

Article

Deriving Thermospheric Density from GNSS POD: An Orthogonal Polynomial Differentiation Approach for Accelerometer-Free LEO Satellites—A High-Fidelity Simulation Study

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Abstract

Accurate thermospheric mass density is crucial for precision orbit determination, trajectory prediction, and collision warning in the increasingly congested Low Earth Orbit (LEO) region. Traditional high-precision instantaneous inversion methods rely on dedicated onboard accelerometers (e.g., CHAMP and GRACE satellites), which are costly, scarce, and unable to meet the expansive monitoring needs of modern mega-constellations. In recent years, orbit data inversion methods based on high-precision orbit determination data have been extensively studied. Among them, the interpolation–differentiation method, although capable of retrieving instantaneous atmospheric density, severely amplifies position errors and measurement noise, significantly limiting retrieval accuracy. To address this gap, this study proposes a novel atmospheric density retrieval scheme for accelerometer-free LEO satellites, introducing an orthogonal polynomial (Legendre and Chebyshev) sliding-window fitting approach. The proposed method is evaluated using high-precision simulated orbit data. Under instantaneous inversion conditions, the Legendre method yields an average relative error (MRE) of 16.55%, a normalized root mean square deviation (NRMSD) of 12.80%, and a correlation coefficient R of 0.9370. To further evaluate its potential in climatological modeling, this paper also proposes a local solar time (LST) bin-averaging method to assess atmospheric inversion accuracy. Under this LST averaging approach, the inversion performance achieves an MRE of 6.83% and a correlation coefficient R exceeding 0.98. This demonstrates the feasibility of the inversion method for analyzing diurnal effects on atmospheric field evolution.



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1. Introduction

With the rapid expansion of mega-constellations (e.g., Starlink, Planet) and small satellites, low Earth orbit (LEO) is becoming increasingly congested. Precision orbit determination (POD) and orbit prediction are therefore critical for ensuring satellite operational safety, collision warning, and re-entry forecasting [1].

As the primary error source in LEO precision orbit determination and prediction, thermospheric mass density directly determines the magnitude of atmospheric drag and significantly influences the long-term evolution of satellite orbits [2,3]. Even minor errors in estimating atmospheric density can lead to large deviations in orbit predictions, posing serious threats to on-orbit safety—especially for large-scale constellations operating in highly congested orbital regions [4,5].

Thermospheric mass density is also a key parameter reflecting the state of the space environment. Variations in atmospheric density are closely linked to solar activity (e.g., solar flares, coronal mass ejections) and geomagnetic disturbances (e.g., geomagnetic storms) [6–8]. Hence, high-accuracy atmospheric density retrieval serves as an effective means to monitor Sun–Earth interactions and detect space–weather anomalies timely [9–11], providing valuable data resources for space–weather monitoring and early-warning systems.

As LEO mega-constellations continue to proliferate, there is a growing demand for real-time, high-sensitivity detection of short-term atmospheric variations. The long-term stable operation of such constellations urgently requires accurate and timely atmospheric density data as a fundamental guarantee [12–14].

Methods for obtaining thermospheric mass density can be classified into three categories [15], each with its own advantages and limitations. The first comprises empirical models (e.g., NRLMSISE-00 [16], JB2008 [17]), which are simple and fast but can exhibit errors exceeding 100% during storms [18,19]. The second involves onboard accelerometers (e.g., CHAMP [20], GRACE [21], GOCE [22], Swarm [23,24]), providing higher accuracy (~10% error) but are too scarce and costly for sustained global monitoring [25].

The third category comprises orbit-based retrieval, which infers density from precise orbit data. Owing to its low cost, wide coverage, and compatibility with most LEO satellites, this approach has gained increasing attention [20]. Two primary data sources are used: (1) two-line element (TLE) data, which are easily accessible but of low resolution, limiting their use to long-term studies; and (2) GNSS precision orbit determination (POD) data, which offer high precision and temporal resolution, and are preferred for high-accuracy, high-resolution retrieval [26]. Extensive research has been conducted in this area. For example, Doornbos et al. used TLE data with numerical orbit integration for low-cost global monitoring [27], while Andres et al. employed GNSS POD data with spline interpolation to improve accuracy [28]. Orbit-based methods generally follow either an energy-based or a dynamics-based approach [29]. Dynamics-based methods, which retrieve acceleration from orbit data before calculating density, face a key challenge: improving the accuracy of acceleration derived from discrete position sequences. Common techniques include interpolation–differentiation and semi-major axis decay, but the former amplifies noise and the latter lacks short-term precision [30]. Improved methods such as Markov processes [31] or Kalman filtering [32] have been proposed, yet their computational cost hinders onboard, real-time use. Moreover, numerical differentiation of orbit data remains an ill-posed problem that can severely amplify noise and produce unstable results. Figure 1 illustrates the GNSS-based mass density derivation process.

Furthermore, most CubeSats and mega-constellations lack dedicated payloads (e.g., high-precision accelerometers) for tracking atmospheric density [33], thus making cost-effective, autonomous sensing crucial for safety. Additionally, LEO satellites with short periods (e.g., ~90 min) cross the terminator multiple times daily, where solar-irradiation shifts cause large density fluctuations that amplify errors in traditional retrieval methods [34].

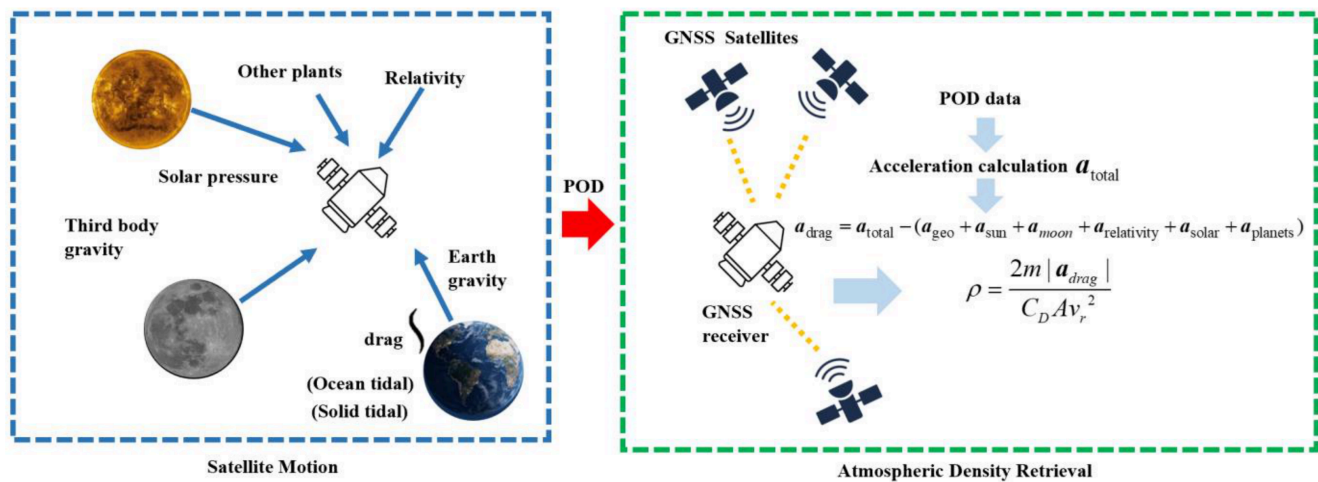


Figure 1. GNSS-based mass density derivation process.

Orthogonal polynomial fitting (e.g., using Chebyshev or Legendre bases) leverages orthogonality to suppress observational noise, mitigate ill-conditioning in normal equations, and improve numerical stability [35,36]. This well-established technique is widely used in astrodynamics for tasks such as interpolating GPS/BeiDou precise ephemerides and performing long-term LEO prediction, effectively overcoming divergence issues common in conventional high-degree polynomial fitting [37–39]. However, its application to atmospheric density retrieval remains underexplored [40]. A key strength of the method is that once polynomial coefficients for position/velocity are fitted, the non-conservative acceleration (including atmospheric drag) can be derived analytically through polynomial differentiation. This avoids the truncation errors and noise amplification inherent in numerical differentiation, making it theoretically better suited for high-precision drag-acceleration retrieval.

Therefore, to address the need for high-resolution, accurate atmospheric density data from accelerometer-less constellations/CubeSats, and to study small-scale thermospheric variations, this paper contributes:

- (1) A systematic error analysis of acceleration-interpolation-based density retrieval methods;
- (2) A novel density retrieval method based on orthogonal polynomial (Legendre/Chebyshev) fitting;
- (3) Optimization of the method's key parameters (polynomial degree, fitting window);
- (4) A local-time-averaging method to mitigate terminator-crossing effects;
- (5) A comparative evaluation demonstrating the proposed method's superior accuracy and stability over conventional interpolation–differentiation.

This work fills the research gap by proposing and validating a new GNSS POD-based density retrieval method suitable for LEO satellites without dedicated accelerometers.

2. Methods

2.1. Overall Technical Approach

The atmospheric density retrieval method proposed in this study comprises five main steps: (1) obtaining high-precision position time series of the LEO satellite via GNSS-based precision orbit determination (POD); (2) applying a sliding-window strategy and using orthogonal polynomials (Legendre or Chebyshev) to fit the position time series, thereby reducing the influence of orbital noise; (3) performing a second-order differentiation on the fitted curve to derive the satellite's total acceleration; (4) subtracting all gravitational forces, solar radiation pressure, relativistic effects, and other contributions from the total

acceleration to isolate the drag acceleration; and (5) inverting the atmospheric density using the drag formula, with the drag acceleration and satellite characteristic parameters (mass, drag coefficient, cross-sectional area) as inputs. A detailed description of each step is provided below.

However, this study employs model simulation data as the reference, which falls under proof-of-concept validation.

2.2. Fundamentals of Atmospheric Density Retrieval

In the Earth-Centered Inertial (ECI) frame, the satellite's equation of motion is given by Equation (1), where $\mathbf{a}_{\text{geo}}$ denotes the gravitational acceleration due to the Earth, $\mathbf{a}_{\text{sun}}$ and $\mathbf{a}_{\text{moon}}$ are the gravitational accelerations from the Sun and the Moon, $\mathbf{a}_{\text{drag}}$ represents the atmospheric drag acceleration, $\mathbf{a}_{\text{relativity}}$ denotes relativistic effects, $\mathbf{a}_{\text{solar}}$ denotes solar radiation pressure, and $\mathbf{a}_{\text{planets}}$ denotes to the gravitational accelerations from other celestial bodies excluding the Earth, Sun, and Moon.

$$\ddot{\mathbf{r}} = \mathbf{a}_{\text{total}} = \mathbf{a}_{\text{geo}} + \mathbf{a}_{\text{moon}} + \mathbf{a}_{\text{sun}} + \mathbf{a}_{\text{drag}} + \mathbf{a}_{\text{relativity}} + \mathbf{a}_{\text{solar}} + \mathbf{a}_{\text{planets}}, \quad (1)$$

Figure 2 illustrates, for a satellite at an altitude of 450 km, the variations and relative magnitudes of these perturbation forces over a 24 h orbital period.

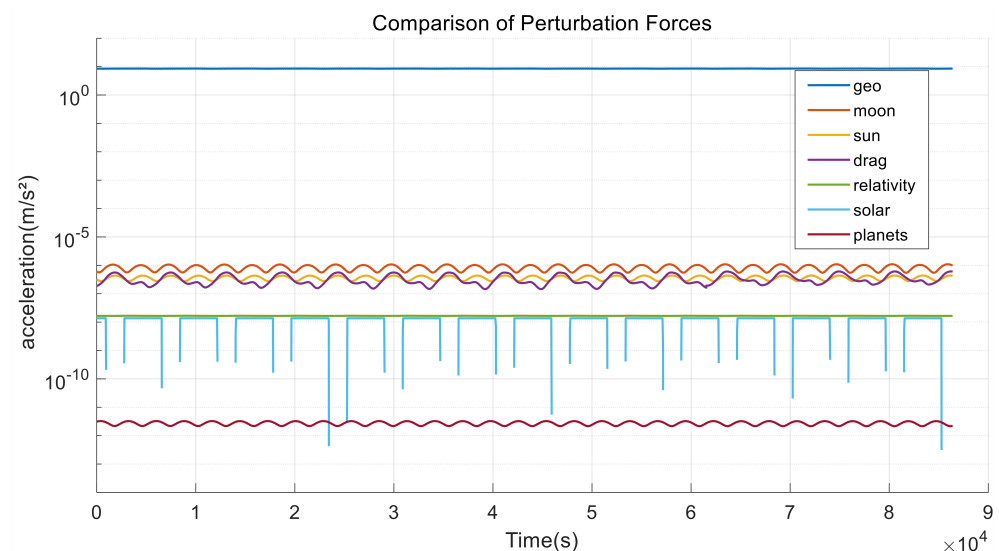


Figure 2. Comparison of perturbation forces. It illustrates the variation in magnitude of various perturbation forces at 450 km altitude over a 24 h period, with the epoch starting at 06:54 on 5 March 2026. The data is presented in a log-scale plot.

As shown in Figure 2 and confirmed by model calculations, within the altitude range of 200–600 km, the atmospheric drag acceleration is the dominant non-gravitational perturbation acceleration. Its magnitude is proportional to the atmospheric density. The atmospheric drag acceleration is given by:

$$\mathbf{a}_{\text{drag}} = \frac{1}{2} \frac{C_D A}{m} \rho \mathbf{v}_r \mathbf{v}_r, \quad (2)$$

where C_D is the drag coefficient; A is the satellite's cross-sectional area; m is the satellite mass; ρ is the atmospheric density (the parameter to be retrieved); $\mathbf{v}_r$ is the satellite's

velocity relative to the atmosphere; and v_r is the magnitude of $\mathbf{v}_r$. The atmospheric density ρ can be retrieved as follows [41]:

$$\rho = \frac{2m|\mathbf{a}_{drag}|}{C_D A v_r^2}, \quad (3)$$

According to Equation (3), the key to atmospheric density retrieval lies in accurately obtaining the atmospheric drag acceleration $\mathbf{a}_{drag}$. As shown in Equation (4), $\mathbf{a}_{drag}$ can be calculated from the $\mathbf{a}_{total}$ and the perturbation accelerations derived from known models. Therefore, the core challenge in orbit-based atmospheric density retrieval is to accurately extract $\mathbf{a}_{total}$ from a discrete series of orbital positions.

$$\mathbf{a}_{drag} = \mathbf{a}_{total} - (\mathbf{a}_{geo} + \mathbf{a}_{sun} + \mathbf{a}_{moon} + \mathbf{a}_{relativity} + \mathbf{a}_{solar} + \mathbf{a}_{planets}). \quad (4)$$

2.3. Orthogonal Polynomial Fitting with a Sliding Window

The conventional method for obtaining the total acceleration from position is interpolation–differentiation, which involves two steps: interpolation and differentiation. Interpolation enhances the temporal resolution of the existing data to improve differentiation accuracy. Common interpolation methods include Lagrange interpolation, Hermite interpolation, and cubic spline interpolation. The differentiation step calculates the instantaneous acceleration of the spacecraft by performing a second-order difference on its position vector. The formula for calculation is given in Equation (5).

$$\mathbf{a}_{total} = \lim_{\Delta t \rightarrow 0} \frac{\dot{\mathbf{r}}_{t_1} - \dot{\mathbf{r}}_{t_{-1}}}{2\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}_{t_2} - 2\mathbf{r}_{t_0} + \mathbf{r}_{t_{-2}}}{(2\Delta t)^2} \quad (5)$$

where $r, \dot{r}$ represent the position and velocity, and $t_{-2} \sim t_2$ denotes five adjacent time points, and Δt represents the time interval between two adjacent time points. Ref. [42] indicates that employing 8th-order Lagrange interpolation with a time interval of 0.05 s can achieve millimeter-level accuracy in the retrieved acceleration. However, the accuracy of this method is highly sensitive to the temporal resolution and noise level of the data. The interpolation–differentiation method using these parameters will be compared with the method proposed in this paper.

Our Method. The principle of the sliding-window orthogonal polynomial fitting method is: Given a discrete series of position observations $(t_i, y_i), i = 1, \dots, m$, where m is the number of data points within a fitting window, the sequence is fitted piecewise (via a sliding window) into a smooth, parameterized polynomial function. Acceleration is then obtained by taking the second derivative of this polynomial function. The specific steps are: time normalization, orthogonal polynomial fitting, acceleration calculation, and optimization of fitting parameters.

(1) Time Normalization

Prior to fitting, the sampling time t must be normalized to the interval $[-1, 1]$ to satisfy the definition domain of the orthogonal polynomials. The normalization formula is Equation (6).

$$t' = \frac{2(t - t_{\min})}{t_{\max} - t_{\min}} - 1 \quad (6)$$

Here, $t_{\min}$ and $t_{\max}$ the minimum and maximum values of the sampling time, and t' denotes the normalized time.

(2) Orthogonal Polynomial Fitting

Orthogonal polynomials possess orthogonality, meaning the integral of the product of two different orthogonal polynomials over a defined interval equals zero. This property renders the normal matrix of the fitting problem diagonal or nearly diagonal, thereby avoiding the ill-conditioned problems associated with non-orthogonal basis functions and improving the stability and accuracy of the fitting. This paper selects two commonly used orthogonal polynomials: Legendre polynomials and Chebyshev polynomials.

Legendre polynomials $L_k(t)$ are defined on the interval $[-1, 1]$. With the weight function $w(t) = 1$, the orthogonality [43] is expressed as Equation (7).

$$\int_{-1}^1 L_m(t)L_n(t)dt = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases} \quad (7)$$

Legendre polynomials satisfy the recurrence relation in Equation (8), with initial conditions $L_0(t) = 1$, $L_1(t) = t$.

$$(n+1)L_{n+1}(t) = (2n+1)tL_n(t) - nL_{n-1}(t) \quad (8)$$

From Equations (7) and (8), the explicit expression for Legendre polynomials can be derived as Equation (9).

$$L_n(x) = \sum_{m=0}^{[n/2]} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m}. \quad (9)$$

Chebyshev polynomials $T_k(t)$ are also defined on the interval $[-1, 1]$ with weighted orthogonality and the explicit expression is given in Equation (10), where $[n/2]$ denotes the floor function.

$$T_n(t) = \sum_{k=0}^{[n/2]} (-1)^k C_n^{2k} t^{n-2k} (1-t^2)^k. \quad (10)$$

Chebyshev polynomials satisfy the recurrence relation in Equation (11), with initial conditions $T_0(t) = 1$, $T_1(t) = t$.

$$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t). \quad (11)$$

Analytical expressions for the first and second derivatives of Chebyshev polynomials can be derived from the recurrence relation. Chebyshev polynomials have the property of uniform distribution of extrema, which effectively suppresses the Runge phenomenon.

Within the normalized time interval, the fitting model for the satellite position vector in the inertial frame is given as Equation (12).

$$\vec{r}(t') = \sum_{i=0}^n c_i p_i(t') \quad (12)$$

In the Equation (12), P_i is the i th order orthogonal polynomial, and c_i is the fitting coefficient. Leveraging the orthogonality and the QR decomposition method, the fitting coefficient matrix $\mathbf{c}$ ($\mathbf{c} \in \mathbb{R}^n$), $\mathbf{c} = (c_0, c_1, c_2, c_3, \dots, c_n)^T$ can be calculated directly, as shown in Equation (13), thereby avoiding the need to solve an ill-conditioned normal matrix and improving fitting stability.

$$\begin{aligned} \mathbf{c} &= (\mathbf{P}^T \mathbf{P})^{-1} \mathbf{P}^T \mathbf{y}, \\ \mathbf{P} &= (p_0(\boldsymbol{\tau}), p_1(\boldsymbol{\tau}), p_2(\boldsymbol{\tau}), p_3(\boldsymbol{\tau}), \dots, p_n(\boldsymbol{\tau}))^T, \\ \mathbf{y} &= (y_0, y_1, y_2, y_3, \dots, y_m)^T, \boldsymbol{\tau} = (t'_0, t'_1, t'_2, t'_3, \dots, t'_m). \end{aligned} \quad (13)$$

Here, τ is the normalized time sequence, $\mathbf{P} \in \mathbb{R}^{m \times n}$, and $\mathbf{y} \in \mathbb{R}^m$ is the sequence of satellite positions during the normalized time sequence.

(3) Acceleration Calculation

Once the fitting polynomial $\vec{r}(t')$ for the position series is obtained, the acceleration can be derived by taking the second derivative of the polynomial. Since orthogonal polynomials have well-defined analytical expressions, their first and second derivatives can be computed analytically, thereby avoiding the error accumulation associated with numerical differentiation.

For Legendre polynomials, the first derivative $P'_k(t')$ and the second derivative $P''_k(t')$ can be calculated using the recurrence relation Equation (14).

$$\begin{aligned} P'_k(t') &= \frac{k}{1-t'^2} [t' P_k(t') - P_{k-1}^{k-1}(t')] \\ P''_k(t') &= \frac{k}{1-t'^2} [P_k(t') + t' P'_k(t') - P'_{k-1}(t')] + \frac{2kt'}{(1-t'^2)^2} [t' P_k(t') - P_{k-1}(t')] \end{aligned} \quad (14)$$

For Chebyshev polynomials of the first kind, the first derivative $T'_k(t')$ and the second derivative $T''_k(t')$ can also be calculated using the recurrence relation Equation (15).

$$\begin{aligned} T'_k(t') &= k U_{k-1}(t') \\ T''_k(t') &= k U'_{k-1}(t') = \frac{k^2 T_k(t') - k(k-2) T_{k-2}(t')}{1-t'^2} \\ U_{k-1}(t') &= \frac{\sin[(k) \arccos t']}{\sqrt{1-t'^2}} \end{aligned} \quad (15)$$

where $U_{k-1}(t')$ represents the Chebyshev polynomial of the second kind of order.

(4) Denormalization

To convert the normalized time t' back to the original time t , and to obtain the analytical expression for acceleration in the original time domain, the second derivative of the position fitting polynomial $a(t)$ is computed with consideration of the time transformation $Tran^2$. The transformation is given by Equation (16).

$$\begin{aligned} Tran &= \frac{2}{t_{\max} - t_{\min}} \\ a(t) = \mathbf{r}''(t) &= \mathbf{r}''(t'(t)) \cdot Tran^2(t', t) \end{aligned} \quad (16)$$

(5) Optimization of Fitting Parameters

The performance of the orthogonal polynomial fitting is influenced by two key parameters: the polynomial degree and the size of the fitting window (number of data points). In this study, a grid search is employed to identify the optimal fitting degree. The search range for the polynomial degree is set from **4 to 10**, and the fitting window size is varied from **6 to 18** data points. The quality of the fit is evaluated using the Mean Relative Error (MRE) and the Standard Deviation (STD) between the retrieved atmospheric density and the reference (true) atmospheric density.

2.4. Average Density Calculation Methods

Analysis of the long-term evolution of atmospheric density typically focuses on statistical trends and periodic variations. Overly pursuing high-precision retrieval at specific locations is not only of limited physical significance but may also introduce observational noise and systematic errors, which can obscure the understanding of overall evolutionary patterns. Therefore, this paper adopts two temporal averaging methods to extract statistical characteristics of density.

The first method calculates the average density based on fixed time intervals. Continuous time is divided into equal-length intervals ΔT (e.g., 1 h), and the arithmetic mean of the retrieved densities $\bar{\rho}_{\Delta T}$ within each interval is computed:

$$\bar{\rho}_{\Delta T} = \frac{1}{N} \sum_{i=1}^N \rho_i. \quad (17)$$

Here, ρ_i is an individual value within the interval, and N is the number of samples in that interval.

For LEO satellites with orbital altitudes below 500 km, their orbital periods are short (approximately 90 min), causing them to cross the terminator multiple times per day. Around the terminator, solar irradiance undergoes abrupt changes, triggering rapid adjustments in atmospheric temperature and density, which lead to significant short-period fluctuations in the orbit-derived atmospheric density. To analyze the accuracy of atmospheric density retrieval under diurnal variations, this paper proposes a second method: calculating the average density based on the satellite's Local Solar Time (LST).

This method primarily focuses on the evolution of average density within a 24 h period, based on the binning approximation of the sub-satellite point longitude. Thus, the simplified formula shown in Equation (18) is sufficient to meet the accuracy requirements for calculating local solar time.

$$T_{\text{LST}} = t_{\text{UTC}} + \frac{\lambda}{15}. \quad (18)$$

where T_{LST} is the local solar time; t_{UTC} is the Coordinated Universal Time; λ is the geographical longitude in degrees.

Using Equation (19), the atmospheric density value during each orbital pass is associated with the corresponding local time. The 24 h of a day in local time are then divided into 24 consecutive bins (each 1 h wide). The arithmetic mean is calculated for all density samples falling within the j -th local time bin:

$$\bar{\rho}_{\text{LST}}(j) = \frac{1}{M_j} \sum_{k=1}^{M_j} \rho_k^{(j)}, \quad j = 1, 2, \dots, 24. \quad (19)$$

where M_j is the number of samples in the j th bin, and $\rho_k^{(j)}$ is the k -th retrieved density value within that bin.

According to Equations (18) and (19), a 24 h average atmospheric density curve with local time as the abscissa can be obtained. This method effectively smooths out instantaneous disturbances captured as the satellite passes different longitudes (i.e., different local times), particularly the dramatic variations near the terminator. The effectiveness of this method can be evaluated by comparing the correlation or consistency between the resulting average density curve and a reference true average density curve.

3. Experimental Design and Data

3.1. High-Fidelity Orbital Data

To establish a rigorous “ground truth” for evaluating the retrieval algorithm and to isolate the analysis from uncontrollable observational noise, we generated high-fidelity simulated benchmark data using the High Precision Orbit Propagator (HPOP). The dynamic force models included the 21×21 EGM2008 gravity field, third-body perturbations (Sun and Moon), solar radiation pressure with a conical shadow model, and atmospheric drag driven by the NRLMSISE-00 model. The satellite parameters are defined as follows: the atmospheric drag coefficient $C_d = 3.4$, the solar radiation pressure (SRP) co-

efficient $C_s = 1.4$, the SRP cross-sectional area $A_{SRP} = 1.01 \text{ m}^2$, the atmospheric drag cross-sectional area $A_d = 1.01 \text{ m}^2$, and the satellite mass $m = 480 \text{ kg}$. At the epoch, $t = 2026 - 03 - 05T06 : 54 : 22.000Z(UTC)$ the orbital elements of the satellite are: semi-major axis $a = 6821.533 \text{ km}$, eccentricity $e = 0.0032$, right ascension of the ascending node $\Omega = 211.8828^\circ$, argument of perigee $\omega = 92.9940^\circ$, true anomaly $f = 156.4804^\circ$, and inclination $i = 88.9132^\circ$. A variable-step Runge–Kutta 7 algorithm (ode113) was employed for numerical integration, with a 1 s step size and a 1 s sampling interval.

To verify the dynamic fidelity of the simulation, the generated orbit was compared with the actual GNSS-based precise orbit determination (POD) data of GRACE-A over the same period. The 3D position root mean square (RMS) difference over a 24 h arc was approximately 10 m. Since orbit perturbations induced by atmospheric drag are typically on the order of hundreds of meters, this 10 m discrepancy is negligible, demonstrating that the simulated orbit effectively mirrors the satellite's dynamic evolution in the real space environment.

The primary advantage of utilizing this high-fidelity simulated data is the provision of a fully controlled validation environment. First, the forward-modeled density sequence from NRLMSISE-00 serves as an absolute reference to quantify retrieval errors, a metric unachievable with empirical observational data. Furthermore, the simulation circumvents the interference of unmodeled dynamic forces (e.g., complex macro model errors) and instrument noise inherent in real data, ensuring that the performance evaluation strictly reflects the algorithm's inherent capability to extract atmospheric signals. By forming a complete “density input–orbit propagation–density retrieval” closed loop, this experimental framework offers a robust theoretical and quantitative validation basis for the novel averaging method proposed in this study.

3.2. Evaluation Metrics

To assess the performance of different atmospheric density retrieval methods, three metrics are employed: the Mean Relative Error (MRE), the Correlation Coefficient (R), and the Normalized Root Mean Square Deviation (NRMSD). The MRE is used to quantify the accuracy of the retrieved results. The correlation coefficient (R) evaluates the consistency in variation trends between the retrieved density series and the reference density series. The NRMSD analyzes the distribution characteristics of errors and systematic biases. The formulas for these metrics are defined as Equation (20).

$$\begin{aligned} \text{MRE} &= \frac{1}{N} \sum_{i=1}^N \left| \frac{\rho_i^{\text{inv}} - \rho_i^{\text{ref}}}{\rho_i^{\text{ref}}} \right| \\ R &= \frac{\sum_{i=1}^N (\rho_i^{\text{inv}} - \bar{\rho}^{\text{inv}})(\rho_i^{\text{ref}} - \bar{\rho}^{\text{ref}})}{\sqrt{\sum_{i=1}^N (\rho_i^{\text{inv}} - \bar{\rho}^{\text{inv}})^2 \cdot \sum_{i=1}^N (\rho_i^{\text{ref}} - \bar{\rho}^{\text{ref}})^2}} \\ \text{NRMSD} &= \frac{\text{RMSE}}{\bar{\rho}^{\text{ref}}} = \frac{1}{\bar{\rho}^{\text{ref}}} \sqrt{\frac{1}{N} \sum_{i=1}^N (\rho_i^{\text{inv}} - \rho_i^{\text{ref}})^2} \end{aligned} \quad (20)$$

In Equation (20), ρ_i^{inv} and ρ_i^{ref} represent the retrieved density value and the reference density value for the i th sample, respectively. $\bar{\rho}^{\text{inv}}$ and $\bar{\rho}^{\text{ref}}$ denote the sample means of the retrieved series and the reference series, respectively.

4. Results

Given the quasi-stationary nature of the dynamical environment within a short arc, we first extracted a 1000 s arc in Section 4.1 and performed a grid search to efficiently determine the optimal fitting parameters (polynomial order and window size). Subse-

quently, in Section 4.2, we applied these optimal parameters to the full 24 h orbital arc to comprehensively evaluate the retrieval accuracy, stability, and generalization capability under long-period, varying dynamical conditions. To assess the long-term characteristics of atmospheric evolution, we presented and analyzed the proposed Local Solar Time (LST) averaging method in Section 4.3.

4.1. Comparison of Retrieval Performance Using Optimized Parameters

Polynomial fitting of discrete position sequences is a common data-smoothing and interpolation method in the precise orbit determination and data processing of GPS satellites. Fitting accuracy is affected by the data sampling interval and the length of the fitting arc, requiring adjustment of the polynomial degree and arc length according to the specific task. To avoid the accumulation of polynomial fitting errors due to the extension of orbital arcs, this study employs a sliding-window strategy, ensuring that the fitting depends solely on the data within the current window. The original orbit data used have a sampling interval of 1 s (corresponding to the standard temporal resolution of GPS precise orbit determination), and each point used in the retrieval corresponds to the data within the truncated sliding window.

For GPS satellites, a polynomial degree of 8 to 10 is typically sufficient to achieve centimeter-level orbit fitting accuracy. However, LEO satellites like GRACE are significantly affected by perturbations such as atmospheric drag, characterized by high orbital velocity, short orbital periods, and complex perturbation variations, potentially requiring a higher polynomial degree to capture subtle dynamical changes. Meanwhile, the length of the fitting window must be carefully selected: a window that is too long may cause the polynomial to fail in capturing high-frequency perturbations, while a window that is too short may not reflect the trending characteristics of orbital motion.

Since, within a relatively short window (less than one orbital period), the variations in the main perturbing forces acting on the satellite (such as atmospheric drag and Earth's non-spherical perturbations) are relatively smooth, the dynamical environment can be considered quasi-stationary, and the linearization error of the local computational model is small. Therefore, if optimal fitting accuracy can be achieved within a certain short arc, the same parameter combination should maintain stable accuracy in other arcs with similar dynamical characteristics, and even over longer periods—implying that locally optimal parameters possess a certain degree of global robustness. Thus, by optimizing parameters within a limited window, the optimal parameters applicable to the entire data arc can be obtained, thereby avoiding the high computational cost associated with performing global optimization over the entire long arc.

In this paper, the sampling interval is set to 1 s. Preliminary validation showed that a 1000 s arc is appropriate. An arc of 1000 s is sufficient to capture the peak-to-valley variations in short-term perturbation acceleration, while using longer arcs (e.g., a full orbital period) for optimization yielded no change in the optimal parameters. The number of points in the sliding window and the range of polynomial degrees are adjusted in a timely manner based on the properties of the polynomials and the fitting performance. Through a grid search, the parameter combination that minimizes the mean relative error is sought under the premise of ensuring stability (minimum standard deviation). The results are as follows:

Figure 3a,b present heatmaps and line plots of the Mean Relative Error (MRE) for the Legendre and Chebyshev methods across different polynomial orders (horizontal axis) and fitting window sizes (vertical axis). As shown in Figure 3a, both methods achieve the minimum MRE of 5.66% with the parameter combination of polynomial order 5 and a window size of 6 data points. A polynomial order of $n = 6$ acts as a dividing line, separating

regions of relatively low and high MRE for both methods. Notably, the Chebyshev method exhibits very low MRE (dark blue areas in the heatmap) for most window sizes when the polynomial order is ≤ 6 . However, when the order increases to 7 or higher—particularly with small window sizes (e.g., 6, 8 points)—the relative error increases sharply. In contrast, the Legendre method shows the opposite trend: it yields higher errors (yellow areas) for orders ≤ 4 , but the error decreases significantly for orders ≥ 7 , especially with larger window sizes. While the Legendre method shows localized high errors for orders ≤ 6 with larger windows, its error rapidly decreases as both the order and window size increase. Figure 3b reflects the same pattern: the error of the Chebyshev method increases for orders > 6 , whereas the Legendre method shows higher errors only for orders ≤ 6 , with error decreasing as the order increases. For orders > 6 , the MRE of the Legendre method stabilizes in the 5–6% range. Therefore, the Chebyshev method performs well for orders ≤ 6 but is highly sensitive to higher orders combined with small windows, leading to a steep error increase. The Legendre method maintains low error across most parameter combinations, with MRE stabilizing for orders ≥ 6 . Synthesizing the information from Figure 3a,b, the optimal polynomial order for both methods is around 6 (e.g., 5, 6, or 7).

Figure 3c illustrates the stability of both methods across the same parameter space. The Chebyshev method exhibits severe numerical instability when high polynomial orders are combined with small window sizes (e.g., 6, 8 points). In contrast, larger window sizes (≥ 10 points) demonstrate good stability across all orders. The Legendre method may show initial instability for low-order, large-window combinations, but increasing the order (to ≥ 5) and window size significantly enhances its stability.

Consequently, considering both stability and error performance, for the Chebyshev method, to avoid numerical instability and error explosion, the polynomial order must remain low. Selecting an order ≤ 6 ensures very low error and high stability for all tested window sizes (from 6 to 18 points). Since the minimum MRE is achieved with order 5 and a window size of 6, this combination (Order = 5, Window Size = 6) is selected as optimal for Chebyshev. For the Legendre method, optimal performance requires an order ≥ 5 , under which both error and stability remain excellent regardless of window size. Balancing computational speed and simplicity, the same combination (Order = 5, Window Size = 6) is also chosen as optimal for the Legendre method.

The convergence of both methods to the same optimal parameters is dictated by the inherent mathematical and physical constraints of the short-arc orbit fitting problem for LEO satellites, rather than the specific choice of orthogonal polynomial basis. Specifically, for the high-frequency (1 s sampling), perturbation-rich orbital data of an LEO satellite, the motion within a very short fitting arc (6 s) can be sufficiently approximated by a low-order (5th-degree) polynomial. Increasing the polynomial order or window size may slightly improve local fitting but significantly increases the risk of overfitting and reduces the model's global generalizability. Therefore, the consistency of the optimal parameters reflects a universal trade-off between model complexity and data information content, given the specific data characteristics and engineering constraints. The fact that the Legendre method can achieve even better accuracy and stability with increased order and window size suggests that its functional basis is particularly well-suited for characterizing the local variational features of this type of orbital data.

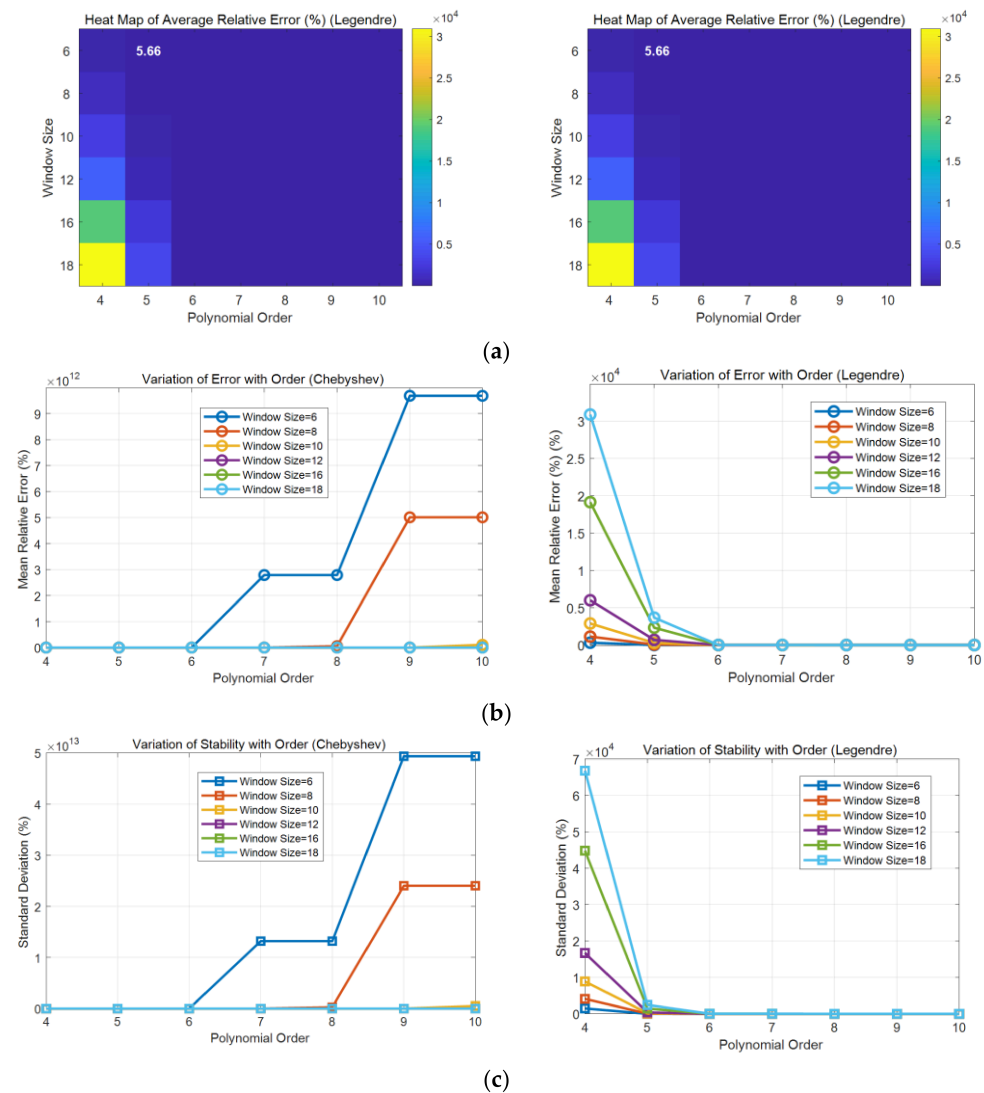


Figure 3. Performance comparison of Chebyshev and Legendre polynomial fitting methods. (a) Heatmap of the MRE (%) for the Chebyshev (left) and Legendre (right) polynomial methods across different window sizes (vertical axis) and polynomial orders (horizontal axis). The color bar indicates the error magnitude, with brighter colors (yellow) representing higher errors. (b) The variation in relative error with increasing polynomial order for the Chebyshev (left) and Legendre (right) methods, with line colors indicating different window sizes. (c) The variation in stability with increasing polynomial order for the Chebyshev (left) and Legendre (right) methods, with line colors indicating different window sizes.

Figure 4 illustrates the retrieval performance of the two polynomial fitting methods after parameter optimization. Within a 1000 s test arc, Figure 4a provides a direct comparison between the atmospheric density time series retrieved by the optimized Chebyshev and Legendre methods and the reference values from the NRLMSISE-00 model. The results show that both retrieved curves closely follow the variations in the reference model, indicating that the optimized parameter configurations effectively capture the main features of atmospheric density changes. Figure 4b further plots the relative error series of both methods relative to the reference model over the entire arc. It can be observed that the dynamic changes in the error curves for both methods are highly consistent, fluctuating around the mean value (5.66%) within a certain range, not exceeding 15%. Both methods exhibit a synchronized error trough around the 300 s mark in the arc, suggesting that parameter optimization has enabled both methods to achieve comparable accuracy levels.

and consistent error patterns. To more precisely quantify the consistency between the two methods, Figure 4c displays the series of point-by-point differences in relative error (Chebyshev minus Legendre). This difference series exhibits random fluctuations with extremely small amplitudes (on the order of approximately $\pm 1 \times 10^{-4}\%$) around the zero line, indicating that the precision of the results retrieved by the two methods is nearly equivalent for this arc, with no significant systematic bias.

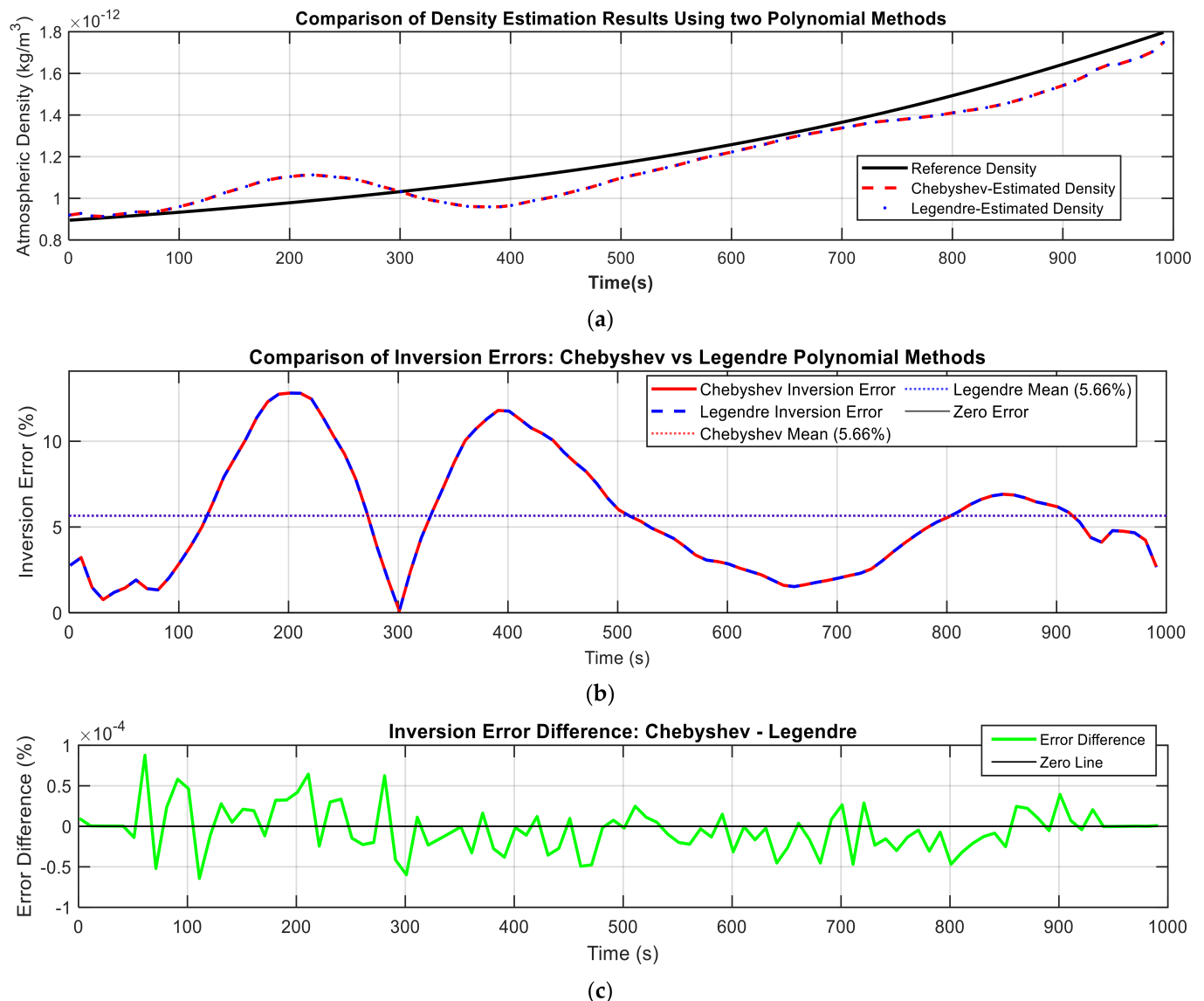


Figure 4. Comparison of inversion performance between Chebyshev and Legendre polynomial methods. (a) Time series of density estimation results using Chebyshev and Legendre polynomial methods. The black solid line represents the reference (true) density, while the red dashed and blue dotted lines depict the density estimated by the Chebyshev and Legendre methods, respectively. (b) Time series of Inversion Errors. The red and blue lines correspond to the instantaneous errors of the Chebyshev and Legendre methods, with their respective mean error values (5.66%) annotated. (c) Time series of the inversion error difference (Chebyshev minus Legendre). The green line shows the instantaneous difference. Positive values indicate that the Chebyshev method yields a larger error than the Legendre method at that epoch.

In summary, the optimized methods demonstrate excellent overall performance on the test arc. The calculated metrics for the retrieval over this arc are as follows: the Mean Relative Error (MRE) is 5.6567%, indicating high retrieval accuracy; the Normalized

Root Mean Square Deviation (NRMSD) is 0.085352, further confirming that the overall deviation between the retrieved results and the reference values is small; and the correlation coefficient is 0.967385, demonstrating a high degree of consistency in the variation patterns between the retrieved series and the reference model. These three quantitative metrics collectively verify that, after targeted parameter optimization, both the Chebyshev and Legendre methods can provide stable, reliable atmospheric density retrieval results that are in strong agreement with the physical model for this application scenario.

4.2. Comparison of 24-Hour Retrieval Accuracy

To compare the performance of Legendre and Chebyshev polynomials in atmospheric density retrieval, evaluate the advantages of orthogonal polynomials over the traditional interpolation–differentiation method over longer arcs, and ensure reproducibility and rigor, the parameter settings for each method are described and justified below.

(1) Implementation of the baseline method. It has been verified in Ref. [42] that the interpolation–differentiation method, employing 8th-order Lagrange interpolation with a fixed step of 0.05 s, can achieve millimeter-level acceleration retrieval accuracy. Further increasing the interpolation order or switching to alternative interpolation polynomials does not yield significant improvements in retrieval accuracy, but instead increases computational complexity. Accordingly, all comparative experiments in this section (e.g., Table 1 and Figure 5) uniformly adopt this parameter configuration without further modification. Specifically, between every two original sampling points, a Lagrange polynomial is constructed using a total of nine nodes (i.e., an 8th-order stencil), and the position sequence at intermediate epochs is interpolated at uniform intervals of 0.05 s. Central differences are then applied to the interpolated high-density sequence to obtain estimates of velocity and acceleration. Although the raw experimental data have a sampling rate of 1 Hz (i.e., a 1 s interval), this baseline achieves an equivalent resampling rate of 20 Hz through the aforementioned fine-grained interpolation, thereby theoretically improving the accuracy of numerical differentiation.

(2) Implementation of our method. Under the sliding-window strategy, each window is independently and dynamically fitted without sharing parameters. Moreover, within any given arc, the perturbations inside the sliding window are nearly constant, making the fitting problem mathematically identical across windows. The fitting accuracy depends solely on the data quality within the window and the window parameters (order and length), and is independent of the total arc length used for parameter optimization. Accordingly, the optimal parameters obtained from the grid search in Section 4.1 are adopted (order = 5, window = 6). Multiple sub-arcs randomly extracted from the 24 h dataset were independently optimized, and the results consistently converged to an order range of 5–6 and a window length of 6–8 s, which is highly consistent with the main optimization outcome, ruling out fortuitous results.

Table 1. Evaluation metrics for the three inversion methods.

	MRE (%)	NRMSD (%)	R
Legendre	16.55	12.80	0.9370
Chebyshev	16.65	12.90	0.9380
Interpolation–Differentiation	1299.05	1267.83	0.1281

The retrieval results are shown in Figure 5, and the evaluation metrics are listed in Table 1.

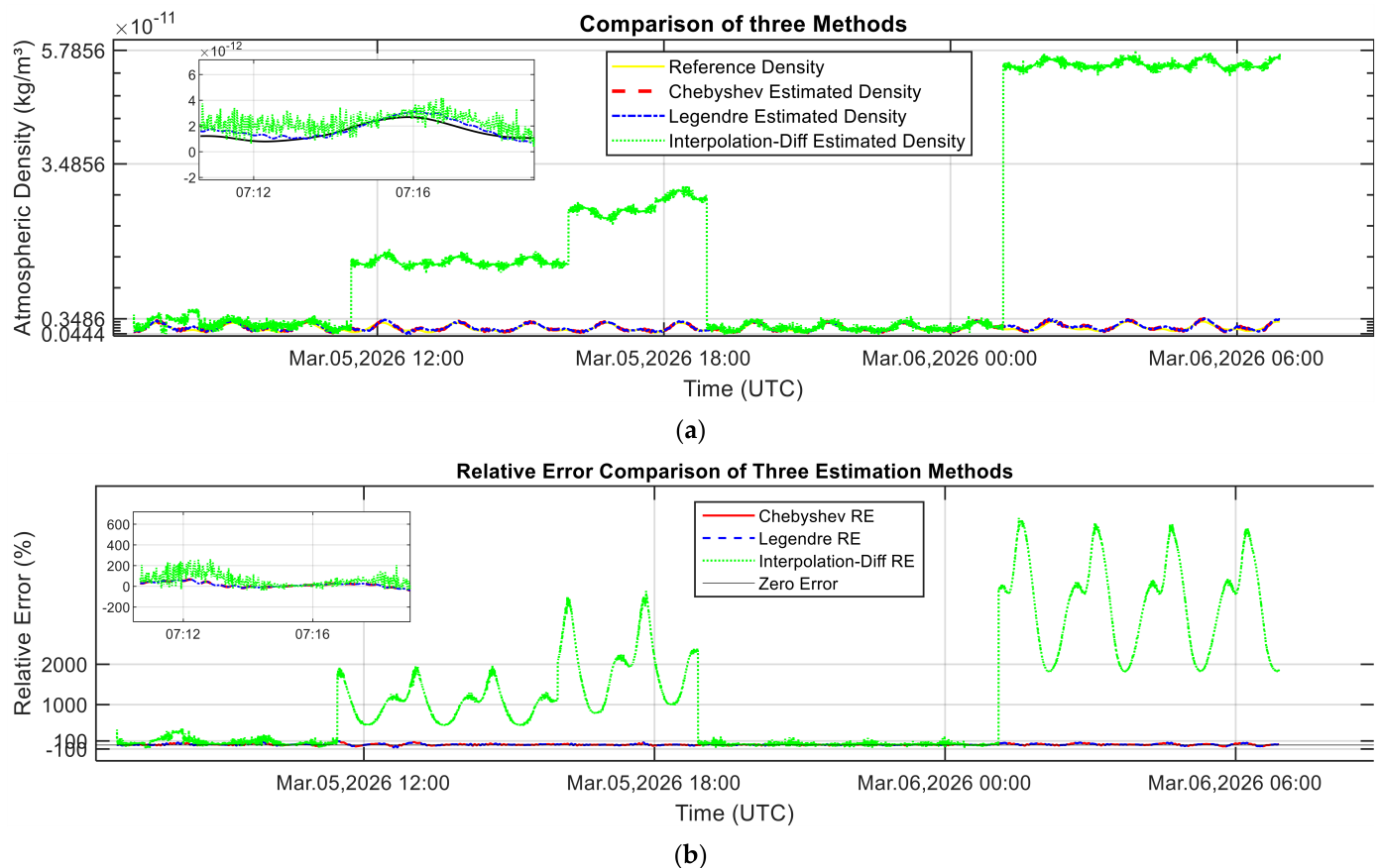


Figure 5. Comparison of density estimation and relative error for three methods. (a) Time series of atmospheric density using Chebyshev, Legendre, and interpolation–differentiation polynomial methods from 12:00 UTC on 5 March to 06:00 UTC on 6 March 2026. (b) Time series of the relative error using Chebyshev, Legendre, and interpolation–differentiation polynomial methods. The black solid line denotes the zero-error baseline. The interpolation–differentiation method shows extreme error peaks exceeding 100%, whereas the Chebyshev and Legendre methods maintain stable errors fluctuating near the baseline. The inset zooms in on a short segment (07:12–07:16 UTC) to illustrate the detailed error behavior.

As shown in Figure 5, over the 24 h test arc, the retrieval performance of both the parameter-optimized Chebyshev and Legendre polynomial methods is significantly superior to that of the traditional interpolation–differentiation method. The traditional method exhibits intense, discontinuous abnormal spikes at multiple intervals (e.g., in the latter part of the arc), severely deviating from the true density trend. This indicates severe numerical instability, rendering it unreliable for retrieving continuous, physically reasonable density values. Its error series appears highly unstable, with outliers exceeding 100%—and even reaching two orders of magnitude—at multiple time points. These correspond to the abnormal spikes observed in Figure 5a, highlighting the method’s lack of reliability for long-arc retrieval.

As indicated in Table 1, for the 24 h arc retrieval at 1 s resolution, the Legendre method achieves the optimal result with an MRE of 16.55%, an NRMSD of 12.80%, and a correlation coefficient R of 0.9370. The metrics for the Chebyshev method are very close to those of Legendre, with an MRE of 16.65%, an NRMSD of 12.90%, and an R of 0.9380. Both far outperform the interpolation–differentiation method in terms of both accuracy and predictive capability. This demonstrates that the proposed methods possess high temporal resolution, enabling them to capture short-term variations in atmospheric density.

Both orthogonal polynomial basis functions exhibit nearly identical capability in fitting the simple underlying trends of the data. However, considering the overall performance and robustness discussed in Sections 4.1 and 4.2, owing to its wider stable parameter range and superior adaptability to higher-order fitting, the Legendre method is selected for the subsequent assessment of the atmospheric retrieval capability of the orthogonal polynomial fitting approach.

4.3. Comparison of Averaged Density Evolution Accuracy

In practical applications, atmospheric density is often used in the form of a time-averaged value over a certain period to reduce the effects of noise. Using a sliding window of 1 h, averages were calculated based on both UTC and Local Solar Time (LST). The retrieval performance for the averaged density is shown in Figure 6.

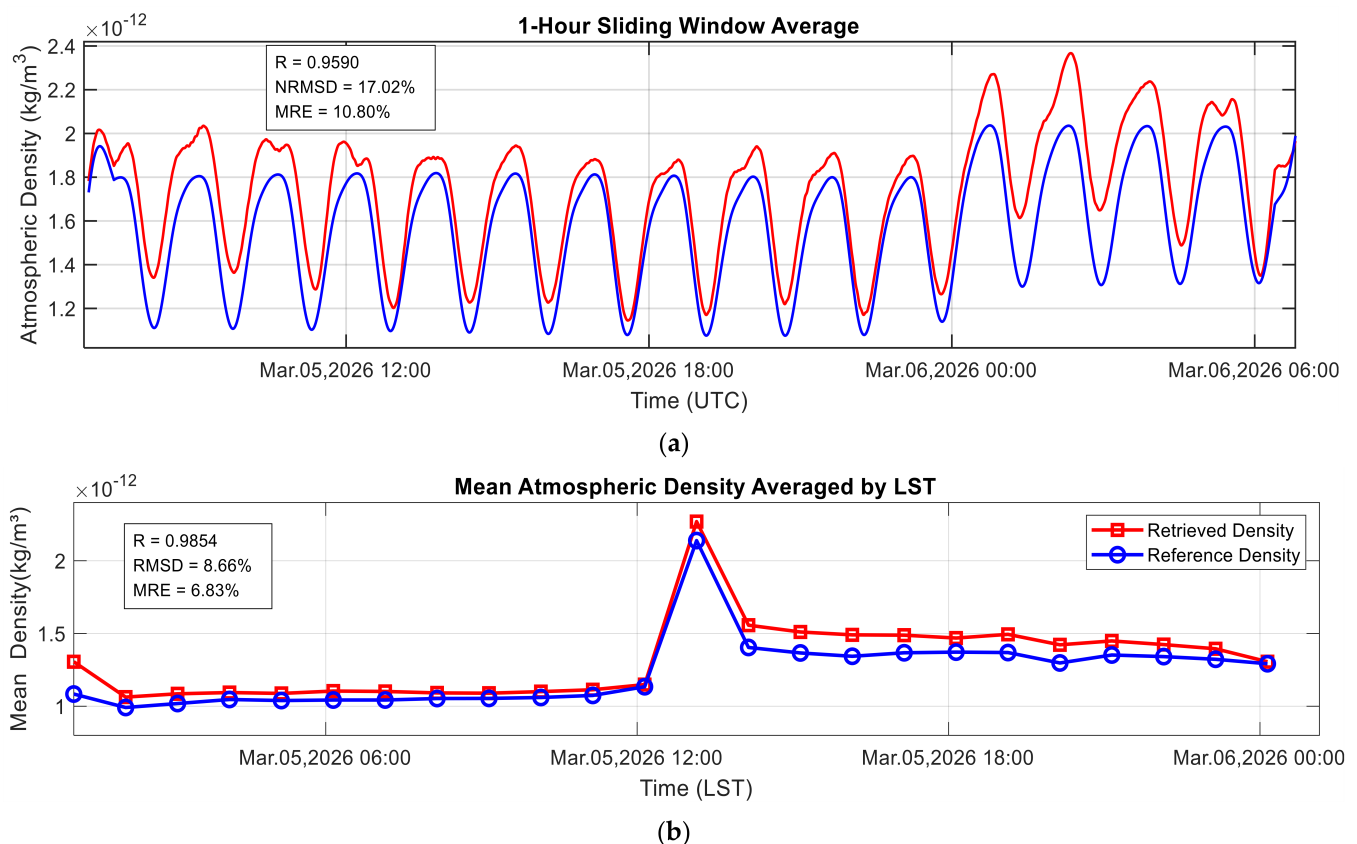


Figure 6. Comparison of retrieved and reference atmospheric density using 2 averaging methods. (a) One-hour sliding window average. Time series of the true density (blue line) and the retrieved density (red line) from 12:00 UTC on 5 March to 06:00 UTC on 6 March 2026. (b) Mean density averaged by local solar time (LST). The true (blue circles) and retrieved (red squares) densities are binned and averaged according to LST, revealing the diurnal variation pattern.

Figure 6a,b illustrate the optimization effects on the atmospheric density retrieval results achieved by two post-processing methods: the “1 h sliding-window average” (using Equation (17)) and the “LST-based average” (using Equation (19)). A comprehensive analysis demonstrates that the LST-based averaging method more effectively improves retrieval accuracy and reveals the inherent physical characteristics of atmospheric density.

First, the LST-based averaging significantly enhances the overall accuracy and stability of the retrieval. Compared to the results 1 h UTC sliding mean in Figure 6a, the metrics for the LST-averaged results show marked improvement: the correlation coefficient (R) increased from 0.9590 to 0.9854, indicating a very high consistency in the variation patterns

between the retrieved and true values; the NRMSD was substantially reduced from 17.02% to 8.66%; and the MRE improved from 10.80% to 6.83% (below the 10% threshold commonly used in engineering applications). This improvement in accuracy primarily stems from the effective suppression by LST averaging of short-term, high-frequency fluctuations and “abrupt change” interference caused by periodic orbital motion and terminator crossings, thereby allowing the retrieved results to more clearly reveal the inherent, smooth variation trend dominated by solar radiation.

Second, the retrieval results accurately reproduce the spatiotemporal distribution characteristics of the diurnal variation in atmospheric density. As clearly shown in Figure 6b, both the true and retrieved densities exhibit a typical diurnal pattern: during the daytime period from approximately 12:00 to 16:00 LST, atmospheric density reaches higher values due to heating by solar extreme ultraviolet radiation; during the nighttime period from approximately 22:00 to 06:00 LST, the density decreases to lower values. The optimized retrieval method not only perfectly replicates this diurnal pattern, but its curve nearly coincides with the true value curve. This provides strong evidence that the retrieval framework proposed in this study possesses excellent spatial adaptability and physical consistency, enabling it to reliably capture the macroscopic evolution of atmospheric density with varying solar illumination conditions.

However, it should be noted that while the fixed-LST-window averaging method improves the accuracy of long-term trends, it inevitably smooths out genuine instantaneous variations. This method effectively mitigates the negative impact of gradient discontinuities near the terminator on overall statistical metrics, but it cannot resolve instantaneous errors caused by rapid dynamical processes shorter than the averaging window (e.g., 1 h). Therefore, this method is more suitable for application scenarios that prioritize trend accuracy and long-term statistical characteristics (e.g., atmospheric model validation, long-term density modeling). For real-time spacecraft orbit prediction tasks that require capturing rapid perturbations, its use requires careful consideration.

In summary, the retrieval method proposed in this study, combined with LST-based post-processing averaging, enables high-accuracy ($\text{MRE} < 7\%$), high-correlation ($R > 0.98$) atmospheric density retrieval, accurately reflects its core spatiotemporal distribution characteristics, and meets the density data accuracy requirements for high-precision atmospheric modeling and most aerospace engineering applications.

4.4. Sensitivity Analysis

To evaluate the robustness of the proposed orthogonal polynomial differentiation method under realistic uncertainties, a series of sensitivity experiments were conducted. The nominal configuration (Legendre polynomial, order = 5, window length = 6, 1 Hz sampling, 450 km altitude, $F10.7 = 140$, $K_p = 3$, $C_d = 3.4$) served as the baseline. Each parameter was perturbed individually while keeping others fixed. The perturbed parameters include sampling interval (1 s, 2 s, 5 s, 10 s), orbital altitude (400 km, 450 km, 600 km), solar activity level ($F10.7 = 70, 140, 200$ sfu), geomagnetic disturbance ($K_p = 0, 3, 6$), drag coefficient ($C_d = 2.72, 3.40, 4.08$) and area-to-mass ratio (nominal $\pm 10\%$). The 24 h density retrieval MRE was used as the primary metric. To further assess the method's tolerance to observation errors, GNSS position noise is also added to each Cartesian component (Gaussian white noise, $\sigma = 3$ cm, 6 cm, 10 cm). The results are summarized in Table 2.

Table 2. Sensitivity analysis results: 24 h density retrieval MRE (%) under various parameter perturbations.

Parameter	Value/Range	MRE (%)
Baseline	—	16.55
Sampling interval	2 s	17.03
	5 s	18.41
	10 s	19.82
	400 km	16.11
Orbital altitude	500 km	15.73
	600 km	23.84
Solar activity (F10.7)	70	14.91
	200	18.32
Kp	0	15.44
	6	19.67
Drag coefficient (Cd)	2.72 (−20%)	19.85
	4.08 (+20%)	15.12
Area/mass	−10%	14.81
	+10%	18.33
GNSS position noise(Added to X/Y/Z)	3 cm	17.65
	6 cm	19.89
	10 cm	26.30

The above sensitivity patterns align with physical expectations. Coarsening the sampling rate degrades accuracy gracefully, with MRE remaining below 20% even at 10 s sampling. The method performs best at 500 km altitude and worst at 600 km altitude, because the sharp decrease in atmospheric density at higher altitudes makes the drag acceleration and noise more difficult to distinguish. Solar maximum and strong geomagnetic storms increase thermospheric variability, raising MRE to 18.32% and 19.67%, respectively. A $\pm 20\%$ variation in Cd causes an MRE change of approximately $\pm 3.5\%$, indicating moderate sensitivity to ballistic coefficient uncertainty. A $\pm 10\%$ change in A/m results in an MRE variation of about $\pm 1.8\%$, showing relatively low sensitivity to satellite parameter errors. Overall, the proposed method maintains a 24 h MRE below 20% across all tested scenarios, confirming its robustness to a wide range of space environments and satellite platform uncertainties. The most influential factors are geomagnetic activity and solar activity, as they alter the spatiotemporal distribution characteristics of atmospheric density, followed by drag coefficient uncertainty. Although the average drag coefficient affects short-term density estimation accuracy, the long-term estimation error remains acceptable.

As for GNSS position noise, the addition of Gaussian white noise with $\sigma = 3$ cm, 6 cm, and 10 cm raises the MRE from the baseline of 16.55% to 17.65%, 19.89%, and 26.30%. Notably, for noise levels typical of LEO GNSS POD accuracy ($\sigma \leq 6$ cm), the MRE remains below 20%, and even under extreme noise ($\sigma = 10$ cm), the error increase remains bounded without catastrophic failure. This demonstrates that the proposed method possesses inherent robustness to random measurement errors, achieving usable accuracy without dedicated preprocessing.

5. Discussion

Compared to the traditional interpolation–differentiation method, the proposed orthogonal polynomial fitting-based method offers the following advantages:

High Stability: The orthogonality of the polynomial basis avoids the ill-conditioning of the normal matrix.

High Accuracy: Orthogonal polynomial fitting effectively suppresses orbital noise, and acceleration is obtained via analytical differentiation, avoiding error accumulation from numerical differentiation. The proposed method achieves a Mean Relative Error (MRE) of only about 16.5% over a 24 h evolution, representing an improvement of at least two orders of magnitude over the traditional method.

Strong Adaptability. The method exhibits strong adaptability to LEO satellites such as GRACE. Traditional methods suffer from time-resolution limitations, where longer sampling intervals introduce significant errors. In contrast, the orthogonal polynomial differentiation method proposed here is more tolerant of longer intervals. As shown in Table 2, when the sampling interval is relaxed from 1 s to 2 s, 5 s, and 10 s, the 24 h averaged density retrieval accuracy (MRE) remains below 20% using the Legendre method. Furthermore, the method demonstrates similar robustness to other sources of uncertainty: the MRE stays below 20% under moderate variations in orbital altitude (400–500 km), solar activity ($F_{10.7} = 70\text{--}200$ sfu), geomagnetic disturbance ($K_p = 0\text{--}6$), drag coefficient ($\pm 20\%$), and area-to-mass ratio ($\pm 10\%$), as well as under GNSS position noise up to 6 cm without any dedicated preprocessing. Only under extreme conditions, such as an altitude of 600 km or noise levels of 10 cm, does the MRE exceed 20%. These results collectively confirm that the proposed method offers robust and adaptable performance across a wide range of operational scenarios, making it a promising candidate for accelerometer-free thermospheric density retrieval on LEO satellites.

Ease of Implementation: Fitting coefficients for orthogonal polynomials can be directly calculated using their orthogonality, simplifying implementation. For the same retrieval arc, the traditional method significantly increases data volume, demanding more computational resources. The proposed method can perform near-real-time retrieval as long as the number of data points meets the fitting requirements, without data volume expansion.

Analyzing the method's onboard real-time applicability from computational workload and storage. Each sliding window (6 points, 5th-order polynomials) requires a single QR-based least-squares fit and analytical differentiation, yielding extremely low floating-point operations. In contrast, interpolation–differentiation demands $20\times$ data upsampling (interpolating 1 s to 0.05 s), the energy method requires numerical integration, and the Kalman filter involves recursive covariance updates and long-arc storage—all incurring higher costs. Thus, the proposed method offers clear advantages in efficiency and storage. However, this analysis is based on simulated data. Actual onboard observations may contain outliers, residual ionospheric delays, and receiver clock jumps, potentially causing fitting distortion. Preprocessing steps such as outlier rejection and adaptive smoothing may be necessary before engineering deployment. Whether the method can achieve stable autonomous real-time operation remains to be tested in realistic GNSS data.

However, the method has some limitations: (1) It is currently based on HPOP simulation data and requires further validation with real satellite orbit data (e.g., from GRACE). (2) While it reduces data requirements, offline parameter optimization is needed, and parameters are sensitive to the environment, necessitating intelligent adjustment methods. (3) The Local Time (LST) averaged density method can mitigate the impact of terminator crossings but cannot completely eliminate errors caused by instantaneous atmospheric density variations.

Future work will focus on: (1) Validating the method using real LEO satellite orbit and accelerometer data to enhance its reliability. (2) Extending the method to multi-satellite constellations (e.g., Swarm, Starlink) to achieve high spatiotemporal resolution global atmospheric density retrieval. (3) Integrating the proposed method with empirical atmospheric models to improve retrieval accuracy under complex space weather conditions.

6. Conclusions

This paper proposes and evaluates a novel atmospheric density retrieval method based on GNSS precise orbit determination and orthogonal polynomial differentiation. Through systematic simulation studies, the following main conclusions are drawn:

Both Legendre and Chebyshev orthogonal polynomials demonstrate excellent accuracy and stability in extracting acceleration from GNSS position sequences for density retrieval, significantly outperforming the interpolation–differentiation method and offering greater computational simplicity than the energy method.

Experimental results show that under point-by-point inversion conditions, the proposed method achieves a mean relative error (MRE) of about 16.5% over a 24 h arc. After applying a 1 h moving average, the error can be further reduced to below 11%. Following local solar time (LST) bin-averaging, the statistical accuracy improves to an MRE of 6.83% with a correlation coefficient R reaching 0.9854, demonstrating promising capability for climatological density modeling. It should be emphasized that the accuracy improvement under LST averaging stems from statistical smoothing effects and does not represent the method's instantaneous inversion performance. For short-arc orbit prediction or real-time space weather applications, the point-by-point inversion accuracy (MRE ~16.5%) should be taken as the benchmark.

This study provides a high-accuracy, stable, and low-cost atmospheric density retrieval solution for LEO satellites without dedicated accelerometers. It holds significant importance for improving satellite precise orbit determination accuracy and supporting space environment research. The method is widely applicable to various LEO satellites and shows promising application prospects. Future work will focus on applying this method to actual measured GNSS data and addressing related engineering implementation challenges.

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Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

LEO	Low Earth Orbit
POD	Precise orbit determination
TLE	Two-Line Element
GNSS	Global Navigation Satellite System
GRACE	Gravity Recovery and Climate Experiment
GOCE	Gravity field and steady-state Ocean Circulation Explorer
MRE	Mean Relative Error
STD	Standard Deviation
UTC	Coordinated Universal Time
LST	Local Solar Time
R	Correlation Coefficient
NRMSD	Normalized Root Mean Square Deviation

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