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Article

Pricing Multidimensional American Options

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Abstract: A new explicit form is provided for the solution of optimal stopping problems involving a multidimensional geometric Brownian motion. A free-boundary value approach is adopted and the value function is obtained via fundamental solution methods. There are many applications for the valuation of perpetual options of American style, which are of interest for finance and managerial decisions.

Keywords: American options; optimal exercise boundary; multidimensional stochastic processes; optimal stopping

1. Introduction

Despite the huge progress on the pricing methods for American options, an explicit solution for their value is known only for options with infinite maturity written on a single underlying asset. Under classical assumptions on the underlying assets, the problem can be formulated as a free-boundary value for a Black–Scholes-type PDE. When the underlying stochastic processes are geometric Brownian motions (hereafter GBM) the value-matching and smooth-fit conditions apply, but finding the exercise threshold in explicit form remains an elusive research question. Early exercise boundary approximation techniques have been developed in many articles to obtain the option price and hedge ratio (see the Refs. Ait-Sahlia and Lai (2001); Huang et al. (1996); Kim (1990), among others). It is almost impossible to summarize all approaches to approximate the price of American options, starting from the first attempts to replace the exercise boundary with a finite set of discrete points (Geske and Johnson 1984) or to resort to finite-difference schemes (Brennan and Schwartz 1977).

While multi-asset European options have been priced extensively in the literature even in a non-Gaussian Black–Merton–Scholes framework (see the Ref. Agliardi (2012), for a comprehensive approach), there are few studies on the American option written on several assets. For these options, an explicit solution is not available even in the perpetual case. Various payoff functions for American options on two assets have been considered in the Ref. Broadie and Detemple (1997), which remains a fundamental work for the multiasset case. On the other hand, the regularity of the free boundary for multi-asset American options has been studied in the Ref. Laurence and Salsa (2009).

Recently, some traditional numerical schemes and simulation-based approaches have been applied to price these American-style options. In particular, finite difference schemes are used to discretize the complementarity problem involving Black–Scholes-type equations (see, e.g., the Ref. Nielsen et al. 2008; O'Sullivan and O'Sullivan 2011, and a comonotonic finite difference method in the Ref. Hanbali and Linders 2019), radial basis techniques are used in the Ref. Egorova et al. (2018), while least-square Monte Carlo algorithms extending the Ref. Longstaff and Schwartz (2001) are another popular approach in this field (see the Ref. Chan et al. 2006; Samimi and Mehrdoust 2018). Overall, the pricing of multi-asset American options is recognized in the literature as a quite difficult issue from a

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computational point of view; on the other hand, the lack of analytical expressions for the solution renders the judgement of the accuracy of the approximation a hard task.

For single-asset models, there exist several articles studying the pricing problem under more general stochastic processes than GBM, such as Lévy processes (Boyarchenko and Levendorskiĭ 2002; Gukhal 2001; Levendorskiĭ 2004; Mordecki 2002). For multi-asset options the pricing problem has no explicit solution in general, even in the GBM case. In some special cases (e.g., two-dimensional exchange options) an explicit solution can be obtained in an elementary way. When the options are written on a basket of risky assets and the payoff function is non-homogeneous, only numerical methods are available (e.g., finite difference schemes). The most difficult task remains to identify the optimal exercise boundary. Thus, in this paper, we concentrate on this case, which still deserves investigation. We use an analytical approach to solve the pricing problem for perpetual American options in the multidimensional case. We obtain an explicit expression for the option value which is written as an integral involving modified Bessel functions of the second kind (see Section 3). A proof relating modified Bessel functions and the n-dimensional modified Helmoltz equation is provided in the Appendix A, which is of theoretical interest for other possible applications.

Although perpetual options are rare, they can be used in efficient numerical procedures to approximate the price of American options with finite maturity, for example, Carr's randomization method (Carr 1998). More importantly, perpetual options provide a technical tool to attack several problems arising in real option theory (see the Ref. Boyarchenko and Levendorskiĭ 2007; Dixit and Pindyck 1994), and thus their scope of application spans far beyond financial derivatives.

2. Problem-Setting and Motivating Examples

In what follows, we assume that the value of the underlying asset, X_t, follows an $n-$dimensional geometric Brownian motion with respect to a given filtration, $\mathfrak{F}$, representing the available information. (See Section 3 for the detailed notation).

Given a positive payoff function $G : \mathbf{R}^n_+ \longrightarrow [0, +\infty)$, usually a continuous function, and a risk-free interest rate $r > 0$, the rational price of a perpetual American option with instantaneous payoff $G(X_t)$ is obtained as

$$V_0(x) = \sup_t E_x[e^{-rt}G(X_t)] = E_x[e^{-rt^*}G(X_{t^*})] \tag{1}$$

where the supremum is taken over all stopping times w.r.t. $\mathfrak{F}$. Here, E_x denotes the expectation operator conditional on $X_0 = x$. The optimal stopping time t^* represents the optimal exercise time. In the Ref. Shiryaev (2008), existence of V_0 is obtained when $E_x[\sup_{t\geq 0} e^{-rt}G(X_t)] < \infty$ and t^* is the first entry time of X_t into a stopping region, S (or exercise region) where V_0 equals G, that is, it is optimal to exercise the option immediately. On the contrary, $V_0 > G$ is in the so-called continuation region, C. The boundary between the two regions is the optimal exercise boundary, ∂C. The boundary-value formulation of this optimal stopping problem is as follows:

$$\begin{cases} [r-L]V = 0 & \text{in } C \\ V = G & \text{in } S \end{cases} \tag{2}$$

with $V \geq G$ in C and $[r-L]V \geq 0$ in S. Here, L is the differential operator associated with X_t, which is defined in (6).

Some examples of payoff functions for multi-asset American options are reported below. Let $G(x) = \max\left\{\widehat{G}(x), 0\right\}$ and let $\omega = \pm 1$ denote the call/put attribute, with $+1$ denoting a call option and -1 a put option.

Basket (or index) options: $\widehat{G}(x_1, \ldots, x_n) = \omega[\sum_{i=1}^n w_i x_i - K]$

where the $w_i's$ are the weights in the basket.

A special case is represented by the

Arithmetic average option: $\widehat{G}(x_1, \ldots, x_n) = \omega[\frac{1}{n}\sum_{i=1}^{n} x_i - K]$

Spread options: $\widehat{G}(x_1, x_2) = \omega[x_2 - x_1 - K]$.

The special case of *exchange options* is obtained setting $K = 0$.

Option on the product with random exercise price:
$\widehat{G}(x_1, x_2) = \omega[x_1 x_2 - K x_2]$[1].

Power-product options: $\widehat{G}(x_1, \ldots, x_n) = \omega[(\prod_{i=1}^{n} x_i)^p - K]$ for some $p > 1$.

Options on the max: $\widehat{G}(x_1, \ldots, x_n) = \omega[\max(x_1, \ldots, x_n) - K]$.

Multiple strike options: $\widehat{G}(x_1, \ldots, x_n) = \max[\omega_1(x_1 - K_1), \ldots \omega_n(x_n - K_n)]$.

Note that in (1) we can safely replace G with $\widehat{G}$ because it is not optimal to exercise the option when its payoff is negative. Moreover, arguing as in the Ref. Boyarchenko and Levendorskiĭ (2002), we will assume that $\widehat{G}(X_t)$ can be written in terms of a cash flow $g(X_t)$, so that the valuation problem reduces to maximizing the following expected value:

$$E_x[\int_t^{+\infty} e^{-r\tau} g(X_\tau) d\tau] \tag{3}$$

where g represents the revenue flow which is acquired by exercising the option. In the case of regular payoff functions, this method can be easily motivated as follows. If $\widehat{G}$ is $\mathcal{C}^1$ with absolutely continuous first derivatives (and other technical conditions), Ito's formula implies that

$$e^{-rt}\widehat{G}(X_t) = \widehat{G}(x) + \int_0^t e^{-r\tau}(L - r)\widehat{G}(X_\tau) d\tau$$

and we denote $(L - r)\widehat{G}$ by $-g$. Note that g is a distribution rather than a function if $\widehat{G}$ is not sufficiently regular, so this method can be used also for less regular functions. If we assume that $E_x[\int_0^\infty e^{-r\tau} g(X_\tau) d\tau] < \infty$, then

$$E_x[\int_t^\infty e^{-r\tau} g(X_\tau) d\tau] = E_x[\int_0^\infty e^{-r\tau} g(X_\tau) d\tau] + E_x[\int_0^t e^{-r\tau}(-g(X_\tau)) d\tau].$$

Thus the expected value of $e^{-rt}\widehat{G}(X_t)$ equals $\widehat{G}(x) - E_x[\int_0^\infty e^{-r\tau} g(X_\tau) d\tau] + E_x[\int_t^\infty e^{-r\tau} g(X_\tau) d\tau]$ and the optimal exercise problem reduces to maximizing (3). Alternatively, one can maximize

$$E_x[\int_0^t e^{-r\tau}(-g(X_\tau)) d\tau] \tag{4}$$

Then we can confine the analysis to the optimal stopping problem for (4). If V is the value function for the optimal stopping problem for (4), then $V_0 = V + G$ is the option price. The boundary-value formulation of problem (4) is as follows:

$$\begin{cases} [L - r]V + g = 0 & \text{in } C \\ V = 0 & \text{in } S \end{cases}$$

A typical assumption on g is that g is measurable and satisfies an integrability condition of the form:

$$E_x[\int_0^{+\infty} e^{-rt}|g(X_t)| dt] < \infty \text{ for any } x.$$

Note that if $g \leq 0$, then the optimal stopping time is ∞, while if $g \geq 0$ it is optimal to stop immediately. To avoid such trivial situations one needs to assume that g changes sign.

Let δ_i denote the instantaneous dividend yield which is paid on the ith risky asset. Then, in view of the argument above, some examples of payoff streams, g, associated with American options are:

$\omega[\sum_{i=1}^{n} \delta_i w_i x_i - rK]$ for a basket option;
$\omega[\delta_2 x_2 - \delta_1 x_1 - rK]$ for a spread option;
$\omega[\sum_{i=1}^{2} \delta_i x_i 1_{\{x_i > x_j, j \neq i\}} - rK - \frac{1}{2}\sum_{i,j} \sigma_{ij} x_1^2 v]$ for a max-option in R^2 where v is the Lebesgue measure on the line $x_1 = x_2$.

For convex payoff streams, the following property can be proved.

Proposition 1. *Let g be a convex function. Then the stopping set for*

$$\sup_t E_x[\int_0^t e^{-r\tau} g(X_\tau) d\tau]$$

is convex.

Proof. Let us prove that V, the value function for this problem, is convex. Let $x' = (x_1', \dots x_n')$, $x'' = (x_1'', \dots x_n'')$, $\theta \in [0,1]$ and denote $\theta x' + (1-\theta)x''$ by $\overline{x}$. Let $X_t^i(x_i)$ denote the stochastic process X_t^i started at x_i. In view of convexity of g for any stopping time τ we have:

$$\begin{aligned}
&E[\int_0^\tau e^{-rt} g(X_t^1(\overline{x}_1), \dots X_t^n(\overline{x}_n))dt] \\
&\leq \theta E[\int_0^\tau e^{-rt} g(X_t^1(x_1'), \dots)dt] + (1-\theta)E[\int_0^\tau e^{-rt} g(X_t^1(x_1''), \dots)dt] \\
&\leq \theta V(x') + (1-\theta)V(x'').
\end{aligned}$$

Then taking the sup of the left-hand side we get

$$V(\overline{x}) \leq \theta V(x') + (1-\theta)V(x''),$$

which implies that V is convex. Let us show that the stopping set is convex arguing by contradiction. If $x', x'' \in S$ and $\overline{x} \notin S$ then $0 < V(\overline{x}) \leq \theta V(x') + (1-\theta)V(x'') = 0$ which yields a contradiction. $\square$

3. Valuation Formula

Assume that the option is written on n risky assets and let r denote the risk-free interest rate. Each risky asset pays a continuously compounded dividend yield, δ_i, per unit of time. The risk factors are modelled throughout an n-dimensional geometric Brownian motion, X_t, where the dimension represents the number of components in the underlying portfolio. Its components are of the form

$$X_t^{(i)} = X_0^{(i)} \exp[(r - \delta_i - \frac{\sigma_i^2}{2})t + \sigma_i W_t^{(i)}] \tag{5}$$

where $(W_t^{(1)}, \dots, W_t^{(n)})$ is an n-dimensional Wiener process with respect to a given filtration, $\mathfrak{F}$, and with $E[W_t^{(i)} W_t^{(j)}] = \rho_{ij} t$ where the non-negative definite matrix $(\rho_{ij})_{i,j=1,\dots n}$ has $\rho_{ii} = 1$. Let σ_{ij} denote $\sigma_i \sigma_j \rho_{ij}$ for all $i, j = 1, \dots n$. Assume that $\Sigma = (\sigma_{ij})_{i,j=1,\dots,n}$ is a Hermitian positive definite matrix. In what follows, $r - \delta_i - \frac{\sigma_i^2}{2}$ will be denoted by m_i.

The differential operator associated with the process X_t is the following differential operator in $\mathbf{R}_+^n$:

$$L = \frac{1}{2}\sum_{i,j=1}^{n} \sigma_{ij} x_i x_j \partial^2_{x_i x_j} + \sum_{i=1}^{n} (r - \delta_i) x_i \partial_{x_i} \tag{6}$$

where $x_j \geq 0, 1 \leq j \leq n$.

Let g denote the payoff stream associated with a perpetual option. The boundary-value formulation of the problem is as follows:

$$\begin{cases} [L - r]V + g = 0 & \text{in } C \\ V = 0 & \text{in } S \end{cases} \tag{7}$$

where $C \subset R^n$ denotes the continuation region, where it is not optimal to exercise the option, while $S \subset R^n$ denotes the early exercise region. The early exercise boundary, ∂S, separating the two regions should be determined as part of the problem. In the sequel we will consider explicitly the case where it can be represented as a function $x_n = b(x_1, \ldots, x_{n-1})$ and we refer to Peskir (2019) and to Peskir and Shiryaev (2006) for the continuity property of the exercise boundary.

In order to solve problem (7), first we make the following change of variables:

$$y_j = \ln x_j, j = 1, \ldots n,\ V(x_1, \ldots, x_n) = e^{-\alpha_1 y_1 - \ldots - \alpha_n y_n} v(y_1, \ldots y_n),$$

$$g(x_1, \ldots x_n) = -f(y_1, \ldots y_n) e^{-\alpha_1 y_1 - \ldots \alpha_n y_n},$$

where we choose $\alpha = \begin{pmatrix} \alpha_1 \\ \ldots \\ \alpha_n \end{pmatrix}$ such that $\Sigma\alpha = m = \begin{pmatrix} m_1 \\ \ldots \\ m_n \end{pmatrix}$.

The continuation region is mapped into $\widetilde{C}$ and problem (7) becomes:

$$\begin{cases} \widetilde{L}v = f & \text{in } \widetilde{C} \\ v = 0 & \text{in } \widetilde{S} \end{cases} \tag{8}$$

where

$$\tilde{L} = \sum_{i,j=1}^{n} \frac{\sigma_{ij}}{2} \frac{\partial^2}{\partial y_i \partial y_j} - k^2. \tag{9}$$

Here $k^2 = r + \frac{1}{2}\alpha^t \Sigma \alpha \geq 0$ and k is a real nonnegative number.

Let us now consider the Cholesky decomposition for the matrix Σ, that is, $\Sigma = TT^*$ where T is a lower triangular matrix (with positive diagonal entries). For example, in the two-dimensional case we can write:

$$T = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2} \end{pmatrix}.$$

Changing to variables $Y = TZ$ and denoting $v(Y) = u(Z)$ the differential equation $\widetilde{L}v = f$ is transformed into:

$$\frac{1}{2}\Delta_Z u - k^2 u = \varphi(Z), \tag{10}$$

where Δ_Z is the n-dimensional Laplace operator in the variable Z and $\varphi(Z) = f(TZ) = f(Y)$. The continuation region is transformed into C_z. For example, if C is of the form

$$\{(x_1, \ldots x_n) \in R_+^n;\ x_i < x_i^*, i = 1, \ldots, n-1 \text{ and } x_n < b(x_1, \ldots, x_{n-1})\},$$

then C_z takes the form

$$\left\{ Z \in R^n; z_i < \sum_{k=1}^{i} \widehat{t}_{ik} \ln x_k^*, i = 1, \ldots, n-1,\ \ z_n < \sum_{k=1}^{n-1} \widehat{t}_{n-1,k} \ln x_k^* + \widehat{t}_{nn} \ln b((TZ)_{n-1}) \right\}$$

where $\widehat{t}_{ij}$ are the entries of T^{-1} and $(TZ)_{n-1}$ denotes the first $n-1$ rows of TZ.

Then we can write the solution for this modified Helmholtz-type equation in terms of its fundamental solution, $\mathcal{E}_n$, as follows:

$$u(Z) = \int \cdots \int_{C_z} \mathcal{E}_n(\eta)\varphi(Z)d\eta.$$

The n-dimensional Green kernel $\mathcal{E}_n$ can be computed in terms of modified Bessel functions of the second kind, K_ν. (See Appendix A). More precisely:

$$\mathcal{E}_n(\eta) = \frac{-2}{(2\pi)^{n/2}}[\frac{|\eta|}{\sqrt{2}k}]^{1-n/2}K_{1-\frac{n}{2}}(\sqrt{2}k|\eta|).$$

Changing back variables, we can write:

$$V(x_1,\ldots x_n) = \int \cdots \int_{C_x} -\mathcal{E}_n(\eta)g(x_1e^{t_{11}\eta_1},\ldots x_ne^{\Sigma_{k=1}^n t_{kn}\eta_k})$$

$$\exp[\alpha_1 t_{11}\eta_1 + \ldots \alpha_n \Sigma_{k=1}^n t_{kn}\eta_k]d\eta$$

where, in the above-mentioned case,

$$C_x = \Bigg\{\eta \in R^n;\ \eta_i < \sum_{k=1}^i \widehat{t}_{ik}\ln(\frac{x_k^*}{x_k}), i = 1,\ldots,n-1,$$

$$\eta_n < \sum_{k=1}^{n-1}\widehat{t}_{n-1,k}\ln(\frac{x_k^*}{x_k}) + \widehat{t}_{nn}\ln\frac{b(x_1e^{t_{11}\eta_1},\ldots x_ne^{\Sigma_{k=1}^n t_{kn}\eta_k})}{x_n}\Bigg\}.$$

If we insert the optimal boundary $x_n = b(x_1,\ldots,x_{n-1})$ into the above-obtained expression for the value function then we get

$$V(x_1,\ldots x_{n-1}, b(x_1,\ldots,x_{n-1})) = 0,$$

that is, we get an equation for the unknown function $b(x_1,\ldots,x_{n-1})$. We need a guess for b and then we improve it through numerical iteration. For example, one can start with a hyperplane joining the points (if any) where the optimal boundary meets the Cartesian axes, as these points can be obtained solving one-dimensional problems in explicit form.

The advantage of having the equation for the unknown threshold derived above lies in the possibility of controlling for the approximated expression for b, which provides guidance for the numerical procedure. As Bessel functions are included in built-in-function packages in various numerical softwares, the formula above can be easily computed numerically and thus the procedure can be implemented in practice. In the next section we connect our analytical procedure to its probabilistic interpretation.

4. Probabilistic Interpretation

In this Section we provide a probabilistic interpretation for the expression of the value function obtained in Section 3. It is known (see the Ref. Revuz and Yor 1999) that the transition density of a Bessel process of order ν can be expressed throughout modified Bessel functions of order ν, that is:

$$p_t^{(\nu)}(x,y) = \frac{y^{\nu+1}}{t(x)^\nu}e^{-(x^2+y^2)/(2t)}I_\nu(\frac{xy}{t})\mathbf{1}_{\{y>0\}}.$$

On the other hand, Bessel processes can be connected to multi-dimensional Wiener processes. More precisely, in view of a relationship between n-dimensional Brownian motions and Bessel processes of order $\nu = \frac{n}{2} - 1$, one can compute the resolvent kernel for the operator in (10) in terms of modified Bessel functions of order ν. Consider an n-

dimensional Wiener process, $\{W_t\}$, where $f_t(z) = \frac{1}{(2\pi t)^{n/2}} e^{-|z|^2/(2t)}$ denotes the probability density function of W_t. Then the Green kernel

$$G_{k^2}(z, \eta) = \int_0^\infty e^{-k^2 t} f_t(z) dt$$

can be computed as

$$\frac{2}{(2\pi)^{n/2}} \left(\frac{|z-\eta|}{\sqrt{2}k} \right)^{1-n/2} K_{1-\frac{n}{2}}(\sqrt{2}k|z-\eta|),\ k > 0,$$

using an identity for this integral transform which is found, for instance, in the Ref. Erdélyi et al. (1953).

As this Green kernel is the inverse of the operator $k^2 - \frac{1}{2}\Delta_Z u$, we obtain the expression for the fundamental solution $\mathcal{E}_n(z)$ which is derived in Section 3 through a different method. (See the Appendix A).

5. Discussion and Concluding Remarks

The problem of finding an exact solution for American options on a multiple underlying asset has never been solved, even in the case of a perpetual option. As Firth (2005) claims, "Pricing single asset American options is a hard problem in mathematical finance.. . . Pricing multi–asset (high-dimensional) American options is still more difficult". As we discussed in the introduction, numerous simulation-based methods and numerical schemes have been proposed to find an approximate solution to this challenging pricing problem. The problem can be formulated starting from the stochastic differential equations modelling the underlying assets and then employing Monte Carlo simulations to derive approximate solutions; alternatively, one can consider the problem in its partial differential formulation and construct a discretization with good properties and accuracy. These approximation methods are based on traditional techniques which have been successfully employed with European-type derivatives, but need extra effort when applied to American-type derivatives, which require additional information about the future paths of all underlying assets.

One popular methodology employs the least-square Monte Carlo approach introduced by Longstaff and Schwartz (2001). While Monte Carlo methods are widely known and easy to implement in principle, they require simulation of the paths of each single stock; the combination with least-square estimations for American options further slows down the algorithm. Thus, the execution can become prohibitively time-consuming when the basket size underlying the option is large and various technical improvements have been developed to reduce the computational effort (e.g., multi-level Monte Carlo simulation along with proper numerical schemes, regression at the end of each time step, splitting of the simulation into several processes and parallel implementation, quantization methods).

Another class of approximation methodologies is based on finite difference approximations of the differential operator which also contains second-order mixed derivative terms. Thus, the classical finite difference methods lead to some off-diagonal entries in the matrix associated with the discretized operator, which determines oscillations in the computed solution and the difference approximation may become unstable. When the size of the underlying basket is large, the numerical scheme may become unstable, the results inaccurate, and sometimes pricing multi-dimensional derivatives may even be impossible. Some improvements have been devised to cope with these deficiencies related to multi-dimensionality (see Hanbali and Linders 2019; Wu 2013).

In any case, in the absence of an analytical solution, benchmarking the numerically computed solutions is a hard task and testing the accuracy of the approximation requires ad hoc theorems for each algorithm.

In this paper, instead of proposing another numerical procedure, we follow an analytical approach to solve the pricing problem for perpetual American options in the multidimensional case. We obtain an explicit expression for the option value which is written as an integral involving modified Bessel functions of the second kind. Although these functions are non-elementary ones, they are included in built-in-function packages in various numerical softwares, and thus can be computed numerically in practice.

At the end of Section 3 we provided an equation to determine the optimal exercise threshold. One limitation of this result is that this equation requires an n-dimensional integration, which makes effective evaluation difficult to perform for high dimensions. For low dimensions, the results is of practical use; for higher dimensions, it can be combined with a numerical solution and may serve as a benchmark to test the reliability of the numerically computed solution.

Finally, our analysis is limited to the case of perpetual options. Although they are of limited usage in the financial markets, the analysis of perpetual options suggests methods for pricing American options with finite maturity (Carr 1998). More importantly, our results are applicable to real options where usually no expiration date is prescribed to the decision process. For the real option theory, our result is valuable as it solves a long-standing open problem: how to express the value function of a multi-factor optimal decision problem in a mathematically rigorous form. This opens the way to further applications in management science.

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Appendix A. Bessel Functions and Modified Helmoltz Equation

Let K_ν denote a modified Bessel function of the second kind, which satisfies

$$K_\nu(x) = \int_0^\infty e^{-x\cosh(t)} \cosh(\nu t)dt, x > 0.$$

K_ν solves the differential equation:

$$y'' + \frac{1}{x}y' - (1 + \frac{\nu^2}{x^2})y = 0. \tag{A1}$$

Moreover, the following properties hold:

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}}e^{-x} \text{ for } x \to +\infty \tag{A2}$$

$$K_\nu(x) \sim \frac{\Gamma(|\nu|)}{2}\left(\frac{2}{x}\right)^{|\nu|} \text{ and } K_0(x) \sim ln\frac{1}{x} \text{ for } x \to +0 \tag{A3}$$

$$2K'_\nu(x) = -K_{\nu+1}(x) - K_{\nu-1}(x). \tag{A4}$$

Now we show that $\mathcal{E}_n(x) = \frac{-2}{(2\pi)^{n/2}}[\frac{|x|}{\sqrt{2}k}]^{1-n/2}K_{1-\frac{n}{2}}(\sqrt{2}k|x|)$ is the fundamental solution of

$$\frac{1}{2}\Delta_x u - k^2 u = 0,\ x \in R^n.$$

Let $\psi \in \mathcal{C}_0^\infty(R^n)$. Then

$$\left\langle [\frac{1}{2}\Delta_x - k^2]\mathcal{E}_n, \psi \right\rangle = \frac{c_n}{2}\int_{S_{n-1}}\int_0^\infty \rho^{1-\frac{n}{2}} K_{1-\frac{n}{2}}(\sqrt{2}k\rho)[\frac{\partial}{\partial\rho} - 2k^2](\rho^{n-1}\frac{\partial\psi}{\partial\rho})d\rho d\Theta$$

where the integral is written in polar coordinates, $c_n = \frac{-2}{(2\pi)^{n/2}}[\frac{1}{\sqrt{2}k}]^{1-n/2}$, $d\Theta = \sin^{n-2}\theta_1 \ldots \sin\theta_{n-1} d\theta_1 \ldots d\theta_{n-1}$, and the contribution of $\Delta_{S_{n-1}}$ is null as the function $\mathcal{E}_n$ is radial. In view of Equation (A1) we have

$$\frac{\partial}{\partial\rho}[\rho^{n-1}\frac{\partial}{\partial\rho}(\rho^{1-\frac{n}{2}}K_{1-\frac{n}{2}}(\sqrt{2}k\rho)] = 2k^2\rho^{\frac{n}{2}}K_{1-\frac{n}{2}}(\sqrt{2}k\rho)$$

and thus integration by parts in $\int d\rho$ yields:

$$\left\langle [\frac{1}{2}\Delta_x - k^2]\mathcal{E}_n, \psi \right\rangle = \frac{-1}{2}\int_{S_{n-1}}[\rho^{n-1}\frac{\partial\mathcal{E}_n}{\partial\rho}\psi]_0^\infty d\Theta.$$

Using (A2) and (A3) the last integral can be written as

$$\frac{-c_n}{2}(\frac{2}{\sqrt{2}k})^{n/2-1}\Gamma(\frac{n}{2})\psi(0)\int_{S_{n-1}}d\Theta = \psi(0).$$

Then $[\frac{1}{2}\Delta_x - k^2]\mathcal{E}_n$ is the Dirac distribution.

Note

[1] see the Ref. Broadie and Detemple (1997), for a real-world interpretation

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