong Kong, curriculum developers suggest using constructivist learning theory to inform changes [34]. There are five learning dimensions in the revised primary mathematics curriculum—Number, Shape and Space, Measures, Data Handling, and Algebra [35]. The curriculum places a strong emphasis on developing primary students' higher-order thinking skills and lifelong learning abilities [36]. Considering the appropriate implementation of a reform curriculum in Hong Kong, Lo's [31,32] analytical framework was employed as a theoretical orientation for the present study (See Figure 1). Three mathematical knowledge constructs and four types of mathematical beliefs of the pre-service primary teachers were interpreted and analyzed.

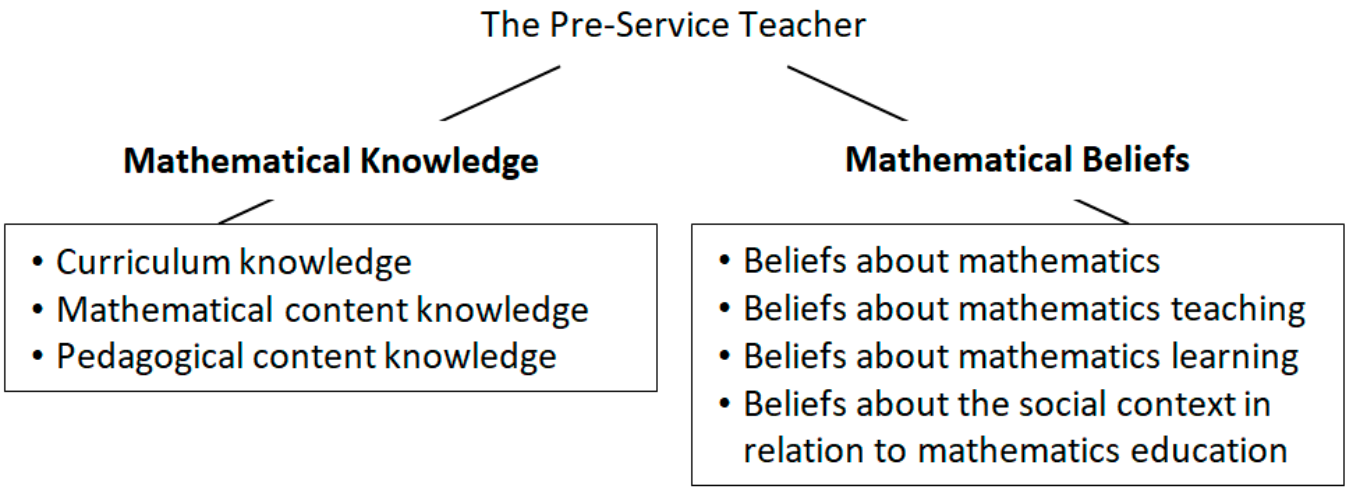


Figure 1. A conceptual framework for the study of pre-service teachers' mathematical knowledge and beliefs [31] (adapted from pp. 36, 45, 57, and 79).

This paper addresses the following research questions:

1. What is the nature of pre-service primary teachers' curriculum knowledge, mathematical content knowledge, and pedagogical content knowledge?
2. What are the characteristics of pre-service primary teachers' beliefs about: mathematics, mathematics teaching, mathematics learning, as well as beliefs about the social context in relation to mathematics education?

2. Materials and Methods

2.1. Research Design

A cross-sectional design was adopted to investigate the mathematical knowledge and beliefs of the pre-service primary teachers in a four-year undergraduate teacher education program in Hong Kong. Under the program structure, a total of 20 mathematics modules were offered for Mathematics major students while non-Mathematics major students could avoid studying any mathematics module. Mathematics major students had to do all 20 mathematics modules across four academic years. Those who had selected mathematics as their major were invited to participate in this research.

The selection of cases for this investigation was based on four criteria. First, only the participants who completed the interviews and provided at least one lesson plan were considered. Nine lesson plans were voluntarily provided by eight interviewees (two from each of Year 2 and Year 3; four from Year 4). Thus, the cases were selected from these eight pre-service teachers.

Second, it was desirable that all year groups from the teacher education program were included to represent a potential spread of mathematical knowledge and teaching experiences since they only had practicum placements during Years 3 and 4. The Year 1 pre-service teachers did not produce any lesson plans as they only did classroom observations

in different education settings. While the Year 2 pre-service teachers needed to attend a 10-day attachment over 10 weeks in a primary school, Year 3 and Year 4 pre-service teachers were required to complete a six-week practicum and an eight-week practicum respectively. Hence the Year 2 pre-service teachers had much less field experience than those in Years 3 and 4. To address this difference, the cases were selected from Years 2 to 4 in the teacher education program.

Third, it was also desirable that the selected cases represented a range of mathematical beliefs. During the interviews, the eight pre-service teachers revealed support for their reform-oriented beliefs about mathematics teaching and learning. However, some of their responses also reflected support for more conventional beliefs about mathematics teaching. According to the interview analysis, five pre-service teachers supported more reform-oriented pedagogical beliefs. The other three appeared to support a more mixed set of beliefs about mathematics teaching. Besides, the interview results also indicated that these pre-service teachers held various views about the important learning outcomes of primary mathematics. As a result, it was decided to select three pre-service teachers who appeared to support slightly different sets of beliefs.

Fourth, another consideration for selection was the content of lesson plans. Out of nine lesson plans provided by the eight interviewees, eight were considered to be well-structured. In general, each well-structured lesson plan contained a set of instructional practices involving three elements—instructions, questions, and statements. Instructions included classroom commands and activity briefings. It was found that closed questions were frequently used. These questions were followed by the correct answers that students were expected to provide. Statements related to mathematical facts, definitions, and explanations also appeared in the lesson plans. These particular elements provided some evidence about the nature of pre-service teachers' mathematical content knowledge and pedagogical content knowledge.

After consideration of the diversity of pre-service teachers' teaching experience, mathematical knowledge, and beliefs, as well as the richness of content of their lesson plans, three cases of pre-service teachers from different stages of the teacher education program were selected for in-depth analysis.

2.2. Data Collection and Ethics

The data collection commenced after obtaining permission from the participating teacher education institute. All participants were identified through the teacher education institute in order to protect their rights to privacy.

The researcher was able to meet all targeted groups of pre-service teachers to provide further information about the study, to go through the participant information sheet and the consent form, and to invite them to complete the questionnaire. Interviewees were chosen from the respondents who recorded their names on the questionnaire and agreed to attend an individual interview. The interviewees were invited to provide one video-record or one lesson plan of their favorite mathematics lesson they had delivered during their most recent practicum.

To ensure the anonymity of participants and the confidentiality of data, pseudonyms were employed for the analysis and reporting of the data derived from questionnaires, interviews, and lesson plans. For the selected three cases in the study, the following pseudonyms were assigned: Jodie (a Year 2 pre-service teacher), Kelly (a Year 3 pre-service teacher), and Natalie (a Year 4 pre-service teacher).

2.3. Data Analysis

Computer techniques were adopted to facilitate the processes of qualitative data analysis. This supported the development of an open coding scheme to analyze the data sets. A concept map was constructed by the researcher for each case to make connections between all data sources and assist with the holistic approach to data analysis and reporting (see Figures 2–4).

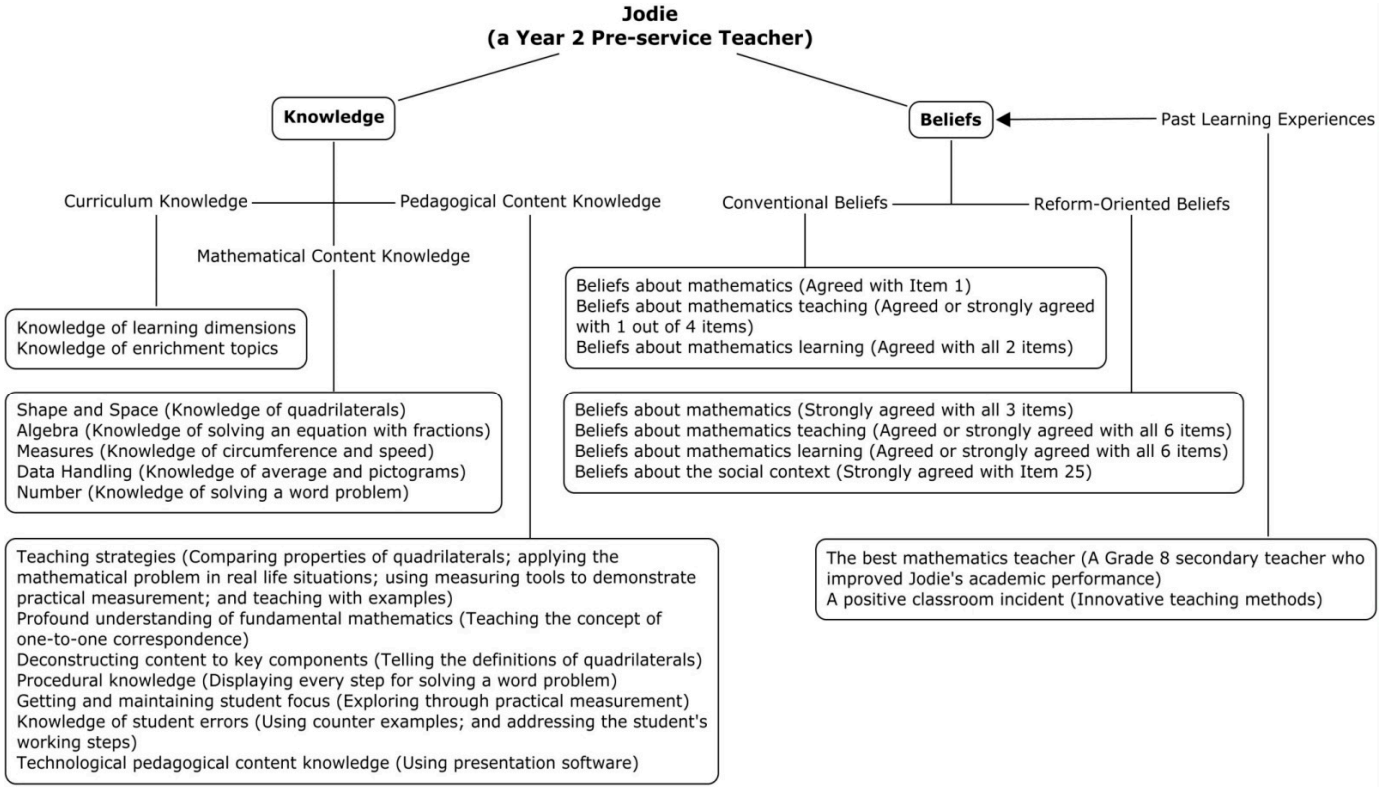


Figure 2. Synthesis of Jodie's reported mathematical knowledge and beliefs.

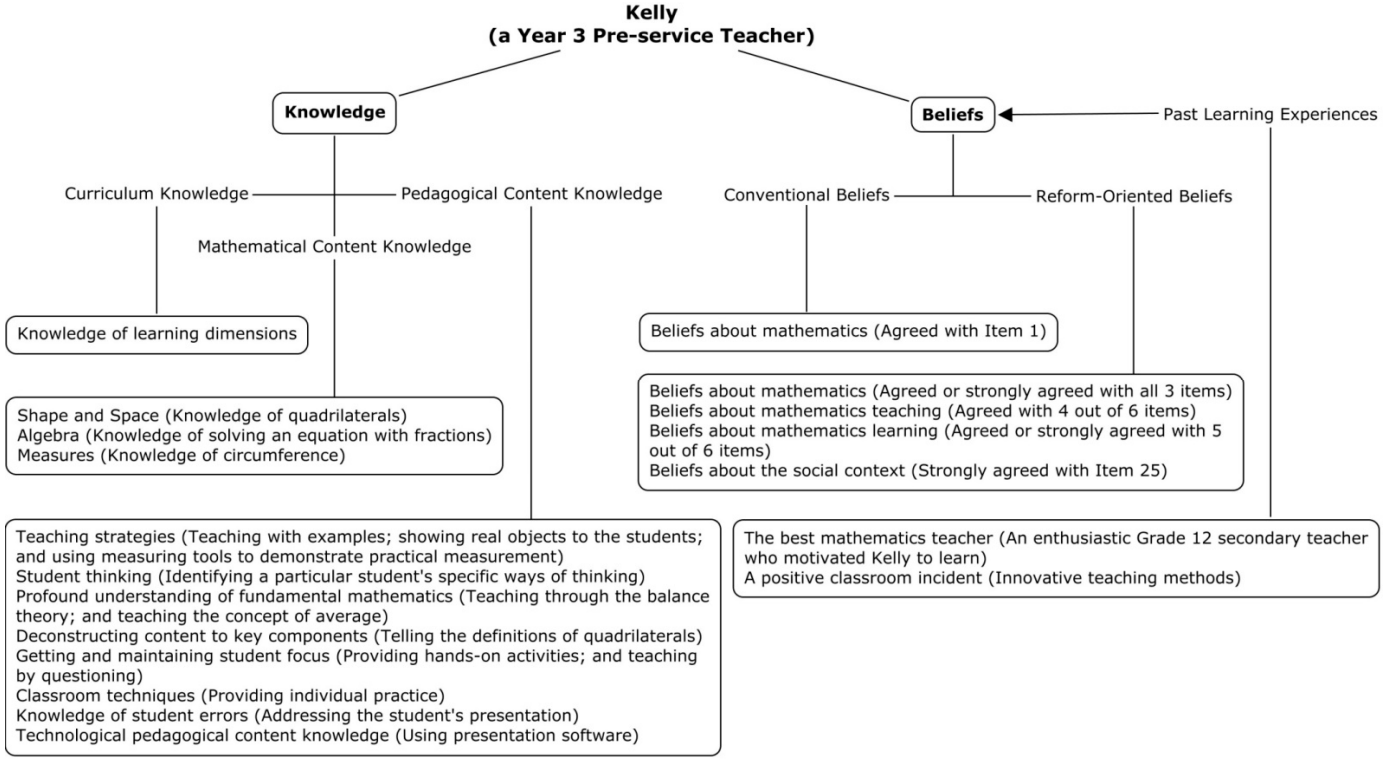


Figure 3. Synthesis of Kelly's reported mathematical knowledge and beliefs.

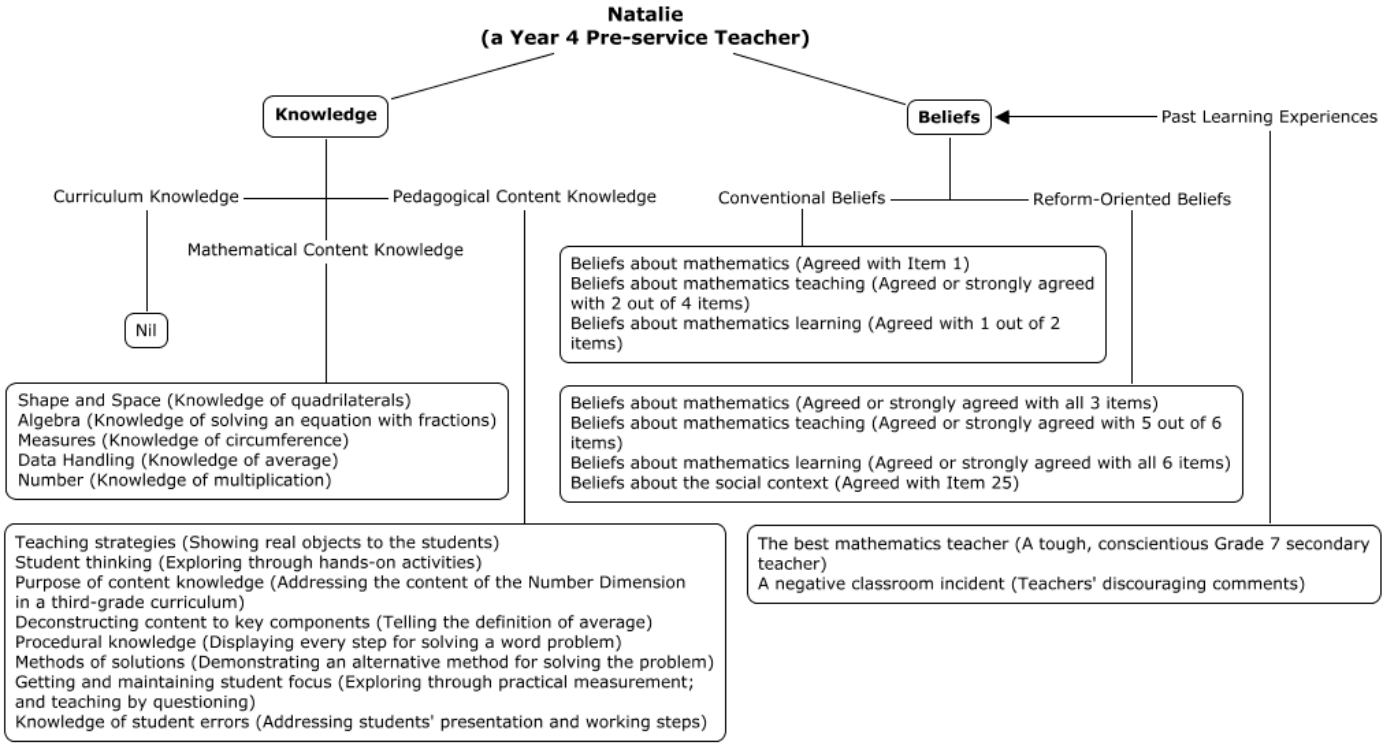


Figure 4. Synthesis of Natalie's reported mathematical knowledge and beliefs.

3. Results

The selected three cases are described in the following sub-sections respectively. Each sub-section begins with a description of the pre-service teacher's inferred mathematical beliefs. This is followed by a separate discussion about the nature of the pre-service teacher's mathematical knowledge which included curriculum knowledge, mathematical content knowledge, and pedagogical content knowledge. Their questionnaire and interview responses are discussed, and the results from the lesson plan analysis are presented.

3.1. The Case of Jodie

At the time of the data collection, Jodie was enrolled in Year 2 of the teacher education program and she had not had a school-based practicum experience. She chose mathematics as her major because she was "interested in mathematics". She had not experienced the revised curriculum since she completed her primary education in Hong Kong before the launch of the revised primary mathematics curriculum. According to Jodie, her performance in mathematics examinations had significantly improved after she had her best mathematics teacher in Grade Eight. Jodie had a very strong mathematics background since she studied Mathematics, Additional Mathematics, and Pure Mathematics in her secondary school and passed these subjects with one A, one B, and one C in the public examinations.

Jodie provided two lesson plans from two different mathematics topics, one for teaching sixth graders about "speed" and another for teaching second graders about "pictograms". These two topics involved the two learning dimensions of Measures and Data Handling. Two sets of PowerPoint slides and a worksheet were attached to each of her lesson plans. Jodie's questionnaire and interview responses together with data obtained from her lesson plans are summarized in the concept map in Figure 2. The following sub-sections elaborate on each of the components of the concept map.

3.1.1. Jodie's Mathematical Beliefs

Jodie started her teacher education after positive experiences with mathematics learning since she could not recall any negative incidents in mathematics lessons from her school

days. During the interview, Jodie recalled the friendly attitude of a newly qualified teacher and positive learning experiences of mathematics through innovative teaching methods.

Jodie's reported mathematical beliefs generally tended towards reform-oriented approaches although there were some inconsistencies in her beliefs. She agreed or strongly agreed with all 16 reform-oriented belief statements in the questionnaire. However, she also agreed with four out of nine conventional belief statements:

- Mathematics is computation.
- The role of the mathematics teacher is to transmit mathematical knowledge and to verify that learners have received this knowledge.
- Mathematics learning is being able to get the right answers quickly.
- Being able to memorize facts is critical in mathematics learning.

While Jodie agreed with these four statements, the analysis of interview responses placed Jodie in the category of 'reform-oriented beliefs about mathematics teaching'. She wanted to become a primary mathematics teacher who "could get kids to love mathematics" and "engage them in learning". Her ultimate aim was to prepare students for lifelong learning. She said that:

> *It is important for primary school students to explore mathematics by themselves. [. . .] For me, mathematics is a way of thinking rather than a school subject. I will take a role to develop students' self-directed learning skills through mathematics. These skills might serve as a foundation of their whole life.*

From the interview comments, Jodie's beliefs about mathematics teaching were student-centered. She used "wind" as a metaphor to represent the role of a primary mathematics teacher and explained that "the teacher should encourage students to be proactive in learning". She seemed to regard mathematics teachers as facilitators to motivate students' initiatives.

As revealed by the questionnaire and represented in Figure 2, it can be seen that Jodie agreed with some conventional belief statements such as the importance of being able to obtain the right answers quickly and memorize facts in mathematics learning. However, the interview results also indicated that she considered "logical thinking skills" as an important learning outcome of mathematics. She stated that:

> *Many primary school students had logical problems in their speaking and writing. So I hoped my future students could develop their reasoning abilities in my mathematics lessons.*

Without negative learning experiences of mathematics, Jodie tended to hold more reform-oriented beliefs although some of her interview responses were inconsistent with the questionnaire results. As revealed by the questionnaire, she agreed or strongly agreed with all the reform-oriented beliefs statements (see Figure 2). Her beliefs about mathematics teaching and learning were generally in line with the notion of the reform curriculum.

3.1.2. Jodie's Mathematical Knowledge

Jodie's mathematical curriculum knowledge was classified as belonging to the medium level during the questionnaire analysis. As indicated in Figure 2, Jodie was able to correctly name the five learning dimensions in primary mathematics, and to properly nominate "divisibility" as an enrichment topic in the syllabus. Nonetheless, she had inadequate knowledge of the three curriculum objectives. In her responses to a questionnaire item regarding the revised primary mathematics curriculum in Hong Kong, some generic skills such as "problem-solving abilities", "creative thinking", and "critical thinking" were incorrectly regarded as the main objectives of the revised curriculum. She did not know that 'knowledge' and 'attitudes' were the other two curriculum objectives besides 'skills' although the skills she nominated are highly desirable outcomes of mathematics education.

Jodie's questionnaire responses confirmed her high level of mathematical content knowledge. She provided correct answers to all five mathematics questions, demonstrating her sound knowledge of quadrilaterals, circumference, and averages, as well as her ability

to solve an equation with fractions and a word problem. Based on Chick, Baker, et al.'s [21] pedagogical content knowledge framework, the analysis of the questionnaire and interview responses placed Jodie in five sub-categories: 'teaching strategies', 'profound understanding of fundamental mathematics', 'deconstructing content to key components', 'procedural knowledge', and 'getting and maintaining student focus' (see Figure 2). Because Jodie provided two lesson plans from the topics of "speed" and "pictograms", her knowledge of content and pedagogy in the dimensions of Measures and Data Handling are further discussed.

Jodie's superior knowledge of measurement was inferred from her questionnaire responses. She was able to use correct mathematical language to explain why the perimeter of a square was longer than the circumference of a circle drawn inside the square. Her questionnaire responses indicated her profound understanding of the relationship between the circumference and diameter of a circle. She explained:

> *It was because:*
>
> *perimeter of the square = 4 × length of each side of the square*
>
> *circumference = π × diameter, ($\pi \approx 3.14$)*
>
> *diameter = length of each side of the square*
>
> *∴ perimeter of the square > circumference*

Evidence of her deep understanding of measurement was further exposed in the lesson plan analysis. In her lesson plan, Jodie gave a clear and accurate explanation for introducing the formula for computing speed, "speed = distance ÷ time". She stated that "speed indicates the distance traveled by an object in an interval of time". This confirmed that Jodie not only knew the measurement formulas but also understood their mathematical meanings.

In the Measure Dimension, Jodie's questionnaire responses were located in the category of 'clearly PCK'. She suggested using measuring tools to demonstrate practical measurement for teaching sixth graders to measure the circumference of a circle. She preferred using specific materials for teaching particular measurement concepts, and this was confirmed in the analysis of her lesson plans. A walking game was included in Jodie's lesson plan for teaching second graders about the concept of speed. She planned to select three students for the game. Each student needed to walk different distances. Jodie decided to utilize a "stopwatch" to measure the time used by each student and then recorded the results in a table. Finally, she demonstrated the method for calculating the actual speed for each student.

These data indicate that Jodie elaborated upon teaching strategies in order to increase student motivation through practical measurement activities (see Figure 2). Jodie demonstrated sound knowledge of content and pedagogy in the Measures Dimension. She was able to suggest learner-centered approaches to instruction, aligning with the philosophy of the revised primary mathematics curriculum in Hong Kong.

Moreover, Jodie's questionnaire responses indicated her ability to use the mean value theorem to solve an average problem. Averages and statistical graphs are major mathematics topics in the Data Handling Dimension. Unfortunately, no data from her lesson plans could provide additional information or evidence of her knowledge about averages. In spite of that, the analysis of Jodie's lesson plan gives us an insight into her knowledge of graph theory. In Jodie's second lesson plan on pictograms, she planned to introduce the concept of one-to-one correspondence using pictures and stated that:

> *There are a set of apples and a set of pears on the screen. To compare the numbers of apples and pears without counting, we can match the apples and the pears one-to-one.*

This was an appropriate example for second graders because pictures or concrete objects made mathematics easier for younger students to understand. One-to-one correspondence is a basic mathematical concept in Data Handling. As Jodie addressed this

concept in her lesson plan, she exhibited her profound knowledge of fundamental mathematics in the area of statistics.

In the Data Handling Dimension, Jodie's questionnaire responses were placed in the category of 'clearly PCK'. She preferred using examples for teaching sixth graders about the concept of average. She suggested a teaching example of "calculating students' average marks in their final examination". This approach was classified as a general teaching strategy based on Chick, Baker, et al.'s [21] pedagogical content knowledge sub-categories. The analysis of lesson plans provided additional information about her pedagogical content knowledge in this learning area.

In Jodie's lesson plan, she included a classroom activity for teaching second graders to create a pictogram. She planned to collect data on snack preferences in the class, and then used a worksheet to provide step-by-step instructions helping students to display the statistical information in a pictogram. This teaching method connected students' mathematical knowledge to their real-life experiences—a recommended approach in the revised primary mathematics curriculum.

Jodie also addressed possible student misconceptions about making pictograms. She used an example to illustrate students' common mistakes in creating a pictogram. According to Chick, Baker, et al. [21], this approach was identified as general student thinking in relation to misconceptions. The evidence obtained from her lesson plans revealed Jodie's pedagogical content knowledge in the Data Handling Dimension not only involved instructional content knowledge but also knowledge of student errors.

The lesson plan analysis affirmed Jodie's pedagogical content knowledge in Measures and Data Handling as revealed by the questionnaire. Her lesson plans involved different interactive classroom activities and games. In order to hold students' attention, she prepared engaging PowerPoint slides for the lessons.

According to her mathematical knowledge and beliefs as summarized in Figure 2, Jodie generally demonstrated solid knowledge of mathematics and pedagogy although she lacked actual teaching experiences. A small degree of conflict appeared between her reform-oriented beliefs about mathematics teaching and pedagogical content knowledge because Jodie suggested some traditional teaching methods such as a 'telling' approach in her lesson plans.

3.2. The Case of Kelly

Kelly was enrolled in Year 3 of the teacher education program at the time of questionnaire completion and had completed one in-school practicum experience. She reported that she was "interested in mathematics" and "loved mathematics", and cited these as reasons for choosing mathematics as a major in her teacher education. She had not been exposed to the reform mathematics curriculum in Hong Kong during her primary education. Kelly had a strong mathematics background as she studied Mathematics, Additional Mathematics, and Pure Mathematics in her secondary school and passed these subjects with one B and two C grades in the public examinations.

Kelly provided a lesson plan with PowerPoint slides for teaching fourth graders about "fitting and dissecting shapes" from the Shape and Space Dimension. A summary of Kelly's mathematical knowledge and beliefs as revealed by the questionnaire, interview, and lesson plan analysis is presented in the concept map in Figure 3 and discussed in two sub-sections that follow.

3.2.1. Kelly's Mathematical Beliefs

When Kelly prepared for the final public examination in Grade 12, she met her best mathematics teacher Miss Cheung in a Pure Mathematics class. Kelly nominated this teacher to be the best because of her clear instructions and understandable explanations. She described Miss Cheung as enthusiastic about teaching and having a good relationship with students not only in school but also outside school. During the interview, Kelly did

not recall any negative incidents in the mathematics lessons but remembered the teacher encouraging students with awards.

Unlike Jodie, Kelly's reported mathematical beliefs were more mixed as she supported most reform-oriented statements as well as some conventional components. She disagreed or strongly disagreed with all the conventional belief statements except the one regarding mathematics as computation (see Figure 3). However, she also disagreed with three reform-oriented belief statements:

- Mathematics teachers should negotiate social norms with the students in order to develop a co-operative learning environment in which students can construct their knowledge.
- Good mathematics teachers should provide particular training for students to participate in mathematics competitions.
- Young students are capable of much higher levels of mathematical thought than has been suggested traditionally.

The analysis of interview responses confirmed her mixed beliefs about mathematics teaching. When asked what kind of primary mathematics teacher she wanted to be, she answered that she wanted to be a teacher who would "try to understand different learning styles among students" but "students had to follow her every instruction to learn mathematics in the class". She explained that:

> *Because each class period has its time limit, I have to discipline students in the classroom and teach them through effective classroom practices. [. . .] Students are allowed to talk in the class but they must focus on a particular mathematics topic.*

Kelly used "a compass" as a metaphor to represent her identity as a primary mathematics teacher. She believed that the teacher was comparable to a guide providing instructions for solving problems. Although Kelly agreed with mathematics as computation in the questionnaire, her interview responses to the questions concerning beliefs about mathematics teaching seemed not to corroborate this conventional belief about mathematics. She emphasized that "mathematics teachers are responsible for developing students' mathematical thinking skills, not only computation skills."

Kelly regarded "specified content knowledge" as an important learning outcome of mathematics. Apart from knowledge, she also addressed students' feelings about mathematics because she wanted to make her future students love mathematics. In general, Kelly's questionnaire and interview responses reflected her mixed beliefs about mathematics teaching and learning although some of her mathematical beliefs were influenced by traditional Chinese culture such as student discipline and teacher-directed instructions.

3.2.2. Kelly's Mathematical Knowledge

Kelly's responses to the questionnaire items revealed a medium level of mathematical curriculum knowledge. She knew about the five learning dimensions of the revised primary mathematics curriculum, but she did not know the three curriculum objectives and could not nominate any enrichment topic from the new syllabus (see Figure 3). The analysis of questionnaire responses classified Kelly's mathematical content knowledge at the medium level. She did not provide correct answers to the mathematics questions in the Data Handling and Number Dimensions. Although she was identified as demonstrating a medium level of mathematical content knowledge in the questionnaire analysis, she was able to provide correct answers for the mathematics problem in the Shape and Space Dimension (see Figure 3). Kelly recognized the properties of squares and rhombuses in the questionnaire. Data from her lesson plan also indicated her sound knowledge of geometry. In her lesson plan, Kelly listed all the characteristics of rhombuses based on Euclidean geometry. She also pointed out that "rectangles, trapezoids, squares, parallelograms, rhombi, and kites are quadrilaterals", and clearly illustrated their properties using a classification table.

In the Shape and Space Dimension, Kelly's questionnaire responses were located at the sub-categories of 'teaching strategies' and 'classroom techniques'. For teaching sixth

graders to classify quadrilaterals, she advocated "showing real objects to the students and guiding the students to summarize their properties", "asking the students to give examples and non-examples", and "using individual practice to enhance and evaluate students' understanding" (see Figure 3). Kelly suggested taking a student-centered approach to teaching geometry. Additional evidence was obtained from her lesson plan.

Data from her lesson plan offered a deeper insight into the nature of her pedagogical content knowledge in primary geometry. Kelly included a shape game in her lesson plan. She preferred using presentation software such as PowerPoint to visually represent shapes in the classroom. She intended to have the students build particular shapes with a tangram (an ancient Chinese geometric puzzle with seven pieces), and then planned to display the method of dissection with the assistance of PowerPoint slides.

By providing a hands-on activity, Kelly endeavored to create opportunities for students to creatively use geometrical shapes through information and communication technology. Since Kelly combined the learning activity with her computer skills, she demonstrated an understanding of the interplay between pedagogy, mathematical content, and technology knowledge. She seemed to have developed technological pedagogical content knowledge in the Shape and Space Dimension.

The analysis of questionnaire responses and lesson plans confirmed that Kelly preferred using student-centered approaches in teaching geometry. Even though she demonstrated her mixed beliefs about mathematics teaching during the interview, her teaching suggestions encouraging students to participate in class were learner-focused. These teaching strategies were consistent with the reform-oriented approaches recommended in the revised primary mathematics curriculum but might not be able to address the individual needs of fourth graders.

3.3. The Case of Natalie

Natalie was a graduating pre-service teacher in the teacher education program at the time of the data collection and had completed two practicum placements. She reported that she was "interested in mathematics" and "loved mathematics" so she chose to specialize. Similar to the cases of Jodie and Kelly, she had not experienced the reform curriculum in school. Natalie studied Mathematics, Additional Mathematics, and Pure Mathematics in her secondary school and passed these subjects with one B, one D, and one E in the public examinations. Natalie provided a lesson plan and a worksheet for teaching second graders about "multiplication: by 5 and 10" from the Number Dimension. Data collected from Natalie's questionnaire, interview, and lesson plan are summarized in the concept map in Figure 4 and presented in the following sub-sections.

3.3.1. Natalie's Mathematical Beliefs

Natalie did not recall any positive mathematics learning experiences during the interview and had been affected by a teacher's discouraging comments. She felt threatened when the teacher said "being too confident, you will fail in the mathematics examination". She began her teacher education program without good memories of mathematics learning. Natalie met her best mathematics teacher in Grade Seven when she started her secondary education. She described this teacher as conscientious and tough, and believed that an effective mathematics teacher should possess these personal qualities. Natalie agreed with her teacher's traditional teaching methods because she believed that:

> *Practice was the only way to mathematical success. I do not think I can improve my academic performance without doing daily practice.*

Natalie agreed or strongly agreed with all the reform-oriented belief statements except the belief about the integration of mathematics into other subjects in the primary curriculum. She also agreed or strongly agreed with four out of nine conventional belief statements in the questionnaire (see Figure 4). Her interview responses further reflected her support for more conventional beliefs about mathematics teaching and learning.

The analysis of interview responses placed Natalie in a category of 'conventional beliefs about mathematics teaching' because of her teacher-centered descriptions of the identity of a primary mathematics teacher. She intended to be "a tough teacher" who required students to follow her instructions in class. She believed that it was the best way for students to learn as "systemic teaching enhanced their mathematics learning". It should be noted that the best mathematics teacher she nominated at the beginning of the interview also had a tough character.

During the interview, Natalie showed her passion for teaching. She hoped her students "would love mathematics after the class" and also "could apply mathematical skills in lifelong learning". She used "a road map" as a metaphor to represent the role of a primary mathematics teacher. However, her explanation for the metaphor reflected her support for more conventional beliefs about mathematics teaching. She regarded the teacher as a guide to "instruct students to get the right answers". The analysis indicated that most of her interview responses were consistent with her questionnaire responses when she agreed or strongly agreed with the following conventional belief statements:

- Mathematics is computation.
- The role of the mathematics teacher is to transmit mathematical knowledge and to verify that learners have received this knowledge.
- Mathematics problems given to students should be quickly solvable in a few steps.
- Mathematics learning is being able to get the right answers quickly.

3.3.2. Natalie's Mathematical Knowledge

As indicated in Figure 4, Natalie lacked mathematical curriculum knowledge because she provided incorrect answers to all questionnaire items regarding the reform curriculum. She was able to name Shape and Space, Data Handling, and Number as learning dimensions of the revised primary mathematics curriculum, but she did not know Measures and Algebra were also included. Besides, she also incorrectly regarded "problem-solving questions" as an enrichment topic of the syllabus.

Natalie's mathematical content knowledge was identified at a high level in the questionnaire analysis. However, she did not achieve full marks for the mathematics problem in the Number Dimension because her presentation was unclear. She was able to solve the mathematics word problem using a reasonable method and provided a correct solution but she did not include the units in her answer.

Interview responses suggested that Natalie had limited knowledge of mathematics language. When dealing with a sixth grader's error, Natalie indicated that the student's work involved "a wrong concept" and "an inverse of two numbers" but she could not use 'dividend' and 'divisor' as the mathematical terms to explain the mathematical error of a student. The data obtained from her lesson plan helped to substantiate this finding. Natalie mentioned the terms 'multiplicand' and 'multiplier' in her lesson plan but there was no definition given for these two mathematical words. When she planned a lesson for teaching arithmetic to lower graders, she paid less attention to the definitions of mathematical terms she mentioned than classroom activities. It may have been caused by her limited knowledge of mathematics language particularly in the learning area of Number.

The interview responses indicated that Natalie preferred teaching by questioning. For guiding lower and upper graders to solve mathematics word problems, Natalie suggested the same method. She said she would like to ask students some questions such as "how many items?" and "how many people?" in order to identify the fundamental mathematical components. According to Chick, Baker, et al.'s [21] analyzing framework, her suggestions were identified as a sub-category of 'getting and maintaining student focus' under a category of 'pedagogical knowledge in content context'.

Data from Natalie's lesson plan disclosed more details about her choices of teaching strategies in the Number Dimension. At the beginning of the lesson, Natalie planned to "ask the whole class to recite the multiplication tables," and then "select an individual student to do the recitation again". A follow-up activity was introduced:

Three to four students are grouped together. Each group gets a bag of slips. An answer is written on each slip. When the teacher asks a question, students should find out the correct answer from the bag.

The analysis of interview responses and lesson plans affirmed that Natalie preferred using conventional approaches for teaching students about arithmetic. Recitation is regarded as a traditional teaching method and is not recommended in the revised curriculum guide, but a synthesis of memorizing and understanding may lead to the excellent academic performance of students in Confucius heritage culture [37].

Moreover, closed questions are less likely to develop students' higher-order thinking skills. The activity she proposed may increase students' motivation but most likely could not enhance their learning in mathematics. Her lesson plan attached a worksheet with five mathematics word problems without providing any instruction. It seemed that the worksheet was designed for training students' computational skills rather than consolidating students' understanding of multiplication. Natalie was the only interviewee who did not provide a positive learning experience in mathematics. Her limited mathematical knowledge and conventional beliefs about mathematics teaching and learning were seemingly caused by her prior mathematics learning experience.

4. Discussion

The results of this research generally align with previous studies, indicating limited knowledge and a mixture of conventional and reform-oriented beliefs among pre-service primary mathematics teachers [38,39]. However, the current study adds value by specifically investigating the Hong Kong context, thereby filling a gap in the existing literature. These findings highlight certain issues that should be further considered in mathematics teacher education and development. They are discussed here as implications for both theory and practice.

Two issues that arose from this study provide contributions to the theory of mathematical competencies among teachers. First, there is an interactive relationship between mathematical knowledge and beliefs during the teacher preparation process. Second, culture plays a pivotal role in shaping beliefs and knowledge, potentially reshaping the comprehensive understanding of teacher competencies. Understanding the impact of these issues represents an initial step towards the potential development of theories and models in order to advance mathematical competencies within the field of primary teacher education.

4.1. Relationship between Knowledge and Beliefs in Mathematics Education

The relationship between knowledge and beliefs has been explored in research on mathematics education, but it has not been clearly defined [40–43]. One theoretical implication of the study is to extend the relationship between mathematical knowledge and beliefs to the pre-service teacher population. According to the organizing framework for the TEDS-M study [27], both knowledge and beliefs are learning outcomes of teacher education programs although previous research mainly studied their development separately.

Results of the current study indicated that pre-service teachers' pedagogical content knowledge in primary mathematics was competing with their mathematical beliefs. While many pre-service teachers agreed with the reform-oriented belief statements and supported the notion of reform-oriented approaches to mathematics teaching and learning, data also revealed that the pre-service teachers always suggested some traditional teaching methods such as teaching by telling for teaching primary mathematics. This finding contributes to the research literature by adding a new perspective on the relationship between knowledge and beliefs in mathematics education. It also confirms that it is necessary to study knowledge and beliefs together in order to improve teacher preparation.

4.2. Theories of Teacher Competencies

Several knowledge constructs and different types of beliefs were investigated in this qualitative study. Another theoretical implication of the study is to provide a clear picture demonstrating pre-service primary teachers' mathematical competences in the Hong Kong context.

Hong Kong, along with other East Asian countries, shares a Confucian-heritage culture that influences the values of the participants in the study. These values are reflected in the five constant virtues of Confucianism: humanity, righteousness, ritual, knowledge, and integrity [44]. They are deeply rooted in collective values [45]. In Confucian philosophy, education holds great significance as the pursuit of knowledge is considered to be a path to self-improvement, and it is assumed that everyone has the potential to be educated [46]. The distinctive features of learning within the Confucian-heritage culture include a focus on social achievement, an emphasis on diligence and practice, attributing success to effort, and fostering a competitive spirit [47]. These cultural values influence learning styles in mathematics. This suggests that culture plays a vital role in shaping the mathematical knowledge and beliefs of pre-service teachers [48].

Additionally, the current study utilized the pedagogical content knowledge framework proposed by Chick, Baker, et al. [21] to analyze the data and assess the strengths and weaknesses of the participants. However, it does not adequately capture the distinction between constructivist and traditional teaching methods while their framework was not specifically designed to analyze the pedagogical content knowledge of mathematics teachers from non-Western countries. For instance, within the category of 'pedagogical knowledge in a content context,' a sub-category such as 'getting and maintaining student focus' may encompass constructivist approaches to teaching in Western countries, whereas in Hong Kong, it may encompass traditional teaching methods such as "teaching by questioning".

Acknowledging the crucial role of culture in shaping teachers' knowledge and beliefs, it is essential to take the cultural context into account when applying the conceptual framework of teacher competencies depicted in Figure 1 [31]. Since teaching is a cultural activity [49], researchers must consider cultural variables and dimensions when utilizing the existing theories and models in diverse educational settings.

4.3. Concluding Remarks

In conclusion, the analysis of the three cases presented in this paper offers valuable insights into the nature of mathematical knowledge among pre-service primary teachers with varying mathematical beliefs. It is surprising to note that Natalie, a graduating pre-service teacher, exhibited more conventional beliefs regarding mathematics teaching and displayed weaker mathematical knowledge compared to Jodie and Kelly, who were in Years 2 and 3 respectively. In contrast to Kelly, both Jodie and Natalie displayed more conflicting beliefs pertaining to mathematics teaching and learning (see Figures 2–4). While these three cases may not fully represent their respective year groups, these findings raise concerns regarding the effectiveness of teacher preparation programs.

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Appendix A

Table A1. The 25 items in a mathematical beliefs survey [32] (p. 113).

Belief Statement
1. Mathematics is computation.
2. Mathematics is the dynamic searching for order and pattern in the learner's environment.
3. Mathematics is an interesting subject.

Belief Statement
4. Mathematics is a beautiful, creative and useful human endeavor that is both a way of knowing and a way of thinking.
5. Mathematics teachers should provide instructional activities which result in problematic situations for learners.
6. The role of the mathematics teacher is to transmit mathematical knowledge and to verify that learners have received this knowledge.
7. Mathematics teachers should recognize that what seems like errors and confusions from an adult point of view are students' expressions of their current understanding.
8. Mathematics teachers should negotiate social norms with the students in order to develop a co-operative learning environment in which students can construct their knowledge.
9. Good mathematics teachers should love mathematics.
10. Good mathematics teachers should provide particular training for students to participate in mathematics competitions.
11. Mathematics teachers or the textbook—not the student—are the authorities for what is right or wrong.
12. Mathematics problems given to students should be quickly solvable in a few steps.
13. Right answers are much more important in mathematics than the ways in which you get them.
14. It is possible to integrate mathematics into other subjects in primary curriculum.
15. Mathematics learning is enhanced by challenge within a supportive environment.
16. Mathematics knowledge is the result of the learner interpreting and organizing the information gained from experiences.
17. Students are rational decision makers capable of determining for themselves what is right and wrong.
18. Mathematics learning is being able to get the right answers quickly.
19. Periods of uncertainty, conflict, confusion, and surprise are a significant part of the mathematics learning process.
20. Mathematics learning is enhanced by activities which build upon and respect students' experiences.
21. Being able to memorize facts is critical in mathematics learning.
22. Young students are capable of much higher levels of mathematical thought than has been suggested traditionally.
23. English and Chinese Language are more important subjects than Mathematics in primary education.
24. Learning language is more useful than learning mathematics for a primary student.
25. The idea of specialist mathematics teacher should be promoted in primary education.

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