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Control Limits for an Adaptive Self-Starting Distribution-Free CUSUM Based on Sequential Ranks

Michael Lang

Graduate School of Excellence Computational Engineering, Technische Universität Darmstadt, Dolivostraße 15, 64293 Darmstadt, Germany; michael.lang@ieee.org; Tel.: +49-6151-1624401; Fax: +49-6151-1624404

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Abstract: Since their introduction in 1954, cumulative sum (CUSUM) control charts have seen a widespread use beyond the conventional realm of statistical process control (SPC). While off-the-shelf implementations aimed at practitioners are available, their successful use is often hampered by inherent limitations which make them not easily reconcilable with real-world scenarios. Challenges commonly arise regarding a lack of robustness due to underlying parametric assumptions or requiring the availability of large representative training datasets. We evaluate an adaptive distribution-free CUSUM based on sequential ranks which is self-starting and provide detailed pseudo-code of a simple, yet effective calibration algorithm. The main contribution of this paper is in providing a set of ready-to-use tables of control limits suitable to a wide variety of applications where a departure from the underlying sampling distribution to a stochastically larger distribution is of interest. Performance of the proposed tabularized control limits is assessed and compared to competing approaches through extensive simulation experiments. The proposed control limits are shown to yield significantly increased agility (reduced detection delay) while maintaining good overall robustness.

Keywords: cumulative sums; distribution-free; nonparametric; sequential ranks; change point detection

1. Introduction

From a historical perspective, the advent of modern statistical process control (SPC) arose out of the post industrial revolution realization that to yield goods of acceptable quality a manufacturing process ought to operate within prespecified margins of error (in other words it ought to be stable or in control) [1]. In oversimplified terms, control charts are central to SPC and serve to continuously monitor a process to assess whether the observed deviations from the nominal process are due to mere chance (in control) or not (out-of-control) (see generally [1–3]).

Control charts were first introduced by W. A. Shewhart in 1924 and gained widespread popularity following the publication of Shewhart's seminal monograph [4] in 1931. The following decades witnessed a substantial research interest and output resulting in important SPC developments including, but not limited to, cumulative sum (CUSUM) [5] and exponentially weighted moving average (EWMA) [6] control charts as well as Bayesian approaches [7–9]. Only the former will be considered here; the interested reader is referred to [1–3,10] for an exhaustive treatment of the subject matter and to [11–13] for a more concise overview.

Note that, as has been pointed out throughout the years by several prominent scholars [11,14,15], to this date and despite considerable advances in nonparametric approaches, most control charts remain based on the normality assumption. Despite its appeal, the normal distribution clearly is rarely an appropriate model for real-world applications. According to Stoumbos et al. [11] there exists a fundamental disconnect between practitioners and researchers as well as a gap between applied and theoretical research: "The existence of these gaps is disturbing, because it means that most practitioners have received little of the potential benefit from the technical advances made in SPC over the last half-century." (see [11] at 993).

The present work aims to shrink the above-mentioned gap by providing ready-to-use tables of control limits for an adaptive self-starting distribution-free CUSUM suitable to a wide variety of applications where a process is monitored for a departure from the underlying sampling distribution to a stochastically larger distribution. While this procedure has previously briefly been outlined and used by this author in [16,17], respectively, it is first thoroughly proposed and assessed in the current work.

Following a review of some pertinent fundamentals in Section 2 we proceed by reviewing the adaptive distribution-free CUSUM, providing a simple, yet effective calibration algorithm and obtaining a set of control limits suitable for a wide variety of scenarios. The performance of the control limits obtained as outlined in Section 3 is then assessed through extensive simulation experiments, whose results are outlined and discussed in Section 4; it will be shown that the proposed control limits yield a significantly reduced detection delay while maintaining good overall robustness. Finally, our concluding remarks set out in Section 5 complete this work.

2. Parametric and Nonparametric Univariate CUSUM Control Charts

The following subsections concisely restate the parametric (normal) univariate CUSUM and McDonald's sequential ranks CUSUM (SRC) [18]. All considerations will be limited to the most basic task of detecting a positive shift in the mean of a sequentially observed process using one-sided control charts.

2.1. Conventional Parametric CUSUM

Let F and G denote normal distributions given as $F \sim \mathcal{N}(\mu_0, 1)$ and $G \sim \mathcal{N}(\mu_0 + \delta, 1)$. Furthermore, for the sake of simplicity, let $\mu_0 = 0$ and $\delta = 1$. Consider observing a sequence of independent random variables $\{x_n, n \geq 1\}$ such that $\{x_1, \ldots, x_{\tau-1}\} \sim F$ and $\{x_\tau, x_{\tau+1}, \ldots\} \sim G$, i.e., a distributional shift $F \to G$ occurs at time instance τ. Assuming perfect knowledge of all parameters describing F and G (i.e., μ_0 and δ) Page's CUSUM [5] represents the gold standard change detection technique and can be computed sequentially as

$$C_0 = 0, \quad C_n = \max\{0, C_{n-1} + x_n - k_C\}, \quad n \geq 1\,. \tag{1}$$

The CUSUM signals, thereby declaring a distributional shift to have occurred, if

$$C_n > h_C\,, \tag{2}$$

with h_C and k_C being the prespecified control limit and reference constant, respectively.

The CUSUM's in-control average run length (ARL) is defined as the expected time until a change is signaled under F, i.e.,

$$ARL = E_F \inf\{n > 0 : C_n > h_C\}\,. \tag{3}$$

Note that this is akin to a nominal type-I-error level in the realm of hypothesis testing and that hence the closeness of the actual in-control ARL to ARL_0 is commonly regarded as an indicator of the control chart's robustness [15,19]. Accordingly, h_C and k_C are chosen such that ARL_0 is (at least approximately) attained when the observed process is in-control (see, e.g., [3,20]). It is well known that choosing $k_C = \delta/2$ is optimal [21] (see also [22,23]).

2.2. Sequential Ranks CUSUM (SRC)

Consider again the sequence $\{x_n, n \geq 1\}$; the sequential rank of x_n is defined as

$$R_n = 1 + \sum_{r=1}^{n-1} (x_n - x_r)^+ \ , \tag{4}$$

where $(x)^+$ is 1 for $x > 0$ and 0 otherwise. The SRC is then

$$C_{\mathrm{SRC}_n} = \max\{0, C_{\mathrm{SRC}_{n-1}} + \frac{R_n}{n+1} - k_{\mathrm{SRC}}\}\ ,\ n \geq 1\,, \tag{5}$$

with $C_{\mathrm{SRC}_0} = 0$ and k_{SRC} some reference constant. Akin to Equation 2, the SRC signals if $C_{\mathrm{SRC}_n} > h_{\mathrm{SRC}}$.

A crucial advantage of the SRC stems from the fact that, given the observed process is in-control, it can be shown (see [18] and references therein) that the quantities $\frac{R_n}{n+1}$ are independent and discrete uniform on $\{\frac{1}{n+1}, \frac{2}{n+1}, \cdots, \frac{n}{n+1}\}$. Hence, in addition to the approach followed in [18], h_{SRC} for a fixed k_{SRC} can be obtained through a straightforward Monte Carlo procedure (see, e.g., [24] at 12) without requiring any historical training data.

3. Adaptive Control Limit SRC (AC-SRC)

As will be shown in detail in Section 4, the actual applicability of the SRC is often hampered due to its virtually unacceptable performance in certain scenarios. More specifically, the SRC suffers from a lack of agility, i.e., given a distributional shift actually occurred the SRC may require an undue amount of time to signal (i.e., it exhibits a large detection delay); as will be shown, this is especially pronounced if a change occurs soon after monitoring commenced. In such case, the amount of data gathered by the SRC may be grossly insufficient, thus resulting in a prolonged time to signal. It should be noted that, as McDonald correctly points out (see [18], pg. 628–629), the above mentioned lack of agility (compared to an optimal parametric approach) and the poor detection of changes occurring after a relatively small number of observations is to some degree inherent to all nonparametric procedures.

The idea behind the adaptive control limit SRC (AC-SRC) proposed by this author [16,17] is to mitigate the SRC's drawbacks while maintaining its ease-of-use, robustness, and the ability to obtain generally valid control limits ahead of time. This is facilitated by the AC-SRC being inspired by and incorporating large parts of a distribution-free bootstrap based CUSUM proposed by Chatterjee and Qiu [19]. Said authors in 2009 proposed an elegant procedure where the conventional fixed control limit is swapped for a sequence of control limits obtained from the conditional distribution of the test statistic (i.e., the CUSUM) given the last time it was zero. Chatterjee and Qiu estimate these conditional distributions by means of bootstrapping; note that among other things this implies the need of a large amount of representative training data as well as a high computational burden. However, transferring the key idea of the approach by Chatterjee and Qiu to the SRC results in the AC-SRC described and analyzed in the following.

Akin to the SRC described in Section 2.2 let R_n and $C_{\text{AC-SRC}_n}$ denote the sequential rank of x_n and the respective SRC as provided by Equations (4) and (5), respectively. Furthermore, let $Y_{\text{AC-SRC}_j}$ be a random variable following the conditional distribution

$$Y_{\text{AC-SRC}_j} \sim [C_{\text{AC-SRC}_n} | T_{\text{AC-SRC}_n} = j] \,, \tag{6}$$

where $T_{\text{AC-SRC}_n}$, also referred to as *sprint length*, denotes the time elapsed since $C_{\text{AC-SRC}_n}$ was last zero, i.e.

$$\begin{aligned} T_{\text{AC-SRC}_n} &= 0 \quad \text{if} \quad C_{\text{AC-SRC}_n} = 0 \\ T_{\text{AC-SRC}_n} &= j \quad \text{if} \quad C_{\text{AC-SRC}_n} \neq 0, \ldots, C_{\text{AC-SRC}_{n-j+1}} \neq 0\,, \\ & \qquad\qquad C_{\text{AC-SRC}_{n-j}} = 0\,; \quad j = 1, \ldots, n\,. \end{aligned}$$

Central to the method by Chatterjee and Qiu is the fact that the conditional distributions in Equation (6) depend only on j and F but not on n [19]. Then for any positive integer $j_{\max} \leq n$ the (unconditional) distribution of $C_{\text{AC-SRC}_n}$ can be approximated by means of the conditional distributions in Equation (6) as

$$C_{\text{AC-SRC}_n} \sim \sum_{j=1}^{j_{\max}} Y_{\text{AC-SRC}_j} I_{T_{\text{AC-SRC}_n}=j} + Y^* I_{T_{\text{AC-SRC}_n} > j_{\max}}\,, \tag{7}$$

with I being the common indicator function and $Y^* \sim [C_{\text{AC-SRC}_n} | T_{\text{AC-SRC}_n} > j_{\max}]$. Since the AC-SRC is based on sequential ranks which, given the process is in control, are independent and discrete uniform on $\{\frac{1}{n+1}, \frac{2}{n+1}, \cdots, \frac{n}{n+1}\}$ (see Section 2.2) the sequence of control limits $\{h_j\}$ can be determined (ahead of time) without the need for training data by means of Monte Carlo simulations as outlined in Algorithm 1.

The AC-SRC then signals if $T_{\text{AC-SRC}_n} = j$ and $C_{\text{AC-SRC}_n} > h_j$ for $1 \leq j \leq j_{\max}$ or if $T_{\text{AC-SRC}_n} > j_{\max}$ and $C_{\text{AC-SRC}_n} > h_{j_{\max}}$. Note that following recommendations by Chatterjee and Qiu the h_j are only calculated up to a reasonably small $j_{\max}$ after which, if the test statistic does not bounce back to zero, they are kept fixed at $h_{j_{\max}}$. Furthermore, $k_{\text{AC-SRC}}$ is linked to $j_{\max}$ such that a desired sprint length tET_n, which is set to be proportional to $j_{\max}$ (see Section 3.1), e.g., as $tET_n = \lfloor \frac{3 j_{\max}}{4} \rfloor$, is approximately attained by the average sprint length. That is, $k_{\text{AC-SRC}} : \bar{T}_{\text{AC-SRC}_n} \approx tET_n = \lfloor \frac{3 j_{\max}}{4} \rfloor$.

Algorithm 1: Adaptive Control Limit SRC (AC-SRC)

Input: B, B_1 : number of Monte Carlo runs for main/fine-tuning procedure
tET_n : desired expected sprint length $E\{T_{\text{AC-SRC}_n}\}$, default is $tET_n = \lfloor \frac{3j_{\max}}{4} \rfloor$
Δ : fraction (e.g., $\Delta = \frac{1}{200}$) for fine-tuning

Part I: *calibrate* $k_{AC\text{-}SRC}$

while $\bar{T}_{AC\text{-}SRC_n} \neq tET_n \pm 1\%$ **do**
  for $i = 1$ **to** B_1 **do**
    for $n = 1$ **to** $N_{AC\text{-}SRC}$ **do**
      calculate $C_{\text{AC-SRC}_n} = \max\{0, C_{\text{AC-SRC}_{n-1}} + Y_n - k_{\text{AC-SRC}}\}$ with $Y_n \sim \text{unif}\{\frac{1}{n+1}, \frac{2}{n+1}, \cdots, \frac{n}{n+1}\}$;
      if $C_{AC\text{-}SRC_n} = 0$ *for first time* **then** save $T_{\text{AC-SRC}_{n\,i}} = n$ and break ;
    end
  end
  $\bar{T}_{\text{AC-SRC}_n} = \frac{1}{B_1}\sum_{b=1}^{B_1} T_{\text{AC-SRC}_{n\,b}}$
  if $\bar{T}_{AC\text{-}SRC_n} < tET_n \pm 1\%$ **then** increase $k_{\text{AC-SRC}}$ by $\Delta \cdot k_{\text{AC-SRC}}$;
  else decrease $k_{\text{AC-SRC}}$ by $\Delta \cdot k_{\text{AC-SRC}}$;
end

Part II: *calculate* $\{h_j\}_{j=1}^{j_{max}}$ *for fixed* $k_{AC\text{-}SRC}$ *and* j_{max}

for $j = 1$ **to** j_{max} **do**
  Initialization: $C^*_{\text{AC-SRC}_{\text{old}}} = 0,\ T^*_{\text{AC-SRC}_{\text{old}}} = 0,\ b = 0$;
  Step 0: set $b = b + 1$;
  Step 1: update $C^*_{\text{AC-SRC}_{\text{new}}} = \max\{C^*_{\text{AC-SRC}_{\text{old}}} + Y^*_n - k_{\text{AC-SRC}}, 0\}$;
    if $C^*_{AC\text{-}SRC_{new}} > 0$ **then** set $T^*_{\text{AC-SRC}_{\text{new}}} = T^*_{\text{AC-SRC}_{\text{old}}} + 1$;
    else if $C^*_{AC\text{-}SRC_{new}} = 0$ **then** set $T^*_{\text{AC-SRC}_{\text{new}}} = 0$;
  Step 2: **if** $T_{AC\text{-}SRC_{new}} = j$ **then** record $Y_{\text{AC-SRC}_{j:b}} = C^*_{\text{AC-SRC}_{\text{new}}}$ and return to Step 0 if $b < B$;
    else update $C^*_{\text{AC-SRC}_{\text{old}}} = C^*_{\text{AC-SRC}_{\text{new}}}$ and $T^*_{\text{AC-SRC}_{\text{old}}} = T^*_{\text{AC-SRC}_{\text{new}}}$ and return to Step 1;
end
set h_j as $B(1 - ARL_0{}^{-1})$ ordered value of $Y_{\text{AC-SRC}_{\text{j:1}}}, Y_{\text{AC-SRC}_{\text{j:2}}}, \ldots, Y_{\text{AC-SRC}_{\text{j:B}}}$;
set $h_j = h_{j_{\max}}$ for $j > j_{\max}$;

Part III: *calibrate the ARL (by fine-tuning* $\{h_j\}_{j=1}^{j_{max}}$ *)*

while $\overline{ARL} \neq ARL_0 \pm 5\%$ **do**
  calculate B_1 $C_{\text{AC-SRC}_n}$ as in Part I and estimate $\overline{ARL}$ based on (new) $\{h_j\}_{j=1}^{j_{\max}}$;
  if $\overline{ARL} < ARL_0 \pm 1\%$ **then** increase all entries in $\{h_j\}_{j=1}^{j_{\max}}$ by $\Delta \cdot \{h_j\}_{j=1}^{j_{\max}}$;
  else decrease all entries in $\{h_j\}_{j=1}^{j_{\max}}$ by $\Delta \cdot \{h_j\}_{j=1}^{j_{\max}}$;
end

3.1. Remarks on and Suggestions for the Selection of AC-SRC Parameters

The aim of this Section is twofold: first, to complete the description of the proposed procedure by justifying some seemingly completely arbitrary design choices of Section 3 , and second, to provide guidance to practitioners in order to facilitate the applicability of our method.

Average in-control and out-of-control run lengths, which are commonly referred to as ARL_0 and ARL_1 in the SPC literature, play a crucial role in the design and use of control charts. The ARL_0 characterizes the chart's propensity to false alarms in terms of the average number of samples they are separated by, whereas ARL_1 describes what we referred to as the control chart's agility, i.e., the average delay between the occurrence of an actual change and its detection. Clearly, then, there exists an inherent trade off between the objectives of low false alarm rates (large ARL_0) and small detection delays (small ARL_1). In this paper, we assume that the common approach of choosing an acceptable ARL_0 followed by attempts to minimize ARL_1 is pursued. The suitability of an ARL_0 highly depends on the particular problem at hand and is influenced, among other things, by crucial aspects such as weighting the aforementioned conflicting objectives to ensure compliance with requirements as well as detailed knowledge of the specific application. Accordingly, we find further discussions pertaining ARL_0 to be beyond the scope of this paper and, again, refer the interested reader to selected representatives of the established SPC literature [1–3].

Recall that, given a fixed and pre-determined $j_{\max}$, Algorithm 1 starts out by calibrating $k_{\text{AC-SRC}}$ such that the average sprint length $\bar{T}_{\text{AC-SRC}_n}$ equals the desired sprint length tET_n within a reasonable margin of error. Following Chatterjee and Qiu [19] ,we fix the desired sprint length tET_n as a in theory arbitrary ratio of $j_{\max}$; throughout this work $tET_n = \lfloor \frac{3j_{\max}}{4} \rfloor$ is used. Note that, although the rationale for linking $k_{\text{AC-SRC}}$ and $j_{\max}$ is compelling, doing so is not required.

The behavior of the AC-SRC's test statistic $C_{\text{AC-SRC}_n}$ is crucially influenced by the specific choice of $k_{\text{AC-SRC}}$ in that the propensity of $C_{\text{AC-SRC}_n}$ bouncing back to zero decreases for smaller $k_{\text{AC-SRC}}$ (and vice versa for larger values of the reference constant). In other words, the average sprint length $\bar{T}_{\text{AC-SRC}_n}$ increases with a reduction of $k_{\text{AC-SRC}}$, whereas increasing the reference value results in smaller sprint lengths. Furthermore, the sensible constraint of choosing $tET_n \ll j_{\max}$ restricts the computational burden of Algorithm 1 and reasonably ensures its algorithmic stability. In fact, in the absence of constraints on $j_{\max}$ and $k_{\text{AC-SRC}}$, 'inappropriate' combinations such as, e.g., (very) large $k_{\text{AC-SRC}}$ and $j_{\max}$ could easily result in the inability to evaluate Equation (6) which, in turn, is required in Part II of Algorithm 1 . While we find the aforementioned to establish sufficient and convincing justification for the choice of calibrating $k_{\text{AC-SRC}}$ such that $\bar{T}_{\text{AC-SRC}_n} \approx tET_n = \lfloor \frac{3j_{\max}}{4} \rfloor$ holds, it is arbitrary in that other reasonable but not necessarily superior design choices are readily discernible (see [19]).

As expected, and consistent with the considerations expressed by Chatterjee and Qiu [19] pertaining to their bootstrap-based method, we observed diminishing returns with increasing the length $j_{\max}$ of the sequence of adaptive control limits $\{h_j\}_{j=1}^{j_{\max}}$.

While we are unable to provide specific guidelines pertaining to the selection of AC-SRC's pertinent tuning parameters and further research in this area is required, we advocate the use of rather short sequences $\{h_j\}_{j=1}^{j_{\max}}$ with $6 \leq j_{\max} \ll 30$. Based on our current understanding and evidence, we recommend to set up the AC-SRC as discussed above and to adjust it to the requirements of the specific scenario by means of choosing either smaller or larger $j_{\max}$.

As will be corroborated by simulation results in Section 4.2, a reasonably consistent degree of fine-tuning is attainable with smaller $j_{\max}$ allowing for good agility, whereas using slightly larger values for $j_{\max}$ yields improved robustness at the expense of an increased detection delay.

4. Results and Discussion

4.1. Control Limits and Reference Values for the AC-SRC

Ready-to-use sets of reference constants $k_{\text{AC-SRC}}$ and respective sequences of control limits $\{h_j\}_{j=1}^{j_{\max}}$ for combinations of ARL_0 and $j_{\max}$ have been determined following the calibration procedure described in Algorithm 1. Again we emphasize that the main contribution of this work is in providing practitioners with a wide choice of predetermined control limits to be used out-of-the-box without requiring any further adjustments.

We used values for $k_{\text{AC-SRC}}$ calibrated such that $\bar{T}_{\text{AC-SRC}_n} \approx tET_n = \lfloor \frac{3j_{\max}}{4} \rfloor$ and $N_{\text{AC-SRC}} = 5000$, $B = 5 \cdot 10^4$, $B_1 = 5000$, $\Delta = \frac{1}{200}$. All result were further averaged over 200 Monte Carlo runs.

To improve readability the tabularized sets of control limits $\{h_j\}_{j=1}^{j_{\max}}$ and reference values $k_{\text{AC-SRC}}$ for $ARL_0 = \{100, 200, 300, 370, 400, 500, 600, 700, 800, 900, 1000\}$ and $j_{\max} = \{6, 8, 10, 12, 14, 16, 18\}$ are deferred to Tables A1–A11 in Appendix A. $ARL_0 = 370$ was included due to its popularity among practitioners, which stems from Shewhart $\bar{x}$ control charts using three-sigma limits having an in-control ARL of 370 (see generally [1]).

4.2. Performance Evaluation of the Proposed AC-SRC

To obtain an accurate representation of the proposed AC-SRC's performance and put it into perspective we conducted simulation experiments to ascertain a control chart's detection delay (DD), in-control ARL, and false alarm rate (FAR). A shift in the process distribution from $F \sim \mathcal{N}(0,1)$ to $G \sim \mathcal{N}(1,1)$ occurring at various time instances τ was simulated. FAR in this context refers to instances in which a particular control chart signaled although the actual shift at time instance τ had not occurred yet. Results were obtained for $\tau = \{10, 20, 30, 40, 50\}$, $j_{\max} = \{6, 8, 10, 12, 14, 16, 18\}$, $ARL_0 = \{100, 500, 1000\}$ and compared with optimal values for the parametric CUSUM (as provided in [3]) and the conventional SRC (as provided in [18]) for the respective ARL_0 as illustrated in Table 1. All results were averaged over $2 \cdot 10^5$ Monte Carlo runs.

Table 1. Optimal control limits h and reference values k for the parametric cumulative sum (CUSUM) (C) and the conventional sequential ranks CUSUM (SRC) for $ARL_0 = \{100, 500, 1000\}$ (ARL = average run length).

	$ARL_0 = 100$		$ARL_0 = 500$		$ARL_0 = 1000$	
	C	**SRC**	**C**	**SRC**	**C**	**SRC**
k	0.5	0.6428	0.5	0.6425	0.5	0.6428
h	2.8497	0.798	4.3891	1.2031	5.0708	1.382

Furthermore, the robustness of all three control charts to deviations from the normal distribution was assessed by simulating an impulsive noise environment through the use of a two component Gaussian mixture model, as is often done in related work (see [25], pg. 176; see also [26,27]). Accordingly, instead of $F \sim \mathcal{N}(0,1)$, the in-control are modeled as

$$F \sim (1-\eta)\, \mathcal{N}(0,1) + \eta\, \mathcal{N}(0,\kappa) \tag{8}$$

with $0 \le \eta \le 1$ expressing the probability that contamination with the heavy-tailed component modeled using $\kappa \gg 1$ occurs. Thus, again, at time instance τ a shift in distribution from $F \sim (1-\eta)\, \mathcal{N}(0,1) + \eta\, \mathcal{N}(0,\kappa)$ to $G \sim (1-\eta)\, \mathcal{N}(1,1) + \eta\, \mathcal{N}(1,\kappa)$ occurs. All reported results claiming impulsive noise contamination were obtained using $\eta = 0.1$ and $\kappa = 100$.

4.2.1. Performance under Normality

Tables 2–4 show results of simulation experiments as outlined in Section 4.2 for the normal use case, i.e., a shift in distribution from $F \sim \mathcal{N}(0,1) \rightarrow G \sim \mathcal{N}(1,1)$ occurs at time instance τ.

Table 2. $ARL_0 = 100$, 0% contamination.

				$j_{\max}$ (AC-SRC)						
τ		**C**	**SRC**	**6**	**8**	**10**	**12**	**14**	**16**	**18**
	DD	5.5798	33.1970	28.9016	29.5565	34.0757	36.7935	40.1121	42.0636	45.7209
10	**ARL**	100.1217	118.7456	99.9766	99.9764	99.6454	99.8677	99.9687	99.2931	99.6043
	FAR	0.0655	0.0056	0	0	0	0.0001	0.0001	0.0002	0.0003
	DD	4.5829	12.4038	12.3431	12.7814	13.4354	14.7583	16.1362	16.9636	18.2698
20	**ARL**	100.1221	118.7666	99.8431	100.1864	99.5062	100.0187	100.0108	99.4962	99.5444
	FAR	0.1566	0.0607	0.0148	0.0084	0.0064	0.0280	0.0518	0.0642	0.0779
	DD	4.5789	8.2788	8.7000	9.3913	10.4685	11.2349	12.0707	12.4557	13.2802
30	**ARL**	100.0206	118.8567	99.7885	100.2029	99.4369	100.1514	99.7156	99.4519	99.5012
	FAR	0.2390	0.1306	0.0702	0.0875	0.1068	0.1164	0.1314	0.1403	0.1587
	DD	4.5749	6.9814	7.9473	8.5354	9.3278	9.8141	10.4779	10.9331	11.5477
40	**ARL**	100.0611	118.8167	99.9674	100.3011	89.7641	100.0303	89.8406	99.3637	99.3303
	FAR	0.3141	0.2013	0.1799	0.1818	0.1878	0.2029	0.2186	0.2268	0.2405
	DD	4.5802	6.3752	7.6181	7.9070	8.6649	9.2004	9.6475	10.0578	10.5800
50	**ARL**	100.1531	118.8285	99.8177	100.2513	99.3827	99.9977	99.7287	99.4817	99.6277
	FAR	0.3810	0.2695	0.2668	0.2620	0.2823	0.2889	0.2951	0.3072	0.3224

Clearly the parametric CUSUM's exceptional performance comes as no surprise considering its optimality if, as is the case here, the monitored process is actually Gaussian. Questions of greater interest concern whether or not a substantial performance difference between the SRC and the proposed AC-SRC can be observed.

Our qualitative assessment of performance differences will focus on differences among the examined control charts pertaining to:

- Detection delay (DD)
  - One of if not the major objective in practical applications is to detect a change as quickly as possible; hence, DD should be small (see also Section 3.1).
- Average run length (ARL)
  - Recall that the ARL describes the average time or run length until the control chart signals under in-control conditions, i.e., without a change having occurred. The ARL is, loosely speaking, akin to the type-I error level in hypothesis testing. Rather than setting a false alarm rate control charts are typically designed by choosing a desired ARL_0. The actual in-control ARL determined in our simulation experiment should be reasonably close to the nominal ARL_0 and we interpret this closeness as indicating the control chart's robustness.
- False alarm rate (FAR)
  - Moreover, recall that even if the monitored process is in-control any CUSUM chart will eventually signal. Clearly there is a relation between FAR and ARL; however, since said relation and false alarm properties of CUSUMs in general are neither well explored nor straightforward, especially for rather small ARLs, a discussion is deemed beyond the scope of this work. The

interested reader is referred to, e.g., [28]. Coming back to the issue at hand, as fas as our performance assessment is concerned FAR values should be as small as possible (ideally zero).

Table 3. $ARL_0 = 500$, 0% contamination.

τ		C	SRC	$j_{\max}$ (AC-SRC) 6	8	10	12	14	16	18
10	DD	7.4845	268.0231	127.1618	115.8420	114.1003	113.0386	118.7457	122.5208	134.0196
	ARL	500.3259	532.0541	489.4462	487.0666	483.9462	486.4092	500.5751	504.5539	527.5008
	FAR	0.0088	0.0001	0	0	0	0	0	0	0
20	DD	7.4774	89.3780	26.3227	36.4466	27.1662	28.6116	30.5613	32.5018	35.2079
	ARL	499.8330	531.6094	484.7291	484.6379	484.0727	486.1253	500.1508	504.9982	526.9485
	FAR	0.0280	0.0067	0.0012	0.0003	0.0001	0	0	0	0
30	DD	7.4762	36.6425	15.4354	16.9657	17.7711	19.1151	20.2779	21.8232	23.2656
	ARL	500.0630	531.3130	489.2917	484.4814	484.2848	486.0503	499.7263	505.1260	526.9345
	FAR	0.0473	0.0190	0.0094	0.0044	0.0030	0.0016	0.0010	0.0006	0.0003
40	DD	7.4691	20.2862	13.0108	14.5117	15.2444	16.3110	17.1893	18.4266	19.5790
	ARL	500.2418	531.3190	487.2419	486.2570	484.3922	486.3174	500.3466	505.4857	527.5350
	FAR	0.0661	0.0336	0.0233	0.0144	0.0117	0.0078	0.0056	0.0038	0.0025
50	DD	7.4804	14.5257	12.0488	13.4507	14.0356	14.9430	15.6976	16.7346	17.6902
	ARL	499.8426	531.3030	487.4462	484.9996	483.3978	486.6009	500.5158	504.9296	527.0213
	FAR	0.0849	0.0493	0.0410	0.0285	0.0252	0.0187	0.0147	0.0109	0.0077

Table 4. $ARL_0 = 1000$, 0% contamination.

τ		C	SRC	$j_{\max}$ (AC-SRC) 6	8	10	12	14	16	18
10	DD	8.8165	648.1897	362.1763	324.2384	318.3756	317.0301	318.4782	316.5780	316.4286
	ARL	998.8042	1045.7143	1003.2	996	993.8	997	991.7	992.7	993.7
	FAR	0.0030	0	0	0	0	0	0	0	0
20	DD	8.7880	254.2445	60.9372	54.4992	54.1869	57.0418	59.4549	62.6949	64.8553
	ARL	1003.3	1045.5714	1003.7	1001.1	993.1	996.5	992	992.8	993.8
	FAR	0.0130	0.0023	0.0001	0	0	0	0	0	0
30	DD	8.7839	102.8965	24.4908	25.4678	27.1288	29.9713	31.4247	34.1553	35.4113
	ARL	999.4	1044.6429	104.8	998.3	993.4	993.7	997.4	994.1	992.9
	FAR	0.0229	0.0080	0.0021	0.0003	0.0002	0	0	0	0
40	DD	8.7871	47.5679	17.5591	20.3572	21.7860	24.2974	25.4796	27.6644	28.7028
	ARL	1000.4571	1045.3143	1002.9	998.6	993.2	996.4	993.9	991.9	993.4
	FAR	0.0328	0.0148	0.0061	0.0022	0.0014	0.0006	0.0003	0.0001	0.0001
50	DD	8.7878	26.7629	15.8388	18.3891	19.6391	21.7982	22.8171	24.7035	25.5985
	ARL	1000.5	1045.7429	1007.6	997.3	994.2	996.9	995.7	993	993.3
	FAR	0.0426	0.0222	0.0130	0.0065	0.0043	0.0023	0.0016	0.0009	0.0006

Examining the entries of Tables 2–4 it can generally be observed that the proposed AC-SRC performs well, especially keeping in mind that the error margins allowed for in Algorithm 1 (namely up to 5% deviation from ARL_0) are fairly relaxed and could easily be tightened at the expense of an increased computational burden. Still, the proposed AC-SRC more often than not outperforms the conventional SRC in all aspects. More specifically, it is offers substantial benefits especially for larger ARLs and small to medium τ.

However, it ought also be pointed out that the AC-SRC does struggle to outperform the conventional SRC for $ARL_0 = 100$ as shown in Table 2. Its overall performance however still appears acceptable. A likely cause stems from the fact that the sequential ranks approximation by an independent uniformly distributed random variable strictly speaking only holds asymptotically and convergence appears to be somewhat slow. Note that despite a deviation of up to 5% was allowed in the determination of the AC-SRC control limits the AC-SRC's actual ARL is remarkably close to ARL_0 and the FARs are consistently lower than both C and SRC. Finally, focusing on Tables 3 and 4 it is evident that the AC-SRC indeed results in an increased agility, as evidenced by substantially reduced detection delays.

4.2.2. Performance under Impulsive Noise Contamination

The second part of our performance analysis focused on assessing the performance of C, SRC, and AC-SRC when subjected to impulsive noise contamination (as described in Section 4.2). Recall that at time instance τ the distributional shift now occurs from $F \sim (1-\eta)\ \mathcal{N}(0,1) + \eta\ \mathcal{N}(0,\kappa)$ to $G \sim (1-\eta)\ \mathcal{N}(1,1) + \eta\ \mathcal{N}(1,\kappa)$ with $\eta = 0.1$ and $\kappa = 100$.

The breakdown of the parametric CUSUM is hardly surprising and not worthy of further discussion; rather the interest lies in whether or not the benefits shown by the SRC in the uncontaminated use case persist if the underlying process is heavier tailed. Examining Tables 5–7 we answer in the affirmative. More specifically, while a slight increase in both DD and ARL deviation is observed, all material arguments raised in Section 4.2.1 apply mutatis mutandis.

In conclusion, we would like to re-emphasize the reasonably consistent degree of fine-tuning attainable by means of sensibly choosing $j_{\max}$, wherein smaller $j_{\max}$ yield reduced detection delays at the expense of an increased type-I error rate, whereas larger $j_{\max}$ result in improved robustness and decreased agility.

Table 5. $ARL_0 = 100$, 10% contamination.

				$j_{\max}$ (AC-SRC)						
τ		**C**	**SRC**	**6**	**8**	**10**	**12**	**14**	**16**	**18**
10	**DD**	3.8910	52.2870	46.3682	47.2927	52.3379	55.5305	59.8147	62.2322	66.5670
	ARL	24.7372	118.8468	99.6766	100.5088	99.6951	100.2877	99.7397	99.4211	99.5998
	FAR	0.2834	0.0057	0	0	0	0	0.0001	0.0002	0.0004
20	**DD**	3.8894	26.8525	23.5949	23.2869	24.8711	26.7932	29.3518	30.8005	33.2495
	ARL	24.7353	118.8606	100.0215	100.4114	99.8287	100.1349	99.7336	99.2912	99.6710
	FAR	0.5369	0.0610	0.0147	0.0089	0.0066	0.0277	0.0529	0.0640	0.0778
30	**DD**	3.8831	18.4651	15.8380	16.2014	18.1705	19.2565	20.6417	21.3250	22.9193
	ARL	24.7227	118.9065	99.9930	100.2403	99.7697	99.7791	99.8651	99.4605	99.5673
	FAR	0.7007	0.1306	0.0696	0.0882	0.1088	0.1161	0.1334	0.1377	0.1585
40	**DD**	3.9057	14.8526	13.8203	14.0883	15.1973	15.9524	17.1175	17.9106	18.9459
	ARL	24.7271	118.8372	99.9986	100.3971	99.7328	100.1732	99.6427	99.3361	99.6082
	FAR	0.8071	0.2014	0.1821	0.1811	0.1890	0.2012	0.2189	0.2286	0.2415
50	**DD**	3.8800	13.0198	12.6930	12.7257	13.9105	14.5288	15.3389	15.9883	16.9118
	ARL	24.7468	118.8856	99.9125	100.5026	99.7612	100.0074	99.6560	99.5814	99.4769
	FAR	0.8756	0.2697	0.2665	0.2649	0.2843	0.2876	0.2937	0.3087	0.3216

Table 6. $ARL_0 = 500$, 10% contamination.

				$j_{\max}$ (AC-SRC)						
τ		**C**	**SRC**	**6**	**8**	**10**	**12**	**14**	**16**	**18**
10	**DD**	6.3947	353.9665	225.9275	209.4729	205.9059	204.0480	213.5904	214.6237	233.6764
	ARL	65.5479	531.4316	488.0331	484.4087	488.4807	488.2045	499.9793	506.4905	526.8804
	FAR	0.0931	0.0001	0	0	0	0	0	0	0
20	**DD**	6.3823	201.6783	81.3509	71.7558	70.5015	70.3994	72.9243	75.0408	82.1264
	ARL	65.5938	531.5086	488.6755	483.7614	485.5939	487.7366	501.0609	506.4544	527.0465
	FAR	0.2273	0.0068	0.0015	0.0003	0.0001	0	0	0	0
30	**DD**	6.3845	123.0448	40.4024	37.0531	36.8415	37.7073	39.4376	41.2816	44.6164
	ARL	65.6212	531.4936	488.4589	482.7787	485.7063	487.1630	488.9567	507.7213	526.9891
	FAR	0.3423	0.0192	0.0100	0.0043	0.0029	0.0017	0.0010	0.0005	0.0004
40	**DD**	6.3847	81.4736	27.3471	26.5232	27.0461	28.0854	29.3648	30.9860	33.1040
	ARL	65.5715	531.2075	488.4647	485.5648	485.1457	487.8131	500.0233	506.9858	527.3921
	FAR	0.4401	0.0338	0.0245	0.0152	0.0114	0.0080	0.0056	0.0036	0.0023
50	**DD**	6.3891	57.7347	22.3869	22.5064	23.1674	24.1505	25.1836	26.6830	28.3272
	ARL	65.5934	531.5276	488.1271	484.9405	485.9766	487.5572	498.3469	506.6913	526.9864
	FAR	0.5233	0.0493	0.0417	0.0297	0.0250	0.0191	0.0149	0.0108	0.0076

Table 7. $ARL_0 = 1000$, 10% contamination.

				$j_{\max}$ (AC-SRC)						
τ		**C**	**SRC**	**6**	**8**	**10**	**12**	**14**	**16**	**18**
10	**DD**	7.5806	794.075	571.8691	533.4575	521.0822	520.3675	519.0203	508.9829	512.4341
	ARL	97.3143	1044.2857	1007.3	996.5	995.5	998.7	997.2	991.8	992.1
	FAR	0.0554	0	0	0	0	0	0	0	0
20	**DD**	7.5627	509.1507	223.2311	187.2751	180.3638	179.9559	175.3016	177.2089	178.9137
	ARL	97.3286	1044.3714	1008.3	997.5	994.8	999.6	997.5	995.3	997.6
	FAR	0.1511	0.0023	0.0001	0	0	0	0	0	0
30	**DD**	7.5561	331.2999	93.7481	76.7901	75.5415	75.9972	76.4086	78.6915	80.2621
	ARL	97.3571	1044.9714	1006.5	996.5	998.6	1002.3	997.9	990.1	997.2
	FAR	0.2375	0.0079	0.0018	0.0004	0.0002	0	0	0	0
40	**DD**	7.5618	221.1639	50.4663	44.4734	44.9347	47.2327	48.3099	51.1874	52.2784
	ARL	97.3	1044.4857	1006.6	996.6	994.7	998.5	997.2	994.3	994.9
	FAR	0.3152	0.0149	0.0063	0.0025	0.0015	0.0006	0.0003	0.0002	0.0001
50	**DD**	7.5575	152.8864	35.1211	33.7103	34.8390	37.2713	38.4121	41.0106	42.1451
	ARL	97.3429	1044.6571	1006.7	996.2	994.1	996.6	996.8	992.8	995.1
	FAR	0.3848	0.0224	0.0131	0.0064	0.0043	0.0023	0.0016	0.0009	0.0007

5. Conclusions

In the present work we evaluated an adaptive self-starting distribution-free CUSUM based on sequential ranks and for the first time provided detailed pseudo-code of a simple, yet effective calibration algorithm. The main original contribution of this work, however, is in providing precomputed control limits and reference values for a wide variety of AC-SRC configurations, thus allowing practitioners to apply the procedure off-the-shelf without further adjustments and irrespective of the data generating model underlying their specific use case. Performance and robustness of the proposed tabularized control limits were assessed and compared to both parametric CUSUM and conventional SRC through extensive simulation experiments. While far from optimal, we were able to show that the proposed control limits result in a substantially decreased detection delay, while maintaining good overall robustness properties and allowing for easy and intuitive fine-tuning.

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Abbreviations

The following abbreviations are used in this manuscript:

AC-SRC	Adaptive Control Limit SRC
ARL	Average Run Length
CUSUM	Cumulative Sum Control Chart
DD	Detection Delay
EWMA	Exponentially Weighted Moving Average Control Chart
FAR	False Alarm Rate
SPC	Statistical Process Control
SRC	Sequential Ranks CUSUM

Appendix A. Tabularized AC-SRC Control Limits and Reference Values

Table A1. Adaptive control limit SRC (AC-SRC) for $ARL_0 = 100$.

ARL_0	100	100	100	100	100	100	100
j_{max}	6	8	10	12	14	16	18
$k_{AC\text{-}SRC}$	0.5486	0.5318	0.5267	0.5209	0.5180	0.5142	0.5131
h_1	0.4168	0.4274	0.4250	0.4247	0.4221	0.4199	0.4165
h_2	0.8487	0.8410	0.8331	0.8308	0.8251	0.8209	0.8144
h_3	1.2013	1.2080	1.2012	1.1910	1.1775	1.1697	1.1598
h_4	1.4961	1.5056	1.4885	1.4765	1.4627	1.4510	1.4378
h_5	1.7470	1.7605	1.7395	1.7283	1.7110	1.6984	1.6826
h_6	1.9664	1.9825	1.9652	1.9542	1.9375	1.9233	1.9053
h_7		2.1859	2.1675	2.1558	2.1388	2.1223	2.1042
h_8		2.3741	2.3520	2.3437	2.3244	2.3089	2.2895
h_9			2.5270	2.5167	2.4970	2.4818	2.4627
h_{10}			2.6886	2.6807	2.6632	2.6467	2.6268
h_{11}				2.8359	2.8157	2.8016	2.7772
h_{12}				2.9803	2.9638	2.9473	2.9238
h_{13}					3.1035	3.0865	3.0624
h_{14}					3.2351	3.2216	3.1945
h_{15}						3.3497	3.3252
h_{16}						3.4718	3.4474
h_{17}							3.5652
h_{18}							3.6794

Table A2. AC-SRC for $ARL_0 = 200$.

ARL_0	200	200	200	200	200	200	200
$j_{\max}$	6	8	10	12	14	16	18
$k_{\text{AC-SRC}}$	0.5486	0.5318	0.5269	0.5207	0.5180	0.5145	0.5130
h_1	0.4409	0.4557	0.4552	0.4586	0.4570	0.4570	0.4549
h_2	0.8964	0.9173	0.9095	0.9112	0.9071	0.9070	0.9020
h_3	1.2875	1.3083	1.3054	1.3049	1.2942	1.2925	1.2848
h_4	1.6139	1.6366	1.6257	1.6265	1.6155	1.6105	1.6011
h_5	1.8911	1.9201	1.9076	1.9136	1.9015	1.8981	1.8865
h_6	2.1345	2.1715	2.1616	2.1657	2.1553	2.1488	2.1392
h_7		2.3988	2.3878	2.3947	2.3837	2.3791	2.3673
h_8		2.6087	2.5963	2.6087	2.5930	2.5904	2.5795
h_9			2.7919	2.8033	2.7903	2.7884	2.7725
h_{10}			2.9716	2.9882	2.9766	2.9725	2.9578
h_{11}				3.1640	3.1468	3.1487	3.1324
h_{12}				3.3281	3.3123	3.3123	3.2966
h_{13}					3.4691	3.4709	3.4557
h_{14}					3.6205	3.6227	3.6075
h_{15}						3.7701	3.7512
h_{16}						3.9081	3.8927
h_{17}							4.0295
h_{18}							4.1614

Table A3. AC-SRC for $ARL_0 = 300$.

ARL_0	300	300	300	300	300	300	300
$j_{\max}$	6	8	10	12	14	16	18
$k_{\text{AC-SRC}}$	0.5490	0.5318	0.5266	0.5205	0.5180	0.5143	0.5129
h_1	0.4534	0.4678	0.4666	0.4706	0.4696	0.4713	0.4696
h_2	0.9374	0.9482	0.9445	0.9519	0.9486	0.9526	0.9496
h_3	1.3456	1.3714	1.3611	1.3605	1.3508	1.3553	1.3496
h_4	1.6910	1.7255	1.7138	1.7148	1.7049	1.7073	1.7001
h_5	1.9880	2.0307	2.0157	2.0219	2.0084	2.0103	2.0058
h_6	2.2470	2.2975	2.2822	2.2905	2.2776	2.2836	2.2728
h_7		2.5398	2.5260	2.5360	2.5230	2.5290	2.5178
h_8		2.7641	2.7492	2.7598	2.7476	2.7539	2.7459
h_9			2.9559	2.9689	2.9586	2.9645	2.9548
h_{10}			3.1476	3.1676	3.1502	3.1638	3.1522
h_{11}				3.3505	3.3367	3.3494	3.3405
h_{12}				3.5272	3.5136	3.5292	3.5158
h_{13}					3.6829	3.6986	3.6892
h_{14}					3.8398	3.8602	3.8491
h_{15}						4.0177	4.0039
h_{16}						4.1662	4.1569
h_{17}							4.2997
h_{18}							4.4396

Table A4. AC-SRC for $ARL_0 = 370$.

ARL_0	370	370	370	370	370	370	370
$j_{\max}$	6	8	10	12	14	16	18
$k_{\text{AC-SRC}}$	0.5489	0.5316	0.5267	0.5208	0.5178	0.5144	0.5130
h_1	0.4822	0.5002	0.4747	0.4773	0.4764	0.4785	0.4744
h_2	0.9830	1.0085	0.9522	0.9539	0.9505	0.9545	0.9527
h_3	1.4209	1.4516	1.3758	1.3710	1.3600	1.3616	1.3602
h_4	1.7870	1.8226	1.7224	1.7251	1.7161	1.7210	1.7170
h_5	2.1002	2.1490	2.0337	2.0357	2.0246	2.0302	2.0252
h_6	2.3789	2.4335	2.3032	2.3066	2.2951	2.3016	2.2936
h_7		2.6923	2.5494	2.5554	2.5423	2.5512	2.5462
h_8		2.9309	2.7765	2.7809	2.7690	2.7788	2.7749
h_9			2.9873	2.9948	2.9819	2.9936	2.9876
h_{10}			3.1838	3.1934	3.1799	3.1936	3.1881
h_{11}				3.3806	3.3685	3.3837	3.3782
h_{12}				3.5565	3.5472	3.5600	3.5547
h_{13}					3.7132	3.7353	3.7295
h_{14}					3.8774	3.8995	3.8938
h_{15}						4.0602	4.0521
h_{16}						4.2078	4.1982
h_{17}							4.3509
h_{18}							4.4898

Table A5. AC-SRC for $ARL_0 = 400$.

ARL_0	400	400	400	400	400	400	400
$j_{\max}$	6	8	10	12	14	16	18
$k_{\text{AC-SRC}}$	0.5486	0.5316	0.5267	0.5206	0.5180	0.5142	0.5129
h_1	0.4935	0.5121	0.4817	0.4826	0.4785	0.4809	0.4794
h_2	1.0109	1.0346	0.9718	0.9706	0.9607	0.9647	0.9619
h_3	1.4632	1.4870	1.3973	1.3911	1.3722	1.3779	1.3723
h_4	1.8388	1.8735	1.7536	1.7491	1.7297	1.7348	1.7287
h_5	2.1609	2.2105	2.0728	2.0667	2.0436	2.0475	2.0394
h_6	2.4470	2.5056	2.3480	2.3429	2.3170	2.3226	2.3141
h_7		2.7729	2.6001	2.5929	2.5664	2.5744	2.5657
h_8		3.0145	2.8307	2.8272	2.7992	2.8061	2.7965
h_9			3.0449	3.0424	3.0084	3.0214	3.0094
h_{10}			3.2445	3.2425	3.2114	3.2261	3.2125
h_{11}				3.4314	3.3986	3.4169	3.4044
h_{12}				3.6114	3.5831	3.6012	3.5891
h_{13}					3.7542	3.7728	3.7589
h_{14}					3.9180	3.9374	3.9242
h_{15}						4.0984	4.0863
h_{16}						4.2527	4.2403
h_{17}							4.3870
h_{18}							4.5315

Table A6. AC-SRC for $ARL_0 = 500$.

ARL_0	500	500	500	500	500	500	500
$j_{\max}$	6	8	10	12	14	16	18
$k_{\text{AC-SRC}}$	0.5485	0.5314	0.5265	0.5208	0.5182	0.5142	0.5131
h_1	0.5208	0.5440	0.5122	0.5059	0.4900	0.4909	0.4846
h_2	1.0788	1.1047	1.0372	1.0241	0.9914	0.9933	0.9804
h_3	1.5573	1.5953	1.4967	1.4720	1.4230	1.4251	1.4062
h_4	1.9657	2.0181	1.8898	1.8564	1.7922	1.7919	1.7678
h_5	2.3154	2.3803	2.2343	2.1982	2.1222	2.1218	2.0951
h_6	2.6225	2.7034	2.5356	2.4945	2.4104	2.4119	2.3772
h_7		2.9926	2.8089	2.7667	2.6718	2.6698	2.6350
h_8		3.2611	3.0592	3.0123	2.9119	2.9129	2.8743
h_9			3.2905	3.2440	3.1390	3.1376	3.1000
h_{10}			3.5081	3.4586	3.3519	3.3483	3.3045
h_{11}				3.6607	3.5419	3.5462	3.5036
h_{12}				3.8516	3.7302	3.7389	3.6848
h_{13}					3.9112	3.9227	3.8699
h_{14}					4.0841	4.0928	4.0387
h_{15}						4.2556	4.2060
h_{16}						4.4243	4.3662
h_{17}							4.5165
h_{18}							4.6654

Table A7. AC-SRC for $ARL_0 = 600$.

ARL_0	600	600	600	600	600	600	600
$j_{\max}$	6	8	10	12	14	16	18
$k_{\text{AC-SRC}}$	0.5486	0.5320	0.5268	0.5205	0.5181	0.5142	0.5132
h_1	0.5482	0.5680	0.5379	0.5378	0.5144	0.5150	0.4981
h_2	1.1295	1.1626	1.0953	1.0912	1.0411	1.0418	1.0075
h_3	1.6341	1.6843	1.5875	1.5711	1.4977	1.4969	1.4460
h_4	2.0644	2.1303	2.0031	1.9906	1.8963	1.8942	1.8294
h_5	2.4385	2.5102	2.3621	2.3501	2.2444	2.2388	2.1667
h_6	2.7623	2.8489	2.6882	2.6741	2.5530	2.5514	2.4636
h_7		3.1570	2.9789	2.9677	2.8323	2.8265	2.7325
h_8		3.4369	3.2453	3.2338	3.0898	3.0840	2.9809
h_9			3.4938	3.4808	3.3257	3.3265	3.2161
h_{10}			3.7211	3.7120	3.5470	3.5446	3.4283
h_{11}				3.9335	3.7595	3.7615	3.6345
h_{12}				4.1429	3.9641	3.9624	3.8278
h_{13}					4.1480	4.1512	4.0177
h_{14}					4.3312	4.3343	4.1938
h_{15}						4.5149	4.3631
h_{16}						4.6828	4.5280
h_{17}							4.6861
h_{18}							4.8414

Table A8. AC-SRC for $ARL_0 = 700$.

ARL_0	**700**	**700**	**700**	**700**	**700**	**700**	**700**
$j_{\mathbf{max}}$	**6**	**8**	**10**	**12**	**14**	**16**	**18**
$k_{\mathbf{AC\text{-}SRC}}$	0.5485	0.5318	0.5269	0.5208	0.5185	0.5146	0.5129
h_1	0.5729	0.5981	0.5655	0.5654	0.5399	0.5408	0.5253
h_2	1.1772	1.2201	1.1495	1.1459	1.0942	1.0956	1.0609
h_3	1.7063	1.7644	1.6641	1.6521	1.5692	1.5707	1.5224
h_4	2.1557	2.2336	2.1034	2.0897	1.9904	1.9890	1.9284
h_5	2.5459	2.6385	2.4855	2.4705	2.3538	2.3501	2.2775
h_6	2.8870	2.9966	2.8256	2.8117	2.6776	2.6768	2.5981
h_7		3.3249	3.1313	3.1167	2.9722	2.9705	2.8835
h_8		3.6161	3.4151	3.4005	3.2427	3.2436	3.1445
h_9			3.6734	3.6624	3.4919	3.4926	3.3935
h_{10}			3.9141	3.9049	3.7263	3.7323	3.6206
h_{11}				4.1381	3.9465	3.9507	3.8318
h_{12}				4.3565	4.1583	4.1654	4.0471
h_{13}					4.3633	4.3671	4.2385
h_{14}					4.5495	4.5640	4.4302
h_{15}						4.7462	4.6082
h_{16}						4.9226	4.7855
h_{17}							4.9518
h_{18}							5.1086

Table A9. AC-SRC for $ARL_0 = 800$.

ARL_0	**800**	**800**	**800**	**800**	**800**	**800**	**800**
$j_{\mathbf{max}}$	**6**	**8**	**10**	**12**	**14**	**16**	**18**
$k_{\mathbf{AC\text{-}SRC}}$	0.5487	0.5316	0.5269	0.5206	0.5181	0.5144	0.5134
h_1	0.5900	0.6251	0.5884	0.5935	0.5703	0.5702	0.5459
h_2	1.2147	1.2666	1.1924	1.1997	1.1524	1.1523	1.1024
h_3	1.7631	1.8362	1.7274	1.7317	1.6618	1.6580	1.5869
h_4	2.2318	2.3283	2.1874	2.1918	2.1045	2.0984	2.0080
h_5	2.6372	2.7569	2.5894	2.5957	2.4860	2.4804	2.3743
h_6	2.9928	3.1350	2.9422	2.9539	2.8334	2.8273	2.7040
h_7		3.4745	3.2697	3.2773	3.1406	3.1403	3.0023
h_8		3.7846	3.5575	3.5746	3.4295	3.4231	3.2787
h_9			3.8327	3.8521	3.6982	3.6907	3.5340
h_{10}			4.0909	4.1112	3.9451	3.9418	3.7774
h_{11}				4.3554	4.1809	4.1773	3.9958
h_{12}				4.5831	4.4028	4.4008	4.2094
h_{13}					4.6160	4.6165	4.4223
h_{14}					4.8154	4.8222	4.6167
h_{15}						5.0125	4.8086
h_{16}						5.2057	4.9797
h_{17}							5.1611
h_{18}							5.3314

Table A10. AC-SRC for $ARL_0 = 900$.

ARL_0	900	900	900	900	900	900	900
j_{max}	6	8	10	12	14	16	18
$k_{AC\text{-}SRC}$	0.5491	0.5318	0.5267	0.5204	0.5181	0.5148	0.5131
h_1	0.6016	0.6433	0.6097	0.6173	0.5909	0.5892	0.5726
h_2	1.2415	1.3066	1.2347	1.2472	1.1946	1.1904	1.1571
h_3	1.8051	1.8973	1.7927	1.8051	1.7256	1.7173	1.6715
h_4	2.2886	2.4069	2.2714	2.2870	2.1874	2.1766	2.1170
h_5	2.7038	2.8523	2.6929	2.7108	2.5906	2.5776	2.5060
h_6	3.0740	3.2459	3.0660	3.0866	2.9494	2.9373	2.8504
h_7		3.5973	3.3982	3.4314	3.2756	3.2617	3.1673
h_8		3.9184	3.7021	3.7371	3.5752	3.5587	3.4569
h_9			3.9949	4.0333	3.8552	3.8354	3.7322
h_{10}			4.2626	4.3011	4.1154	4.0952	3.9796
h_{11}				4.5601	4.3648	4.3439	4.2216
h_{12}				4.7992	4.5941	4.5710	4.4496
h_{13}					4.8169	4.7958	4.6658
h_{14}					5.0292	5.0140	4.8737
h_{15}						5.2138	5.0763
h_{16}						5.4120	5.2603
h_{17}							5.4477
h_{18}							5.6263

Table A11. AC-SRC for $ARL_0 = 1000$.

ARL_0	1000	1000	1000	1000	1000	1000	1000
j_{max}	6	8	10	12	14	16	18
$k_{AC\text{-}SRC}$	0.5488	0.5315	0.5267	0.5204	0.5180	0.5145	0.5131
h_1	0.6210	0.6627	0.6288	0.6365	0.6123	0.6133	0.5929
h_2	1.2830	1.3521	1.2773	1.2897	1.2401	1.2415	1.2002
h_3	1.8660	1.9650	1.8565	1.8668	1.7946	1.7942	1.7343
h_4	2.3673	2.4839	2.3538	2.3700	2.2772	2.2754	2.2008
h_5	2.7970	2.9581	2.7936	2.8136	2.7032	2.6998	2.6098
h_6	3.1839	3.3639	3.1819	3.2012	3.0771	3.0784	2.9711
h_7		3.7312	3.5328	3.5546	3.4182	3.4149	3.3030
h_8		4.0695	3.8544	3.8793	3.7291	3.7279	3.6101
h_9			4.1454	4.1850	4.0249	4.0271	3.8824
h_{10}			4.4204	4.4694	4.2975	4.2991	4.1504
h_{11}				4.7359	4.5586	4.5521	4.4033
h_{12}				4.9908	4.7935	4.8017	4.6367
h_{13}					5.0253	5.0380	4.8617
h_{14}					5.2440	5.2558	5.0762
h_{15}						5.4759	5.2897
h_{16}						5.6774	5.4903
h_{17}							5.6778
h_{18}							5.8687

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