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# An Improved Sample-Aggregation Method for Weibull Estimation of Bushing Maximum Friction Torque Under Small-Sample Conditions

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**Abstract**

This study addresses the instability of statistical modeling for small-sample maximum friction torque data under multiple temperature conditions. Within the Weibull distribution framework, a sample-aggregation method is proposed, and a unified modeling scheme separating central tendency from dispersion structure is established. This approach enables equivalent aggregation of data across different temperature levels while preserving structural consistency, thereby improving parameter estimation stability and statistical efficiency. To overcome the tendency of single-criterion optimization to fall into local optima under small-sample conditions, a secondary identification criterion combining residual minimization with a Levene-based statistical consistency test is introduced, and a dual-level search strategy is used to obtain a more robust global optimal solution. The parameter estimation results indicate that direct estimation based on small samples produces unstable parameters, with the coefficient of variation of the shape parameter reaching approximately 7.4%. In contrast, the sample-aggregation method shows that the scale parameter increases with temperature, while the location parameter first decreases and then increases due to the combined influence of central tendency and dispersion. The parameters obtained by the aggregation method exhibit more stable and regular variation trends with temperature. The results demonstrate that the proposed method significantly improves parameter stability and statistical efficiency for small-sample maximum friction torque data and provides a practical statistical modeling approach for multi-condition small-sample engineering data.

**Keywords:** sample aggregation; small size sample; Weibull distribution; maximum friction torque

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## 1. Introduction

Variable stator vane (VSV) systems in aero-engines operate under high-temperature environments, cyclic mechanical loading, and frequent actuation conditions [1–3]. Sliding bushings, typically made of high-performance polymeric materials such as polyimide, play a crucial role in ensuring stable motion transmission and positioning accuracy [4,5]. During long-term service, friction and wear inevitably occur, leading to variations in friction torque, clearance evolution, and surface degradation [6]. Among these factors, changes in the maximum friction torque directly affect actuation stability, control precision, and the jamming reliability of the VSV system [7]. Therefore, understanding and quantitatively

characterizing the statistical behavior of maximum friction torque data under different temperature conditions is of significant engineering importance [8,9]. Due to the characteristics of aerospace engineering, the available sample sizes are usually limited, which leads to unstable parameter estimation and reduces the reliability of statistical inference.

To address similar small-sample problems, the concept of sample aggregation was introduced in fatigue analysis for P–S–N curve fitting [10–13]. Xie [14] proposed a theoretical framework for sample aggregation based on the principle of quantile consistency. The framework assumes that fatigue life data obtained at different stress levels originate from the same statistical population and differs mainly in location and scale parameters. Under this assumption, small samples from multiple stress levels can be equivalently transformed and aggregated to a reference stress level, forming an equivalent large sample and thereby improving the stability of parameter estimation. Based on the backwards statistical inference method, Bai [15] introduced the assumption of consistency in the coefficient of variation, which significantly simplifies the procedural framework while achieving favorable application outcomes. Li [13] systematically reviewed existing studies and further improved the sample-aggregation method by revising the assumed relationship between stress and the standard deviation of fatigue life, and proposed stricter sample size requirements and engineering application recommendations. Ma [16] introduced a particle swarm optimization algorithm into the parameter identification process of the sample-aggregation method, and the global optimization strategy improved the stability and accuracy of parameter estimation under small-sample conditions. Luo [17] applied the sample-aggregation method to the fatigue performance analysis of alloy steel railway axles for high-speed trains and verified its effectiveness in fatigue life assessment of engineering materials. Song [18] extended the sample-aggregation method to fatigue life modeling for data following the Weibull distribution and validated the effectiveness of the method in probabilistic fatigue life analysis using aluminum alloy fatigue test data. Mu [19] discussed the scenario of fully small-sample conditions and applied it to the construction of P–S–N curves for vibration fatigue data.

In addition, several studies have extended the concept of sample aggregation from different perspectives. Pan [20] and Liu [21] investigated sample-aggregation methods incorporating the Bootstrap approach. Wu [22] introduced correction factors such as load type factor, surface quality factor, and size factor, enabling the sample-aggregation method to be better applied to fatigue life assessment of engineering structural components. Zu [23] proposed a method based on uncertainty theory, in which equivalent aggregation of small-sample data from multiple stress levels was achieved through confidence consistency transformation and coefficient of variation adjustment, and a probabilistic fatigue life model was established based on the maximum entropy distribution assumption. Wang [24] addressed the problem of probabilistic fatigue life prediction for rubber materials under small-sample conditions and proposed a method based on the Weibull distribution assumption and the principle of life quantile consistency to construct P–S–N curves. Wang [25] combined the strain energy density normalization assumption with the principle of quantile consistency and proposed a P–S–N curve fitting method suitable for small-sample conditions. He [26] addressed the small-sample fatigue problem of notched specimens under multiaxial loading by combining data augmentation techniques with the sample-aggregation method and incorporating stress field intensity theory, which improved the accuracy of parameter estimation.

Existing sample-aggregation methods have been mainly applied to fatigue life modeling. When one stress level contains a relatively large dataset that can be used as a reference level, the methods are relatively mature. However, in fully small-sample scenarios, the parameter identification process may easily fall into local optimal solutions, and effective

solutions are still limited. In addition, existing studies mainly focus on normal and lognormal distributions, while studies on sample-aggregation methods for the three-parameter Weibull distribution are still insufficient.

Based on these considerations, this study extends the sample-aggregation method from fatigue life analysis to friction–wear data processing and establishes a unified statistical framework applicable to normal (lognormal) and three-parameter Weibull distributions. Furthermore, a variance homogeneity test is introduced as a second criterion to improve the stability of aggregation results under fully small-sample conditions.

## 2. Theoretical Framework and Improved Sample-Aggregation Method

### *2.1. Theoretical Foundation of the Sample Aggregation Principle*

The concept of sample aggregation originates from fatigue life statistics and P–S–N curve analysis. Its primary objective is to structurally integrate sample information obtained under multiple operating conditions in small-sample scenarios through the construction of equivalent mapping relationships, thereby improving the stability and reliability of statistical parameter estimation.

The theoretical basis of this approach is the probability quantile consistency principle in fatigue life analysis, which states:

For the same specimen tested under different operating conditions, the corresponding random life values should occupy identical probability quantiles in their respective parent distributions.

This principle forms the theoretical starting point of the sample-aggregation method.

Let the random variable under operating condition $S_j$ be denoted as $X_j$, with cumulative distribution function (CDF):

$$F_j(x) = P(X_j < x) \tag{1}$$

where $S_j$ is the $j$-th operating condition level; $X_j$ is the random variable under this condition; $F_j(\cdot)$ is the cumulative distribution function.

For specimen $i$, the observation under condition $S_j$ is $x_{ji}$, and its probability quantile is:

$$p_i = F_j(x_{ji}) \tag{2}$$

According to the probability quantile consistency principle, if the same specimen is tested under another condition $S_k$ with observed lifetime $x_{ki}$, then

$$p_i = F_j(x_{ji}) = F_k(x_{ki}) \tag{3}$$

This relationship indicates that sample points across different operating conditions exhibit a one-to-one correspondence in terms of probability structure.

If the random variable follows a normal distribution, $X_j \sim N\left(\mu_j, \sigma_j^2\right)$ and $X_k \sim N\left(\mu_k, \sigma_k^2\right)$, then quantile consistency can be expressed as

$$\frac{x_{ji} - \mu_j}{\sigma_j} = \frac{x_{ki} - \mu_k}{\sigma_k} \tag{4}$$

where $\mu_j$ is the mean lifetime under condition $S_j$, and $\sigma_j$ is the corresponding standard deviation; $\mu_k$ is the mean lifetime under condition $S_k$, and $\sigma_k$ is the corresponding standard deviation.

This equality demonstrates that the standardized variable remains invariant across operating conditions, which forms the mathematical basis of equivalent transformation.

### *2.2. Method and Procedure of Sample Aggregation*

From the standardized consistency relationship, the equivalent transformation formula can be derived as

$$x_{ki} = \mu_k + \frac{\sigma_k}{\sigma_j}(x_{ji} - \mu_j) \tag{5}$$

This transformation consists of two components:

- Location shift (correction of mean differences);
- Scale transformation (rescaling by standard deviation ratios).

Under multi-condition scenarios with small sample sizes at each level, the implementation procedure of the sample-aggregation method is as follows:

1. Collect lifetime data under different operating conditions.
2. Fit the mean (or median)–condition relationship (e.g., an S–N curve).
3. Select a reference condition $S_1$ and define a candidate range for the reference standard deviation $\sigma_1$.
4. Define the search range for the dispersion parameter $K$.
5. Perform equivalent transformation and compute the standard deviation of the aggregated samples.
6. Determine the optimal parameter combination through error minimization and statistical testing.

### *2.3. Extension and Refinement of the Sample Aggregation Principle*

#### 2.3.1. Quantile Consistency Under the Three-Parameter Weibull Distribution

Consider a random variable under condition $S_j$ that follows a three-parameter Weibull distribution $X_j \sim \text{Weibull}(\beta_j, \eta_j, \gamma_j)$, where $\beta_j$ is the shape parameter, $\eta_j$ is the scale parameter under condition $S_j$, and $\gamma_j$ is the location parameter.

The cumulative distribution function is

$$F_j(x) = 1 - \exp\left[-\left(\frac{x - \gamma_j}{\eta_j}\right)^{\beta_j}\right] \tag{6}$$

According to the quantile consistency principle,

$$F_j(x_{ji}) = F_k(x_{ki}) \tag{7}$$

Substituting the CDF expression yields

$$\left(\frac{x_{ji} - \gamma_j}{\eta_j}\right)^{\beta_j} = \left(\frac{x_{ki} - \gamma_k}{\eta_k}\right)^{\beta_k} \tag{8}$$

In the three-parameter Weibull distribution, the shape parameter $\beta$ is usually associated with the failure mechanism, the scale parameter $\eta$ represents the characteristic level of the random variable, and the location parameter $\gamma$ represents the threshold or minimum value of the distribution. Under the dominance of the same failure mechanism, the stochastic evolution pattern and dispersion structure of the system remain consistent. The shape parameter $\beta$ in the Weibull distribution primarily reflects the variation law of the failure rate and the structural characteristics of the distribution; therefore, it can be assumed to remain invariant when the failure mechanism does not change. Different operating conditions mainly influence the characteristic scale or level of the response rather than

altering the intrinsic distribution form. If the shape parameter $\beta$ is assumed to remain invariant across operating conditions, the equation simplifies to

$$\frac{x_{ji} - \gamma_j}{\eta_j} = \frac{x_{ki} - \gamma_k}{\eta_k} \tag{9}$$

For the three-parameter Weibull distribution, the mean and standard deviation are given by

$$\mu_j = \gamma_j + \eta_j \Gamma\left(1 + \frac{1}{\beta}\right) \tag{10}$$

$$\sigma_j = \eta_j C(\beta) \tag{11}$$

where $C(\beta) = \sqrt{\Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma^2\left(1 + \frac{1}{\beta}\right)}$.

From these relationships,

$$\eta_j = \frac{\sigma_j}{C(\beta)} \tag{12}$$

$$\gamma_j = \mu_j - \frac{\sigma_j}{C(\beta)} \Gamma\left(1 + \frac{1}{\beta}\right). \tag{13}$$

It follows that

$$\frac{x_{ji} - \mu_j + \frac{\sigma_j}{C(\beta)} \Gamma\left(1 + \frac{1}{\beta}\right)}{\sigma_j / C(\beta)} = \frac{x_{ki} - \mu_k + \frac{\sigma_k}{C(\beta)} \Gamma\left(1 + \frac{1}{\beta}\right)}{\sigma_k / C(\beta)}. \tag{14}$$

By rearranging both the numerator and denominator and eliminating the common constant terms, the above expression can be simplified to

$$\frac{x_{ji} - \mu_j}{\sigma_j} = \frac{x_{ki} - \mu_k}{\sigma_k} \tag{15}$$

The equivalent transformation is thus

$$x_{ki} = \mu_k + \frac{\sigma_k}{\sigma_j}(x_{ji} - \mu_j) \tag{16}$$

Therefore, when the shape parameter $\beta$ remains invariant across operating conditions, the probability quantile consistency of the three-parameter Weibull distribution ultimately reduces to the standardized-variable consistency form.

#### 2.3.2. Dispersion Model and Parameter Search Interval

To characterize the dispersion structure under different operating conditions, a linear dispersion model is introduced:

$$\sigma_j = \sigma_1 + K(S_j - S_1), \quad j = 1, 2, \ldots, G \tag{17}$$

where $\sigma_j$ is the true population standard deviation under condition $S_j$; $\sigma_1$ is the true population standard deviation at the reference condition $S_1$; $K$ is the dispersion growth coefficient, representing the rate at which the standard deviation varies with the operating variable; $G$ is the total number of operating condition levels.

Let the sample standard deviation at the reference condition be

$$s_1 = \mathrm{std}(X_1 - \hat{\mu}_1) \tag{18}$$

It is important to distinguish between population parameters and sample statistics: $\sigma_j$ denotes the unknown true population standard deviation, whereas $s_j$ denotes the sample standard deviation calculated from experimental data under condition $S_j$. Thus, $s_j$ serves as a statistical estimator of $\sigma_j$, and may exhibit noticeable fluctuation under small-sample conditions. Since $\sigma_1$ is an unknown population parameter and should not be directly replaced by $s_1$ under small-sample conditions, its search interval is defined as $0 \leq \sigma_1 \leq 2s_1$.

This interval is determined based on statistical rationality and engineering considerations: the standard deviation must be non-negative; the upper bound $2s_1$ provides sufficient flexibility while preventing excessive inflation of dispersion; and under small-sample normal or near-normal conditions, the true population standard deviation is unlikely to exceed twice the sample estimate unless extreme outliers are present. Therefore, the interval $[0, 2s_1]$ achieves a balance between statistical plausibility and computational stability.

To reduce the influence of mean fluctuations caused by small sample sizes, all mean values involved in the calculations in this study are replaced by the sample median.

Let $S_G$ denote the highest operating condition, with corresponding sample standard deviation $s_G$. To ensure that the predicted dispersion remains within a statistically reasonable range, the search interval for $K$ is defined as

$$0 \leq K = \frac{\sigma_G - \sigma_1}{T_G - T_1} \leq \frac{2\hat{\sigma}_G - 0}{T_G - T_1} = \frac{2\hat{\sigma}_G}{T_G - T_1} \tag{19}$$

2.3.3. Sample Aggregation and Parameter Determination Criteria

Given a parameter combination ($\sigma_1$, $K$), the samples from different operating conditions are mapped to the reference condition $S_1$ using the following transformation:

$$X_{ij}^{(1)} = \hat{\mu}_1 + (X_{ij} - \hat{\mu}_j)\frac{\sigma_1}{\sigma_j} = \hat{\mu}_1 + (X_{ij} - \hat{\mu}_j)\frac{\sigma_1}{\sigma_1 + K(S_j - S_1)} \tag{20}$$

Here, $X_{ij}$ denotes the $i$-th observation under condition $S_j$, and $\sigma_j$ is determined by the dispersion model. After transformation, all samples are statistically converted into a unified dispersion structure relative to the reference condition.

The standard deviation of the aggregated samples is

$$\sigma_{calc} = \mathrm{std}(X_{ij}^{(1)} - \hat{\mu}_1) \tag{21}$$

An internal matching objective function is constructed as

$$J(\sigma_1, K) = |\sigma_{calc} - \sigma_1| \tag{22}$$

For each given $\sigma_1$, the value of $K$ within its admissible search interval that minimizes $J$ is selected, yielding the functional relationship $K^*(\sigma_1)$. This step ensures numerical consistency between the calculated dispersion of the aggregated samples and the prescribed dispersion parameter, thereby achieving internal structural matching.

In conventional approaches, the global minimum of $J(\sigma_1, K)$ is directly adopted as the sole criterion for parameter selection. However, such a strategy focuses only on numerical proximity between the aggregated standard deviation and the target value, without verifying whether the variance structures across operating conditions are statistically homogeneous. Under small-sample conditions, the minimum of the objective function may be influenced by random fluctuations, potentially resulting in dispersion structures that are not truly homogeneous in a statistical sense.

To overcome this limitation, the present study introduces the Levene test for homogeneity of variances as a second statistical criterion in addition to the internal matching

condition. The ideal aggregation outcome should produce statistically consistent variance structures across operating conditions. The Levene test evaluates variance homogeneity from a statistical inference perspective. In this study, the Brown–Forsythe modification of the Levene test is adopted, in which absolute deviations from the group median are used instead of deviations from the mean, thereby improving robustness under small-sample and non-normal conditions. This method requires weaker normality assumptions and is constructed based on absolute deviations, making it more robust under small-sample conditions and more suitable for engineering experimental data.

Let the test statistic follow $F \sim F_{G-1,N-G}$, with the corresponding significance level defined as $P_1 = P(F_{G-1,N-G} \geq F_{\text{obs}})$, where $F_{\text{obs}}$ denotes the observed value of the Levene test statistic computed from the aggregated samples. The final parameter combination is determined as

$$(\sigma_1^*, K^*) = \text{argmax}_{\sigma_1} P_1(\sigma_1, K^*(\sigma_1)) \tag{23}$$

By combining minimization of the internal objective function with maximization of the variance homogeneity significance level, the proposed dual-criterion framework ensures not only numerical consistency of the dispersion structure but also statistical validation of variance homogeneity, thereby improving the robustness and reliability of parameter identification.

#### 2.3.4. Equivalent Large Sample at the Reference Level and Distribution Parameter Estimation

After obtaining the optimal parameter combination $(\sigma_1^*, K^*)$, the dispersion structure among different operating conditions is fully determined. Based on the identified dispersion model, all original samples can be equivalently mapped to the reference operating level $S_1$, thereby constructing an equivalent "large sample" at the reference condition. Compared with a single small-sample dataset, this equivalent sample integrates information from multiple operating levels and effectively increases the sample size while preserving probability quantile consistency. As a result, the influence of small-sample random fluctuations on parameter estimation is significantly reduced.

For the equivalent large sample constructed at the reference level, the parameters of the three-parameter Weibull distribution are estimated using the aggregated data. Due to the increased effective sample size, the parameter estimates obtained from the aggregated sample exhibit improved stability and statistical efficiency. Consequently, the resulting parameter estimates are more accurate and reliable than those obtained directly from a single small sample. Under the assumption that the shape parameter $\beta$ remains invariant across all operating levels, the estimated $\beta$ at the reference level can be treated as a global shape parameter. Subsequently, by combining the central tendency function and the identified dispersion model, the corresponding scale and location parameters for other operating levels can be derived, thereby establishing consistent three-parameter Weibull distribution models for all operating conditions and achieving unified probabilistic modeling for multi-level small-sample data.

To systematically illustrate the overall procedure of the proposed sample aggregation statistical modeling method, a methodological framework is presented in Figure 1.

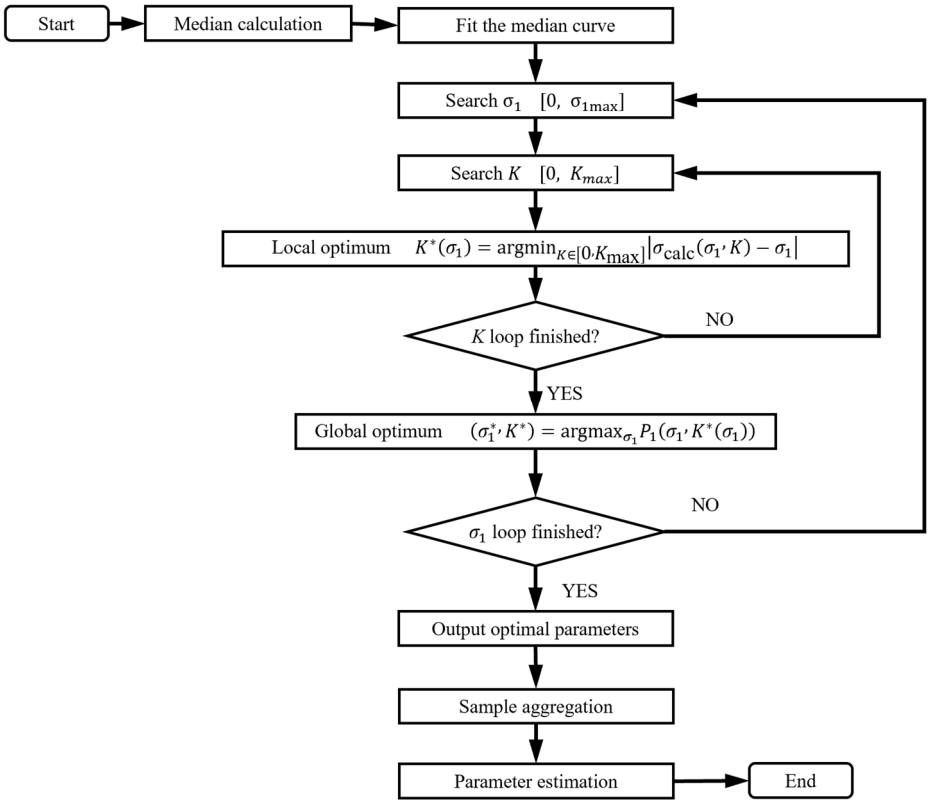



**Figure 1.** Flowchart of the proposed statistical modeling framework based on sample aggregation.

## 3. Case Study Analysis and Experimental Data Validation

### *3.1. Multi-Level Test Data*

The friction–wear experimental data of a certain bushing are selected as a case study. The test specimen is a polyimide-based composite bushing used in the stator blades of an aircraft engine, with graphite added as a solid lubricant. The tests were conducted under three temperature levels, namely 250 °C, 300 °C, and 350 °C. Multiple repeated experiments were performed at each temperature level. The response variable considered in this study is the maximum friction torque (unit: N·m).

The total number of samples is 21. The original experimental data are listed in Table 1.

**Table 1.** Maximum friction torque under different temperature levels.

| Temperature T (°C) | Maximum Friction Torque M (N·m) |
|---|---|
| 250 | 1.00, 1.27, 1.31, 1.33, 1.54, 1.57 |
| 300 | 1.34, 1.45, 1.52, 1.60, 1.65, 1.76 |
| 350 | 1.51, 2.00, 2.13, 2.30, 2.37, 2.65, 2.73, 2.90, 3.17 |

If the data at each temperature level are modeled independently, the limited sample size may lead to unstable parameter estimation. Therefore, it is necessary to introduce the equivalent sample-aggregation method proposed in Section 2 for unified data processing.

### *3.2. Central Trend Modeling*

A three-parameter logarithmic model is adopted to establish the functional relationship between temperature and the median friction torque:

$$T = a\ln(M + c) + b \tag{24}$$

The model parameters are identified using nonlinear least squares regression, yielding the fitted equation:

$$T = 40.756 \ln(M - 1.222) + 344.241 \tag{25}$$

For subsequent analysis, the model is rewritten in an equivalent form with temperature as the independent variable:

$$\hat{M}(T) = \exp\left(\frac{T - 344.241}{40.756}\right) + 1.222 \tag{26}$$

Figure 2 presents the sample points, the median values at each temperature level, and the fitted curve. The fitted model captures the monotonic increasing trend of friction torque with temperature and provides a unified reference baseline for the subsequent dispersion modeling.

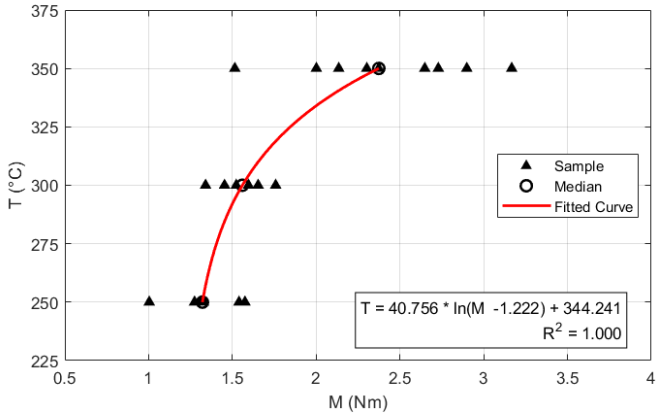



**Figure 2.** Central trend fitting of maximum friction torque versus temperature.

### *3.3. Identification of Dispersion Parameters and Search Result Analysis*

Based on the equivalent sample aggregation and dual-layer search method proposed in Section 2, the dispersion parameters $\sigma_1$ and $K$ were identified for the present case study.

Figure 3 illustrates the relationship between $\sigma_1$ and the corresponding locally optimal $K$. It can be observed that, as $\sigma_1$ increases, the locally optimal $K$ decreases monotonically. When $\sigma_1$ is relatively small, a larger $K$ is required to compensate for the dispersion growth in the high-temperature range in order to satisfy the self-consistency condition after aggregation. As $\sigma_1$ increases, the baseline dispersion level becomes larger, and the required growth rate in the high-temperature region correspondingly decreases, resulting in a smaller optimal $K$. This trend reflects the intrinsic coupling between the baseline dispersion level and the dispersion growth rate. The two horizontal stages of $K$ in Figure 3 correspond to the upper and lower boundary values of the admissible range of $K$, as defined by Equation (19).

Figure 4 presents the relationship between $\sigma_1$ and the Levene test $p$-value $P_1$. It can be seen that $P_1$ exhibits a "first increasing and then decreasing" trend as $\sigma_1$ increases. When $\sigma_1$ is small, the compression effect of dispersion is strong, and the high-temperature group may be excessively amplified, thereby introducing additional variance differences. As $\sigma_1$ increases, the aggregation process approaches a balanced state, and the variance differences among groups are gradually reduced, leading to an increase in $P_1$. However, when $\sigma_1$ continues to increase, the dispersion correction effect weakens, and the original inter-group differences reappear, resulting in a decrease in $P_1$. Therefore, the $\sigma_1 - P_1$ curve possesses a unique maximum point.

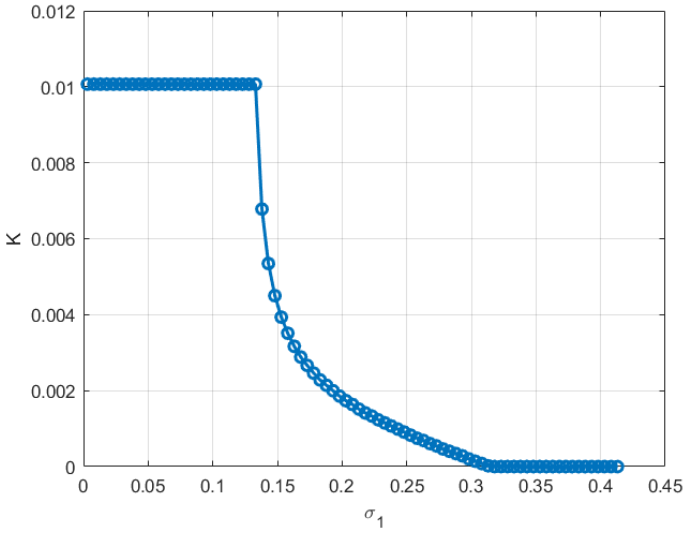



**Figure 3.** Relationship between $\sigma_1$ and the locally optimal K.

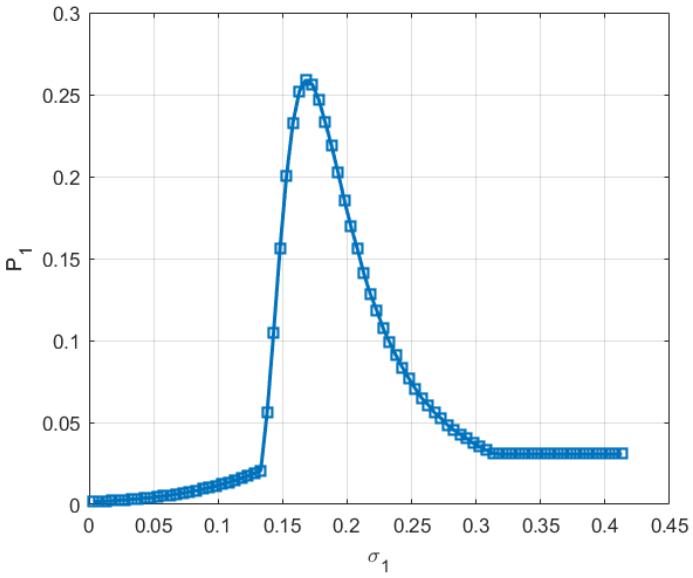



**Figure 4.** Relationship between $\sigma_1$ and the Levene test $p$-value $P_1$.

The final identified parameters are: $\sigma_1^* = 0.1681$, $K^* = 0.0029$, $P_1^{\max} = 0.2589$. This parameter combination corresponds to the maximum $P_1$ and constitutes the optimal dispersion model for the present case.

To further demonstrate the necessity of the second criterion, Figure 5 shows the search result when only the self-consistency error minimization is adopted as the selection criterion. It can be observed that, under the traditional approach, the self-consistency error curve tends to be relatively flat near its minimum, leading to many candidate points with nearly identical values. Since these near-minimum points are numerically indistinguishable, the algorithm may prematurely converge within a local region, resulting in a local optimal solution and potential parameter bias.

In contrast, by introducing the variance homogeneity test as a global criterion, the selected parameters not only satisfy the self-consistency requirement but also maximize inter-group variance consistency in a statistical sense. This significantly reduces the risk of being trapped in local optima and improves the stability and reliability of the parameter identification process.

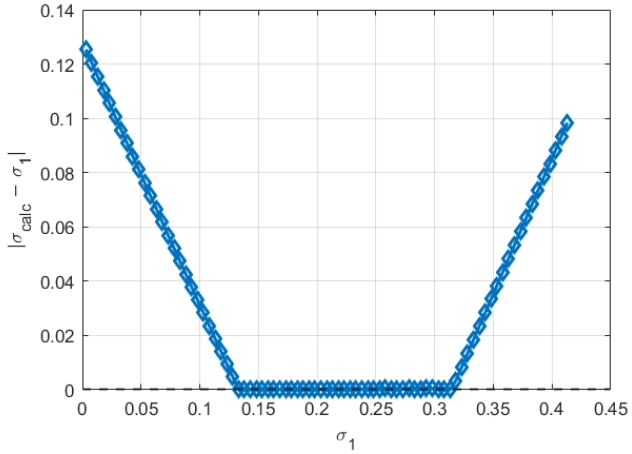



**Figure 5.** Optimality criterion of the traditional method.

### *3.4. Construction of Equivalent Samples and Weibull Parameter Estimation Results*

After obtaining the optimal dispersion parameters $(\sigma_1^*, K^*)$, the equivalent sample-aggregation method proposed in Section 2 was applied. The original samples at different temperature levels were equivalently transformed to the reference level, thereby constructing an equivalent large sample at the baseline condition. The constructed equivalent sample is listed in Table 2.

**Table 2.** Equivalent large sample constructed at the reference level.

| Temperature T (°C) | Maximum Friction Torque M (N·m) |
|---|---|
| 250 | 1.00, 1.00, 1.18, 1.20, 1.23, 1.26, 1.27, 1.29, 1.30, 1.31, 1.32, 1.33, 1.34, 1.37, 1.42, 1.43, 1.45, 1.51, 1.54, 1.57, 1.61 |

### *3.5. Weibull Parameter Estimation Results and Discussion*

The three-parameter Weibull distribution parameters at the reference level were estimated using the Correlation Coefficient Method (CCM). The parameters at the remaining temperature levels were subsequently derived according to the procedure described in Section 2. In addition, the Weibull parameters were also directly estimated from the small samples at each temperature level using the CCM method for comparison. The estimated shape, scale, and location parameters for all temperature levels obtained by different methods are summarized in Table 3.

**Table 3.** Comparison of Weibull parameter estimates obtained by the sample-aggregation method and the CCM method.

| Temperature T (°C) | Sample Aggregation | CCM |
|---|---|---|
| 250 | Weibull (2.48, 0.38, 0.95) | Weibull (2.48, 0.41, 0.92) |
| 300 | Weibull (2.48, 0.82, 0.83) | Weibull (2.55, 0.36, 1.23) |
| 350 | Weibull (2.48, 1.20, 1.31) | Weibull (2.21, 0.95, 1.28) |

In engineering practice, it is generally assumed that the shape parameter of the Weibull distribution remains invariant when the failure mechanism does not change, which is also a basic assumption of the sample-aggregation method. When the Weibull parameters are directly estimated from small samples at each temperature level using the CCM method, three different shape parameters are obtained, with a coefficient of variation of 7.4%. This indicates that the estimation of the shape parameter under small-sample conditions is subject to noticeable fluctuations, whereas the sample-aggregation method provides more

stable shape parameter estimates, which is consistent with the engineering assumption that the shape parameter remains constant when the failure mechanism is unchanged.

The scale parameter mainly reflects the dispersion of the data, and the dispersion is usually expected to vary monotonically with the operating level. The results obtained by the sample-aggregation method show that the scale parameter increases gradually with temperature. However, the scale parameter estimated directly using the CCM method shows that the value at the intermediate temperature level is the smallest, which is inconsistent with the expected trend of dispersion variation. This discrepancy is mainly caused by the small sample size, where individual abnormal data points can significantly influence the parameter estimation results, leading to unreasonable estimates.

The location parameter is influenced by both the central tendency and the dispersion of the data. The location parameter estimated by the sample-aggregation method shows a trend of first decreasing and then increasing with temperature. This indicates that in the temperature range of 250–300 °C, the increase in central tendency is smaller than the increase in dispersion, resulting in the minimum value of the higher-level data being smaller than that of the lower-level data. In the temperature range of 300–350 °C, the change in central tendency is larger than the increase in dispersion, which is theoretically consistent with the fitted central trend curve. In contrast, the location parameters obtained by direct CCM estimation are derived from small samples at individual temperature levels and are strongly affected by small-sample variability, making it difficult to perform a systematic analysis of the parameter variation pattern.

## 4. Conclusions, Innovations and Outlook

### *4.1. Conclusions*

This study investigated the statistical modeling of small-sample maximum friction torque data under multiple temperature conditions and proposed a sample aggregation modeling method for fully small-sample scenarios within the Weibull distribution framework. By introducing a secondary criterion based on the Levene test (Brown–Forsythe modification), the stability and uniqueness of the optimal parameter estimation under fully small-sample conditions were improved. The results show that the sample-aggregation method can equivalently transform and aggregate data from different temperature levels while maintaining structural consistency across operating conditions, thereby improving the stability and consistency of parameter estimation under small-sample conditions.

The parameter estimation results indicate that direct estimation based on small samples is strongly affected by sample fluctuations, resulting in unstable parameter estimates, with the coefficient of variation of the shape parameter reaching approximately 7.4%. In contrast, based on the assumption of invariant shape parameter, the sample-aggregation method reveals that the scale parameter increases with temperature, while the location parameter shows a trend of first decreasing and then increasing due to the combined influence of central tendency and dispersion. Overall, the parameters obtained by the sample-aggregation method exhibit more stable and regular variation trends with temperature, which are more consistent with engineering practice.

Overall, the proposed sample-aggregation method improves statistical efficiency, parameter stability, and engineering interpretability of small-sample data modeling. The proposed framework provides an effective statistical modeling approach for multi-condition small-sample engineering data and can be extended to other reliability analysis and engineering statistical modeling problems.

### *4.2. Innovations*

A unified sample aggregation framework applicable to normal, lognormal, and three-parameter Weibull distributions is established, which extends the applicability of sample-aggregation methods beyond a single distribution model and improves the generality of the statistical modeling framework.

A secondary criterion based on the Levene test (Brown–Forsythe modification) is introduced into the parameter identification process. By combining the self-consistency criterion with statistical variance homogeneity testing, a dual-criterion optimization strategy is developed, which effectively improves the stability and uniqueness of parameter estimation under fully small-sample conditions.

The sample-aggregation method is extended from traditional fatigue life statistical analysis to maximum friction torque data analysis, providing a new methodological approach for multi-condition statistical modeling under small-sample constraints in tribological engineering.

### *4.3. Outlook*

Future research can further expand the dispersion modeling framework by exploring nonlinear or multi-parameter coupled structures, enabling more accurate characterization of the dispersion evolution of maximum friction torque under complex operating conditions.

In multi-factor coupled environments, incorporating temperature, load, and other operational parameters into a unified statistical framework represents an important direction for extending the sample aggregation concept to more complex engineering systems. Such development would contribute to constructing a more comprehensive statistical analysis methodology for friction and wear behavior.

In addition, integrating the aggregated equivalent samples with reliability modeling and uncertainty quantification methods would provide a stronger statistical foundation for long-term stability assessment and engineering life prediction of maximum friction torque systems.

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