

Article

Model Predictive Control for an SMA Actuator Based on an SGPI Model

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Abstract

Shape memory alloy (SMA) actuators possess unique advantages for aerospace applications, including significant deformation, a high work-to-weight ratio, and structural simplicity. However, SMA actuators exhibit inherently strongly saturated and asymmetric hysteresis characteristics, which cause significant hysteresis in the output response. These hysteresis nonlinearities, compounded by process and measurement noise, severely degrade control precision. To overcome these issues, this study proposes a Smoothed Generalized Prandtl–Ishlinskii (SGPI) model to characterize such hysteresis behavior. Based on the SGPI model, we developed a state-space representation for the SMA actuator. Furthermore, an Extended Kalman Filter (EKF) is employed to estimate unmeasurable internal hysteresis states, and these estimates are subsequently utilized as input states for Model Predictive Control (MPC). The simulation results demonstrate that the proposed EKF-MPC approach achieves both rapid dynamic response and high-precision tracking control, effectively compensating for hysteresis nonlinearity while rejecting noise disturbances.

Keywords: shape memory alloy actuator; hysteresis model; Extended Kalman Filter; Model Predictive Control

1. Introduction

Shape memory alloy (SMA) actuators possess unique advantages for aerospace applications, including large deformation, a high work-to-weight ratio, and structural simplicity, leading to broad application prospects in areas such as space-deployable antenna structures [1], spacecraft attitude and thermal control [2,3], morphing wings [4], and aero-engine nozzle control [5]. However, their inherent strongly saturated and asymmetric hysteresis characteristics lead to significant nonlinearity, output oscillations, and closed-loop instability, which limits their application in high-precision and high-reliability aerospace actuation systems.

Researchers have developed numerous models to characterize SMA hysteresis based on micromechanics [6], constitutive modeling [7–9], and phenomenological approaches [10–14]. Current phenomenological hysteresis modeling approaches can be categorized into three distinct classes. The first class consists of operator-based models, which describe hysteresis via the weighted superposition of elementary operators. Common models include the Preisach [15–18], Prandtl–Ishlinskii (PI) [19,20], and Krasnosel’skii–Pokrovskii [21,22] models. The GPI model further characterizes the asymmetric saturation of SMAs by incorporating envelope functions. However, a common limitation



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of these models is the non-differentiability at switching points due to their piecewise-linear characteristics. This lack of smoothness complicates the computation of system Jacobians, which are essential for many modern control and estimation frameworks, thereby hindering the development of a unified and continuous mathematical representation for closed-loop analysis. The second class involves differential equation-based models, such as the Bouc-Wen [23,24] and Duhem [25,26] models. Although they offer the smoothness required for control design, these models have limitations in describing strong saturation and asymmetric hysteresis, making them less suitable for the complex characteristics of SMA actuators. The third class leverages data-driven approaches, utilizing deep learning architectures such as recurrent neural networks [27] or neural operators [28] to achieve high-precision nonlinear mapping. However, despite their high accuracy, data-driven models impose heavy computational burdens and are difficult to deploy in resource-constrained embedded systems, facing challenges in real-time control. Given the respective trade-offs between mathematical smoothness, structural flexibility, and computational cost in existing approaches, there is an urgent need for a modeling approach that preserves the structural flexibility of operator models while providing analytical differentiability. Based on this, the present study establishes a continuously differentiable model suitable for direct state-space analysis by smoothing the operators of the GPI model.

Most existing hysteresis models primarily focus on characterizing the relationship between temperature and displacement or output force [29]. However, such direct mapping is highly dependent on external load conditions. Models identified under specific loads may suffer from significant degradation in accuracy when the load changes, thereby limiting their practical applicability. As a temperature-driven material, the SMA spring undergoes variations in martensite volume fraction during temperature changes. According to the Tanaka model [7], the Young's modulus of SMA exhibits a linear relationship with the martensite volume fraction. Consequently, the phase transformation process directly influences the mechanical properties of the SMA spring, resulting in different stiffness coefficients for SMA in different phases. Therefore, the relationship between temperature and stiffness coefficient—an intrinsic material property governed by phase transformation—demonstrates stronger independence from external loads. By developing a hysteresis model for the temperature–stiffness relationship, the resulting model achieves enhanced generalization capability under varying operational conditions.

Research on hysteresis compensation strategies is extensive in the literature. Quintanar-Guzmán et al. [30] evaluated multiple controllers for the position tracking of an SMA-driven robotic arm, while Bil et al. [31] compared the performance of different compensators for SMA morphing wings through wind tunnel tests. Schmidt et al. [32] proposed a predictive model for antagonistic SMA spring actuators to optimize bistable designs. Currently, mainstream architectures predominantly rely on feedforward compensation based on inverse hysteresis models. For piezoelectric actuators or specific SMA wires where the response is directly driven by an electric field or input current, such input–output mapping remains feasible [33,34]. However, stiffness regulation in SMA springs is primarily temperature-dependent rather than a direct mapping of current. In practical applications, Joule heating via current alters the temperature, but the heating and cooling processes are constrained by ambient convective heat transfer, exhibiting significant time constants and thermal hysteresis. Existing inverse compensation schemes often employ a cascaded structure [29] that calculates a reference temperature via a static inverse model. Such open-loop feedforward approaches typically ignore thermal inertia, leading to severe dynamic phase lags and a lack of robustness against environmental disturbances.

To address the limitations of open-loop inverse compensation, it is necessary to employ a predictive control framework capable of explicitly handling system delays and

unobservable nonlinearities. Model Predictive Control (MPC) [35–37] has emerged as an effective method for such complex systems, as its multi-step optimization over a prediction horizon can effectively compensate for the phase lags induced by thermal inertia. Furthermore, since the internal hysteresis states of SMA are not directly observable, the integration of an Extended Kalman Filter (EKF) [38,39], a recursive estimation method, is essential for providing accurate state feedback under noisy environments. By combining the predictive characteristics of MPC with the state estimation capabilities of EKF, a closed-loop state-feedback control structure can be established to handle the complex hysteresis nonlinearities of SMA actuators.

In this study, we propose a Smoothed Generalized Prandtl–Ishlinskii (SGPI) model by modifying the hysteresis operators of the GPI model to be smooth differentiable functions. This preserves the ability to characterize the strongly saturated and asymmetric hysteresis characteristics of SMA actuators while enabling the design of state estimation-based controllers. Specifically, the state-space representation based on the SGPI model allows the design of an EKF for real-time estimation of unmeasurable hysteresis states under process and measurement noise. These state estimates serve as essential inputs to a MPC algorithm, which performs multi-step prediction and rolling optimization to compute optimal control signals that compensate for hysteresis effects, achieving high precision stiffness coefficients tracking.

The rest of this paper is organized as follows. Section 2 elaborates on the mathematical modeling of the SMA actuator, including thermodynamic modeling, SGPI hysteresis modeling, and model parameter identification. Section 3 presents the design of the EKF and MPC based on the SGPI model. Section 4 presents the simulation results. Finally, Section 5 concludes the paper.

2. Mathematical Modeling of SMA Actuator

As shown in Figure 1, SMAs are widely used in aerospace applications, such as motion control of space soft-bodied crawling robot and adaptive wings [40]. The core actuation of these applications relies on the reliable performance of SMA actuators. These applications fundamentally require an accurate description and control of the dynamic relationship among the input current, temperature state, and output performance of the SMA actuator. Therefore, the focus of this study is to establish the current–temperature–stiffness coefficient correlation model of the SMA actuator and achieve stiffness control of the SMA-driven system. As shown in Figure 2, the mathematical model of the SMA actuator system consists of two components: a thermal model and an SGPI phenomenological mathematical model. The thermal model characterizes the relationship between the input current and the spring temperature, while the SGPI model describes the relationship between the spring temperature and the stiffness coefficient of the SMA actuator.

2.1. SMA Spring Thermodynamic Model

The thermodynamic model of the SMA spring primarily considers three factors [41]: thermal absorption during heating, convective heat exchange with ambient air, and resistive heating governed by Joule’s law:

$$c_p m \frac{dT}{dt} = I^2 R - hA(T - T_\infty), \quad (1)$$

where c_p is the specific heat capacity coefficient of the SMA spring; m is the mass of the spring; I is the current applied to the spring; R is the electrical resistance of the spring; h is the heat convection coefficient; A is the circumferential surface area of the spring; T and T_∞

represent the temperature of the spring and ambient temperature, respectively. Dividing both sides of the thermodynamic model equation by $c_p m$:

$$\frac{dT}{dt} = -\frac{hA}{c_p m} T + \frac{I^2 R}{c_p m} + \frac{hA}{c_p m} T_\infty. \quad (2)$$

Assuming $a = -\frac{hA}{c_p m}$, $b = \frac{R}{c_p m}$, $c = \frac{hA}{c_p m} T_\infty$, and $u = I^2$, the thermodynamic model can be expressed by the following state equation:

$$\dot{T} = aT + bu + c. \quad (3)$$

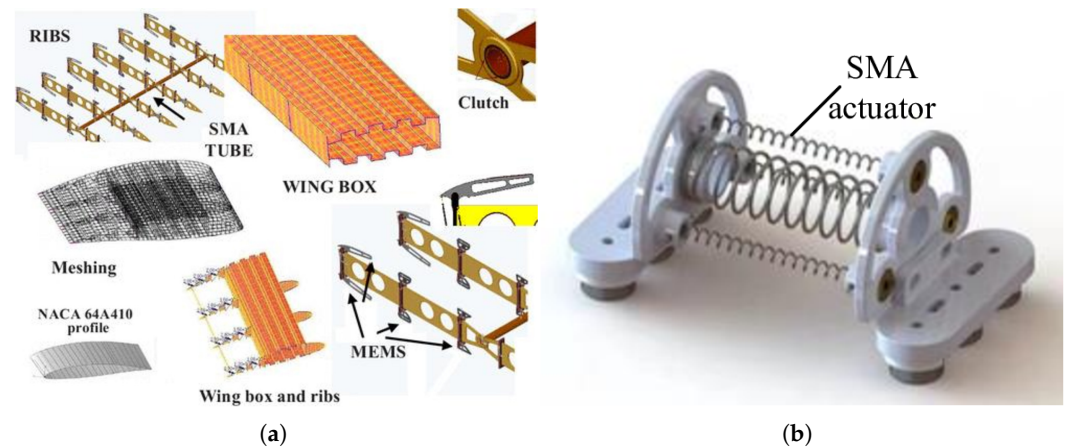


Figure 1. SMA actuators in aerospace applications: (a) Architecture of the adaptive wing structure and SMA actuators [40]; (b) SMA space soft-bodied crawling robot.

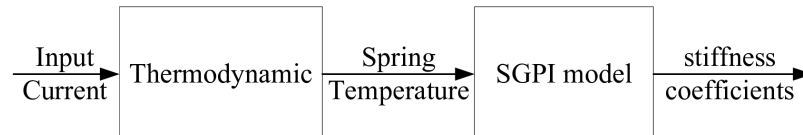


Figure 2. Mathematical modeling of SMA actuator.

2.2. GPI Hysteresis Model

The PI hysteresis model is a nonlinear modeling approach based on hysteresis operator superposition and is widely employed for characterizing hysteresis behavior in smart materials, such as shape memory alloy actuators and piezoelectric actuators. However, although the PI model exhibits symmetric loading or unloading paths, the actual SMA springs demonstrate complex asymmetric hysteresis with output saturation characteristics during thermal cycling. To address these limitations, an envelope function is incorporated into the PI framework, resulting in a Generalized Prandtl–Ishlinskii (GPI) model. The GPI approach fundamentally applies input-signal nonlinearization and characterizes complex hysteresis phenomena through the weighted superposition of multiple play operators. The architecture primarily consists of three cascaded components: envelope function, play hysteresis operators, and density function, as shown in Figure 3a.

For a piecewise monotonic input function $u(t) \in C[0, T]$, where $u(t) \in C[0, T]$ denotes the space of continuous functions in the interval $C[0, T]$. The output $y(t)$ of the GPI model can be expressed as

$$y(t) = GPI[u](t) = PI[v](t) = PI \circ \gamma(t), \quad (4)$$

where γ is the envelope function, $v(t)$ is the system output of the envelope function, and PI denotes the PI hysteresis model.

The PI hysteresis model is characterized by multiple hysteresis play operators. For a piecewise monotonic envelope input function $v(t) \in C[0, T]$, a play hysteresis, as shown in Figure 3b, operator with a fixed threshold r_i , denoted as Γ_i , is defined as follows.

Let, $0 = t_0 < t_1 < \dots < t_k = T$ be a partition of the time domain $[0, T]$ such that $v(t)$ is monotonic on each subinterval $[t_{m-1}, t_m]$, where $m = 0, 1, \dots, k$. Then, the play hysteresis operator $\Gamma_i[v](t)$ and its corresponding hysteresis state $w_i(t)$ can be given by when, $t = t_0 = 0$:

$$\begin{aligned} w_i(0) &= \Gamma_i[v](0) \\ &= \max\left(\gamma_f[u](0) - r_i, \min(\gamma_l[u](0) + r_i, 0)\right), \end{aligned} \quad (5)$$

when, $t_{m-1} \leq t \leq t_m$:

$$\begin{aligned} w_i(t) &= \Gamma_i[v](t) \\ &= \max\left(\gamma_f[u](t) - r_i, \min(\gamma_l[u](t) + r_i, w(t_{m-1}))\right). \end{aligned} \quad (6)$$

To characterize the saturation and asymmetric hysteresis properties of the SMA, the envelope function γ can be represented using various methods such as dead-zone functions, hyperbolic tangent functions, or polynomials. This study employs a hyperbolic tangent function to describe output saturation and asymmetry under specific inputs. The hyperbolic tangent function, as shown in Figure 3c, is expressed as follows:

$$v(t) = \begin{cases} \gamma_f[u](t) = a_0 \tanh(a_1 u(t) + a_2) + a_3 & \dot{u}(t) \geq 0 \\ \gamma_l[u](t) = b_0 \tanh(b_1 u(t) + b_2) + b_3 & \dot{u}(t) < 0 \end{cases} \quad (7)$$

where $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3$ are parameters to be identified.

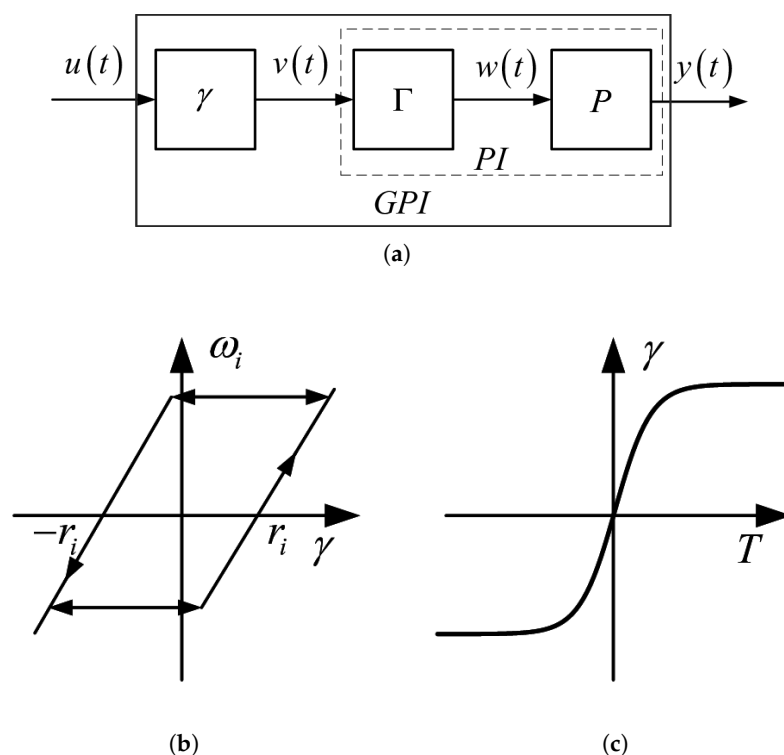


Figure 3. Generalized Prandtl–Ishlinskii model: (a) Cascaded Structure of the GPI hysteresis model; (b) the play hysteresis operator; (c) hyperbolic tangent envelope function.

The output of the GPI hysteresis model consists of a weighted linear combination of multiple play operators with distinct fixed thresholds r_i :

$$y(t) = GPI[v](t) = \sum_{i=1}^n p_i w_i(t), \quad (8)$$

where n is the number of play operators and p_i are positive weighting coefficients (also referred to the density function). The thresholds r_i and weighting coefficients p_i are defined as follows:

$$r_i = ci, \quad (9)$$

$$p_i = \rho e^{-\tau r_i}, \quad (10)$$

where c, ρ, τ are parameters to be identified.

2.3. SGPI Hysteresis Model

Although the GPI model can characterize the strongly saturated and asymmetric hysteresis behaviors of SMA actuators, its hysteresis play operators are constructed using non-smooth $\max()/\min()$ functions with piecewise linearity. These operators induce non-differentiable dynamics, posing significant challenges for direct application in model-based controller synthesis.

To leverage the representational capabilities of the GPI model while enabling controller design, we propose the Smoothed Generalized Prandtl–Ishlinskii (SGPI) model. Specifically, the non-smooth $\max()/\min()$ operators are approximated by differentiable functions $\text{smax}()/\text{smin}()$ as follows:

$$\text{smax}(a, b) = \frac{a + b + \sqrt{(a - b)^2 + \varepsilon}}{2}, \quad (11)$$

$$\text{smin}(a, b) = \frac{a + b - \sqrt{(a - b)^2 + \varepsilon}}{2}. \quad (12)$$

The smoothing parameter $\varepsilon > 0$ introduces a trade-off between model fidelity and numerical differentiability. As $\varepsilon \rightarrow 0$, the SGPI operators converge to the standard non-smooth $\max/\min$ operators, preserving the structural characteristics of the GPI model. However, an excessively small ε results in abrupt gradient changes, known as numerical stiffness, which can compromise the stability of gradient-based optimization and state estimation frameworks. Conversely, a larger ε ensures smoother derivatives that are beneficial for algorithmic convergence but may lead to deviations from the desired hysteresis envelope, potentially distorting the model's output characteristics. In this work, $\varepsilon = 10^{-3}$ is selected to balance approximation accuracy with the numerical robustness required for subsequent computational tasks. The resulting smoothed play hysteresis operator is depicted in Figure 4. In contrast to the conventional piecewise-linear operator shown in Figure 3b, the SGPI formulation ensures smooth dynamics over the entire domain. This transition from piecewise-linear to smooth dynamics establishes a mathematically tractable foundation for model-based control and observation schemes. Consequently, the transformed hysteresis state $w_i(t)$ is described as follows:

when, $t = t_0 = 0$:

$$\begin{aligned} w_i(0) &= \Gamma_i[v](0) \\ &= \text{smax}\left(\gamma_f[u](0) - r_i, \text{smin}(\gamma_l[u](0) + r_i, 0)\right), \end{aligned} \quad (13)$$

when, $t_{m-1} \leq t \leq t_m$:

$$\begin{aligned} w_i(t) &= \Gamma_i[v](t) \\ &= \text{smax}\left(\gamma_f[u](t) - r_i, \text{smin}(\gamma_l[u](t) + r_i, w(t_{m-1}))\right), \end{aligned} \quad (14)$$

where the hyperbolic tangent envelope function is still selected for γ_f and γ_l .

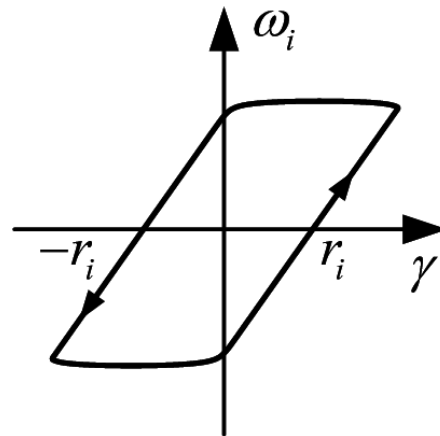


Figure 4. The smoothed play hysteresis operator.

The SGPI model retains the capability of the GPI model to characterize the strongly saturated and asymmetric hysteresis in SMA actuators by establishing differentiable dynamics. This enables the formulation of state-space feedback control laws.

2.4. Identification the SGPI Model Parameters

The SMA actuator system is shown in Figure 5, with Figure 5a illustrating the structural schematic of the SMA actuator and Figure 5b showing the experimental setup of the SMA actuator. The system comprises six components: a SMA spring actuator, clamping slider, programmable power supply, computer, laser displacement sensor, and a thermocouple temperature module. One end of the SMA spring was clamped, whereas the opposite end was connected to the slider via a hook weight. The slider was moved vertically along the linear guide. The laser displacement sensor emitted a beam onto the slider surface, continuously measuring the distance between the slider and overhead beam to compute the real-time SMA spring length. A programmable DC power supply applies voltage across both SMA spring terminals. The thermocouple probe was affixed to the SMA spring to measure real-time temperature variations.

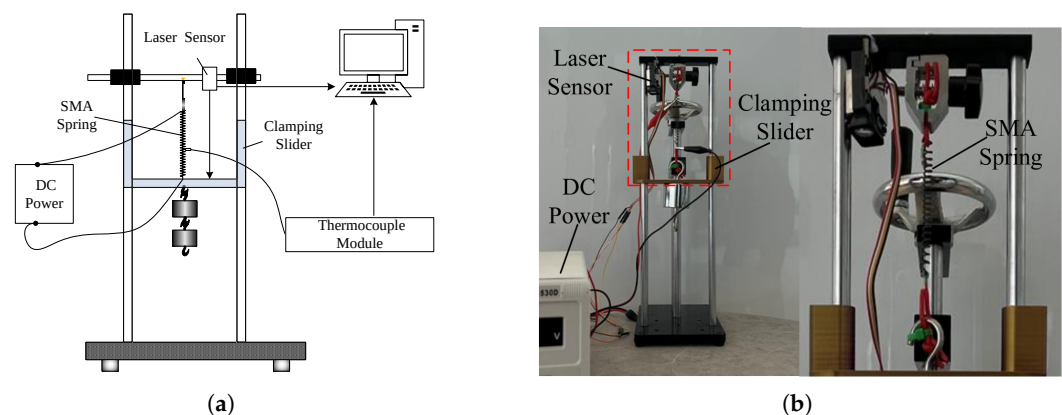


Figure 5. SMA actuator: (a) SMA actuator structure schematic; (b) SMA actuator experimental setup.

To comprehensively account for the relationship between the SMA temperature and output stiffness coefficients, and to effectively identify the parameters of the SGPI hysteresis model, the system was actuated with a constant current of 1 A under a constant load of 250 g. After sustained heating, the SMA spring underwent thermal expansion and contraction. When the SMA spring temperature reached the austenite finish temperature, the spring contracted to its minimum length. At this point, the constant-current power supply was shut off to allow natural cooling, restoring the initial length, while the experimental displacement data were collected.

The measured temperature and displacement data were subsequently processed to determine the stiffness coefficient. Based on the mechanical definition, the stiffness coefficient $y(T)$ was calculated as follows:

$$y_o(T) = \frac{F_l}{l_s(T) - l_{s0}}, \quad (15)$$

where, F_l represents the constant load force applied to the spring, $l_s(T)$ denotes the real-time length of the spring at temperature T , and l_{s0} is the original length of the spring at room temperature before loading.

The parameters of the SGPI hysteresis model were identified through an optimization algorithm with the following mean square error (MSE) objective function:

$$J(X) = \frac{1}{N} \sum_{i=1}^N (y_o(T_i) - y(T_i))^2, \quad (16)$$

where N is the number of sampling points, $y_o(T)$ is the experimentally measured output, $y(T)$ is the output of the SGPI model, and the parameter vector X is defined as

$$X = (a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3, \rho, \tau, c).$$

Particle Swarm Optimization (PSO) and its advanced variants (e.g., Q-learning based PSO [42]) are widely recognized as efficient approaches for nonlinear system identification. In this study, parameter identification was performed using the PSO algorithm as shown in Figure 6. The identification results are presented as follows.

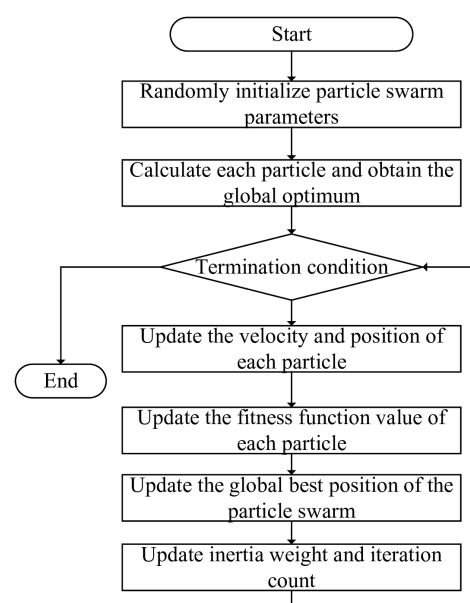


Figure 6. The parameter identification process of the SGPI model based on the particle swarm optimization algorithm.

Table 1 presents the parameter identification results for the SGPI model. Figure 7a compares the identification results of both the SGPI and GPI models against the experimental data, while Figure 7b illustrates the corresponding tracking errors. The results demonstrate that the SGPI model effectively characterizes the hysteresis behavior of the SMA actuator, exhibiting smoother transitions at phase transition points and higher accuracy than the GPI model. To evaluate the modeling precision, three error metrics are defined in Table 2: Root Mean Square Error (RMSE), Average Absolute Error (AAE), and Maximum Relative Displacement Error (MRDE). The proposed SGPI model achieved reductions of 18.9% in RMSE, 15.9% in AAE, and 36.0% in MRDE compared to the GPI model. These quantitative metrics further validate the effectiveness of the proposed SGPI model.

Table 1. Model parameter identification results.

Parameters	Value
a_0	0.3301
a_1	0.1034
a_2	−7.2809
a_3	1.1976
b_0	1.2183
b_1	0.0509
b_2	−3.5333
b_3	1.6089
ρ	10.0000
τ	0.1000
c	0.0252

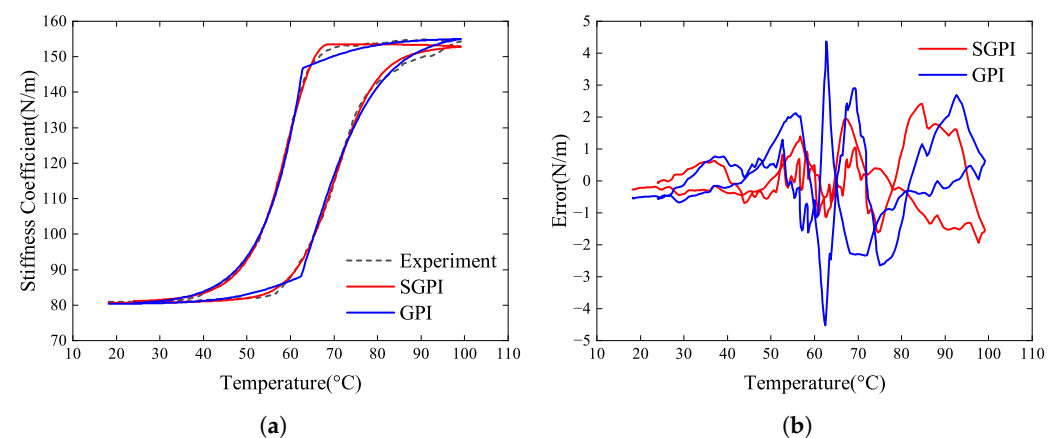


Figure 7. Comparison of output characteristics between SGPI/GPI models and SMA actuator experimental data: (a) Model-experimental output characteristics; (b) model prediction error.

Table 2. Definition of error functions and identification error indices.

Error Function	Definition	SGPI Model	GPI Model
RMSE	$\sqrt{\frac{\sum_{i=1}^N (y_0(t_i) - y(t_i))^2}{N}}$	0.8200	1.3690
AAE	$\frac{\sum_{i=1}^N y_0(t_i) - y(t_i) }{N}$	0.6270	1.0328
MRDE	$\frac{\max(y(t_i) - y_0(t_i))}{\max(y_0) - \min(y_0)} \times 100\%$	3.27%	6.12%

3. Extended Kalman Filter and Model Predictive Control Design

SMA actuators exhibit strongly saturated and asymmetric hysteresis behaviors with a path dependency. Specifically, their current stiffness coefficients depends not only on current input temperature but also on historical stiffness coefficients. Such complex hysteretic nonlinearity hinders precise control using conventional linear controllers. In contrast, Model Predictive Control (MPC) offers a promising approach through receding-horizon optimization and future-state prediction capabilities. However, MPC implementation requires full-system states as prediction initial conditions, whereas only the temperature and stiffness coefficients outputs are measurable in SMA actuators. The virtual hysteresis state, which represents the core hysteretic behavior, is unmeasurable. To address this limitation, this study proposes an Extended Kalman Filter (EKF) state observer that estimates the complete system states in real time using limited measurements, thereby providing reliable initialization for MPC.

3.1. Electro-Thermo-Mechanical State-Space Model

To establish a unified framework for state estimation and control, the continuous-time thermodynamic model (Equation (3)) and SGPI hysteresis model are integrated into a discrete-time state-space representation. The thermodynamic model is discretized using forward Euler method with sampling period T_s :

$$T(k+1) = (1 + aT_s)T(k) + (bT_s)u(k) + cT_s, \quad (17)$$

Defining coefficients $a_c = 1 + aT_s$, $b_c = bT_s$, $c_c = cT_s$, the discrete thermodynamic equation for the SMA actuator is obtained:

$$T(k+1) = a_c T(k) + b_c u(k) + c_c. \quad (18)$$

Concurrently, the hysteresis nonlinearity between temperature and stiffness coefficients is characterized by the SGPI model

$$\begin{cases} w_i(k+1) = \text{smax}(\gamma_f[T](k) - r_1, \text{smmin}(\gamma_l[T](k) + r_1, w_i(k))) \\ y(k) = \sum_{i=1}^n p_i w_i(k) \end{cases} \quad (19)$$

Let, $\mathbf{x}_k = [T(k), w_1(k), \dots, w_n(k)]^T \in \mathbb{R}^{n+1}$ comprises the temperature state variable $T(k)$ and n virtual hysteresis state variables w_i . The electro-thermo-mechanical nonlinear state-space representation of the SMA actuator is expressed as:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{f}(\mathbf{x}_k) + \mathbf{B}\mathbf{u}_k + \boldsymbol{\omega}_k \\ &= \begin{bmatrix} f_T(T(k)) \\ f_w(w_1(k)) \\ \vdots \\ f_w(w_n(k)) \end{bmatrix} + \mathbf{B}\mathbf{u}_k + \boldsymbol{\omega}_k \\ &= \begin{bmatrix} a_c T(k) + b_c u(k) + c_c + \omega_0(k) \\ \text{smax}(\gamma_f[T](k) - r_1, \text{smmin}(\gamma_l[T](k) + r_1, w_1(k))) + \omega_1(k) \\ \vdots \\ \text{smax}(\gamma_f[T](k) - r_n, \text{smmin}(\gamma_l[T](k) + r_n, w_n(k))) + \omega_n(k) \end{bmatrix}, \end{aligned} \quad (20)$$

where $u_k \in \mathbb{R}$ is the system voltage input and $\mathbf{B} = [b_c, 0, \dots, 0]^\top \in \mathbb{R}^{(n+1) \times 1}$ is the input matrix. The process noise vectors $\boldsymbol{\omega}_k \sim \mathcal{N}(0, \mathbf{Q}_a) \in \mathbb{R}^{n+1}$ characterize the external disturbances and SGPI model mismatch.

In SMA actuator system, the measurable system outputs $\mathbf{z}_k = [T_a, y_a]^\top \in \mathbb{R}^2$, where T_a and y_a denote the measured temperature and measured stiffness coefficients, can be expressed as

$$\begin{aligned} \mathbf{z}_k &= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \rho e^{-\tau r_1} & \rho e^{-\tau r_2} & \cdots & \rho e^{-\tau r_n} \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} v_1(k) \\ v_2(k) \end{bmatrix} \\ &= \mathbf{H}_m \mathbf{x}_k + \mathbf{v}_k \end{aligned} \quad (21)$$

where $\mathbf{H} \in \mathbb{R}^{2 \times (n+1)}$ is the observation matrix, and $\mathbf{v}_k \sim \mathcal{N}(0, \mathbf{R}_a) \in \mathbb{R}^2$ represents the measurement noise vector.

The electro-thermo-mechanical nonlinear state-space representation of the SMA actuator is expressed as

$$\begin{cases} \mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k) + \mathbf{B}u_k + \boldsymbol{\omega}_k \\ \mathbf{y}_k = \mathbf{H}_m \mathbf{x}_k \\ \mathbf{z}_k = \mathbf{y}_k + \mathbf{v}_k \end{cases} \quad (22)$$

This unified representation provides the mathematical foundation for subsequent EKF state estimation and MPC controller design.

3.2. Extended Kalman Filter Design

Because of the inherent hysteresis nonlinearity of the system, the standard Kalman filter is inapplicable. Consequently, an Extended Kalman Filter (EKF) was employed to linearize the hysteresis nonlinearity and estimate the augmented state vector, which comprises the temperature and multiple virtual hysteresis states. For the nonlinear state function given in (22), linearization is performed via Taylor series expansion around the posterior state estimate $\hat{\mathbf{x}}_{k-1}$:

$$\begin{aligned} \mathbf{x}_k &= \mathbf{f}(\mathbf{x}_{k-1}) + \mathbf{B}u_{k-1} + \boldsymbol{\omega}_{k-1} \\ &\approx \mathbf{f}(\hat{\mathbf{x}}_{k-1}) + \mathbf{F}_{k-1}(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1}) + \mathbf{B}u_{k-1} + \boldsymbol{\omega}_{k-1}, \end{aligned} \quad (23)$$

where $\mathbf{F}_{k-1}$ is the Jacobian matrix of the system evaluated at the posterior state estimate $\hat{\mathbf{x}}_{k-1}$:

$$\begin{aligned} \mathbf{F}_{k-1} &= \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\hat{\mathbf{x}}_{k-1}} \\ &= \begin{bmatrix} \frac{\partial f_T}{\partial T} & 0 & \cdots & 0 \\ \frac{\partial f_{w_1}}{\partial T} & \frac{\partial f_{w_1}}{\partial w_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{w_n}}{\partial T} & 0 & \cdots & \frac{\partial f_{w_n}}{\partial w_n} \end{bmatrix}_{\hat{\mathbf{x}}_{k-1}}. \end{aligned} \quad (24)$$

The explicit gradients for the thermodynamic function f_T and the hysteresis state function f_w are derived based on the SGPI model definitions. Specifically, the diagonal elements $\lambda_{i,k} = \partial f_{w_i} / \partial w_i$ represent the instantaneous stiffness retention rate of each operator. Due to the smoothing parameter ε in the $\text{smax}()$ and $\text{smin}()$ functions, these derivatives remain continuous, ensuring numerical stability during the Jacobian update.

A key issue in this observer design is the structural observability of the high-dimensional hysteresis state vector $w = [w_1, \dots, w_n]^\top$ when using a single scalar stiffness measurement y_k . As proven in Lemma A1 (see Appendix A), the augmented state vector is locally observable if the input temperature signal satisfies the Persistently Exciting (PE) condition. This observability stems from the diverse distribution of thresholds r_i assigned to each operator. As the input evolves, each operator undergoes the transition from the elastic regime (derivative ≈ 1) to the saturated regime (derivative ≈ 0) at a unique input magnitude. This staggered activation ensures that the operators do not saturate simultaneously, maintaining the linear independence of the columns in the local observability matrix over time. Consequently, the individual contribution of each hysteresis state can be distinguished from the aggregate measurement, ensuring the EKF estimation is well-posed.

The EKF algorithm is implemented in the following recursive steps.

Step 1: State Prediction. Predict the a priori state estimate $\hat{x}_{k|k-1}$ using the posterior estimate $\hat{x}_{k-1}$ and the control input u_{k-1} :

$$\hat{x}_{k|k-1} = f(\hat{x}_{k-1}) + Bu_{k-1}. \quad (25)$$

Step 2: Covariance Prediction. Define the a priori state estimation error as $e_{k|k-1} = x_k - \hat{x}_{k|k-1}$. The a priori error covariance matrix $P_{k|k-1}$ is defined as the expectation $E[e_{k|k-1}e_{k|k-1}^\top]$. It is computed using the Jacobian F_{k-1} :

$$P_{k|k-1} = F_{k-1}P_{k-1}F_{k-1}^\top + Q_a, \quad (26)$$

where Q_a is the process noise covariance matrix representing model uncertainties.

Step 3: Gain Computation. Compute the optimal Kalman gain K^* that minimizes the trace of the posterior covariance:

$$K^* = P_{k|k-1}H_m^\top (H_mP_{k|k-1}H_m^\top + R_a)^{-1}, \quad (27)$$

where H_m is the linearized measurement matrix and R_a is the measurement noise covariance matrix.

Step 4: Measurement Update. Fuse the prediction with the actual sensor measurement z_k to obtain the *a posteriori* state estimate:

$$\hat{x}_k = \hat{x}_{k|k-1} + K^*(z_k - H_m\hat{x}_{k|k-1}). \quad (28)$$

Step 5: Covariance Update. Define the *a posteriori* estimation error as $e_k = x_k - \hat{x}_k$. Update the posterior error covariance matrix $P_k = E[e_k e_k^\top]$:

$$P_k = (I - K^*H_m)P_{k|k-1}. \quad (29)$$

The EKF algorithm comprises two core stages: time update (Step 1–3) and measurement update (Step 4–6), as illustrated in Figure 8. Initialization requires setting the initial state estimate $\hat{x}_0$ and initial error covariance matrix P_0 . The initial state estimate $\hat{x}_0$ should be as close as possible to the true system state, whereas the initial error covariance matrix P_0 may be initialized as a sufficiently large matrix to accelerate convergence.

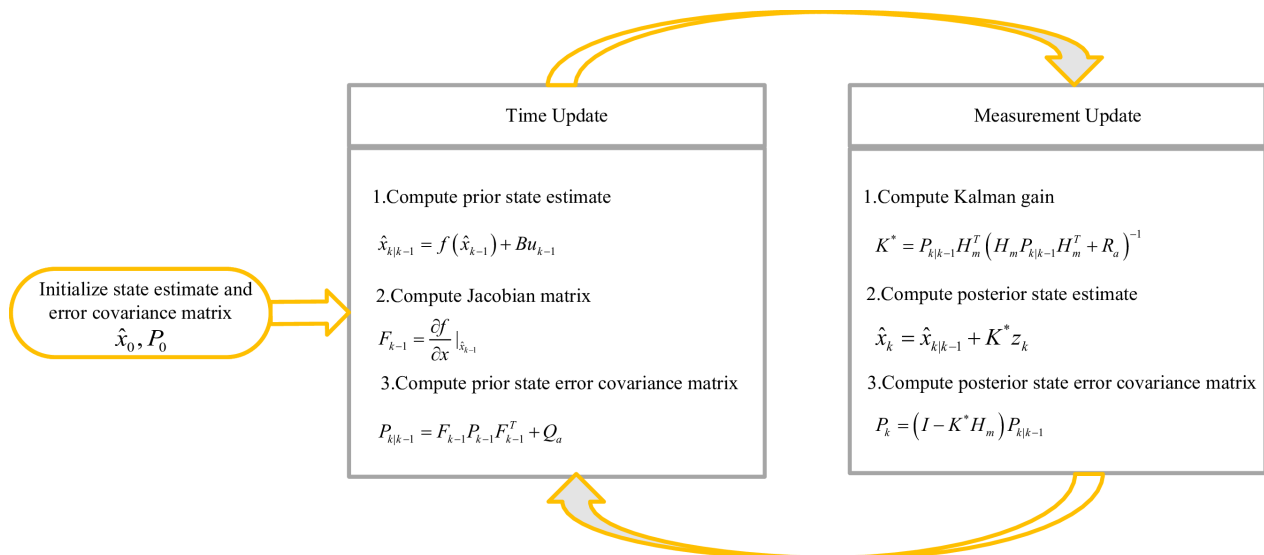


Figure 8. The structure of the Extended Kalman Filter.

3.3. Model Predictive Control Design

Based on the real-time state estimates provided by the EKF state observer in Section 3.2, we propose a Model Predictive Controller (MPC) for the trajectory tracking of the SMA actuator. Utilizing the discrete state-space model of the SMA actuator defined in (22), we adopt the control increment $\Delta u(k)$ as the MPC optimization variable:

$$\Delta u_{k+i} = u_{k+i} - u_{k+i-1}.$$

Adopting the control increment rather than the absolute input as the decision variable effectively introduces an inherent integral action into the closed-loop system. This mechanism effectively compensates for model mismatches and parameter drifts, significantly enhancing the robustness of the control system against uncertainties by eliminating steady-state errors.

The optimization variable vector is defined as $\Delta \mathbf{U} = [\Delta u_k, \Delta u_{k+1}, \dots, \Delta u_{k+N-1}]^T$. The prediction horizon of the MPC controller is N_p , and the control horizon is N_c . Combined with the current EKF state estimate $\hat{x}_{k|k}$, the future state trajectory is predicted as:

$$x_{k+i|k} = f(x_{k+i-1|k}) + Bu_{k+i-1}, \quad i = 1, 2, \dots, N, \quad (30)$$

where the predicted input accumulates as $u_{k+i|k} = u_{k-1} + \sum_{j=0}^i \Delta u_{k+j}$.

The control objective is to ensure that the stiffness coefficient output y_a of the SMA actuator tracks the reference trajectory y_{ref} , while satisfying the following constraints:

$$\begin{aligned} \Delta u_{\min} &\leq \Delta u_k \leq \Delta u_{\max} \\ u_{\min} &\leq u_k \leq u_{\max} \\ y_{\min} &\leq y_k \leq y_{\max}, \end{aligned}$$

These hard constraints introduced in the optimization problem strictly confine the system states within the physically safe region. Combined with the intrinsic dissipative nature of the SMA system, where states naturally decay in the absence of input, this design ensures system stability in the Bounded-Input–Bounded-Output (BIBO) sense.

In MATLAB, the symbolic optimization framework CasADi [43] is employed to formulate the MPC problem as a Nonlinear Programming (NLP) problem:

$$\begin{aligned}
\min_{\Delta U} J &= \sum_{i=1}^N \|y_{k+i|k} - y_{\text{ref}}\|_Q^2 + \sum_{i=0}^{N-1} \|\Delta u_{k+i}\|_R^2, \quad i = 1, 2, \dots, N \\
\text{subj. to} \\
x_{k+i|k} &= f(x_{k+i-1|k}) + Bu_{k+i-1|k} \\
y_{k+i|k} &= H_m x_{k+i|k} \\
u_{k+i|k} &= u_{k-1} + \sum_{j=0}^i \Delta u_{k+j} \\
\Delta u_{\min} &\leq \Delta u_{k+i|k} \leq \Delta u_{\max} \\
u_{\min} &\leq u_{k+i|k} \leq u_{\max} \\
y_{\min} &\leq y_{k+i|k} \leq y_{\max},
\end{aligned} \tag{31}$$

where $Q \in \mathbb{R}^{p \times p}$ and $R \in \mathbb{R}^{m \times m}$ are symmetric positive-definite weighting matrices. Leveraging the state estimates provided by the EKF, the NLP is solved in real time using the IPOPT solver to compute the optimal control inputs, thereby achieving precise tracking of the SMA actuator's stiffness coefficients.

As illustrated in Figure 9, the EKF-MPC control architecture adopts a cascaded configuration. The EKF generates real-time state estimates $\hat{x}_k$ based on the measurable system output z_k , while the incremental MPC utilizes these estimates as initial conditions to iteratively solve the finite-horizon optimal control problem. Based on the local observability established in Lemma A1, the estimation error covariance of the EKF remains bounded under the persistent excitation condition. This implies that the prediction initialization of the MPC is always situated within a bounded neighborhood of the true state. Since the cost function of the NLP optimization problem is continuous, this bounded initial error results only in a bounded sub-optimality of the control inputs, without compromising the stability of the control loop.

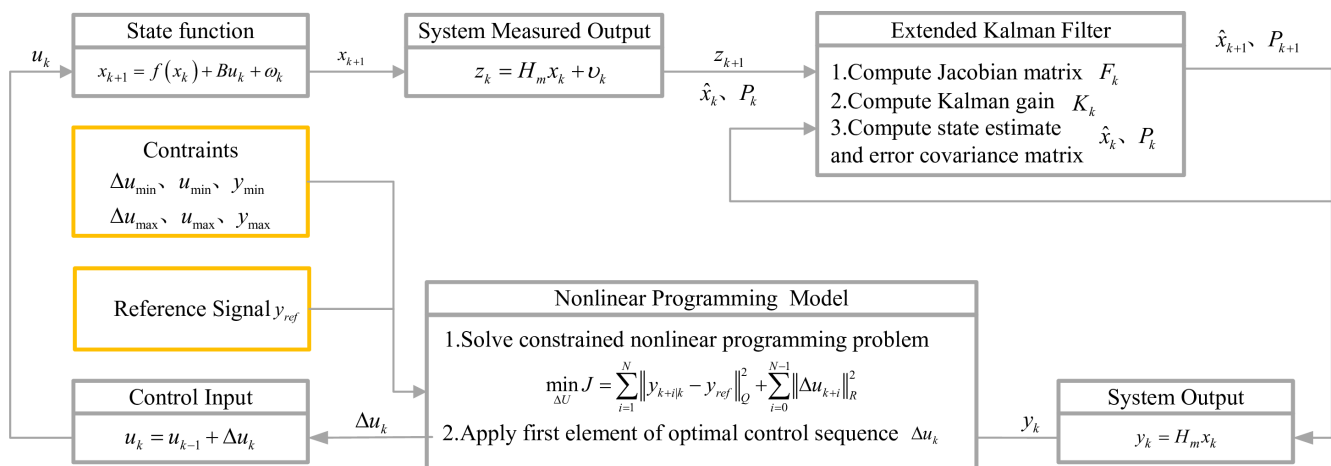


Figure 9. The structure of the Model Predictive Control.

4. Simulation

To validate the effectiveness of the proposed EKF and MPC frameworks, numerical simulations were conducted using an SMA actuator model identified from the experimental tests. The numerical simulations were performed on a personal computer equipped with an Intel Core i7-14650H processor (2.20 GHz) and 16 GB of RAM. The algorithms were implemented using MATLAB R2023b, and the nonlinear programming problems were solved using the CasADi framework version 3.5.5. Under this computational setup, the average execution time per time step was approximately 17 ms.

The simulation environment was configured with a sampling time of $T_s = 0.1$ s and initialized with the SMA thermodynamic parameters $a_c = -0.0602$, $b_c = 5.0830$, and $c_c = 1.2040$. The MPC horizons were set to $N_p = 15$ and $N_c = 5$. This selection provides a 1.5 s look-ahead window, which is sufficient to cover the dominant thermal time constant of the SMA for phase lag compensation, while the smaller N_c ensures that the optimization can be completed within the sampling interval. The initial conditions consisted of a spring temperature $T_0 = 20$ °C and the initial stiffness coefficients $y_0 = 80.6190$ N/m. The hysteresis virtual state vector $w \in \mathbb{R}^{11}$ was initialized using the operator definition in (13):

$$w_0 = \{0.8675, 0.8423, 0.8171, 0.7919, 0.7667, 0.7415, 0.7163, 0.6911, 0.6659, 0.6408, 0.6156\}^T$$

Realistic perturbations were incorporated by injecting zero-mean Gaussian process noise $\omega_k \sim \mathcal{N}(0, Q_c)$ and measurement noise $v_k \sim \mathcal{N}(0, R_c)$ with diagonal covariance matrices:

$$Q_c = \text{diag}([0.1^2, 0.01^2, \dots, 0.01^2]) \in \mathbb{R}^{12 \times 12},$$

$$R_c = \text{diag}([0.25^2, 0.25^2]) \in \mathbb{R}^{2 \times 2}.$$

To validate the estimation capability of the EKF for both measurable outputs and virtual hysteresis states under process and measurement noise, a low-frequency sinusoidal excitation signal $u(t) = 0.5(\sin(0.005t) + 1)$ is applied to the system for the first 50 s, followed by a period of zero input for the subsequent 50 s. The process noise covariance matrix Q_a and the measurement noise covariance matrix R_a in the EKF were configured to match the true noise covariance matrices Q_c and R_c of the simulation environment. The estimation performance and quantitative analysis results are shown in Figure 10 and Table 3.

Table 3. Mean squared error comparison of measured and estimated values.

Variable	Measured MSE	Estimated MSE
Temperature	0.2499	0.1383
Stiffness coefficients	0.2409	0.2068

Figure 10a shows the estimated temperature profile of the actuator, Figure 10b shows the estimated stiffness coefficients trajectory against the measured and true values, and Figure 10c presents the Euclidean norm evolution of the virtual hysteresis state estimate. Table 3 presents quantitative performance metrics. The simulation results demonstrate that under measurement noise, the EKF data fusion mechanism achieves a 44.66% reduction in temperature estimation standard deviation, a 14.16% reduction in stiffness coefficients estimation standard deviation, and a virtual hysteresis state estimation error norm that is consistently below 0.08. This results validate the effectiveness of the EKF in estimating both the measurable outputs and unmeasurable hysteretic states under noisy conditions.

To analyze the sensitivity of the proposed Extended Kalman Filter (EKF) to initial state errors and noise parameter tuning, two sets of comparative simulation experiments were conducted. First, to assess the convergence performance under uncertain initialization, three experimental cases were established with the noise covariance matrices held constant. These cases corresponded to initial temperature estimates of 20 °C, 40 °C, and 60 °C, respectively. As illustrated in Figure 11, despite the significant discrepancies in the initial states, the state estimates in all cases rapidly converged to the true trajectory within approximately 0.7 s. This demonstrates that the algorithm possesses strong robustness to initial errors and exhibits fast convergence capabilities. Second, to evaluate the sensitivity of the filter to process noise parameters, three different filter configurations were selected

while maintaining accurate initial states. Specifically, the process noise covariance matrix Q_a was scaled by factors of 0.1, 1, and 10, respectively. For these three scenarios, the Mean Squared Error (MSE) of the estimation errors for temperature, stiffness, and the Euclidean norm of the virtual hysteresis state vector w were calculated. The quantitative comparisons are summarized in Table 4. The results indicate that while the baseline parameter configuration yields the lowest MSEs, the estimation errors remain within a low and bounded range even under significant parameter mismatch. This confirms that the proposed algorithm maintains stable tracking performance across different noise parameter configurations.

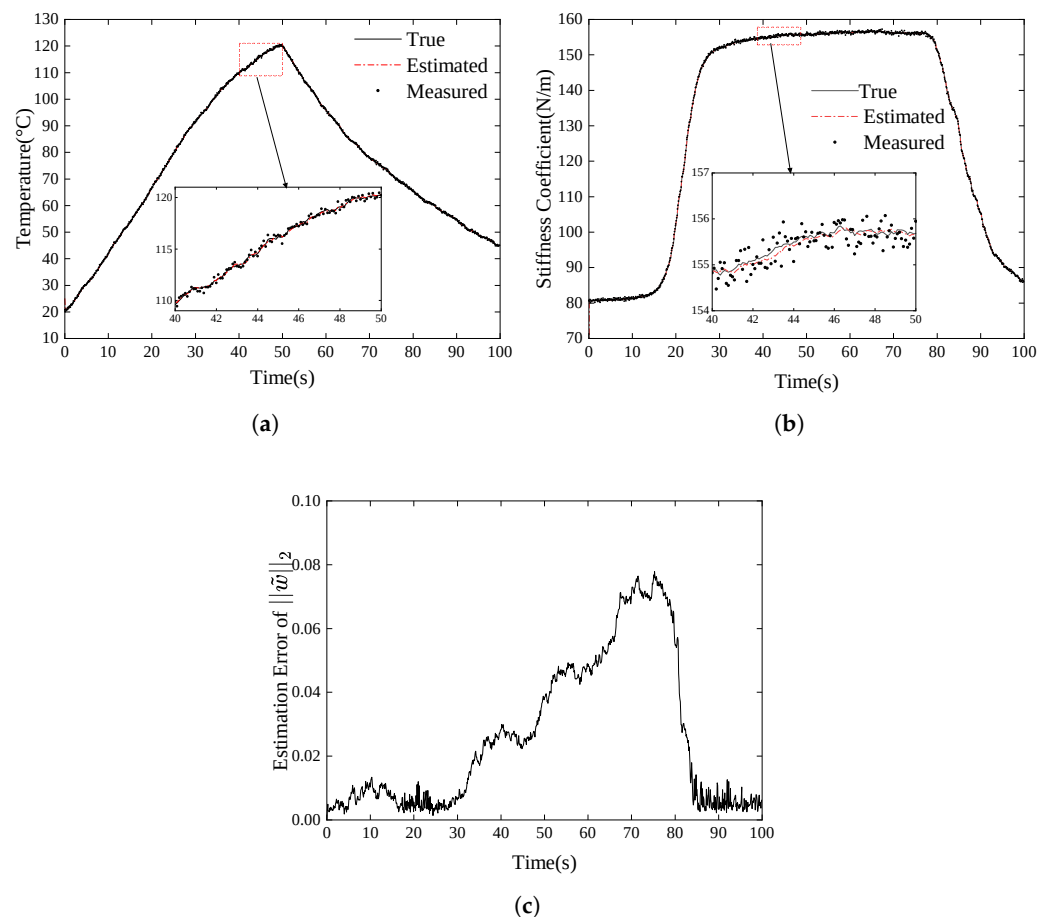


Figure 10. Simulation results of the Extended Kalman Filter: (a) temperature estimation; (b) stiffness coefficients estimation; (c) Euclidean norm of the estimation error in the virtual hysteresis state.

To validate the effectiveness of the MPC algorithm, simulation studies were conducted for multi-setpoint tracking under disturbance conditions. Comparative analyses were performed against the traditional PID control and feedforward compensation control. The PID parameters were initially determined using the Ziegler–Nichols frequency response method and subsequently fine-tuned to achieve an optimal balance between transient response speed and overshoot suppression. The feedforward compensation strategy, developed in reference to [29], consists of a static inverse-GPI model to cancel hysteresis nonlinearities, integrated with a Model Reference Adaptive Control (MRAC) for thermal regulation and a PID loop to compensate for the modeling errors of the inverse-GPI model. Figure 12 presents the modeled stiffness coefficients tracking and error curves of the three control strategies, and Table 5 provides quantitative comparison results of the key performance metrics.

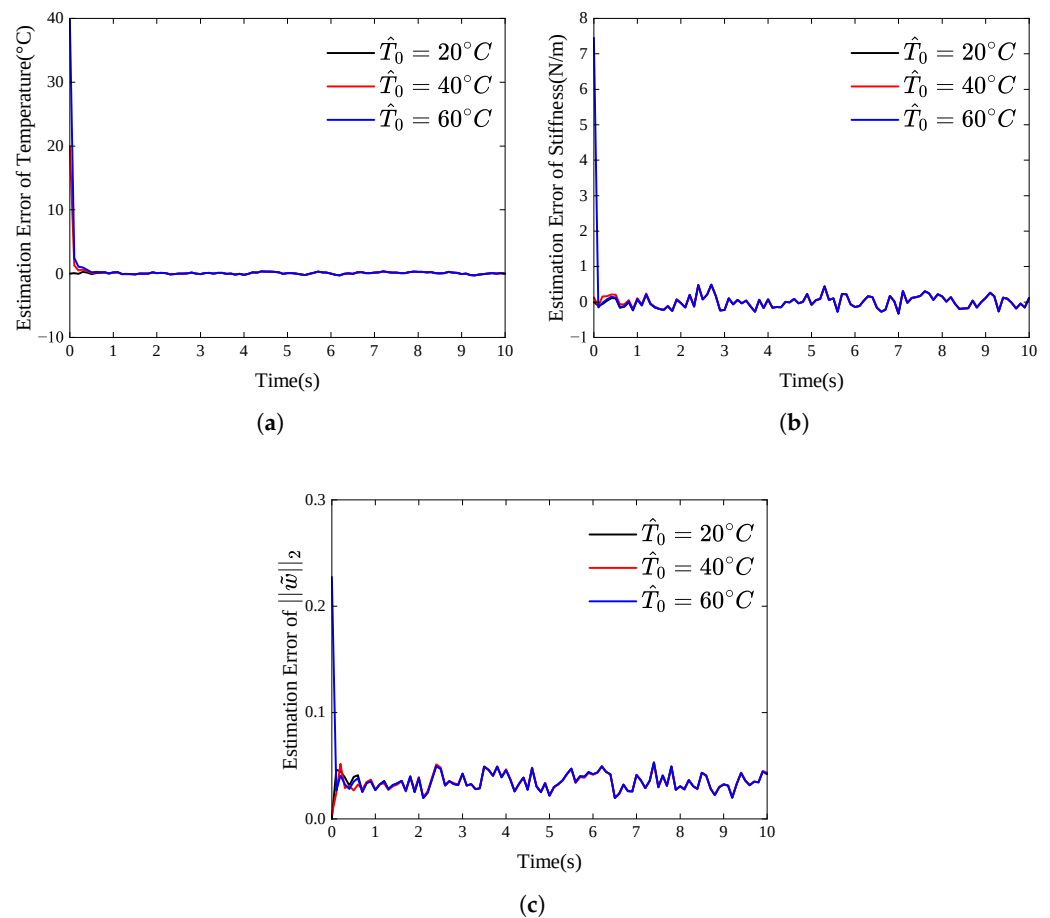


Figure 11. Estimation errors under different initial state guesses: (a) temperature estimation; (b) stiffness coefficients estimation; (c) Euclidean norm of the estimation error in the virtual hysteresis state.

Table 4. MSE of estimation errors for temperature, stiffness, and virtual hysteresis state under different process noise parameter settings.

Parameter Settings	MSE of Temp. ($^\circ\text{C}$)	MSE of Stiff. (N/m)	MSE of $\ \tilde{w}\ _2$
Under-estimated $Q_a = 0.1Q_c$	0.3943	0.6102	0.0206
Baseline Tuning $Q_a = Q_c$	0.1383	0.2068	0.0115
Over-estimated $Q_a = 10Q_c$	0.2352	0.2396	0.0116

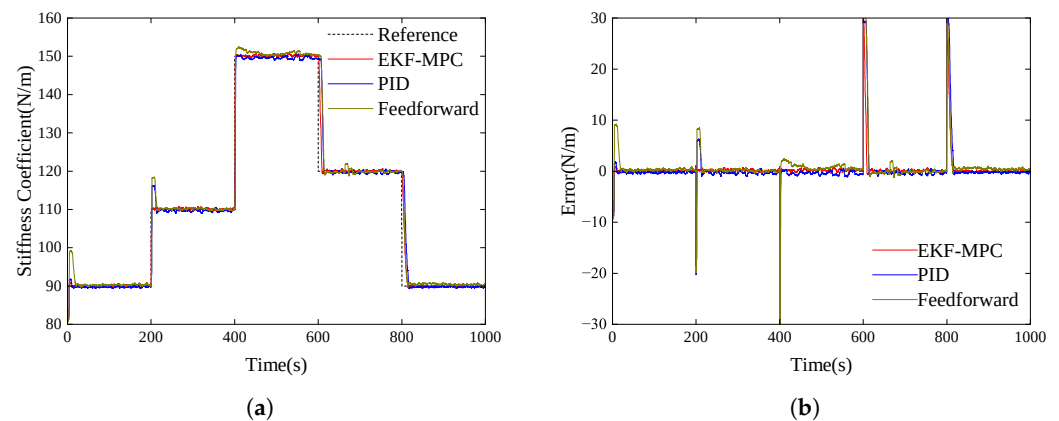


Figure 12. Simulation results of the Model Predictive Control: (a) stiffness coefficients tracking performance under different controllers; (b) tracking error comparison.

Table 5. Performance comparison of different controllers across step transitions.

Controller	Overshoot (%)	Settling Time (s)	RMSE	Steady Error (N/m)
Step 1: 80 → 90 N/m				
EKF-MPC	11.31	8.3	1.2239	0.0915
PID	18.01	10.1	1.2740	0.1697
Feedforward	91.99	25.8	2.3078	0.3165
Step 2: 90 → 110 N/m				
EKF-MPC	4.53	1.0	0.9123	0.1990
PID	31.41	13.5	2.1161	0.3413
Feedforward	42.30	13.6	2.3681	0.1602
Step 3: 110 → 150 N/m				
EKF-MPC	1.83	1.6	1.9744	0.1774
PID	1.39	2.8	3.5985	0.6002
Feedforward	6.05	20.8	3.3020	0.3649
Step 4: 150 → 120 N/m				
EKF-MPC	2.41	8.5	4.2379	0.1047
PID	3.24	13.3	6.3494	0.3675
Feedforward	4.07	13.6	6.5506	0.1047
Step 5: 120 → 90 N/m				
EKF-MPC	1.75	12.5	4.3793	0.1047
PID	2.40	15.8	5.9443	0.2449
Feedforward	1.36	13.1	4.8037	0.3242
Overall Average Performance				
EKF-MPC	4.37	6.38	2.5455	0.1355
PID	11.29	11.10	3.8564	0.3447
Feedforward	29.15	17.38	3.8664	0.2541

Note: Steady Error represents the mean absolute error after the system reaches a stable state; RMSE evaluates the dynamic tracking precision throughout each step period.

The simulation results indicate that the proposed EKF-MPC controller exhibits superior overall performance across all evaluated metrics. In terms of overshoot suppression, the EKF-MPC demonstrates a significant advantage. Simulation data from Steps 3 to 5 show that its overshoot is maintained within 2%, outperforming both the PID and feedforward-feedback control schemes. Notably, EKF-MPC achieves a faster response during the stiffness reduction phases (Steps 4–5). Fundamentally, both PID and feedforward-feedback control are reactive. They require the perception of a tracking error or a discrete change in the reference stiffness to trigger a reduction in the heating current. Specifically, the PID controller must wait for an error signal to emerge before adjusting its output, while the feedforward component remains static until the reference stiffness trajectory shifts. Consequently, these methods rely on a slow integration process to compensate for model mismatches. In contrast, by leveraging the receding horizon optimization mechanism, the MPC is capable of pre-adjusting the control input based on stiffness variations within the prediction horizon. This proactive control action allows for the early modulation of the input current, effectively compensating for the thermal inertia and passive cooling limitations that hinder rapid stiffness adjustment in SMA actuators. Regarding settling time, the EKF-MPC exhibits the fastest response across all steps, with an average settling time of 6.38 s, representing a substantial improvement over the 11.1 s recorded for the PID control. Furthermore, the EKF-MPC achieves the lowest average steady-state error (0.1355 N/m) among the three methods. These findings confirm that the integration of EKF state estimation with the MPC disturbance rejection mechanism effectively enhances the control precision and stability of the system under complex hysteric nonlinear conditions.

5. Conclusions

In this study, a SGPI model was proposed by reconstructing hysteresis operators within the GPI framework to characterize the strongly saturated and asymmetric hysteresis in SMA actuators. The model parameters were identified using the particle swarm optimization method. Owing to the differentiability of the SGPI model, state-space functions were established to implement an EKF-MPC strategy, achieving real-time estimation of unmeasurable virtual hysteresis states and precise positioning control. Simulations demonstrated rapid and effective reference tracking under process and measurement noise disturbances. While the EKF-MPC framework is designed to robustly compensate for parametric uncertainties via integral action, the current validation was limited to a nominal load. Future work will focus on rigorous experimental validation under varying dynamic loads.

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Appendix A

Lemma A1 (Local Observability of Hysteresis States). *Consider the hysteresis system described by the Smoothed Generalized Prandtl-Ishlinskii (SGPI) model, defined as follows:*

$$\begin{cases} w_i(k+1) = \text{smax}(u_k - r_i, \text{smin}(u_k + r_i, w_i(k))) \\ y(k) = \sum_{i=1}^n p_i w_i(k) \end{cases} \quad (\text{A1})$$

where w_i represents the state of the i -th operator with threshold r_i , and p_i are the weights. Assume the input signal u_k satisfies the Persistently Exciting (PE) condition, implying that the variation range of the input signal is sufficient to cover the threshold ranges of all hysteresis operators. Then, the hysteresis state vector $\mathbf{w}_k = [w_{1,k}, \dots, w_{n,k}]^\top$ satisfies the local observability condition when observed through the stiffness measurement y_k .

Proof. The local observability of the system depends on the full rank condition of the observability matrix derived from the linearized error dynamics. According to the measurement equation presented in the manuscript, the linearized observation matrix $\mathbf{H}_w \in \mathbb{R}^{1 \times n}$ is given by $\mathbf{H}_w = [p_1, \dots, p_n]$, where $p_i = \rho e^{-\tau r_i}$ denote non-zero and distinct weighting coefficients.

Consider the Jacobian matrix of the hysteresis subsystem, denoted as $\Phi_k \in \mathbb{R}^{n \times n}$. Based on the definitions of the smax and smin functions in the SGPI model, Φ_k is a diagonal matrix:

$$\Phi_k = \text{diag}(\lambda_{1,k}, \lambda_{2,k}, \dots, \lambda_{n,k}) \quad (\text{A2})$$

where the diagonal element $\lambda_{i,k}$ is defined as the partial derivative of the i -th SGPI operator with respect to its own state at time step k . Considering the loading ($\Delta u_k > 0$) and unloading ($\Delta u_k < 0$) phases, the explicit expression for $\lambda_{i,k}$ is derived as

$$\lambda_{i,k} = \begin{cases} \frac{1}{2} \left(1 + \frac{w_{i,k} - (u_k - r_i)}{\sqrt{(w_{i,k} - (u_k - r_i))^2 + \varepsilon}} \right), & \text{if } \Delta u_k > 0 \\ \frac{1}{2} \left(1 - \frac{w_{i,k} - (u_k + r_i)}{\sqrt{(w_{i,k} - (u_k + r_i))^2 + \varepsilon}} \right), & \text{if } \Delta u_k < 0 \end{cases} \quad (\text{A3})$$

As indicated by the equation above, $\lambda_{i,k}$ varies continuously within the interval $(0, 1)$. Specifically,

- Elastic Region: When the effective input $u_k \pm r_i$ is far from the current state $w_{i,k}$, $\lambda_{i,k} \approx 1$.
- Saturated Region: When the operator is fully driven by the input, $\lambda_{i,k} \approx 0$.
- Transition Region: When $w_{i,k} \approx u_k \pm r_i$, $\lambda_{i,k}$ transitions smoothly between 0 and 1 due to the smoothing parameter ε .

Construct the local observability matrix $\mathcal{O}_w \in \mathbb{R}^{n \times n}$ within an n -step time window:

$$\mathcal{O}_w = \begin{bmatrix} \mathbf{H}_w \\ \mathbf{H}_w \Phi_k \\ \vdots \\ \mathbf{H}_w \prod_{j=0}^{n-2} \Phi_{k+j} \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & \cdots & p_n \\ p_1 \lambda_{1,k} & p_2 \lambda_{2,k} & \cdots & p_n \lambda_{n,k} \\ \vdots & \vdots & \ddots & \vdots \\ p_1 \prod_{j=0}^{n-2} \lambda_{1,k+j} & p_2 \prod_{j=0}^{n-2} \lambda_{2,k+j} & \cdots & p_n \prod_{j=0}^{n-2} \lambda_{n,k+j} \end{bmatrix} \quad (\text{A4})$$

To prove the system is observable, we must demonstrate that $\mathcal{O}_w$ has full rank. This is equivalent to proving that all column vectors v_i of $\mathcal{O}_w$ are linearly independent.

We proceed by contradiction. Assume the column vectors are linearly dependent; then, there exists a set of constants $c_1, c_2, \dots, c_n$, not all zero, such that for any input state, the following linear combination vanishes:

$$\sum_{i=1}^n c_i v_i = \mathbf{0} \quad (\text{A5})$$

We examine the row in this vector equation containing first-order derivative information (i.e., the second row of $\mathcal{O}_w$). This implies that the weighted sum of the derivatives of each operator must be zero for any arbitrary input u :

$$\sum_{i=1}^n \tilde{c}_i \lambda_{i,k}(u) = 0, \quad \forall u \quad (\text{A6})$$

where $\tilde{c}_i = c_i p_i$ are the modified coefficients. Differentiating the above equation with respect to input u , and defining the input sensitivity function $\phi_i(u)$ for the i -th operator:

$$\phi_{i,k}(u) = \frac{\partial \lambda_{i,k}(u)}{\partial u} = \frac{\varepsilon}{\left[(w_{i,k} - (u \pm r_i))^2 + \varepsilon \right]^{3/2}} \quad (\text{A7})$$

The function $\phi_i(u)$ behaves as a bell-shaped function with local compact support characteristics. It achieves a strict maximum if and only if the input u satisfies $w_{i,k} - (u \pm r_i) = 0$. This indicates that the peak sensitivity corresponds precisely to the state switching point of the operator.

Since the SGPI model assigns a unique threshold r_i to each operator, such that $r_1 < r_2 < \dots < r_n$, for any two distinct operators i and j , the peak positions of their sensitivity functions $\phi_i(u)$ and $\phi_j(u)$ are strictly separated along the input axis. We select a

specific test input u^* located at the switching point of the j -th operator, i.e., $u^* = w_{j,k} \pm r_j$. Substituting this into the derived linear combination equation $\sum \tilde{c}_i \phi_i(u) = 0$ yields

$$\tilde{c}_j \cdot \phi_j(u^*) + \sum_{i \neq j} \tilde{c}_i \cdot \phi_i(u^*) = 0 \quad (\text{A8})$$

At this specific input, $\phi_j(u^*)$ attains a significant non-zero maximum value. Conversely, for other terms $i \neq j$, due to the threshold difference $r_i - r_j \neq 0$, the input u^* is far from the switching point of the i -th operator. Given a sufficiently small smoothing parameter ε , it follows that $\phi_i(u^*) \approx 0$. Therefore, to satisfy the equilibrium, we must have $\tilde{c}_j = 0$.

Iterating this logic through all $j \in \{1, \dots, n\}$, it is proven that all coefficients $\tilde{c}_j$ must be zero. Since the weights $p_i \neq 0$, the original coefficients c_i must all be zero. This confirms that the column vectors of the observability matrix $\mathcal{O}_w$ are linearly independent in the real domain, i.e., $\text{rank}(\mathcal{O}_w) = n$.

Consequently, the hysteresis state vector satisfies the local observability condition. $\square$

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