International Journal of
Financial Studies

MDPI

Article

Applications of the Shapley Value to Financial Problems

Olamide Ayodele [1,*,†], Sunday Timileyin Ayodeji [2,†] and Kayode Oshinubi [3,*]

[1] Institute of Financial Services Analytics, University of Delaware, Newark, DE 19713, USA
[2] Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA; su652868@ucf.edu
[3] Black in Mathematics Association, Pretoria 0001-0039, South Africa
* Correspondence: oayodele@udel.edu (O.A.); oshinubik@gmail.com (K.O.)
† These authors contributed equally to this work.

Abstract: Managing risk, matching resources efficiently, and ensuring fair allocation are fundamental challenges in both finance and decision-making processes. In many scenarios, participants contribute unequally to collective outcomes, raising the question of how to distribute costs, benefits, or opportunities in a justifiable and optimal manner. This paper applies the Shapley value—a solution concept from cooperative game theory—as a principled tool in the following two specific financial settings: first, in tax cooperation games; and second, in assignment markets. In tax cooperation games, we use the Shapley value to determine the equitable tax burden distribution among three firms, A, B, and C, which operate in two countries, Italy and Poland. Our model ensures that countries participating in coalitions face a lower degree of tax evasion compared to non-members, and that cooperating firms benefit from discounted tax liabilities. This structure incentivizes coalition formation and reveals the economic advantage of joint participation. In assignment markets, we use the Shapley value to find the optimal pairing in a four-buyers and four-sellers housing market. Our findings show that the Shapley value provides a rigorous framework for capturing the relative importance of participants in the coalition, leading to more balanced tax allocations and fairer market transactions. Our theoretical insights with computational techniques highlights the Shapley value's effectiveness in addressing complex allocation challenges across financial management domains.

Keywords: Shapley value; tax cooperation; firms; assignment game

check for updates

Academic Editors: Zied Ftiti and Zaghum Umar

Received: 27 March 2025
Revised: 21 April 2025
Accepted: 6 May 2025
Published: 7 May 2025

Citation: Ayodele, O., Ayodeji, S. T., & Oshinubi, K. (2025). Applications of the Shapley Value to Financial Problems. *International Journal of Financial Studies*, *13*(2), 80. https://doi.org/10.3390/ijfs13020080

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

Game theory, as formulated by Von Neumann and Morgenstern (1947), provides a fundamental framework for analyzing strategic interactions among rational decision makers. Within this broad field, cooperative game theory investigates situations where players can form coalitions to improve their collective outcomes (Branzei et al., 2008; Shapley, 1997; Shapley & Shubik, 1954). An example of such cooperation is the airport game introduced by Littlechild and Owen (1976), where cost-sharing among agents must be resolved fairly. A central challenge in cooperative games is the equitable allocation of gains or costs among participants. The concept of transferable utility (TU) games, introduced in early cooperative game theory research, has provided a formalized approach to analyzing such allocation problems (Branzei et al., 2008).

Among the numerous solution concepts proposed to address fair allocation in cooperative games,[1] the Shapley value, introduced by Lloyd Shapley in 1951, stands out as a pivotal contribution (Shapley & Shubik, 1954). The Shapley value provides an axiomatic method for distributing surplus based on each player's marginal contribution across all possible

coalitions (Shapley, 1997). While it has been extensively studied in theoretical contexts, its applications extend across a range of domains, including economics, political science, network analysis, and cost-sharing problems (Algaba et al., 2019; Almaraz et al., 2025; Béal et al., 2018; Mahdiraji et al., 2021; Ni & Wang, 2007; Saavedra-Nieves & Fiestras-Janeiro, 2022). In this work, we apply the Shapley value to contemporary financial contexts that have received limited attention in the existing literature, namely corporate tax sharing and real estate assignment markets.

In the domain of tax cooperation games, the Shapley value offers a principled framework for allocating the collective gains or obligations among participating agents—namely, firms and governments. In practice, multinational firms often operate across several jurisdictions, each with different tax laws and enforcement capabilities. Without coordinated enforcement, these firms can exploit legal arbitrage, shifting profits to low-tax jurisdictions or evading taxes entirely. Meanwhile, governments face the dilemma of under-enforcement due to budgetary constraints or lack of cross-border authority. This results in substantial revenue losses and the distortion of global investment flows. To address this, countries may choose to form coalitions—sharing enforcement costs and coordinating tax rules—while firms, in return, may receive benefits such as reduced tax liability in exchange for compliance. The question that arises is how to allocate both the costs of enforcement and the tax revenue among coalition members in a way that is both fair and stable. This is precisely the kind of allocation problem that cooperative game theory—and in particular, the Shapley value—is designed to solve.

By modeling the tax cooperation scenario as a transferable utility game, we treat the interaction between firms and governments as a coalition formation problem. Our approach extends traditional applications of the Shapley value by incorporating both enforcement costs and discounted tax liabilities as functions of the coalition structure. This enables us to evaluate how tax obligations can be fairly distributed among firms and governments based on their contributions to the collective effort. In addition to the theoretical formulation, we propose a scalable algorithm capable of computing coalition values and Shapley allocations efficiently, even in settings with a large number of agents. This contributes not only to the literature on cooperative taxation mechanisms (Boadway & Tremblay, 2012; Keen & Konrad, 2013; Martens-Weiner, 2006), but also provides practical tools for policymakers seeking to improve compliance while maintaining fairness in multi-jurisdictional tax systems.

In our second application, we demonstrate how the Shapley value can be applied to assignment markets—particularly buyer–seller markets such as real estate—to determine equitable pricing in competitive two-sided environments. In these markets, each buyer may assign different values to the available goods, and each seller may have a unique reservation price or minimum acceptable offer. The central challenge lies in not only determining which buyers should be matched with which sellers to maximize total surplus, but also how to divide that surplus in a manner that is perceived as fair by all participants. This issue is highly relevant in practical settings such as housing allocation, labor markets, and school admissions, where fairness in pricing and allocation directly affects participation, satisfaction, and long-term system stability.

While traditional assignment models—such as those introduced by Shapley and Shubik (1971)—emphasize allocative efficiency, they often leave the division of surplus unaddressed. The Shapley value offers a compelling solution by allocating the total surplus generated by optimal matchings based on each participant's marginal contribution to various coalitions. This ensures that buyers and sellers are rewarded proportionally to the value they bring to the market, addressing concerns of equity and incentive compatibility. In this work, we further prove the condition under which the Shapley value of the assignment game coincide with the Becker solution (Martínez-de Albéniz et al., 2019). This

reformulation enhances computational tractability and allows the method to be applied in larger, real-world markets while retaining its axiomatic rigor.

Our approach complements and extends the existing research on matching and market design. Roth (2008) underscores the importance of fairness and stability in allocation mechanisms, particularly in school choice and labor assignment systems. Hatfield and Kominers (2012) examines the role of surplus division in networks with bilateral contracts, emphasizing the need for cooperative mechanisms in two-sided markets. Our contribution builds on this foundation by embedding the Shapley value within the framework of assortative assignment problems, as described by Becker (1974), and demonstrating its role as both a theoretical and computational tool. This application, alongside our work on tax cooperation, broadens the scope of Shapley value use in applied economics and highlights its versatility in resolving fairness and efficiency tensions across diverse financial settings.

2. Shapley Value for Tax Cooperation Games

Effective tax cooperation between firms and governments is essential for maintaining economic stability and fair fiscal policies. Without such cooperation, countries often engage in aggressive tax competition, where they continuously lower tax rates to attract businesses. This "race to the bottom" erodes public revenue and can lead to inefficient policy decisions. On the corporate side, a lack of cooperation creates incentives for tax evasion, contributing to the growth of the underground economy, as discussed in Feige (2016); Halkos et al. (2020); Schneider (2018); Schneider and Enste (2000). These challenges highlight the importance of developing a fair and transparent approach to tax burden allocation—one that encourages compliance while maintaining equity.

Several models have been proposed to address the issue of tax cooperation. One framework, introduced in Meca and Varela-Peña (2018), considers governments solely as benefactors, offering tax reductions to firms that comply with regulations while imposing penalties on those that attempt to evade taxes. However, this model does not fully capture the complexity of real-world tax systems, where governments also act as beneficiaries of tax revenues. The corporate tax model presented in Meca et al. (2019) expands on this idea by recognizing the dual role of governments as both benefactors and beneficiaries. In this setting, countries are indispensable to the coalition—meaning that if one country is excluded, the financial burden on the remaining members increases.

To address these complexities, a systematic and equitable method for distributing tax liabilities is necessary. The Shapley value, a fundamental concept in cooperative game theory, provides a robust framework for allocating tax burdens based on each participant's contribution to the coalition. This section applies the Shapley value to a corporate tax game involving two countries and three firms, demonstrating how it ensures a fair distribution of tax responsibilities. Additionally, we propose an algorithm capable of handling tax cooperation games with a larger number of participants, offering a scalable solution for more complex tax-sharing arrangements.

2.1. Definition of Terms in Tax Games

Next, we denote $P = \{1, 2, \ldots, p\}$ as a set of firms and $Q = \{1, 2, \ldots, q\}$ as a set of countries. We denote $N = P \cup Q$. The tax which firm $p \in P$ pays to country $q \in Q$ will be denoted as S_p^q and the reduced tax as $\bar{S}_p^q$. If there are at least two countries in a coalition $T \subseteq N$, we assume that these countries must share information with one another, so as to reduce the degree of tax evasion. By our assumption, we note that, for a country $q \in T \subseteq N$, the more countries that are in coalition T with country q, the more relevant information country q gathers, and this directly implies that the degree of tax evasion and underground economy goes smaller.

Definition 1. *The degree of tax evasion and underground economy when country q is a member of coalition $T \subseteq N$ shall be denoted as d_q^T. A consequence of our compulsory assumption for countries to gather information follows as, for any two coalitions T, U such that $T \subseteq U \subseteq N$, we have that*

$$\begin{aligned} d_q^U < d_q^T \quad if \quad (U \setminus T) \cap Q \neq \emptyset \\ d_q^U = d_q^T \quad if \quad (U \setminus T) \cap Q = \emptyset. \end{aligned} \tag{1}$$

These inequalities imply that the degree of the tax evasion and underground economy becomes smaller as the number of countries who share information together in the coalition increase.

Definition 2. *Meca et al. (2019) define the cost of a firm p in coalition $T \subseteq N$ as follows:*

$$c_p^T = \sum_{q:q \in Q \cap T} \bar{S}_p^q + \sum_{q:q \in Q \setminus (Q \cap T)} S_p^q \quad for\ all \quad p \in T \cap P \tag{2}$$

It is not far-fetched to see the cost of firms in the coalition as the tax obligation due to countries in the coalition. Firm $p \in T$ pays a tax of $\bar{S}_k^q$ if country q is in coalition T and pays S_k^q if country q does not belong to the coalition T. Hence, the total cost of firm $p \in T$ is given in Equation (2).

Before proceeding further, we would like to note what constitute as the cost of a country in coalition.

The cost of a country in a coalition stems from its role in tax enforcement and regulatory oversight within the cooperative framework. This cost primarily includes tax enforcement expenditures, which encompass administrative expenses related to auditing, compliance monitoring, and legal enforcement to mitigate tax evasion. Additionally, information-sharing costs arise as countries invest in technological infrastructure and regulatory frameworks to facilitate cross-border data exchange and coordination. Another significant factor is the opportunity cost of cooperation, where a country may need to adjust its tax policies or enforcement mechanisms to align with coalition agreements, potentially affecting its fiscal autonomy.

Definition 3. *Now, we define the cost of a country Meca et al. (2019) q in coalition $T \subseteq N$ as*

$$c_q^T = f_q(d_q^T) \quad for\ all \quad q \in T \cap Q \tag{3}$$

We note that the function f_q is a strictly increasing function. The concept of this increasing functions is that, the larger the number of countries in coalition T, the smaller the degree of tax evasion. So, for $q, q' \in T \cap Q$, and $T \subseteq N$, we have that

$$z_{qq'} = f_q(d_q^{T \setminus \{q'\}}) - f_q(d_q^T) > 0. \tag{4}$$

Definition 4. *A member $\bar{b} \in T \subseteq N$ is a benefactor if the cost inequality*

$$c_b^{T \setminus \{\bar{b}\}} > c_b^T \tag{5}$$

holds for atleast one of the members $b \in T$. Members in a coalition whose cost decreases due to the presence of a benefactor are called beneficiaries.

Theorem 1. *A player $a \in N$ is a benefactor if it is a country Algaba et al. (2019).*

Proof.
Suppose player a is a country q'. We will study the effect of $q' \in T$ and $q' \notin T$ on the cost of the members of the coalition, i.e., $p, q \in T \subseteq N$

- For all (other) countries $q \in T \cap Q$,

$$c_q^T = f_q(d_q^T) \quad and \quad c_q^{T\setminus\{q'\}} = f_q(d_q^{T\setminus\{q'\}}),$$

where $d_q^{T\setminus\{q'\}} > d_q^T$, since q' is a country in the coalition. Hence,

$$c_q^{T\setminus\{q'\}} = f_q(d_q^{T\setminus\{q'\}}) > f_q(d_q^T) = c_q^T. \tag{6}$$

- For all firms $p \in T \cap P$

$$c_p^T = \sum_{q:q\in Q\cap T} \bar{S}_p^q + \sum_{q:q\in Q\setminus(Q\cap T)} S_p^q \tag{7}$$

and

$$\begin{aligned} c_p^{T\setminus\{q'\}} &= \sum_{q:q\in Q\cap(T\setminus\{q'\})} \bar{S}_p^q + \sum_{q:q\in Q\setminus(Q\cap(T\setminus\{q'\}))} S_p^q \\ &= \sum_{q:q\in(Q\cap T)\setminus\{q'\}} \bar{S}_p^q + \sum_{q:q\in(Q\setminus(Q\cap T))\cup\{q'\}} S_p^q \\ &= \sum_{q:q\in Q\cap T} \bar{S}_p^q - \bar{S}_p^{q'} + \sum_{q:q\in Q\setminus(Q\cap T)} S_p^q + S_p^{q'}. \end{aligned} \tag{8}$$

Since $S_p^{q'} > \bar{S}_p^{q'}$[2], then it indeed follows from Equations (7) and (8) that

$$c_p^{T\setminus\{q'\}} > c_p^T. \tag{9}$$

Suppose player a is a firm p'.

- For all countries $q \in T \cap Q$,

$$c_q^T = f_q(d_q^T) \quad and \quad c_q^{T\setminus\{p'\}} = f_q(d_q^{T\setminus\{p'\}}),$$

where $d_q^{T\setminus\{p'\}} = d_q^T$[3]. Then,

$$c_q^{T\setminus\{p'\}} = f_q(d_q^{T\setminus\{p'\}}) = f_q(d_q^T) = c_q^T. \tag{10}$$

- For all firms $p \in T \cap P$

$$c_p^T = \sum_{q:q\in Q\cap T} \bar{S}_p^q + \sum_{q:q\in Q\setminus(Q\cap T)} S_p^q \tag{11}$$

and

$$\begin{aligned} c_p^{T\setminus\{p'\}} &= \sum_{q:q\in Q\cap(T\setminus\{p'\})} \bar{S}_p^q + \sum_{q:q\in Q\setminus(Q\cap(T\setminus\{p'\}))} S_p^q \\ &= \sum_{q:q\in Q\cap T} \bar{S}_p^q + \sum_{q:q\in Q\setminus(Q\cap T)} S_p^q = c_p^T. \end{aligned} \tag{12}$$

It follows from Equations (11) and (12) that

$$c_p^{T\setminus\{p'\}} = c_p^T. \tag{13}$$

Equation (13) is a contradiction to the definition of a benefactor defined in Equation (5), so p' cannot be a benefactor. However, inequality (9) satisfies the condition for a benefactor defined in (5), hence, a country $q' \in T \subseteq N$ is a benefactor. □

2.2. Corporation Tax Games

We begin with the definition of a *cost coalitional vector* as the pair (N, c^N), where $c^N = (c_k^T)_{k\in T, \emptyset \neq T \subseteq N}$ represents the vector of individual agent k costs in all possible coalitions T. We next prove that the cost coalitional vector c^N is *cost monotonic*, i.e., $c_k^S \geq c_k^T$ for all $k \in T$ and $S \subseteq T \subseteq N$.

Theorem 2. *The cost coalitional problem (N, c^N) satisfies cost monotonicity Meca et al. (2019).*

Proof. Let S, T be any two sets such that $S \subseteq T \subseteq N$. A member of S can either be a country or a firm.
Suppose country $q \in S \cap Q$, then

$$c_q^T = f_q(d_q^T) \quad \text{and} \qquad c_q^S = f_q(d_q^S).$$

Since $d_q^S \geq d_q^T$ and f_q is a strictly increasing function, then

$$c_q^S = f_q(d_q^S) > f_q(d_q^T) = c_q^T.$$

On the other hand, suppose firm $p \in S \cap P$, then

$$c_p^T = \sum_{q:q\in Q\cap T} \bar{S}_p^q + \sum_{q:q\in Q\setminus(Q\cap T)} S_p^q = \sum_{q:q\in Q\cap S} \bar{S}_p^q + \sum_{q:q\in Q\cap(T\setminus S)} \bar{S}_p^q + \sum_{q:q\in Q\setminus(Q\cap T)} S_p^q,$$
$$c_p^S = \sum_{q:q\in Q\cap S} \bar{S}_p^q + \sum_{q:q\in Q\setminus(Q\cap S)} S_p^q = \sum_{q:q\in Q\cap S} \bar{S}_p^q + \sum_{q:q\in Q\cap(T\setminus S)} S_p^q + \sum_{q:q\in Q\setminus(Q\cap T)} S_p^q.$$

If $T \setminus S$ contains atleast one country, then

$$c_p^S > c_p^T, \quad \text{since} \quad S_p^q > \bar{S}_p^q.$$

Otherwise,

$$c_p^S = c_p^T,$$

since

$$\sum_{q:q\in Q\cap(T\setminus S)} S_p^q = \sum_{q:q\in Q\cap(T\setminus S)} \bar{S}_p^q.$$

In any case,

$$c_p^S \geq c_p^T.$$

□

Definition 5. *We define the multiple corporation tax game Meca et al. (2019) as a pair (N, v), where*

$$v(T) = \sum_{p\in T} c_p^T \quad \textit{for all} \quad T \subseteq N, \quad \textit{and} \quad v(\emptyset) = 0. \tag{14}$$

Note: $p \in T$ implies p can either be a firm p or a country q[4].

We now take a moment to differentiate between a replaceable and irreplaceable benefactor in a coalition.

Definition 6. *Let $\bar{b}$ be a single benefactor in a coalition T. If $\bar{b}$ leaves the coalition and all other members in T see their cost increase, then $\bar{b}$ is called an irreplaceable benefactor. On the other hand, if there are several benefactors in a coalition T, and the cost of members of the coalition remains the same despite benefactor $\bar{b}$ leaving the coalition, then we say that each benefactor is replaceable.*

Theorem 3. *Let (N, c^N) be a cost coalitional problem with multiple agents and irreplaceable benefactors and (N, v) be the associated multiple corporation tax game. Then,*

i. For all $T \subseteq N$ and firm $p' \in T \cap P$,

$$v(T) - v(T \setminus \{p'\}) = c^T_{p'}$$

ii. For all $T \subseteq N$ and country $q \in T \cap Q$,

$$v(T) - v(T \setminus \{q\}) = c^T_q - \sum_{p:p\in T\cap P} (S^q_k - \bar{S}^q_k) - \sum_{p':p'\in Q\cap(T\setminus\{q\})} \left[f_{p'}(d^{T\setminus\{q\}}_{p'}) - f_{p'}(d^T_{p'}) \right].$$

Proof.

i. For any coalition $T \subseteq N$ and firm $p' \in T \cap P$, we have

$$\begin{aligned} v(T) - v(T \setminus \{p'\}) &= \sum_{p:p\in T} c^T_p - \sum_{p:p\in T\setminus\{p'\}} c^{T\setminus\{p'\}}_p \\ &= c^T_{p'} + \sum_{p:p\in T\setminus\{p'\}} \left[c^T_p - c^{T\setminus\{p'\}}_p \right]. \end{aligned} \tag{15}$$

It is now left to show that $\sum_{p:p\in T\setminus\{p'\}} \left[c^T_p - c^{T\setminus\{p'\}}_p \right] = 0$. Let us first notice that[5]

$$\sum_{p:p\in T\setminus\{p'\}} \left[c^T_p - c^{T\setminus\{p'\}}_p \right] = \sum_{q:q\in Q\cap(T\setminus\{p'\})} \left[c^T_q - c^{T\setminus\{p'\}}_q \right] + \sum_{p:p\in P\cap(T\setminus\{p'\})} \left[c^T_p - c^{T\setminus\{p'\}}_p \right]. \tag{16}$$

Having separated the p that could either be a country or firm in distinct p and q, then we can proceed with no ambiguity. Notice that p' is a firm, and it follows from (1) that

$$d^T_q = d^{T\setminus\{p'\}}_q \quad \text{for all} \quad q \in Q \cap (T \setminus \{p'\}).$$

Hence,

$$c^T_q - c^{T\setminus\{p'\}}_q = 0. \tag{17}$$

So,

$$\begin{aligned} \sum_{p:p\in T\setminus\{p'\}} \left[c^T_p - c^{T\setminus\{p'\}}_p \right] &= \sum_{p:p\in P\cap(T\setminus\{p'\})} \left[c^T_p - c^{T\setminus\{p'\}}_p \right] \\ &= \sum_{p:p\in P\cap(T\setminus\{p'\})} \left[\sum_{q:q\in T\cap Q} \bar{S}^q_p + \sum_{q:q\in Q\setminus(T\cap Q)} S^q_p \right] \\ &- \sum_{p:p\in P\cap(T\setminus\{p'\})} \left[\sum_{q:q\in(T\setminus\{p'\})\cap Q} \bar{S}^q_p + \sum_{q:q\in Q\setminus((T\setminus\{p'\})\cap Q)} S^q_p \right]. \end{aligned}$$

Since p' is a firm, then

$$\sum_{q:q\in T\cap Q} \bar{S}^q_p = \sum_{q:q\in(T\setminus\{p'\})\cap Q} \bar{S}^q_p$$

and

$$\sum_{q:q\in Q\setminus(T\cap Q)} S^q_p = \sum_{q:q\in Q\setminus((T\setminus\{p'\})\cap Q)} \bar{S}^q_p.$$

Therefore,

$$\sum_{p:p\in T\setminus\{p'\}}\left[c_k^T - c_k^{T\setminus\{p'\}}\right] = 0. \tag{18}$$

Hence, from (15)–(18), we have that

$$v(T) - v(T\setminus\{p'\}) = c_{p'}^T.$$

ii. For any $T \subseteq N$ and country $q \in T \cap Q$,

$$\begin{aligned} v(T) - v(T\setminus\{q\}) &= \sum_{p:p\in T} c_p^T - \sum_{p:p\in T\setminus\{q\}} c_p^{T\setminus\{q\}} = c_q^T - \sum_{p:p\in T\setminus\{q\}}\left[c_p^{T\setminus\{q\}} - c_p^T\right] \\ &= c_q^T - \sum_{p':p'\in Q\cap(T\setminus\{q\})}\left[c_{q'}^{T\setminus\{q\}} - c_{q'}^T\right] - \sum_{p:p\in P\cap(T\setminus\{q\})}\left[c_q^{T\setminus\{q\}} - c_q^T\right]. \end{aligned} \tag{19}$$

For $q' \in Q \cap (T \setminus \{q\})$,

$$c_{q'}^{T\setminus\{q\}} - c_{q'}^T = f_{q'}(d_{q'}^{T\setminus\{q\}}) - f_{q'}(d_{q'}^T). \tag{20}$$

Moreover, for every $p \in P \cap (T \setminus \{q\})$,

$$\begin{aligned} c_p^{T\setminus\{q\}} - c_p^T &= \Big[\sum_{q':q'\in Q\cap(T\setminus\{q\})} \bar{S}_p^{q'} + \sum_{q':q'\in Q\setminus(Q\cap(T\setminus\{q\}))} S_p^{q'}\Big] - \Big[\sum_{q'\in Q\cap T} \bar{S}_p^{q'} + \sum_{q':q'\in Q\setminus(Q\cap T))} S_p^{q'}\Big] \\ &= \Big[\sum_{q':q'\in Q\cap(T\setminus\{q\})} \bar{S}_p^{q'} + S_p^q + \sum_{q':q'\in Q\setminus(Q\cap T)} S_j^{q'}\Big] - \Big[\sum_{q':q'\in Q\cap(T\setminus\{q\})} \bar{S}_p^{q'} + \bar{S}_p^q + \\ &\qquad \sum_{q':q'\in Q\setminus(Q\cap T)} S_p^{q'}\Big]. \end{aligned}$$

Hence,

$$c_p^{T\setminus\{q\}} - c_p^T = S_p^q - \bar{S}_p^q. \tag{21}$$

Substituting (20) and (21) into Equation (19), we have,

$$\begin{aligned} v(T) - v(T\setminus\{q\}) &= c_q^T - \sum_{q':q'\in Q\cap(T\setminus\{q\})}\left[f_{q'}(d_{q'}^{T\setminus\{q\}}) - f_{q'}(d_{q'}^T)\right] - \sum_{p:p\in P\cap(T\setminus\{q\})}\left[S_p^q - \bar{S}_p^q\right] \\ &= c_q^T - \sum_{p':p'\in Q\cap(T\setminus\{q\})}\left[f_{p'}(d_{p'}^{T\setminus\{q\}}) - f_{p'}(d_{p'}^T)\right] - \sum_{p:p\in P\cap T}\left[S_p^q - \bar{S}_p^q\right]. \end{aligned}$$

□

2.3. The Shapley Value for Multiple Corporation Tax Games

We begin with the definition of the Shapley value.

Definition 7. *The Shapley value $\phi_i(v)$, for a player $i \in N$ in a TU game (N, v) is defined as the average of the marginal contribution of the game. That is,*

$$\phi_i(v) = \frac{1}{|N|!}\sum_{\pi\in\prod} v(\mathbb{P}(\pi)\cup\{i\}) - v(\mathbb{P}(\pi)). \tag{22}$$

where $\prod$ is the set of all permutations of $\{1, 2, \ldots, n\}$ and $\mathbb{P}(\pi)$ is the set of players in permutation π who have joined the coalition before player i. For readability, let $S = P(\pi)$. Coalition S can be

rearranged in $|S|!$ ways, and the number of orderings of $N \setminus (S \cup \{i\})$ is $(|N| - (|S|+1))! = (|N| - |S| - 1)!$. Hence, Equation (22) can also be seen as follows:

$$\phi_i(v) = \sum_{S: i \notin S \subseteq N} \frac{|S|!(|N| - (|S|+1))!}{|N|!} \Big(v(S \cup \{i\}) - v(S) \Big) \tag{23}$$

Lemma 1. *Let N be a finite set with $|N| = n$, and fix an element $i \in N$. The fraction in the Shapley value defined in Equation (23) sums up to 1, i.e.,*

$$\sum_{S: i \notin S \subseteq N} \frac{|S|!(n - |S| - 1)!}{n!} = 1. \tag{24}$$

Proof. The sum is taken over all subsets S of N that do not contain i. Let us denote $T = N \setminus \{i\}$, so $|T| = n - 1$. Then, for each $k = 0, 1, \ldots, n-1$, there are $\binom{n-1}{k}$ subsets of T of size k. We can group the summation according to the cardinality of S as follows:

$$\begin{aligned} \sum_{S: i \notin S \subseteq N} \frac{|S|!(n - |S| - 1)!}{n!} &= \sum_{k=0}^{n-1} \sum_{\substack{S \subseteq T \\ |S| = k}} \frac{k!(n-k-1)!}{n!} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot \frac{k!(n-k-1)!}{n!}. \end{aligned}$$

Now simplify the binomial coefficient with the following factorial terms:

$$\binom{n-1}{k} \cdot \frac{k!(n-k-1)!}{n!} = \frac{(n-1)!}{k!(n-1-k)!} \cdot \frac{k!(n-k-1)!}{n!} = \frac{(n-1)!}{n!}.$$

Note that this value is independent of k, so we can factor it out of the summation as follows:

$$\begin{aligned} \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot \frac{k!(n-k-1)!}{n!} &= \sum_{k=0}^{n-1} \frac{(n-1)!}{n!} \\ &= \frac{(n-1)!}{n!} \cdot \sum_{k=0}^{n-1} 1 \\ &= \frac{(n-1)!}{n!} \cdot n = 1. \end{aligned}$$

The intuition of this fraction summing to 1 is the weight given to each subset $S \subseteq N \setminus \{i\}$ when computing the marginal contribution of player i in the Shapley value formula. Specifically, it is the probability that player i joins a coalition at the exact moment after the members of S have joined □

Theorem 4. *For any multiple corporation tax game (N, v),*

i. The Shapley value for firms is,

$$\phi_{p'}(v) = c^N_{p'} + \frac{1}{2} \sum_{q: q \in Q} (S^q_{p'} - \bar{S}^q_{p'}) \quad \textit{for all} \quad p' \in P.$$

ii. The Shapley value for countries is,

$$\phi_q(v) = c^N_q - \frac{1}{2} \sum_{p: p \in P} (S^q_p - \bar{S}^q_p) + \frac{1}{2} \sum_{q': q' \in Q \setminus \{q\}} (z_{qq'} - z_{q'q}) \quad \textit{for all} \quad q \in Q.$$

We remind the reader that $z_{qq'}$ is already defined in Equation (4).

Proof.

i. Let us fix $p' \in P$. It follows from the Shapley value defined in (7) that

$$\phi_{p'}(v) = \sum_{T:p' \in T \subseteq N} \frac{(|T|-1)!(|N|-|T|)!}{|N|!} \Big[v(T) - v(T \setminus \{p'\}) \Big].$$

Let

$$\gamma(|T|) = \frac{(|T|-1)!(|N|-|T|)!}{|N|!} \tag{25}$$

We can split the coalitions into two disjoint subsets, such that one contains only firms, and the other contains mixed agents (at least one firm and one country) as follows:

i. $p' \in T \subseteq N$, $T \cap P \neq \emptyset$ and $T \cap Q = \emptyset$.
ii. $p' \in T \subseteq N$, $T \cap P \neq \emptyset$ and $T \cap Q \neq \emptyset$.

We calculate

$$\phi_{p'}(v) = \sum_{\substack{T:p' \in T \subseteq N \\ T \cap P \neq \emptyset, T \cap Q = \emptyset}} \gamma(|T|) \Big[v(T) - v(T \setminus \{p'\}) \Big] + \sum_{\substack{T:p' \in T \subseteq N \\ T \cap P \neq \emptyset, T \cap Q \neq \emptyset}} \gamma(|T|) \Big[v(T) - v(T \setminus \{p'\}) \Big]$$

From Theorem 3, we obtain the equation below, and Equation (2) subsequently applies in the next line of the equation

$$= \sum_{\substack{T:p' \in T \subseteq N \\ T \cap P \neq \emptyset, T \cap Q = \emptyset}} \gamma(|T|) \cdot c^T_{p'} + \sum_{\substack{T:p' \in T \subseteq N \\ T \cap P \neq \emptyset, T \cap Q \neq \emptyset}} \gamma(|T|) \cdot c^T_{p'}$$

$$= \sum_{\substack{T:p' \in T \subseteq N \\ T \cap P \neq \emptyset, T \cap Q = \emptyset}} \gamma(|T|) \Big[\sum_{q:q \in Q} S^q_{p'} \Big] + \sum_{\substack{T:p' \in T \subseteq N \\ T \cap P \neq \emptyset, T \cap Q \neq \emptyset}} \gamma(|T|) \Big[\sum_{q:q \in Q \cap T} \bar{S}^q_{p'} + \sum_{q:q \in Q \setminus (Q \cap T)} S^q_{p'} \Big]$$

From Lemma 1

$$\sum_{T:T \subseteq N, p' \in T} \gamma(|T|) = 1.$$

Hence,

$$\sum_{\substack{T:p' \in T \subseteq N \\ T \cap P \neq \emptyset, T \cap Q = \emptyset}} \gamma(|T|) + \sum_{\substack{T:p' \in T \subseteq N \\ T \cap P \neq \emptyset, T \cap Q \neq \emptyset}} \gamma(|T|) = 1$$

$$\sum_{\substack{T:p' \in T \subseteq N \\ T \cap P \neq \emptyset, T \cap Q = \emptyset}} \gamma(|T|) = 1 - \sum_{\substack{T:p' \in T \subseteq N \\ T \cap P \neq \emptyset, T \cap Q \neq \emptyset}} \gamma(|T|)$$

Then,

$$\begin{aligned}\phi_{p'}(v) &= \Big[\sum_{Q:q\in Q} S^q_{p'}\Big]\cdot\Big[1-\sum_{\substack{T:p'\in T\subseteq N\\ T\cap P\neq\emptyset, T\cap Q\neq\emptyset}}\gamma(|T|)\Big]+\\ &\sum_{\substack{T:p'\in T\subseteq N\\ T\cap P\neq\emptyset, T\cap Q\neq\emptyset}}\gamma(|T|)\Big[\sum_{q:q\in Q\cap T}\bar{S}^q_{p'}+\sum_{q:q\in Q\setminus(Q\cap T)}S^q_{p'}\Big]\\ &= \sum_{q:q\in Q} S^q_{p'}-\sum_{\substack{T:p'\in T\subseteq N\\ T\cap P\neq\emptyset, T\cap Q\neq\emptyset}}\gamma(|T|)\Big[\sum_{q:q\in Q\cap T}S^q_{p'}-\sum_{q:q\in Q\cap T}\bar{S}^q_{p'}\Big]\\ &= \sum_{q:q\in Q} S^q_{p'}-\sum_{q:q\in Q}\Big[(S^q_{p'}-\bar{S}^q_{p'})\cdot\sum_{T:q,p'\in T\subseteq N}\gamma(|T|)\Big]. \end{aligned} \tag{26}$$

We now show that $\sum_{\substack{T\subseteq N\\ q,p'\in T\\ T\cap P\neq\emptyset\\ T\cap Q\neq\emptyset}}\gamma(T)=\frac{1}{2},\quad$ for every $q\in Q$.

Note: First, the given sum $\sum_{\substack{T\subseteq N\\ q,p'\in T\\ T\cap P\neq\emptyset\\ T\cap Q\neq\emptyset}}$ counts the number of subsets T of N that satisfy the following:

i. $q, p'\in T$ (i.e., T must always include these two elements).
ii. $T\cap P\neq\emptyset$ and $T\cap Q\neq\emptyset$, which is always true as $q\in Q$ and $p'\in P$.

Since T must contain q and p', choosing T reduces to selecting the remaining $t-2$ elements from $N\setminus\{q,p'\}$, which has $|N|-2$ elements.
Thus, the number of such subsets of size t is as follows:

$$\binom{|N|-2}{t-2}$$

Summing over all possible sizes t from 2 to $|N|$, we obtain the following:

$$\sum_{\substack{T:q,p'\in T\subseteq N\\ T\cap P\neq\emptyset, T\cap Q\neq\emptyset}}\gamma(|T|)=\sum_{t=2}^{|N|}\binom{|N|-2}{t-2}\gamma(t)$$

Substituting Equation (25), we have

$$\begin{aligned}&=\sum_{t=2}^{|N|}\frac{t-1}{|N|(|N|-1)}\\ &=\frac{1}{|N|(|N|-1)}\sum_{t=1}^{|N|-1}t.=\frac{1}{|N|(|N|-1)}\frac{(|N|-1)|N|}{2}\end{aligned} \tag{27}$$

Hence,

$$\sum_{\substack{T:q,p'\in T\subseteq N\\ T\cap P\neq\emptyset, T\cap Q\neq\emptyset}}\gamma(|T|)=\frac{1}{2}. \tag{28}$$

Then,

$$\phi_{p'}(v) = \sum_{q:q\in Q} S^q_{p'} - \frac{1}{2}\Big[\sum_{q:q\in Q\cap T} S^q_{p'} - \sum_{q:q\in Q\cap T} \bar{S}^q_{p'}\Big] = \sum_{q:q\in Q} S^q_{p'} - \frac{1}{2}\sum_{q:q\in Q} S^q_{p'} + \frac{1}{2}\sum_{q:q\in Q} \bar{S}^q_{p'}$$
$$= \frac{1}{2}\Big[\sum_{q:q\in Q} \bar{S}^q_{p'} + \sum_{q:q\in Q} S^q_{p'}\Big].$$

We reformulate the equation above using Equation (2) to express the following equation as the sum of the first term and a sign change inside the bracket,

$$\phi_{p'}(v) = c^N_{p'} + \frac{1}{2}\sum_{q:q\in Q}\Big[S^q_{p'} - \bar{S}^q_{p'}\Big]. \tag{29}$$

ii. For all countries $q \in Q$, it follows from the Shapley formula that

$$\phi_q(v) = \sum_{T:q\in T\subseteq N} \frac{(|T|-1)!(|N|-|T|)!}{|N|!}\Big[v(T) - v(T\setminus\{q\})\Big].$$

From Theorem 3, we have

$$\phi_q(v) = \sum_{T:q\in T\subseteq N} \gamma(|T|)\Big[c^T_q - \sum_{p':p'\in T\cap P}(S^q_{p'} - \bar{S}^q_{p'}) - \sum_{q':q'\in Q\cap(T\setminus\{q\})}\Big(f_{q'}(d^{T\setminus\{q\}}_{q'}) - f_{q'}(d^T_{q'})\Big)\Big]. \tag{30}$$

Solving each of the addends in Formula (30), we first take into account that, for all $T \subseteq N$, we have

$$c^T_q = c_q - \sum_{q':q'\in Q\cap(T\setminus\{q\})} z_{qq'}.$$

Then,

$$\sum_{T:q\in T\subseteq N}\gamma(|T|)c^T_q = c_q - \sum_{T:q\in T\subseteq N}\gamma(|T|)\sum_{q:q\in Q\cap(T\setminus\{q\})} z_{qq'}$$
$$= c^N_q + \frac{1}{2}\sum_{q:q\in Q\cap(T\setminus\{q\})} z_{qq'}.$$

Secondly,

$$\sum_{T:q\in T\subseteq N}\gamma(|T|)\sum_{p':p'\in T\cap P}(S^q_{p'} - \bar{S}^q_{p'}) = \sum_{T:q\in T\subseteq N}\gamma(|T|)\sum_{p'\in P}(S^q_{p'} - \bar{S}^q_{p'})$$
$$= \frac{1}{2}\sum_{p'\in P}(S^q_{p'} - \bar{S}^q_{p'}).$$

Lastly,

$$\sum_{T:q\in T\subseteq N}\gamma(|T|)\sum_{q':q'\in Q\cap(T\setminus\{q\})}\Big(f_{q'}(d^{T\setminus\{q\}}_{q'}) - f_{q'}(d^T_{q'})\Big) =$$
$$\sum_{q:q\in T\subseteq N}\gamma(|T|)\sum_{q'\in Q\setminus\{q\}}\Big(f_{q'}(d^{T\setminus\{q\}}_{q'}) - f_{q'}(d^T_{q'})\Big)$$

From Equation (4), we have the following:

$$= \frac{1}{2}\sum_{q':q'\in Q\setminus\{q\}} z_{q'q}.$$

Collectively, we obtain

$$\phi_q(v) = c_q^N - \frac{1}{2}\sum_{p:p\in P}(S_p^q - \bar{S}_p^q) + \frac{1}{2}\sum_{q':q'\in Q\setminus\{q\}}(z_{qq'} - z_{q'q}) \quad \text{for all} \quad q \in Q. \tag{31}$$

□

Remarks[6]:

- The cost of a firm $p' \in P$ in grand coalition N is $c_{p'}^N$. Firm p' is a beneficiary in coalition N, and it has benefited from country $q \in Q$ an amount which is $S_{p'}^q - \bar{S}_p^q$. The Shapley value increases the cost of firm p' by half of the amount it benefited from country q.
- The cost of country $q \in Q$ is *reduced* by half of $S_{p'}^q - \bar{S}_p^q$. This reduction explains why the Shapley value always compensates the benefactors.
- Suppose $q, q' \in Q$ are countries in the same coalitions. If q is a benefactor and q' is a beneficiary, then the Shapley value compensates the benefactor by reducing the cost of country q in coalition N by $\frac{1}{2}z_{ll'}$.

2.4. Application on European Corporate Income Tax

European countries require businesses to pay corporate income taxes on their profits. The amount of tax a business pays depend on the tax base and the corporate tax rate. As of 2022, Portugal has the highest corporate income tax rate at 31.5%, Germany at 29.9%, and France at 28.4%.

The application of this section focuses on a model of three firms A, B, and C who operate in two countries, Poland and Italy. Table 1 shows the profit in million EUR for each firm in each of the countries.

Table 1. Annual profit of firms operating in Poland and Italy (in million EUR). The table reports the pre-tax profits earned by each firm in the respective countries. These figures serve as the basis for computing tax obligations under both independent and cooperative tax schemes in our tax cooperation game framework.

Firms	Profit in Poland	Profit in Italy
A	330	550
B	205	310
C	633	405

These firms have to pay tax equivalent to 19% of their profit to Poland and 27.81% to Italy. If any of the firms choose to cooperate with any of the countries, they will be rewarded with a tax reduction of 10% for Poland, and 12% for Italy. As a result, we obtain the tax obligation (Table 2) as follows:

Table 2. Profits and tax liabilities of firms operating in Poland and Italy. The table reports each firm's pre-tax profits in both countries, the corresponding tax liabilities based on national tax rates, and the reduced tax obligations under a cooperative tax agreement. All monetary values are expressed in million EUR. The reduced tax reflects the effect of discount rates applied to incentivize firm participation in tax cooperation schemes.

Firms	Profit in Poland	Profit in Italy	Tax Due in Poland	Tax Due in Italy	Reduced Tax in Poland	Reduced Tax in Italy
A	330	550	62.70	152.96	56.43	134.60
B	205	310	38.95	86.21	35.06	75.86
C	633	405	120.27	112.63	108.24	99.11

To model the cost structure of countries participating in tax cooperation, we assume that each country q incurs a cost composed of two components as follows: a fixed cost

and a variable component that depends on coalition membership. The fixed cost k_1 represents baseline administrative expenses associated with tax enforcement and regulatory compliance, such as domestic auditing infrastructure or staffing. These costs are incurred regardless of whether a country cooperates with others. The second component, $k_2 d_q^T$, captures the marginal cost or savings derived from being part of a coalition. Specifically, d_q^T represents a measure of the enforcement intensity or information-sharing efficiency achieved through cooperation in coalition T, and k_2 scales its impact on the country's total cost. For any country $q \in T \cap Q \subseteq N$, the cost function is given by the following:

$$c_q^T = k_1 + k_2 d_q^T, \quad \text{where} \quad k_1, k_2 \in \mathbb{R}. \tag{32}$$

where d_q^T captures the coalition-dependent factor and is defined as follows:

$$d_q^T = \begin{cases} 1, & \text{if } (T \cap Q) \setminus \{q\} = \emptyset, \\ a, & \text{otherwise.} \end{cases} \tag{33}$$

The cost function presented in Equation (32) reflects the practical structure of tax enforcement costs faced by countries participating in cooperative arrangements. The first term, k_1, represents fixed overhead costs associated with tax enforcement, such as administrative infrastructure, personnel, and domestic audit mechanisms, which are incurred regardless of coalition membership. The second term, $k_2 d_q^T$, introduces a variable component that depends on the composition of the coalition to which the country belongs. This formulation captures the realistic intuition that enforcement costs can be significantly reduced when countries cooperate through mechanisms such as joint audits, shared information systems, and treaty-based data exchanges. In particular, if a country acts alone, it bears the full burden of enforcement. However, when it participates in a coalition, it benefits from collective efforts and shared resources, thereby reducing its effective cost. This assumption aligns with empirical evidence on multilateral tax enforcement effectiveness, as observed in frameworks like the OECD's Base Erosion and Profit Shifting (BEPS) initiative. The discount effect from cooperation is captured by a parameter a, where $0 < a < 1$, which serves to scale down the cost contribution d_q^T in cooperative settings. Thus, our model captures the trade-off countries face between unilateral enforcement and joining coalitions that enhance compliance and lower overall enforcement costs.

For the purpose of our work, we assign specific cost functions to Poland and Italy, reflecting their respective roles in tax cooperation as follows:

$$\begin{aligned} c_{Poland}^T &= 120 + 65 \cdot d_{Poland}^T, \\ c_{Italy}^T &= 210 + 112 \cdot d_{Italy}^T. \end{aligned} \tag{34}$$

Here, the values of a differ between countries to reflect variations in institutional capacity, tax administration efficiency, and the degree to which cooperation reduces enforcement costs. Specifically, we set $a = 0.8$ for Poland and $a = 0.6$ for Italy, indicating that Italy experiences a greater relative reduction in costs when forming coalitions compared to Poland.

This assumption aligns with empirical observations that countries with stronger enforcement mechanisms benefit more from cooperation by reducing redundancy and leveraging shared information. By incorporating these structured cost functions, our model captures the strategic incentives for countries to participate in tax cooperation, making it possible to analyze how coalitional structures influence overall cost allocations and strategic behavior in international tax enforcement.

The pseudocode that implements the theoretical framework of the Shapley value for the tax problem is presented in Algorithm 1.[7] This finds the possible coalitions, cost of each

members in the coalition, and the Shapley value of each member in the grand coalition. Using Algorithm 1, we obtain the cost table (Table 3) for each participant in each of the possible coalitions and the Shapley value of each participants (Table 4).

Algorithm 1 Tax cooperation algorithm

Require: Tax rate of countries R_q; Discount rate of countries to firms for cooperation, $\bar{R}_q$, Firms profit in each country, $Profit_{p,q}$, country cost function c_q^T.

Step 1: Compute Tax Due For Each Firm in Each Country
for each firm profit in each country, $Profit_{p,q}$ **do**
  for each country's tax rate, R_q **do**
    Compute $Tax_Due_{p,q} = \frac{R_q}{100} \times Profit[p,q]$
  end for
end for
Step 2: Compute Reduced Tax
for each firm p in coalition with countries q **do**
  for each country's reduced tax rate, $\bar{R}_q$ **do**
    Compute $Reduced_Tax_{p,q} = Tax_Due_{p,q} - \left(\frac{\bar{R}_q}{100} \times Tax_Due_{p,q}\right)$
  end for
end for
Step 3: Generate the power set $2^{|P \cup Q|} \setminus \{\emptyset\}$, which denote possible coalitions of firms and countries
Step 4: Compute Cost of a Firm in a Coalition
for each firm p **do**
  $\bar{S}_p^q = 0$, $S_p^q = 0$
  for each coalition C **do**
    if country q is in C **then**
      Compute reduced tax obligations of firm p to country q based on coalition membership : $\bar{S}_p^q$
    else if country q is not in C **then**
      Compute tax obligations of firm p to country q based on coalition membership : S_p^q
    end if
    Compute $C_p^T = \bar{S}_p^q + S_p^q$
  end for
end for
Step 5: Compute Cost of a Country in a Coalition
for each country q in Countries **do**
  for each coalition C **do**
    if q is in or not in C **then**
      Compute country cost, C_q^T using the cost function predefined (Equation (34)).
    end if
  end for
end for
Step 6: Compute Coalition Worth
for each coalition C **do**
  Compute total coalition value by summing the cost of members in the coalition
end for
Step 7: Compute Shapley Value for Firms
for each firm p in the grand coalition **do**
  Compute Shapley value using Equation (29)
end for
Step 8: Compute Shapley Value for Countries
for each country c **do**
  Compute Shapley value using Equation (31)
end for
Output: Coalition Cost Table, Shapley Values for members in the grand coalition.

Table 3. Coalition cost allocations for all possible coalitions of firms (A, B, and C) and countries (Poland and Italy). Each row represents a specific coalition, and each column reports the cost borne by a participant under that coalition structure. Black entries denote the cost incurred by a participant who is a member of the coalition, while red entries indicate the cost of participants not in the coalition. This color-coding highlights the incentive for players to join the coalition, as participation results in lower individual costs due to cooperative enforcement benefits or tax discounts. The final column reports the total worth (i.e., aggregate cost) of participants and non-participants of each coalition. This table serves as the foundation for the subsequent Shapley value computations and illustrates how coalition composition affects both individual and collective cost outcomes.

Coalition	A	B	C	Poland	Italy	Total Cost
A	215.66	125.16	232.90	185.00	322.00	1080.72
B	215.66	125.16	232.90	185.00	322.00	1080.72
C	215.66	125.16	232.90	185.00	322.00	1080.72
Poland	215.66	125.16	232.90	185.00	322.00	1080.72
Italy	215.66	125.16	232.90	185.00	322.00	1080.72
A, B	215.66	125.16	232.90	185.00	322.00	1080.72
A, C	215.66	125.16	232.90	185.00	322.00	1080.72
A, Poland	209.39	125.16	232.90	185.00	322.00	1074.45
A, Italy	197.30	125.16	232.90	185.00	322.00	1062.36
B, C	215.66	125.16	232.90	185.00	322.00	1080.72
B, Poland	215.66	121.27	232.90	185.00	322.00	1076.83
B, Italy	215.66	114.81	232.90	185.00	322.00	1070.37
C, Poland	215.66	125.16	220.87	185.00	322.00	1068.69
C, Italy	215.66	125.16	219.38	185.00	322.00	1067.20
Italy, Poland	215.66	125.16	232.90	172.00	277.20	1022.92
A, B, C	215.66	125.16	232.90	185.00	322.00	1080.72
A, B, Poland	209.39	121.27	232.90	185.00	322.00	1070.56
A, B, Italy	197.30	114.81	232.90	185.00	322.00	1052.01
A, C, Poland	209.39	125.16	220.87	185.00	322.00	1062.42
A, C, Italy	197.30	125.16	219.38	185.00	322.00	1048.84
A, Italy, Poland	191.03	125.16	232.90	172.00	277.20	998.29
B, C, Poland	215.66	121.27	220.87	185.00	322.00	1064.80
B, C, Italy	215.66	114.81	219.38	185.00	322.00	1056.85
B, Italy, Poland	215.66	110.92	232.90	172.00	277.20	1008.68
C, Italy, Poland	215.66	125.16	207.35	172.00	277.20	997.37
A, B, C, Poland	209.39	121.27	220.87	185.00	322.00	1058.53
A, B, C, Italy	197.30	114.81	219.38	185.00	322.00	1038.49
A, B, Italy, Poland	191.03	110.92	232.90	172.00	277.20	984.05
A, C, Italy, Poland	191.03	125.16	207.35	172.00	277.20	972.74
B, C, Italy, Poland	215.66	110.92	207.35	172.00	277.20	983.13
A, B, C, Italy, Poland	191.03	110.92	207.35	172.00	277.20	958.50

Table 4. Shapley value allocations for all possible coalition structures among firms (A, B, and C) and countries (Poland and Italy). For each coalition, the table redistributes the total cost of both participating and non-participating members according to their marginal contributions using the Shapley value. Red-colored entries indicate agents who are not part of the coalition but are included for comparative evaluation. Bold asterisk numbers indicate the optimal redistributed cost for each member. The final column, Total, represents the sum of the Shapley values in that row, providing a check on the cost conservation to the total coalition cost in Table 3.

Coalition	A	B	C	Poland	Italy	Total
A	215.66	125.16	232.90	185.00	322.00	1080.72
B	215.66	125.16	232.90	185.00	322.00	1080.72
C	215.66	125.16	232.90	185.00	322.00	1080.72

Table 4. *Cont.*

Coalition	A	B	C	Poland	Italy	Total
Poland	215.66	125.16	232.90	185.00	322.00	1080.72
Italy	215.66	125.16	232.90	185.00	322.00	1080.72
A, B	215.68	125.14	232.90	185.00	322.00	1080.72
A, C	215.90	125.16	232.66	185.00	322.00	1080.72
A, Poland	214.99	125.16	232.90	179.40	322.00	1074.45
A, Italy	204.76	125.16	232.90	185.00	314.54	1062.36
B, C	215.66	125.32	232.74	185.00	322.00	1080.72
B, Poland	215.66	125.47	232.90	180.80	322.00	1076.83
B, Italy	215.66	119.97	232.90	185.00	316.84	1070.37
C, Poland	215.66	125.16	226.51	179.36	322.00	1068.69
C, Italy	215.66	125.16	226.81	185.00	314.57	1067.20
Italy, Poland	215.66	125.16	232.90	156.20	293.00	1022.92
A, B, C	215.86	125.25	232.62	185.00	322.00	1080.72
A, B, Poland	213.65	123.73	232.90	178.28	322.00	1070.56
A, B, Italy	203.38	118.30	232.90	185.00	312.42	1052.01
A, C, Poland	213.02	125.16	224.49	177.75	322.00	1062.42
A, C, Italy	202.73	125.16	224.53	185.00	311.42	1048.84
A, Italy, Poland	209.87	125.16	232.90	149.65	280.71	998.29
B, C, Poland	215.66	123.72	225.14	178.27	322.00	1064.80
B, C, Italy	215.66	118.34	225.33	185.00	312.52	1056.85
B, Italy, Poland	215.66	123.58	232.90	151.80	284.74	1008.68
C, Italy, Poland	215.66	125.16	226.84	149.43	280.28	997.37
A, B, C, Poland	212.41	123.01	223.85	177.25	322.00	1058.53
A, B, C, Italy	201.99	117.49	223.71	185.00	310.29	1038.49
A, B, Italy, Poland	206.86	120.08	232.90	147.51	276.69	984.05
A, C, Italy, Poland	205.07	125.16	221.99	146.23	274.29	972.74
B, C, Italy, Poland	215.66	119.97	223.71	147.37	276.42	983.13
A, B, C, Italy, Poland	**203.34** *	**118.04** *	**220.12** *	**145.00** *	**271.99** *	**958.50** *

2.5. Results Discussion

Our study investigates a cooperative taxation scenario involving three firms—A, B, and C—operating across two jurisdictions—Italy and Poland. Each firm generates distinct profits in both countries and is subject to the corresponding corporate income tax obligations. These tax obligations are computed using the statutory tax rates—19% in Poland and 27.81% in Italy—alongside country-specific cooperation discounts of 10% and 12%, respectively, which are granted when firms form coalitions with tax authorities. Table 2 summarizes the profit and tax liabilities for each firm under the default, non-cooperative regime. The taxation setup, while hypothetical for illustrative purposes, provides a structured environment to apply the Shapley value as a fair and theoretically grounded tool for cost allocation. Our proposed algorithm allows these cooperative scenarios to be evaluated across all possible coalitional structures, offering valuable insight into the potential advantages of coordinated tax enforcement.

Table 3 presents the cost implications of all possible coalitions involving the three firms and the two countries, totaling $2^5 - 1 = 31$ coalition structures. Each row in the table lists the cost borne by participants within a coalition (in black), as well as the costs that non-participating agents would face outside the coalition (in red). These costs reflect, for firms, their total tax obligations across both countries, and for countries, the expenditure on administrative enforcement and infrastructure—with cost reductions applied when coalition-based cooperation is achieved. For instance, the eighth row, which corresponds to the coalition $\{A, \text{Poland}\}$, indicates a cost of 209.39 for Firm A and 185 for Poland, reflecting their discounted obligations. Non-members such as Firm B, Firm C, and Italy continue

to bear their full individual costs. This structure allows us to assess how costs change when different subsets of agents coordinate their efforts and whether such coordination is attractive to all parties.

A key observation from Table 3 is that unilateral action results in the highest individual cost for all agents. When operating independently, Firms A, B, and C incur tax obligations of 215.66, 125.16, and 232.90, respectively—identical to their baseline tax dues in the absence of cooperation (Table 2). Likewise, Poland and Italy bear costs of 185 and 322, respectively, when acting alone. In contrast, joint cooperation consistently reduces costs for the involved members. For example, when Poland and Italy form a coalition, their individual costs decrease to 172 and 277.20, respectively, (see Row 15), illustrating the fiscal benefit of joint tax enforcement. Importantly, even when firms are included in the coalition, the countries' costs do not increase, and firms themselves benefit through reduced tax liabilities. This property aligns with the theoretical notion of benefactors, as formalized in Definition 5, and is supported by Theorem 1, whereby the inclusion of certain agents in a coalition lowers the cost for others without introducing additional burden.

While forming coalitions yields lower total costs, an equally important consideration is how these costs are distributed among participants. An inequitable distribution could disincentivize participation, even when collective gains exist. To address this, we apply the Shapley value to redistribute the total cost among members based on their marginal contributions to each coalition. Table 4 reports the resulting Shapley value allocations for all coalition structures. In the grand coalition involving all five agents, the redistributed tax obligations for Firms A, B, and C are \$203.34, \$118.04, and \$220.12, respectively—each of which is strictly lower than the cost incurred under unilateral or partial cooperation. Likewise, Poland and Italy bear costs of \$145.00 and \$271.98, reflecting significant savings compared to their standalone expenditures. These results indicate that the Shapley value provides a principled mechanism to equitably assign costs in a way that respects each participant's role in achieving overall efficiency.

A natural question is whether smaller coalitions might offer better outcomes for individual agents. Consider, for instance, Firm B's cost in the coalition $\{A, B, C, \text{Italy}\}$ (Row 27 of Table 4), where its redistributed cost is \$117.49—slightly lower than the \$118.04 assigned in the grand coalition. Although this may seem appealing to Firm B, such a coalition imposes a higher cost on Italy (\$310.29) compared to the grand coalition allocation of \$271.98. This imbalance makes the coalition less attractive to Italy, potentially undermining its stability. The Shapley value thus not only ensures fairness but also promotes coalition structures where all participants are better off, reducing the risk of defection. Our results show that while marginal gains might exist in fragmented coalitions for select members, these are typically offset by disproportional burdens on others—underscoring the unique stability of the grand coalition under Shapley value allocation.

Despite the theoretical appeal of the grand coalition, practical limitations may prevent its formation in real-world taxation environments. Coordination among multiple sovereign governments and independent firms requires high levels of trust, transparency, and shared incentives, which are often absent in international tax relations. Information asymmetries, legal restrictions, and strategic behaviors—such as misreporting profits or selectively complying with treaties—that may undermine efforts to implement cooperative schemes. Additionally, political economy considerations, such as the desire to maintain national tax competitiveness, may lead countries to opt out of collective frameworks. While our model assumes full participation and truthful cooperation, actual policy implementation would likely need to address these frictions through credible enforcement mechanisms and robust institutional design. Nonetheless, our results offer a normative benchmark for what can be

achieved under ideal cooperation, guiding future efforts to approximate these outcomes in policy.

From a methodological perspective, our algorithm for computing coalition costs and Shapley value allocations is scalable to moderately large player sets, though its complexity grows exponentially with the number of agents. In high-dimensional settings, approximation techniques or sampling-based Shapley estimators may be required to maintain tractability. Furthermore, the results in our study are based on specific assumptions about tax rates and discount incentives. Variations in these parameters could influence coalition attractiveness and the magnitude of redistributed costs. For instance, a higher discount for coalition participation would make cooperation more appealing, while asymmetries in firm size or profit distribution might shift the burden disproportionately. Although we do not perform a formal sensitivity analysis here, future work could explore how parameter changes affect coalition stability and allocation fairness. Incorporating such robustness checks would further enhance the practical relevance of the Shapley value framework in cooperative tax policy design.

3. Shapley Value for Assignment Game

In many aspects of life, matching problems arise where two distinct groups must be paired efficiently. Students apply to universities, companies hire employees, and buyers negotiate with sellers in markets. These scenarios all involve two-sided markets, where participants on one side seek to form beneficial partnerships with those on the other. However, the challenge lies in determining an optimal and fair way to make these matches, especially when preferences, valuations, or constraints differ among participants.

The study of matching markets has long been a subject of interest in economics and game theory. Gale and Shapley (1962) introduced the concept of stable matching in their work on college admissions, where students and universities rank each other based on preferences. Their findings laid the foundation for understanding how to create stable and incentive-compatible matchings in various settings. While the marriage proposal problem they introduced assumes no transfer of money between participants, Shapley and Shubik (1971) extended the model to assignment games where monetary compensation influences match stability—particularly in markets like real estate or labor allocation.

In this section, we explore assignment markets through the lens of cooperative game theory. Specifically, we examine the assortative assignment game, where agents on both sides are assigned to one another in a way that leads to optimal pairings that maximize total market value. Using a structured valuation matrix, we illustrate how assignments are formed, and we analyze the results through a computational method. Finally, leveraging the known equivalence between the Becker solution (Becker, 1974) and the Shapley value for specific matrix structures, we compute the Shapley value for participants in a real estate market. Our approach not only provides insights into fair revenue distribution but also reinforces the role of cooperative game theory in solving real-world economic matching problems.

3.1. Assignment Market for a Two-Sided Market

Let B be a finite set of buyers and S a finite set of sellers. Each buyer $b \in B$ is willing to buy at most one item and each $s \in S$ has exactly one item for sale. Suppose seller $s_j \in S$ values their item for k_{s_j} and buyer $b_i \in B$ can pay at most $h_{b_i s_j}$ for seller $s_j \in S$. We assume that the item is non-negotiable, that is, there is no trade if $h_{b_i s_j} < k_{s_j}$[8], and we proceed to give definitions to some terms.

Definition 8. *Let $T \subseteq B \cup S$. We define the valuation matrix, $V(T)$, of an assignment game as a matrix of size $|B| \times |S|$, whose elements $v_{b_i s_j}(T)$ is computed as*

$$v_{b_i s_j}(T) = \begin{cases} h_{b_i s_j} - k_{s_j} & \text{if} \quad b_i \in T \cap B, \quad s_j \in T \cap S \quad \text{and} \quad |T| = 2 \\ 0 & \text{otherwise} \end{cases} \tag{35}$$

Our valuation matrix as defined in Equation (35) is consistent with a matching process. A possible matching occurs when there are exactly two distinct members (a seller and a buyer) in coalition T; otherwise, it obtains the value of zero.

Definition 9. *We define the assignment market as the tripple $(B, S, V(T))$.*

Definition 10. *An assignment market $(B, S, V(T))$ is assortative if the valuation matrix $V(T)$ is*

i. Supermodular, that is

$$v_{b_i s_l} + v_{b_i s_j} \leq v_{b_i s_j} + v_{b_k s_l} \quad \text{for all} \quad 1 \leq i < k \leq |B| \quad \text{and} \quad 1 \leq j < l \leq |S|;$$

ii. Monotone, that is

$$v_{b_i s_j} \leq v_{b_k s_l} \quad \text{for all} \quad 1 \leq i \leq k \leq |B| \quad \text{and} \quad 1 \leq j \leq l \leq |S|.$$

Definition 11. *We now define the assignment game as the pair $(B \cup S, w_{V(T)})$, where $w_{V(T)}$ is the worth of possible coalitions $T \subseteq (B \cup S)$ in valuation matrix $V(T)$.*

There is no gainsaying that the assignment markets can be formulated as a standard linear (integer) programming, with the constraint bounded by 1, since we are under the assumption that a buyer can only acquire at most one house, and a seller can only sell one house at most.

$$\begin{aligned} \max : w_{V(T)} &= \sum_{b_i \in B} \sum_{s_j \in S} v_{b_i s_j} x_{b_i s_j} \\ \text{subject to :} \quad & \sum_{b_i \in B} x_{b_i s_j} \leq 1 \quad \text{for} \quad s_j \in S \\ & \sum_{s_j \in S} x_{b_i s_j} \leq 1 \quad \text{for} \quad b_i \in B \\ & x_{b_i s_j} \in \{0, 1\} \end{aligned} \tag{36}$$

We denote $x_{b_i s_j} \in 0, 1$ as the decision variable indicating whether a partnership is formed between buyer b_i and seller s_j. The term $v_{b_i s_j} \in V(T)$ represents the valuation of this transaction within the coalition $T \subseteq B \cup S$. The objective of the assignment market is to identify the optimal matching between buyers and sellers that maximizes the total value of transactions. This is captured through a linear programming formulation, subject to the following two key constraints: each seller may sell at most one house, and each buyer may purchase at most one house.

Having formally linked the assignment game to a linear programming problem that maximizes the total valuation of a coalition T, we next turn to the computation of Shapley values for the participants. This is achieved by leveraging the equivalence between the Becker solution and the Shapley value in assortative assignment games, as established by Martínez-de Albéniz et al. (2019). We begin by presenting the formal definition of the Becker solution, followed by the theorem that demonstrates its equivalence to the Shapley value in this setting.

Definition 12. *Let $(B, S, V(T))$ be a square assortative assignment market. The Becker solution denoted by $\mathbb{B}(V(T))$ is given by the pair $(u^*, v^*) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+$, where*

$$\begin{aligned} u_i^* &= \frac{1}{2}a_{ii} + \frac{1}{2}\sum_{p=1}^{i-1} a_{p+1p} - \frac{1}{2}\sum_{k=1}^{i-1} a_{kk+1}, \\ v_j^* &= \frac{1}{2}a_{jj} - \frac{1}{2}\sum_{p=1}^{j-1} a_{p+1p} + \frac{1}{2}\sum_{k=1}^{j-1} a_{kk+1} \\ &\text{for all} \quad b_i \in B \quad \text{and} \quad s_j \in S. \end{aligned} \tag{37}$$

Theorem 5. *The Shapley value, $\phi(w_{V_T})$, of an assignment game $(B \cup S, w_{V(T)})$ with $|B| = |S| = m$ coincides with the Becker solution $\mathbb{B}(V(T))$ if the valuation matrix $V(T)$ is of the form*

$$\begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \dots & \alpha_1 \\ \alpha_1 & \alpha_2 & \alpha_2 & \dots & \alpha_2 \\ \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_m \end{pmatrix} \quad \text{with} \quad 0 \le \alpha_1 \le \alpha_2 \dots \le \alpha_m. \tag{38}$$

Proof. Let $(B, S, V(T))$ be a square assignment market with $|B| = |S| = m$ and valuation matrix $V(T)$ given by

$$v_{ij} = \alpha_{\min(i,j)}, \quad \text{with } 0 \le \alpha_1 \le \alpha_2 \le \dots \le \alpha_m.$$

The unique optimal matching under $V(T)$ is the identity matching $\mu^* = \{(b_i, s_i)\}_{i=1}^m$ with the total value $w_{V(T)}(B \cup S) = \sum_{i=1}^m \alpha_i$.

The Becker solution allocates this total value via the following:

$$\begin{aligned} u_i^* &= \frac{1}{2}\alpha_i + \frac{1}{2}\sum_{p=1}^{i-1} \alpha_{p+1} - \frac{1}{2}\sum_{k=1}^{i-1} \alpha_k, \\ v_i^* &= \alpha_i - u_i^*, \end{aligned}$$

where u_i^* and v_i^* are the payoffs to b_i and s_i, respectively. Then, $u_i^* + v_i^* = \alpha_i$ and $\sum_{i=1}^m (u_i^* + v_i^*) = \sum_{i=1}^m \alpha_i$.

The Shapley value for each agent in an assignment game is given by

$$\phi_i(w) = \sum_{T \subseteq N \setminus \{i\}} \frac{|T|!(|N| - |T| - 1)!}{|N|!}[w(T \cup \{i\}) - w(T)].$$

By Shapley and Shubik (1971), the Shapley value lies in the core, and under strict assortativity, marginal contributions depend only on rank. Therefore,

$$\phi(w_{V(T)}) = (u_1^*, \dots, u_m^*, v_1^*, \dots, v_m^*) = \mathbb{B}(V(T)).$$

□

3.2. Application of the Shapley Value in Matching Buyers and Sellers

We consider a real estate market consisting of four houses available for sale and four buyers interested in acquiring them. In this setting, each buyer is restricted to purchasing exactly one house. Let the set of sellers be denoted by $S = s_1, s_2, s_3, s_4$ and the set of buyers by $B = b_1, b_2, b_3, b_4$. The sellers' reserve prices (in thousand USD) and the corresponding valuations assigned by the buyers to each house are presented in Table 5.

Table 5. Buyers' valuation and sellers' basis in the real estate market. The table shows the basis price k_{s_j} (in thousand USD) for each seller s_j, alongside the valuation $h_{b_i s_j}$ assigned by each buyer b_i to each seller's house. These valuations reflect the maximum amount each buyer is willing to pay. The objective in the assignment problem is to match buyers to sellers in a way that maximizes total surplus, defined as the difference between the buyer valuation and seller basis.

s_j	k_{s_j}	$h_{b_1 s_j}$	$h_{b_2 s_j}$	$h_{b_3 s_j}$	$h_{b_4 s_j}$
1	30	51	51	51	51
2	40	61	70	70	70
3	35	56	65	73	73
4	42	63	72	80	102

The valuation matrix is based on Equation (35), as follows:

$$V(T) = \begin{pmatrix} 21 & 21 & 21 & 21 \\ 21 & 30 & 30 & 30 \\ 21 & 30 & 38 & 38 \\ 21 & 30 & 38 & 60 \end{pmatrix}.$$

Since the valuation matrix $V(T)$ is supermodular and monotonic, as defined in Definition 10, the triplet $(B, S, V(T))$ constitutes an assortative assignment market. We now proceed to formulate the corresponding linear programming problem that characterizes this market, with the objective of identifying the optimal matching between buyers and sellers. Once the optimal solution is obtained, we apply Theorem 5 to derive the Shapley value for each participant in the grand coalition.

$$\begin{aligned}
\max : w_{V(T)} =& 21x_{b_1s_1} + 21x_{b_1s_2} + 21x_{b_1s_3} + 21x_{b_1s_4} + 21x_{b_2s_1} + 30x_{b_2s_2} + 30x_{b_2s_3} + 30x_{b_2s_4} + \\
& 21x_{b_3s_1} + 30x_{b_3s_2} + 38x_{b_3s_3} + 38x_{b_3s_4} + 21x_{b_4s_1} + 30x_{b_4s_2} + 38x_{b_4s_3} + 60x_{b_4s_4} \\
\text{subject to :} \quad & x_{b_1s_1} + x_{b_1s_2} + x_{b_1s_3} + x_{b_1s_4} \leq 1 \\
& x_{b_2s_1} + x_{b_2s_2} + x_{b_2s_3} + x_{b_2s_4} \leq 1 \\
& x_{b_3s_1} + x_{b_3s_2} + x_{b_3s_3} + x_{b_3s_4} \leq 1 \\
& x_{b_4s_1} + x_{b_4s_2} + x_{b_4s_3} + x_{b_4s_4} \leq 1 \\
& x_{b_1s_1} + x_{b_2s_1} + x_{b_3s_1} + x_{b_4s_1} \leq 1 \\
& x_{b_1s_2} + x_{b_2s_2} + x_{b_3s_2} + x_{b_4s_2} \leq 1 \\
& x_{b_1s_3} + x_{b_2s_3} + x_{b_3s_3} + x_{b_4s_3} \leq 1 \\
& x_{b_1s_4} + x_{b_2s_4} + x_{b_3s_4} + x_{b_4s_4} \leq 1, \\
& x_{b_is_j} \in \{0, 1\}.
\end{aligned} \tag{39}$$

We solved the linear programming in (39) with the Python (*PULP*) library (Jayakumar et al., 2017). The result of this linear programming is as follows:

$$\begin{aligned}
x_{b_1s_1} = 1, x_{b_1s_2} = x_{b_1s_3} = x_{b_1s_4} = 0 \\
x_{b_2s_1} = 0, x_{b_2s_2} = 1, x_{b_2s_3} = x_{b_2s_4} = 0 \\
x_{b_3s_1} = x_{b_3s_2} = 0, x_{b_3s_3} = 1, x_{b_3s_4} = 0 \\
x_{b_4s_1} = x_{b_4s_2} = x_{b_4s_3} = 0, x_{b_4s_4} = 1 \\
w_{V(T)} = 149 = v(B \cup S).
\end{aligned} \tag{40}$$

We recall that $x_{b_is_j}$ denotes the binary decision variable indicating whether a coalition (i.e., a matching) is formed between buyer b_i and seller s_j. The result presented in

Equation (40) implies that the optimal matching consists of buyer 1 paired with seller 1, buyer 2 with seller 2, and buyer 3 with seller 3. This particular matching configuration is the one that maximizes the total surplus in the real estate market, and the worth of each coalition corresponding to the optimal assignment is

$$v_A(\{b_1, s_1\}) = 21, \quad v_A(\{b_2, s_2\}) = 30, \quad v_A(\{b_3, s_3\}) = 38, \quad v_A(\{b_4, s_4\}) = 60.$$

Since the valuation matrix is of the form defined in (38), then, the Shapley value for each agent in the game is calculated based on definition of the Becker solution defined in Equation (37).

$$\begin{aligned} \phi_{b_1}(v_A) = 10.5, \quad \phi_{s_1}(v_A) = 10.5, \quad \phi_{b_2}(v_A) = 15 \quad \phi_{s_2}(v_A) = 15, \\ \phi_{b_3}(v_A) = 19, \quad \phi_{s_3}(v_A) = 19, \quad \phi_{b_4}(v_A) = 30, \quad \phi_{s_4}(v_A) = 30. \end{aligned}$$

The total worth of the grand coalition, $N = \{b_1, b_2, b_3, b_4, s_1, s_2, s_3, s_4\}$, is given by the optimal value of the linear programming formulation, which yields $v(N) = 149$. The sum of the Shapley values across all participants equals this total, confirming the efficiency and consistency of the allocation mechanism.

In the context of this assignment market, the Shapley value does not imply that players individually benefit more by forming the grand coalition than by participating in smaller coalitions—as is the case in the tax cooperation setting. Rather, the Shapley value in this context provides a fair and axiomatic method for allocating the total surplus generated by the optimal matching of buyers and sellers. Each player's Shapley value reflects their average marginal contribution across all possible permutations of the coalition, thereby capturing their relative importance in achieving the socially optimal assignment. This ensures that the surplus from efficient matching is distributed equitably among participants, consistent with cooperative game-theoretic principles.

4. Conclusions

Our work has explored the application of the Shapley value in cooperative game theory across various contexts, notably in multinational tax obligations and the buyer–seller housing market. The detailed analysis demonstrates how the Shapley value not only helps in minimizing individual tax liabilities among corporations and countries but also provides optimal matching in a housing market where multiple buyers and sellers interact. In the tax scenario, we established that the grand coalition—comprising all firms and countries—results in the most advantageous outcomes, reducing individual tax burdens and ensuring stable and equitable cost sharing. The Shapley value calculations indicated that each member benefits from the grand coalition, thus supporting its sustainability and attractiveness. In the housing market scenario, the Shapley value was applied to model and solve the optimal matching problems in a buyer–seller market.

Our findings underscore the Shapley value's utility in providing systematic solutions to potential inequities and enhancing cooperative engagement among disparate entities in both taxation and real estate. The implications for policymakers and multinational corporations are profound, as these strategies optimize economic interactions while adhering to fairness and equity. Future research could expand on these frameworks by incorporating more complex economic conditions and dynamic models that reflect the evolving landscape of global finance and real estate markets. For instance, a future research direction for our housing problem is to consider a negotiable case. Further exploration into other areas of finance and economics, such as resource allocation in public utilities or cost-sharing in environmental projects, could validate the versatility of the Shapley value in broader contexts.

Author Contributions: Conceptualization, O.A. and S.T.A.; methodology, O.A. and S.T.A.; formal analysis, O.A. and S.T.A.; data curation, O.A. and S.T.A.; writing—review and editing, O.A. and K.O. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No real-world data were used in this study. The data were generated based on the tax rates of the concerned countries, assumptions regarding tax rate reductions, and assumptions of firms' profits. Where applicable, the algorithm used to generate the result on the generated data has been provided.

Conflicts of Interest: The authors declare no conflicts of interest.

Notes

1 One such solution is the bargaining model of Harsanyi (1959).
2 The actual tax is higher than the reduced tax as earlier stated. Benefit of coalition.
3 p' is a firm and not suffering effect of tax evasion.
4 Attention to this is highly needed, to clear ambiguity in the proof to Theorem 3.
5 See the attention above!
6 Our numerical result in Section 2.4 emphasize the remarks and interpretation of the formula.
7 Our algorithm is written in pseudocode for clarity's sake. Actual implementation is provided in Python 3.9.6.
8 A future possible direction in this research area would be to consider a negotiable case.

References

Algaba, E., Fragnelli, V., & Sánchez-Soriano, J. (2019). *Handbook of the shapley value*. CRC Press.

Almaraz, S. D.-L., Gelei, A., & Solymosi, T. (2025). Coalition analysis for low-carbon hydrogen supply chains using cooperative game theory. *International Journal of Hydrogen Energy*.

Becker, G. S. (1974). A Theory of Marriage. In *Economics of the family: Marriage, children, and human capital* (pp. 299–351). University of Chicago Press.

Béal, S., Ferrières, S., Rémila, E., & Solal, P. (2018). The proportional shapley value and applications. *Games and Economic Behavior*, *108*, 93–112. [CrossRef]

Boadway, R., & Tremblay, J.-F. (2012). Reassessment of the Tiebout model. *Journal of Public Economics*, *96*(11–12), 1063–1078. [CrossRef]

Branzei, R., Dimitrov, D., & Tijs, S. (2008). *Models in cooperative game theory* (Vol. 556). Springer Science & Business Media.

Feige, E. L. (2016). Reflections on the meaning and measurement of unobserved economies: What do we really know about the 'shadow economy'? *Journal of Tax Administration*, *2*, 5–41. [CrossRef]

Gale, D., & Shapley, L. S. (1962). College admissions and the stability of marriage. *American Mathematical Monthly*, *69*(1), 9–15. [CrossRef]

Halkos, G. E., Papageorgiou, G. J., Halkos, E. G., & Papageorgiou, J. G. (2020). Public debt games with corruption and tax evasion. *Economic Analysis and Policy*, *66*, 250–261. [CrossRef]

Harsanyi, J. C. (1959). *A bargaining model for the cooperative n-person game*. Stanford University.

Hatfield, J. W., & Kominers, S. D. (2012). Matching in networks with bilateral contracts. *American Economic Journal: Microeconomics*, *4*(1), 176–208. [CrossRef]

Jayakumar, A. A., Krishnaraj, C., & Raghunayagan, P. (2017). Optimization of fixed charge problem in Python using PuLP package. *International Journal of Control Theory and Applications*, *10*(2), 443–447.

Keen, M., & Konrad, K. A. (2013). The theory of international tax competition and coordination. In *Handbook of public economics* (Vol. 5, pp. 257–328). Elsevier.

Littlechild, S. C., & Owen, G. (1976). A further note on the nucleous of the "airport game". *International Journal of Game Theory*, *5*, 91–95. [CrossRef]

Mahdiraji, H. A., Razghandi, E., & Hatami-Marbini, A. (2021). Overlapping coalition formation in game theory: A state-of-the-art review. *Expert Systems with Applications*, *174*, 114752. [CrossRef]

Martens-Weiner, J. (2006). *Company tax reform in the European Union: Guidance from the United States and Canada on implementing formulary apportionment in the EU*. Springer Science & Business Media.

Martínez-de Albéniz, F. J., Rafels, C., & Ybern, N. (2019). Solving Becker's assortative assignments and extensions. *Games and Economic Behavior*, *113*, 248–261. [CrossRef]

Meca, A., García-Martínez, J. A., & Mayor-Serra, Á. J. (2019). The shapley value of corporation tax games with dual benefactors. In *Handbook of the shapley value* (pp. 313–330). Chapman and Hall/CRC.

Meca, A., & Varela-Peña, J. C. (2018). Corporation tax games: An application of linear cost games to managerial cost accounting. In *Game theory in management accounting: Implementing incentives and fairness* (pp. 361–378). Springer.

Ni, D., & Wang, Y. (2007). Sharing a polluted river. *Games and Economic Behavior*, *60*(1), 176–186. [CrossRef]

Roth, A. E. (2008). What have we learned from market design? *The Economic Journal*, *118*(527), 285–310. [CrossRef]

Saavedra-Nieves, A., & Fiestras-Janeiro, M. G. (2022). Analysis of the impact of DMUs on the overall efficiency in the event of a merger. *Expert Systems with Applications*, *195*, 116571. [CrossRef]

Schneider, F. (2018). *Size, causes and consequences of the underground economy: An international perspective*. Routledge.

Schneider, F., & Enste, D. H. (2000). Shadow economies: Size, causes, and consequences. *Journal of Economic Literature*, *38*(1), 77–114. [CrossRef]

Shapley, L. S. (1997). A value for n-person games. In *Classics in game theory* (Vol. 69). Princeton University Press.

Shapley, L. S., & Shubik, M. (1954). A method for evaluating the distribution of power in a committee system. *American Political Science Review*, *48*(3), 787–792. [CrossRef]

Shapley, L. S., & Shubik, M. (1971). The assignment game I: The core. *International Journal of Game Theory*, *1*(1), 111–130. [CrossRef]

Von Neumann, J., & Morgenstern, O. (1947). *Theory of games and economic behavior* (2nd ed.) Princeton University Press.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.