

Article

Advanced Results on Fuzzy Output Feedback Control of Nonlinear Systems via Observer-Based Method

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Abstract

This article addresses the output feedback control of nonlinear systems via a fuzzy affine model-based method and observer design technique. To acquire the estimation of the unmeasurable system states, a piecewise affine observer is first devised. By exploiting Young's matrix inequality and its variant, a design approach to the observer-based output feedback controller is proposed under a convex optimization set-up. In particular, only half of the crossing term needs to be upper-bounded, which yields relaxed existence conditions. Moreover, the proposed approach removes the restriction in existing results that the control input matrices of all local affine models must be identical and uncertainty-free. Simulations on a nonlinear navigation electronic circuit are conducted to illustrate the efficacy of the developed approach.

Keywords: output feedback; fuzzy control; nonlinear systems; observer; convex optimization

1. Introduction

A wide range of aerospace systems, including aircraft, on-orbit spacecraft, launch vehicle systems, and navigation electronic circuits, are inherently dominated by strong nonlinear dynamics. Rigorous analysis of such plants requires accurate characterization of complex nonlinear terms. This requirement imposes strict prerequisites on system modeling, controller design, and closed-loop stability assessment [1–5]. As an alternative, fuzzy logic control (FLC) offers a powerful tool for regulating highly complex nonlinear dynamics as well as non-analytic processes encountered in space missions. A variety of fuzzy-based control strategies have been reported for spacecraft attitude stabilization and on-orbit autonomous control tasks, where fuzzy systems are employed to compensate for model uncertainties and external space disturbances [6–10]. Nevertheless, classic FLC approaches still have notable shortcomings for space-borne applications. Owing to complicated on-orbit disturbances and insufficient model information, it is difficult to quantify the closed-loop performance or to synthesize the controller systematically. These difficulties hinder the practical deployment of FLC in intelligent autonomous control of spacecraft.

Among model-based fuzzy approaches, T-S fuzzy models have attracted extensive attention over recent decades [11–14]. It has been proven that T-S fuzzy models are universal function approximators, which can approximate any smooth nonlinear system to an arbitrary degree of accuracy within any convex compact set [15–18]. From this point, T-S fuzzy model-based control for various types of nonlinear systems has drawn rapidly



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growing interest from both academic and industrial communities [19–24]. Specifically, in [20], the model reduction issue was studied for discrete time interval type 2 T-S fuzzy systems. The authors of [24] proposed a network method to design T-S fuzzy models via disjunctive fuzzy neural networks.

In practice, the state variables of aerospace systems are often not fully measurable, and thus the state observer technique becomes a natural tool for estimating the unmeasurable states. Consequently, observer-based output feedback control (OFC) of T-S fuzzy systems has become an active research topic [25–29]. Specifically, a robust controller was designed in [25] for T-S fuzzy time delay systems based on a fuzzy state observer, and the stabilization of T-S fuzzy systems with uncertain measurement noise and disturbance was studied in [29] via interval observers. As a T-S fuzzy affine model operates with substantially enhanced function approximation capability [15,18], the OFC for nonlinear networked systems was further studied in [30] via an asynchronous state observer design method. Nonetheless, the controller and observer synthesis results in [30] were attained by assuming the control input matrices to be uncertainty-free and common for each local affine model, which needs to be relaxed. To the best of our knowledge, the fuzzy OFC problem for nonlinear systems with uncertain and non-identical input matrices remains challenging, which motivates this work.

This paper addresses the observer-based OFC issue for nonlinear systems via a fuzzy affine model-based method. A piecewise affine observer is devised to attain the estimation of the unmeasurable system state variables, and information of the affine terms can be fully utilized simultaneously. The main contributions of this paper are summarized as follows: (1) a piecewise affine observer is devised to estimate the unmeasurable states, and the information of the affine terms is fully utilized; (2) resorting to Young’s matrix inequality with its variant and convexification techniques, new existence conditions for the observer and controller co-design are derived via a linearization procedure; and (3) compared with the existing observer-based OFC results, the requirement that the control input matrices be common and uncertainty-free is removed, and the reduction of conservatism is quantitatively verified in Section 4. Simulation studies on a nonlinear navigation electronic circuit are provided to demonstrate the effectiveness of the proposed approach.

2. Preliminaries

Consider the following T-S fuzzy affine system with r fuzzy **IF-THEN** rules.

Plant Rule $\mathcal{R}^l$: IF $\zeta_1(s)$ is $\mathcal{F}_1^l$ and $\cdots$ and $\zeta_\varphi(s)$ is $\mathcal{F}_\varphi^l$, THEN

$$\begin{cases} \dot{x}(s) = (A_l + \Delta A_l)x(s) + a_l + \Delta a_l + (B_l + \Delta B_l)u(s) + D_{l1}w(s) \\ z(s) = L_l x(s) + N_l u(s) \\ y(s) = C_l x(s) + D_{l2}w(s), \quad l \in \mathcal{L} := \{1, \dots, r\}, \end{cases} \quad (1)$$

in which $\zeta(s) := [\zeta_1(s), \dots, \zeta_\varphi(s)]$ denote the measurable premise variables; $\mathcal{F}_\phi^l$ ($\phi = 1, \dots, \varphi$), $\mathcal{R}^l$, and r are fuzzy sets, the l th fuzzy rule, and the number of fuzzy rules, respectively; $x(s) \in \mathbb{R}^{n_x}$, $y(s) \in \mathbb{R}^{n_y}$, $z(s) \in \mathbb{R}^{n_z}$, and $u(s) \in \mathbb{R}^{n_u}$ represent the system states, the measurement output, the regulated output, and the control input, respectively; $w(s) \in L_2[0, \infty)$ is the exogenous disturbance, while $w(s) \in \mathbb{R}^{n_w}$; and ΔA_l , Δa_l , and ΔB_l denote the uncertainty terms satisfying

$$\begin{bmatrix} \Delta A_l & \Delta a_l & \Delta B_l \end{bmatrix} = U_l \Delta_l(s) \begin{bmatrix} W_{l1} & W_{l2} & W_{l3} \end{bmatrix}, \quad l \in \{1, \dots, r\}, \quad (2)$$

in which W_{l1} , W_{l2} , W_{l3} and U_l are known matrices. $\Delta_l(s) \in \mathbb{R}^{n_1 \times n_2}$ represent time-varying unknown matrices satisfying

$$\Delta_l^T(s)\Delta_l(s) \leq \mathbf{I}. \quad (3)$$

Define $\mu_l[\zeta(s)]$ to be the standardized membership function

$$\mu_l[\zeta(s)] := \frac{\prod_{\phi=1}^p \mu_{l\phi}[\zeta_\phi(s)]}{\sum_{\nu=1}^r \prod_{\phi=1}^p \mu_{\nu\phi}[\zeta_\phi(s)]} \geq 0, \quad \sum_{l=1}^r \mu_l[\zeta(s)] = 1, \quad (4)$$

where $\mu_{l\phi}[\zeta_\phi(s)]$ depicts the grade of membership of $\zeta_\phi(s)$ inside $\mathcal{F}_\phi^l$.

Via a product inference, center-average defuzzifier, and singleton fuzzifier, the global T-S fuzzy affine model is derived as follows:

$$\begin{cases} \dot{x}(s) = (A(\mu) + \Delta A(\mu))x(s) + a(\mu) + \Delta a(\mu) + (B(\mu) + \Delta B(\mu))u(s) + D_1(\mu)w(s) \\ z(s) = L(\mu)x(s) + N(\mu)u(s) \\ y(s) = C(\mu)x(s) + D_2(\mu)w(s) \end{cases} \quad (5)$$

where

$$\begin{bmatrix} a(\mu) & A(\mu) & \Delta a(\mu) & \Delta A(\mu) \\ B(\mu) & \Delta B(\mu) & L(\mu) & N(\mu) \\ D_1(\mu) & D_2(\mu) & C(\mu) & \end{bmatrix} = \sum_{l=1}^r \mu_l \begin{bmatrix} a_l & A_l & \Delta a_l & \Delta A_l \\ B_l & \Delta B_l & L_l & N_l \\ D_{l1} & D_{l2} & C_l & \end{bmatrix}, \quad (6)$$

and $\mu_l := \mu_l[\zeta(s)]$ is used for brevity.

Due to system state-space polyhedral decomposition induced by membership functions (MFs) and fuzzy rules, according to [15,30], the premise variable space can be divided into fuzzy regions ($0 < \mu_l < 1$ for several local models) and crisp regions ($\mu_l = 1$ for several l).

The indices of regions can be decomposed as $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1$, where $\mathcal{I}_0$ denotes the index set of regions containing the origin and $\mathcal{I}_1$ denotes the index set of regions otherwise. The set $\mathcal{S}(i)$ in Equation (7) is involved to depict the indices within region S_i :

$$\mathcal{S}(i) := \{m | \mu_m[\zeta(s)] > 0, \zeta(s) \in S_i, i \in \mathcal{I}, m \in \mathcal{L}\}. \quad (7)$$

Relying on the set $\mathcal{S}(i)$ in Equation (7), we reformulate Equation (5) as

$$\begin{cases} \dot{x}(s) = (\mathcal{A}_i + \Delta \mathcal{A}_i)x(s) + \mathbf{a}_i + \Delta \mathbf{a}_i + (\mathcal{B}_i + \Delta \mathcal{B}_i)u(s) + \mathcal{D}_{i1}w(s), \\ z(s) = \mathcal{L}_i x(s) + \mathcal{N}_i u(s), \\ y(s) = \mathcal{C}_i x(s) + \mathcal{D}_{i2}w(s), \zeta(s) \in S_i, i \in \mathcal{I}, \end{cases} \quad (8)$$

where

$$\begin{bmatrix} \mathbf{a}_i & \mathcal{A}_i & \Delta \mathbf{a}_i & \Delta \mathcal{A}_i \\ \mathcal{B}_i & \Delta \mathcal{B}_i & \mathcal{L}_i & \mathcal{N}_i \\ \mathcal{D}_{i1} & \mathcal{D}_{i2} & \mathcal{C}_i & \end{bmatrix} = \sum_{m \in \mathcal{S}(i)} \mu_m \begin{bmatrix} a_m & A_m & \Delta a_m & \Delta A_m \\ B_m & \Delta B_m & L_m & N_{m1} \\ D_{m1} & D_{m2} & C_m & \end{bmatrix}. \quad (9)$$

with $\mu_m[\zeta(s)] \in (0, 1]$ and $\sum_{m \in \mathcal{S}(i)} \mu_m[\zeta(s)] = 1$.

We can then synthesize a novel piecewise fuzzy affine state observer for the system in Equation (1) in the following form.

Region Rule: IF $\zeta(s) \in S_i$, THEN

Local Observer Rule $\mathcal{R}^m$: IF $\zeta_1(s)$ is $\mathcal{F}_1^m$ and $\dots$ and $\zeta_\varphi(s)$ is $\mathcal{F}_\varphi^m$, THEN

$$\begin{cases} \dot{\hat{x}}(s) = A_m \hat{x}(s) + a_m + B_m u(s) + R_i(y(s) - \hat{y}(s)), \\ \hat{y}(s) = C_m \hat{x}(s), i \in \mathcal{I}, \end{cases} \quad (10)$$

where $R_i \in \mathbb{R}^{n_x \times n_y}$ denote the state observer gains and $\hat{x}(s) \in \mathbb{R}^{n_x}$ is the estimated system states.

Remark 1. Different from conventional linear fuzzy observers that only use the system matrix A_i , the proposed piecewise affine observer explicitly includes the affine term a_m in its dynamics. This design fully retains the affine structure information of the T-S fuzzy affine model, which contributes to higher estimation accuracy for systems with non-zero affine terms.

The state observer in Equation (10) is then reformulated as follows:

$$\begin{cases} \dot{\hat{x}}(s) = A_i \hat{x}(s) + a_i + B_i u(s) + R_i(y(s) - \hat{y}(s)), \\ \hat{y}(s) = C_i \hat{x}(s), i \in \mathcal{I}. \end{cases} \quad (11)$$

We synthesize a piecewise-type controller relying on the state observer in Equation (11) as follows:

$$u(s) = K_i \hat{x}(s), i \in \mathcal{I}, \quad (12)$$

in which $K_i \in \mathbb{R}^{n_u \times n_x}$ is the controller gains.

We denote the augmented vector $\bar{x}(s) = \begin{bmatrix} x^T(s) & e^T(s) \end{bmatrix}^T$, where $e(s) = x(s) - \hat{x}(s)$ is estimation error. The resultant closed-loop system is then attained as follows:

$$\begin{cases} \dot{\bar{x}}(s) = (\bar{A}_i + \Delta \bar{A}_i) \bar{x}(s) + \bar{a}_i + \Delta \bar{a}_i + \bar{D}_i w(s), \\ z(s) = \bar{L}_i \bar{x}(s), \end{cases} \quad (13)$$

where

$$\begin{cases} \bar{A}_i = \begin{bmatrix} A_i + B_i K_i & -B_i K_i \\ \mathbf{0} & A_i - R_i C_i \end{bmatrix}, \Delta \bar{A}_i = \begin{bmatrix} \Delta A_i + \Delta B_i K_i & -\Delta B_i K_i \\ \Delta A_i + \Delta B_i K_i & -\Delta B_i K_i \end{bmatrix}, \\ \bar{a}_i = \begin{bmatrix} a_i \\ \mathbf{0} \end{bmatrix}, \Delta \bar{a}_i = \begin{bmatrix} \Delta a_i \\ \Delta a_i \end{bmatrix}, \bar{D}_i = \begin{bmatrix} D_{i1} \\ D_{i1} - R_i D_{i2} \end{bmatrix}, \bar{L}_i = \begin{bmatrix} L_i + N_i K_i & -N_i K_i \end{bmatrix}. \end{cases} \quad (14)$$

This work attempts to devise an observer-based controller (Equations (10–12)) for the system in Equation (1) such that in an $\mathcal{H}_\infty = \gamma$ set-up, when $w(s) \equiv 0$, the system in Equation (13) is ensured to be with asymptotic stability, and under zero initial condition, we have

$$\int_0^\infty \|z(s)\|^2 ds < \gamma^2 \int_0^\infty \|w(s)\|^2 ds \quad (15)$$

for nonzero $w(s) \in L_2[0, \infty)$.

3. Main Results

Similar to [15], assume that an ellipsoid $\mathcal{E}_i$ can be exploited in this paper to outer approximate of every polyhedral subspace S_i given in Equation (8) as follows:

$$\mathcal{E}_i := \{x \mid \|Q_i x + q_i\| \leq 1\}, S_i \subseteq \mathcal{E}_i. \quad (16)$$

The aforementioned covering is useful especially when S_i is slab subspaces. Then, it can be guaranteed that q_i and Q_i within Equation (16) exist, and then the covering is exact, namely $S_i \subseteq \mathcal{E}_i$, $\mathcal{E}_i \subseteq S_i$.

Specifically, if S_i are slabs with

$$S_i = \{x | \lambda_i \leq \psi_i^T x \leq \vartheta_i\}, \quad (17)$$

where $\psi_i \in \mathbb{R}^{n_x}$, $\lambda_i, \vartheta_i \in \mathbb{R}$, $i \in \mathcal{I}$, then each slab subspace is capable of being depicted by a degenerate ellipsoid (Equation (17)) with

$$Q_i = \frac{2\psi_i^T}{\vartheta_i - \lambda_i} q_i = -\frac{\vartheta_i + \lambda_i}{\vartheta_i - \lambda_i}. \quad (18)$$

The relationship in Equation (19) holds for each subspace based on Equation (17):

$$\begin{bmatrix} 1 \\ x(s) \end{bmatrix}^T \begin{bmatrix} q_i^T q_i - 1 & q_i^T Q_i \\ \star & Q_i^T Q_i \end{bmatrix} \begin{bmatrix} 1 \\ x(s) \end{bmatrix} \leq 0, \quad i \in \mathcal{I}. \quad (19)$$

Theorem 1. The system in Equation (13) is asymptotically stable within an $\mathcal{H}_\infty = \gamma$ framework if scalars $\{\rho_i < 0, i \in \mathcal{I}_1\}$ and matrices $0 < P_1 = P_1^T \in \mathbb{R}^{n_x \times n_x}$, $0 < P_2 = P_2^T \in \mathbb{R}^{n_x \times n_x}$, $K_i \in \mathbb{R}^{n_u \times n_x}$, $R_i \in \mathbb{R}^{n_x \times n_y}$, and $i \in \mathcal{I}$ exist and the subsequent criteria hold:

$$\begin{bmatrix} \Lambda_{11} & \star & \star \\ \Lambda_{21} & -\text{diag}\{\gamma^2 I, I\} & \star \\ \Lambda_{31} & \mathbf{0} & -\text{diag}\{\varepsilon I, \varepsilon^{-1} I\} \end{bmatrix} < 0, \quad i \in \mathcal{I}_0, \quad m \in \mathcal{S}(i), \quad (20)$$

$$\begin{bmatrix} \rho_i(q_i^T q_i - 1) & \star & \star & \star \\ \begin{bmatrix} P_1 a_m \\ \mathbf{0} \end{bmatrix} + \rho_i H^T Q_i^T q_i & \Lambda_{11} + \rho_i H^T Q_i^T Q_i H & \star & \star \\ \mathbf{0} & \Lambda_{21} & -\text{diag}\{\gamma^2 I, I\} & \star \\ \begin{bmatrix} \mathbf{0} \\ W_{m2} \end{bmatrix} & \Lambda_{31} & \mathbf{0} & -\text{diag}\{\varepsilon I, \varepsilon^{-1} I\} \end{bmatrix} < 0, \quad (21)$$

$i \in \mathcal{I}_1, \quad m \in \mathcal{S}(i),$

where $\text{Sym}\{\mathfrak{h}\}$ is short for $\mathfrak{h} + \mathfrak{h}^T$, the notation $\star$ standing for the terms induced by symmetry, the scalar $\varepsilon > 0$ is given, and

$$\begin{cases} \Lambda_{11} = \begin{bmatrix} \text{Sym}\{P_1 A_m + P_1 B_m K_i\} & \star \\ -K_i^T B_m^T P_1 & \text{Sym}\{P_2 A_m - P_2 R_i C_m\} \end{bmatrix}, \\ \Lambda_{21} = \begin{bmatrix} D_{m1}^T P_1 & D_{m1}^T P_2 - D_{m2}^T R_i^T P_2 \\ L_m + N_m K_i & -N_m K_i \end{bmatrix}, \\ \Lambda_{31} = \begin{bmatrix} U_m^T P_1 & U_m^T P_2 \\ W_{m1} + W_{m3} K_i & -W_{m3} K_i \end{bmatrix}, H = \begin{bmatrix} I_{n_x} & -I_{n_x} \end{bmatrix}. \end{cases} \quad (22)$$

Proof. Without loss of generality, only the more complicated circumstance (i.e., $i \in \mathcal{I}_1$) is proven. We construct the Lyapunov function as follows:

$$V(s) = \bar{x}^T(s) \bar{P} \bar{x}(s) \quad (23)$$

with $\bar{P} = \text{diag}\{P_1, P_2\}$, $0 < P_1 = P_1^T \in \mathbb{R}^{n_x \times n_x}$, and $0 < P_2 = P_2^T \in \mathbb{R}^{n_x \times n_x}$.

With the Lyapunov function given in Equation (23), the system in Equation (13) is asymptotically stable under an $\mathcal{H}_\infty = \gamma$ set-up if

$$\dot{V}(s) + z^T(s)z(s) - \gamma^2 w^T(s)w(s) < 0. \quad (24)$$

Recalling the dynamics of the system in Equation (13), we derive Equation (24) as follows:

$$\begin{bmatrix} 1 \\ \bar{x}(s) \\ w(s) \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \star & \star \\ \bar{P}(\bar{a}_i + \Delta \bar{a}_i) & \Sigma_{11} & \star \\ \mathbf{0} & \bar{D}_i^T \bar{P} & -\gamma^2 \mathbf{I} \end{bmatrix} \begin{bmatrix} 1 \\ \bar{x}(s) \\ w(s) \end{bmatrix} < 0, \quad i \in \mathcal{I}_1, \quad (25)$$

where $\Sigma_{11} = \text{Sym}\{\bar{P}(\bar{A}_i + \Delta \bar{A}_i)\} + \bar{\mathcal{L}}_i^T \bar{\mathcal{L}}_i$.

Using the S procedure combined with Equation (19), for scalars $\rho_i < 0$ with $i \in \mathcal{I}_1$, the inequality in Equation (26) implies Equation (25) via a Schur complement:

$$\begin{bmatrix} \rho_i(q_i^T q_i - 1) & \star & \star \\ \bar{P}(\bar{a}_i + \Delta \bar{a}_i) + \rho_i H^T Q_i^T q_i & \Sigma_{11} + \rho_i H^T Q_i^T Q_i H & \star \\ \mathbf{0} & \bar{D}_i^T \bar{P} & -\gamma^2 \mathbf{I} \end{bmatrix} < 0, \quad i \in \mathcal{I}_1, \quad (26)$$

where

$$H = \begin{bmatrix} \mathbf{I}_{n_x} & -\mathbf{I}_{n_x} \end{bmatrix}. \quad (27)$$

By extracting the MFs and employing Lemma 1 given in [30] to tackle the uncertainty terms in the form of Equation (2) and then relying on the Schur complement, Equation (21) indicates Equation (26) for a given scalar $\varepsilon > 0$. This completes the proof. $\square$

Remark 2. The observer and controller gains R_i and K_i in Equations (20) and (21), respectively, are highly coupled with different system matrices and Lyapunov matrices, and solving the inequalities in Theorem 1 with a linearization process is a challenge.

To circumvent the above problems, in the following, two new approaches will be proposed for the purpose of convexification.

Theorem 2. The system in Equation (13) is asymptotically stable within an $\mathcal{H}_\infty = \gamma$ framework if scalars $\{q_i < 0, i \in \mathcal{I}_1\}$ and matrices $0 < \{X_1 = X_1^T, P_2 = P_2^T\} \in \mathbb{R}^{n_x \times n_x}$, $\bar{R}_i \in \mathbb{R}^{n_x \times n_y}$, $\tilde{K}_i \in \mathbb{R}^{n_u \times n_x}$, $i \in \mathcal{I}$, exist and the subsequent criteria hold:

$$\begin{bmatrix} \Gamma_{11} & \star & \star & \star \\ \Gamma_{21} & -\text{diag}\{\gamma^2 \mathbf{I}, \mathbf{I}\} & \star & \star \\ \Gamma_{31} & \mathbf{0} & -\text{diag}\{\varepsilon \mathbf{I}, \varepsilon^{-1} \mathbf{I}\} & \star \\ \begin{bmatrix} -\tilde{K}_i^T B_m^T & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} & \begin{bmatrix} \mathbf{0} & -\tilde{K}_i^T N_m^T \\ \mathbf{0} & \mathbf{0} \end{bmatrix} & \begin{bmatrix} \mathbf{0} & -\tilde{K}_i^T W_{m3}^T \\ \mathbf{0} & \mathbf{0} \end{bmatrix} & -\text{diag}\{\alpha^{-1} X_1, \alpha X_1\} \end{bmatrix} < 0, \quad (28)$$

$i \in \mathcal{I}_0, \quad m \in \mathcal{N}(i),$

$$\begin{bmatrix}
-\varrho_i(\mathbf{I} - \varrho_i \mathbf{q}_i^T) & \star & \star & \\
\begin{bmatrix} \varrho_i a_m \mathbf{q}_i^T + \mathbf{X}_1 \mathbf{Q}_i^T \\ -\mathbf{Q}_i^T \end{bmatrix} & \Gamma_{11} + \text{diag}\{\varrho_i a_m a_m^T, \mathbf{0}\} & \star & \\
\mathbf{0} & \Gamma_{21} & -\text{diag}\{\gamma^2 \mathbf{I}, \mathbf{I}\} & \\
\begin{bmatrix} \mathbf{0} \\ \varrho_i W_{m2} \mathbf{q}_i^T \end{bmatrix} & \Gamma_{31} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \varrho_i W_{m2} a_m^T & \mathbf{0} \end{bmatrix} & \mathbf{0} & \\
\mathbf{0} & \begin{bmatrix} -\tilde{K}_i^T B_m^T & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} & \begin{bmatrix} \mathbf{0} & -\tilde{K}_i^T N_m^T \\ \mathbf{0} & \mathbf{0} \end{bmatrix} & \\
\star & \star & \star & \\
\star & \star & \star & \\
\star & \star & \star & \\
\text{diag}\{-\varepsilon \mathbf{I}, \varrho_i W_{m2} W_{m2}^T - \varepsilon^{-1} \mathbf{I}\} & \star & \star & \\
\begin{bmatrix} \mathbf{0} & -\tilde{K}_i^T W_{m3}^T \\ \mathbf{0} & \mathbf{0} \end{bmatrix} & -\text{diag}\{\alpha^{-1} X_1, \alpha X_1\} & &
\end{bmatrix} < 0, \quad i \in \mathcal{I}_1, \quad m \in \mathcal{N}(i), \quad (29)$$

where $\varepsilon > 0$ and $\alpha > 0$ are and given scalars and

$$\begin{cases} \Gamma_{11} = \text{diag}\{\text{sym}\{A_m X_1 + B_m \tilde{K}_i\}, \text{sym}\{P_2 A_m - \tilde{R}_i C_m\}\}, \\ \Gamma_{21} = \begin{bmatrix} D_{m1}^T & D_{m1}^T P_2 - D_{m2}^T \tilde{R}_i^T \\ L_m X_1 + N_m \tilde{K}_i & \mathbf{0} \end{bmatrix}, \Gamma_{31} = \begin{bmatrix} U_m^T & U_m^T P_2 \\ W_{m1} X_1 + W_{m3} \tilde{K}_i & \mathbf{0} \end{bmatrix}. \end{cases} \quad (30)$$

One can compute the controller and observer gains via

$$\begin{cases} K_i = \tilde{K}_i X_1^{-1}, \\ R_i = P_2^{-1} \tilde{R}_i, \quad i \in \mathcal{I}. \end{cases} \quad (31)$$

Proof. Only the more complicated circumstance ($i \in \mathcal{I}_1$) will be proven without loss of generality. We define

$$\tilde{R}_i = P_2 R_i, \quad \tilde{K}_i = K_i X_1. \quad (32)$$

In noticing that the matrix $\bar{P}$ in Equation (23) has a diagonal structure and performing a congruent transformation to Equation (21) through $\text{diag}\{1, X_1, \mathbf{I}, \mathbf{I}, \mathbf{I}, \mathbf{I}\}$, the subsequent inequality implies Equation (21):

$$\begin{bmatrix}
\rho_i(\mathbf{q}_i^T \mathbf{q}_i - 1) & \star & \star & \star & \\
\bar{\Gamma}_{10} & \bar{\Gamma}_{11} + \rho_i \begin{bmatrix} X_1 \mathbf{Q}_i^T \mathbf{Q}_i X_1 & \star \\ -\mathbf{Q}_i^T \mathbf{Q}_i X_1 & \mathbf{Q}_i^T \mathbf{Q}_i \end{bmatrix} & \star & \star & \\
\mathbf{0} & \Gamma_{21} + \text{diag}\{\mathbf{0}, -N_m K_i\} & -\text{diag}\{\gamma^2 \mathbf{I}, \mathbf{I}\} & \star & \\
\begin{bmatrix} \mathbf{0} \\ W_{m2} \end{bmatrix} & \Gamma_{31} + \text{diag}\{\mathbf{0}, -W_{m3} K_i\} & \mathbf{0} & -\text{diag}\{\varepsilon \mathbf{I}, \varepsilon^{-1} \mathbf{I}\} & \\
& & & &
\end{bmatrix} < 0, \quad (33)$$

$i \in \mathcal{I}_1, \quad m \in \mathcal{N}(i),$

where Γ_{21} and Γ_{31} are given as shown in Equation (30) and

$$\begin{cases} \bar{\Gamma}_{11} = \begin{bmatrix} \text{Sym}\{A_m X_1 + B_m \tilde{K}_i\} & \star \\ -K_i^T B_m^T & \text{Sym}\{P_2 A_m - \tilde{R}_i C_m\} \end{bmatrix}, \\ \bar{\Gamma}_{10} = \begin{bmatrix} a_m \\ \mathbf{0} \end{bmatrix} + \rho_i \begin{bmatrix} X_1 Q_i^T q_i \\ -Q_i^T q_i \end{bmatrix}. \end{cases} \quad (34)$$

By inspecting the terms $\rho_i X_1 Q_i^T q_i$ and $\rho_i X_1 Q_i^T Q_i X_1$ in Equation (33), the scalars ρ_i are multiplied by the Lyapunov matrix X_1 . Moreover, matrices K_i and $\tilde{K}_i$ both exist in Equation (33). The above-mentioned issues indicates that the inequality in Equation (33) is hard to solve through a linearization process. Thus, the first step is to dissociate the Lyapunov matrix X_1 from the scalars ρ_i . Based on the Schur complement, we rewrite the inequality in Equation (33) as follows:

$$\begin{aligned} & \begin{bmatrix} \bar{\Gamma}_{11} + \rho_i \begin{bmatrix} X_1 Q_i^T Q_i X_1 & \star \\ -Q_i^T Q_i X_1 & Q_i^T Q_i \end{bmatrix} & \star & \star \\ \Gamma_{21} + \text{diag}\{\mathbf{0}, -N_m K_i\} & -\text{diag}\{\gamma^2 \mathbf{I}, \mathbf{I}\} & \star \\ \Gamma_{31} + \text{diag}\{\mathbf{0}, -W_{m3} K_i\} & \mathbf{0} & -\text{diag}\{\varepsilon \mathbf{I}, \varepsilon^{-1} \mathbf{I}\} \end{bmatrix} \\ & + \begin{bmatrix} \bar{\Gamma}_{10} \\ \mathbf{0} \\ \mathbf{0} \\ W_{m2} \end{bmatrix} \left(\rho_i (1 - q_i^T q_i) \right)^{-1} (\star) < 0, \quad i \in \mathcal{I}_1, \quad m \in \mathcal{N}(i). \end{aligned} \quad (35)$$

The following matrix inverse lemma

$$(X + YZQ)^{-1} = X^{-1} - X^{-1}Y(Z^{-1} + QX^{-1}Y)^{-1}QX^{-1}, \quad (36)$$

yields

$$\begin{cases} \rho_i^{-1} a_m (1 - q_i^T q_i)^{-1} a_m^T = \rho_i^{-1} a_m a_m^T + \rho_i^{-1} a_m q_i^T (\mathbf{I} - q_i q_i^T)^{-1} q_i a_m^T, \\ a_m (1 - q_i^T q_i)^{-1} q_i^T Q_i X_1 = a_m q_i^T (\mathbf{I} - q_i q_i^T)^{-1} Q_i X_1, \\ \rho_i X_1 Q_i^T q_i (1 - q_i^T q_i)^{-1} q_i^T Q_i X_1 = \rho_i X_1 Q_i^T (\mathbf{I} - q_i q_i^T)^{-1} Q_i X_1 - \rho_i X_1 Q_i^T Q_i X_1, \\ -\rho_i Q_i^T q_i (1 - q_i^T q_i)^{-1} (-q_i^T Q_i) = \rho_i Q_i^T (\mathbf{I} - q_i q_i^T)^{-1} Q_i - \rho_i Q_i^T Q_i, \\ a_m (1 - q_i^T q_i)^{-1} (-q_i^T Q_i) = a_m q_i^T (\mathbf{I} - q_i q_i^T)^{-1} (-Q_i), \\ \rho_i X_1 Q_i^T q_i (1 - q_i^T q_i)^{-1} (-q_i^T Q_i) = \rho_i X_1 Q_i^T (\mathbf{I} - q_i q_i^T)^{-1} (-Q_i) + \rho_i X_1 Q_i^T Q_i, \\ \rho_i^{-1} W_{m2} (1 - q_i^T q_i)^{-1} a_m^T = \rho_i^{-1} W_{m2} a_m^T + \rho_i^{-1} W_{m2} q_i^T (\mathbf{I} - q_i q_i^T)^{-1} q_i a_m^T, \\ W_{m2} (1 - q_i^T q_i)^{-1} q_i^T Q_i X_1 = W_{m2} q_i^T (\mathbf{I} - q_i q_i^T)^{-1} Q_i X_1, \\ W_{m2} (1 - q_i^T q_i)^{-1} (-q_i^T Q_i) = W_{m2} q_i^T (\mathbf{I} - q_i q_i^T)^{-1} (-Q_i), \\ \rho_i^{-1} W_{m2} (1 - q_i^T q_i)^{-1} W_{m2}^T = \rho_i^{-1} W_{m2} W_{m2}^T + \rho_i^{-1} W_{m2} q_i^T (\mathbf{I} - q_i q_i^T)^{-1} q_i W_{m2}^T. \end{cases} \quad (37)$$

We define $q_i = \rho_i^{-1}$. On the basis of Equation (37), we derive Equation (35) as

$$\begin{bmatrix} -q_i(\mathbf{I} - q_i q_i^T) & \star & \star \\ \begin{bmatrix} q_i a_m q_i^T + X_1 Q_i^T \\ -Q_i^T \\ \mathbf{0} \end{bmatrix} & \bar{\Gamma}_{11} + \text{diag}\{q_i a_m a_m^T, \mathbf{0}\} & \star \\ \mathbf{0} & \Gamma_{21} + \text{diag}\{\mathbf{0}, -N_m K_i\} & -\text{diag}\{\gamma^2 \mathbf{I}, \mathbf{I}\} \\ \begin{bmatrix} \mathbf{0} \\ q_i W_{m2} q_i^T \end{bmatrix} & \Gamma_{31} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ q_i W_{m2} a_m^T & -W_{m3} K_i \end{bmatrix} & \mathbf{0} \\ \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \\ -\text{diag}\{\varepsilon \mathbf{I}, \varepsilon^{-1} \mathbf{I} - q_i W_{m2} W_{m2}^T\} & & \end{bmatrix} < 0, \quad i \in \mathcal{I}_1, \quad m \in \mathcal{N}(i). \quad (38)$$

It can be seen that the Lyapunov matrix X_1 has been decomposed from the scalars q_i . As matrices K_i and $\tilde{K}_i$ exist in Equation (38) simultaneously, it is inconvenient to achieve the observer-controller co-design purposes. Now, for two given matrices $\aleph$ and ℓ , through Young's inequality [31], the inequality in Equation (39) holds for any positive definite matrix $Q > 0$ and parameter $\alpha > 0$:

$$\text{Sym}\{\aleph^T \ell\} = \text{Sym}\{\aleph^T Q^{\frac{1}{2}} Q^{-\frac{1}{2}} \ell\} \leq \alpha \aleph^T Q \aleph + \alpha^{-1} \ell^T Q^{-1} \ell. \quad (39)$$

Considering the matrices K_i within Equation (38) yields

$$\text{Sym} \left\{ \begin{bmatrix} 0 & \mathbf{0} \\ 0 & \begin{bmatrix} 0 & -B_m K_i \\ 0 & 0 \end{bmatrix} \\ 0 & \begin{bmatrix} 0 & 0 \\ 0 & -N_m K_i \end{bmatrix} \\ 0 & \begin{bmatrix} 0 & 0 \\ 0 & -W_{m3} K_i \end{bmatrix} \end{bmatrix} \middle| \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \text{Sym} \left\{ \underbrace{\begin{bmatrix} \mathbf{0} \\ -B_m K_i \\ 0 \\ 0 \\ -N_m K_i \\ 0 \\ -W_{m3} K_i \end{bmatrix}}_{\aleph^T} \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_x} \\ \mathbf{0} \end{bmatrix}}_{\ell} \right\}. \quad (40)$$

Based on Equation (39), and then given a scalar $\alpha > 0$, Equation (40) is then derived as follows:

$$\begin{aligned} & \text{Sym} \left\{ \begin{bmatrix} 0 & \mathbf{0} \\ 0 & \begin{bmatrix} 0 & -B_m K_i \\ 0 & 0 \end{bmatrix} \\ 0 & \begin{bmatrix} 0 & 0 \\ 0 & -N_m K_i \end{bmatrix} \\ 0 & \begin{bmatrix} 0 & 0 \\ 0 & -W_{m3} K_i \end{bmatrix} \end{bmatrix} \middle| \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \\ & \leq \alpha \underbrace{\begin{bmatrix} \mathbf{0} \\ -B_m K_i \\ 0 \\ 0 \\ -N_m K_i \\ 0 \\ -W_{m3} K_i \end{bmatrix}}_{\aleph^T} \underbrace{X_1 X_1^{-1} X_1}_{Q} \aleph + \alpha^{-1} \underbrace{\begin{bmatrix} \mathbf{0} \\ 0 \\ \mathbf{I}_{n_x} \\ \text{ine}0 \end{bmatrix}}_{\ell^T} \underbrace{X_1^{-1}}_{Q^{-1}} \ell. \end{aligned} \quad (41)$$

With the Schur complement, the inequality in Equation (29) implies Equation (38) based on Equation (41). This ends the proof. $\square$

Remark 3. From Theorem 2, the controller gains have been decoupled from the crossing terms in the inequalities in Equations (20) and (21) by using the matrix inequality technique in Equation (39). In contrast to the previous OFC results, such as those in [30], which require the control input matrices to be uncertainty-free and common, the proposed co-design scheme is applicable to local models with distinct and uncertain input matrices. This structural relaxation is the first source of the reduced conservatism of the proposed method.

Remark 4. The whole quantity of the term $\text{Sym}\{\mathfrak{N}^T \ell\}$ has been upper-bounded based on Equation (39). Then, a question appears naturally, namely whether a partial quantity of $\text{Sym}\{\mathfrak{N}^T \ell\}$ can be upper-bounded to further reduce controller synthesis conservatism. The answer is positive. Inspired by [32], in the following, we will propose an alternative and enhanced controller synthesis results by means of a more relaxed design procedure.

Theorem 3. The system in Equation (13) has asymptotic stability and a robust $\mathcal{H}_\infty$ performance γ if scalars $\{\rho_i < 0, i \in \mathcal{I}_1\}$ and matrices $0 < \{X_1 = X_1^T, P_2 = P_2^T\} \in \mathbb{R}^{n_x \times n_x}$, $\tilde{R}_i \in \mathbb{R}^{n_x \times n_y}$, and $\tilde{K}_i \in \mathbb{R}^{n_u \times n_x}$, $i \in \mathcal{I}$ exist such that the criteria in Equations (42) and (43) hold:

$$\begin{bmatrix} \Gamma_{11} & \star & \star & \star \\ \Gamma_{21} & -\text{diag}\{\gamma^2 \mathbf{I}, \mathbf{I}\} & \star & \star \\ \Gamma_{31} & \mathbf{0} & -\text{diag}\{\varepsilon \mathbf{I}, \varepsilon^{-1} \mathbf{I}\} & \star \\ \begin{bmatrix} -\alpha \tilde{K}_i^T B_m^T & \mathbf{I} \end{bmatrix} & \begin{bmatrix} \mathbf{0} & -\alpha \tilde{K}_i^T N_m^T \end{bmatrix} & \begin{bmatrix} \mathbf{0} & -\alpha \tilde{K}_i^T W_{m3}^T \end{bmatrix} & -2\alpha X_1 \end{bmatrix} < 0, i \in \mathcal{I}_0, m \in \mathcal{N}(i), \quad (42)$$

$$\begin{bmatrix} -\varrho_i(\mathbf{I} - q_i q_i^T) & \star & \star \\ \begin{bmatrix} \varrho_i a_m q_i^T + X_1 Q_i^T \\ -Q_i^T \end{bmatrix} & \Gamma_{11} + \text{diag}\{\varrho_i a_m a_m^T, \mathbf{0}\} & \star \\ \mathbf{0} & \Gamma_{21} & -\text{diag}\{\gamma^2 \mathbf{I}, \mathbf{I}\} \\ \begin{bmatrix} \mathbf{0} \\ \varrho_i W_{m2} q_i^T \end{bmatrix} & \Gamma_{31} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \varrho_i W_{m2} a_m^T & \mathbf{0} \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix} -\alpha \tilde{K}_i^T B_m^T & \mathbf{I} \end{bmatrix} & \begin{bmatrix} \mathbf{0} & -\alpha \tilde{K}_i^T N_m^T \end{bmatrix} \end{bmatrix} \quad (43)$$

$$\begin{bmatrix} \star & \star \\ \star & \star \\ \star & \star \\ \text{diag}\{-\varepsilon \mathbf{I}, \varrho_i W_{m2} W_{m2}^T - \varepsilon^{-1} \mathbf{I}\} & \star \\ \begin{bmatrix} \mathbf{0} & -\alpha \tilde{K}_i^T W_{m3}^T \end{bmatrix} & -2\alpha X_1 \end{bmatrix} < 0, i \in \mathcal{I}_1, m \in \mathcal{N}(i),$$

where $\varepsilon > 0$ and $\alpha > 0$ are given scalars and $\Gamma_{11}, \Gamma_{21}, \Gamma_{31}$ are given as shown in Equation (30).

One can also compute the observer and controller gains via Equation (31).

Proof. Only the more complex case ($i \in \mathcal{I}_1$) will be proven without loss of generality. We further investigate the Young's inequality, given as shown in Equation (39), which yields

$$\begin{aligned} \text{Sym}\{\aleph^T \ell\} &= \frac{1}{2} \text{Sym}\{\aleph^T \mathcal{Q}^{\frac{1}{2}} \mathcal{Q}^{-\frac{1}{2}} \ell\} + \frac{1}{2} \text{Sym}\{\aleph^T \ell\} \\ &\leq \frac{1}{2} (\alpha \aleph^T \mathcal{Q} \aleph + \alpha^{-1} \ell^T \mathcal{Q}^{-1} \ell) + \frac{1}{2} \text{Sym}\{\aleph^T \ell\} \\ &= (\ell + \alpha \mathcal{Q} \aleph)^T (2\alpha \mathcal{Q})^{-1} (\ell + \alpha \mathcal{Q} \aleph), \end{aligned} \quad (44)$$

where the scalar $\alpha > 0$ is positive. From Equations (44) and (40), we have

$$\begin{aligned} &\text{Sym} \left\{ \begin{bmatrix} 0 & \mathbf{0} \\ 0 & \begin{bmatrix} 0 & -B_m K_i \\ 0 & 0 \end{bmatrix} \\ 0 & \begin{bmatrix} 0 & 0 \\ 0 & -N_m K_i \end{bmatrix} \\ 0 & \begin{bmatrix} 0 & 0 \\ 0 & -W_{m3} K_i \end{bmatrix} \end{bmatrix} \middle| \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \\ &\leq \frac{1}{2} \left(\underbrace{\begin{bmatrix} 0 \\ 0 \\ \mathbf{I}_{n_x} \\ \text{ine}0 \end{bmatrix}}_{\ell^T} + \underbrace{\begin{bmatrix} 0 \\ -B_m K_i \\ 0 \\ -N_m K_i \\ 0 \\ -W_{m3} K_i \end{bmatrix}}_{\aleph^T} \right) \underbrace{\alpha \mathcal{Q}}_{\alpha \mathcal{Q}} \underbrace{(\alpha \mathcal{Q})^{-1}}_{(\alpha \mathcal{Q})^{-1}} \times (\ell + \alpha \mathcal{Q} \aleph). \end{aligned} \quad (45)$$

Through Equation (45) and the Schur complement, the inequality in Equation (43) can be obtained exactly. This concludes the proof. $\square$

Remark 5. Different from the Young's inequality in Equation (39), a more relaxed matrix inequality relation is proposed in Equation (44), as only the half-quantity of $\text{Sym}\{\mathcal{X}^T \mathcal{Y}\}$ is upper-bounded. Resorting to Equation (44), relaxed criteria for the controller and observer co-design are attained within Theorem 3, which further reduces the design conservatism.

Remark 6. Note that in [30], as the control input matrix B is uncertainty-free, common, and in the form of

$$B_l \equiv B = \begin{bmatrix} \mathbb{B} \\ \mathbf{0} \end{bmatrix}, \quad l \in \mathcal{L} \quad (46)$$

together with common measurement output matrices $C_l \equiv C, l \in \mathcal{L}$, $\mathbb{B} \in \mathbb{R}^{n_u \times n_u}$ being a non-singular matrix after a singular value decomposition, the Lyapunov matrix P_1 is required to be a diagonal matrix as $P_1 = \text{diag}\{P_{1(1)}, P_{1(2)}\}$, where $0 < P_{1(1)} \in \mathbb{R}^{n_u \times n_u}$, $0 < P_{1(2)} \in \mathbb{R}^{(n_x - n_u) \times (n_x - n_u)}$. Then, the system matrices A_{m1} and controller gains K_i can be parameterized as follows:

$$A_m = \begin{bmatrix} A_{m(1)} & A_{m(2)} \\ A_{m(3)} & A_{m(4)} \end{bmatrix}, \quad K_i = \begin{bmatrix} K_{i(1)} & K_{i(2)} \end{bmatrix} \quad (47)$$

where $A_{m(1)} \in \mathbb{R}^{n_u \times n_u}$, $A_{m(2)} \in \mathbb{R}^{n_u \times (n_x - n_u)}$, $A_{m(3)} \in \mathbb{R}^{(n_x - n_u) \times n_u}$, $A_{m(4)} \in \mathbb{R}^{(n_x - n_u) \times (n_x - n_u)}$, $K_{i(1)} \in \mathbb{R}^{n_u \times n_u}$, and $K_{i(2)} \in \mathbb{R}^{n_u \times (n_x - n_u)}$. The controller design results in [30] indicate that

$$\text{Sym}\{P_1 A_m + P_1 B K_i\} = \begin{bmatrix} \text{Sym}\{P_{1(1)}(A_{m(1)} + \mathbb{B}K_{i(1)})\} & \star \\ (A_{m(2)} + \mathbb{B}K_{i(2)})^T P_{1(1)} + P_{(2)} A_{m(3)} & \text{Sym}\{P_{1(2)} A_{m(4)}\} \end{bmatrix} < 0, \quad (48)$$

which implies that the matrices $A_{m(4)}$ are required to be Hurwitz-stable due to the positive definite property of the Lyapunov matrix $P_{1(2)}$. From this perspective, the design method given in [30] tends to be conservative. However, our observer-based controller synthesis scheme has released the above-mentioned constraint, and improved design results are obtained via a linearization process.

4. Simulation Studies

As presented within Figure 1, recall a nonlinear circuit system consisting of a nonlinear resistor and parasitic capacitor [33], which has been widely used as a benchmark example in fuzzy control studies. The resistor's I_R - v_c property is given as $I_R = (v_c^3 - v_c)/5$. Via the Kirchhoff current and voltage law, Equation (49) can describe the circuit dynamics as follows:

$$\begin{cases} \dot{v}_c(s) = \frac{1}{C}(-0.2(v_c^3(s) - v_c(s)) + I_L(s) + f(v_c(s))u(s) + \lambda w(s)) \\ \dot{I}_L(s) = \frac{1}{L}(-v_c(s) - I_L(s)R + u(s)) \end{cases} \quad (49)$$

where $f(v_c(s)) = \cos(v_c(s))$; $w(s)$ denotes the exogenous disturbance; and $\lambda = 0.2$, $R = 10\Omega$, $C = 0.2\text{F}$, and $L = 1\text{H}$. We define $x_1(s) = v_c(s)$ as the capacitor's voltage and $x_2(s) = I_L(s)$ as the inductance's current. We then linearize the circuit in Equation (49) around $(\pm 1, 0)^T$ and $(0, 0)^T$, and the circuit in Equation (49) can be depicted by the T-S fuzzy affine dynamic model as follows:

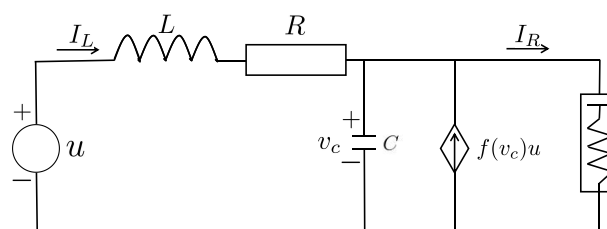


Figure 1. Nonlinear circuit.

Plant Rule $\mathcal{R}^l$: IF $x_1(s)$ is $\mathcal{F}_1^l$, THEN

$$\begin{cases} \dot{x}(s) = (A_l + \Delta A_l)x(s) + a_l + \Delta a_l + (B_l + \Delta B_l)u(s) + D_{l1}w(s) \\ y(s) = C_l x(s) + D_{l2}w(s) \\ z(s) = L_l x(s) + N_l u(s), l \in \{1, 2, 3\} \end{cases} \quad (50)$$

where

$$\begin{cases} A_1 = A_3 = \begin{bmatrix} -2 & 5 \\ -1 & -10 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 5 \\ -1 & -10 \end{bmatrix}, a_1 = \begin{bmatrix} 0.7015 \\ 0 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, a_3 = \begin{bmatrix} -0.7015 \\ 0 \end{bmatrix}, \\ B_1 = B_3 = \begin{bmatrix} 2.7015 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, C_l = \begin{bmatrix} 1 & 0 \end{bmatrix}, D_{l1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, D_{l2} = 1, \\ L_l = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}, N_l = 0.5 \end{cases} \quad (51)$$

and the parameter uncertainties ΔA_l , Δa_l , and ΔB_l are described in terms of Equation (2) with

$$\begin{cases} U_l = \begin{bmatrix} 0.05 \\ 0 \end{bmatrix}, W_{l1} = \begin{bmatrix} 0.1 & 0 \end{bmatrix}, \\ W_{l2} = 0.08, W_{l3} = 0.1. \end{cases} \quad (52)$$

Figure 2 shows the standard MFs, where $d_1 = 0.5$, $d_2 = 5$, and

$$\begin{aligned} S_1 &= \{x \in \mathbb{R}^2 \mid -d_2 \leq x_1 \leq -d_1\}, \\ S_2 &= \{x \in \mathbb{R}^2 \mid -d_1 \leq x_1 \leq d_1\}, \\ S_3 &= \{x \in \mathbb{R}^2 \mid d_1 \leq x_1 \leq d_2\}. \end{aligned} \quad (53)$$

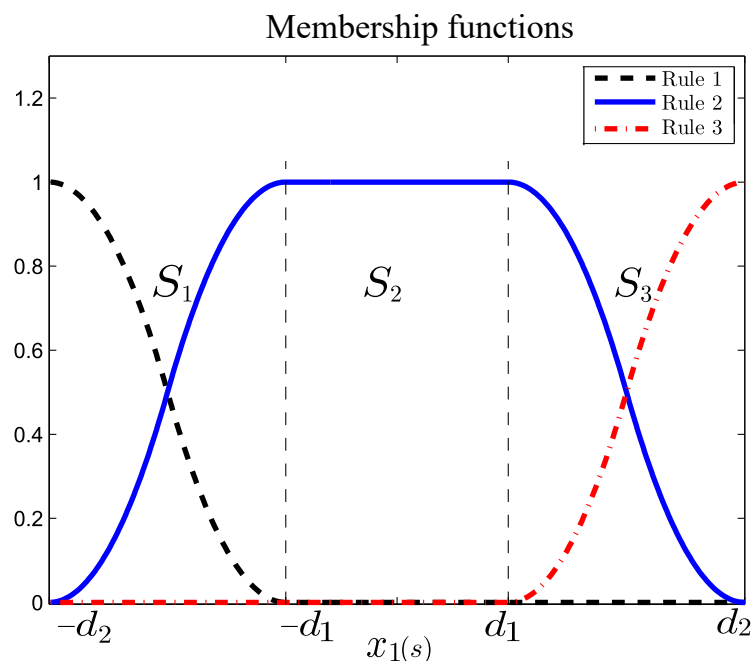


Figure 2. Fuzzy MFs.

In general, the selection of fuzzy rules and partitions follows three principles: (1) the number of rules matches the order of nonlinearity and the number of characteristic operating points; (2) partition boundaries are located between adjacent linearization points to balance regional approximation accuracy; and (3) the number of rules is chosen by trading off the approximation precision and computational complexity of LMIs.

The partition parameters were chosen as follows. Here, $d_2 = 5$ defines the operating range of the capacitor voltage v_c , determined by the physical constraints of the circuit components, and $d_1 = 0.5$ sets the boundary of the central region S_2 , where the nonlinearity is relatively mild near the origin. A sensitivity study with $d_1 \in \{0.3, 1.0\}$ shows that the resulting γ_{min} values were 0.48 and 0.62, respectively, indicating that the adopted partition achieved the best performance among the tested alternatives, while the certified performance varied moderately with the partition. It is also noted that Theorem 3 remained feasible under all the tested partitions, whereas the existing method [30] was inapplicable to this example.

These regions S_1 – S_3 , given in Equation (53), can be depicted by the degenerate ellipsoids precisely with

$$\begin{cases} q_1 = \frac{d_1 + d_2}{d_2 - d_1}, & q_2 = 0, & q_3 = \frac{d_1 + d_2}{d_1 - d_2}, \\ Q_1 = Q_3 = \begin{bmatrix} \frac{2}{d_2 - d_1} & 0 \end{bmatrix}, & Q_2 = \begin{bmatrix} \frac{1}{d_1} & 0 \end{bmatrix}. \end{cases} \quad (54)$$

This work aims to devise an observer-based controller (Equations (10–12)) to guarantee the circuit system in Equation (49) being asymptotically stable with satisfying robust $\mathcal{H}_\infty$ performance γ . Suppose that the circuit states are not completely measurable in this example. In addition, the control input matrices B_i given in Equation (51) are different and contain parameter uncertainties. Therefore, the synthesis approaches previously proposed in [25–28,30] are not applicable. Nonetheless, by choosing $\varepsilon = 0.1$ and $\alpha = 0.5$ and utilizing Theorem 2, the observer and controller gains are

$$\begin{aligned} \left[\begin{array}{c|c|c} R_1 & R_2 & R_3 \end{array} \right] &= \left[\begin{array}{c|c|c} 1.0277 & 1.0313 & 1.0274 \\ \hline 0.0001 & -0.0005 & 0.0002 \end{array} \right], \\ \left[\begin{array}{c} K_1 \\ \text{ine}K_2 \\ \text{ine}K_3 \end{array} \right] &= \left[\begin{array}{cc} -0.3365 & -0.1670 \\ \text{ine} - 0.3352 & -0.1668 \\ \text{ine} - 0.3361 & -0.1667 \end{array} \right] \end{aligned} \quad (55)$$

with an optimal robust $\mathcal{H}_\infty$ index $\gamma_{\min} = 1.0924$. Alternatively, by resorting to Theorem 3 with $\alpha = 0.5$ and $\varepsilon = 0.1$, one has

$$\begin{aligned} \left[\begin{array}{c|c|c} R_1 & R_2 & R_3 \end{array} \right] &= \left[\begin{array}{c|c|c} 1.0099 & 1.0098 & 1.0099 \\ \hline -0.0007 & -0.0007 & -0.0007 \end{array} \right], \\ \left[\begin{array}{c} K_1 \\ \text{ine}K_2 \\ \text{ine}K_3 \end{array} \right] &= \left[\begin{array}{cc} -0.5708 & -0.3088 \\ \text{ine} - 0.5664 & -0.3068 \\ \text{ine} - 0.5695 & -0.3071 \end{array} \right] \end{aligned} \quad (56)$$

with $\gamma_{\min} = 0.3343$. Under the same settings ($\alpha = 0.5, \varepsilon = 0.1$), the minimum admissible H^∞ level provided by Theorem 3 was about 69% smaller than that of Theorem 2, which quantitatively confirms the conservatism reduction established in Remarks 5 and 6. It is also noted that the plant in Equation (51) possesses distinct and uncertain input matrices ($B_1 = B_3 \neq B_2, \Delta B_i \neq 0$), which violates the common-input-matrix requirement of the existing observer-based OFC results [25–28,30] (see Remark 6). Hence, these results are not applicable to the plant considered here.

For a disturbance $w(s) = 60e^{-0.5s} \sin(0.1s)$ and initial states $x_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, while exploiting the observer-based controller in Equations (10–12) with the gains calculated as in Equation (56), Figure 3a–c presents the circuit states, the estimated states, and the control input, respectively. Figure 3d presents the robust $\mathcal{H}_\infty$ performance under a zero initial condition. The ratio $\frac{\sqrt{\int_0^t z^T(s)z(s)ds}}{\sqrt{\int_0^t w^T(s)w(s)ds}}$ was about 0.142, which is lower than $\gamma_{\min} = 0.3343$.

Remark 7. It is emphasized that the certified H^∞ level $\gamma_{\min} = 0.3343$ holds uniformly over the entire admissible uncertainty set $\|\Delta(t)\| \leq 1$ and for all square-integrable disturbances, as the LMI conditions of Theorem 3 are enforced for all admissible uncertainties. The time-domain responses in Figure 3 correspond to one particular realization of uncertainty and disturbance and are presented for illustration only. In addition, the convergence of the estimation error for arbitrary initial conditions

is guaranteed by the positive definiteness of the piecewise Lyapunov matrices, and the scalars α and ε can be tuned to balance feasibility and conservatism for practical use.

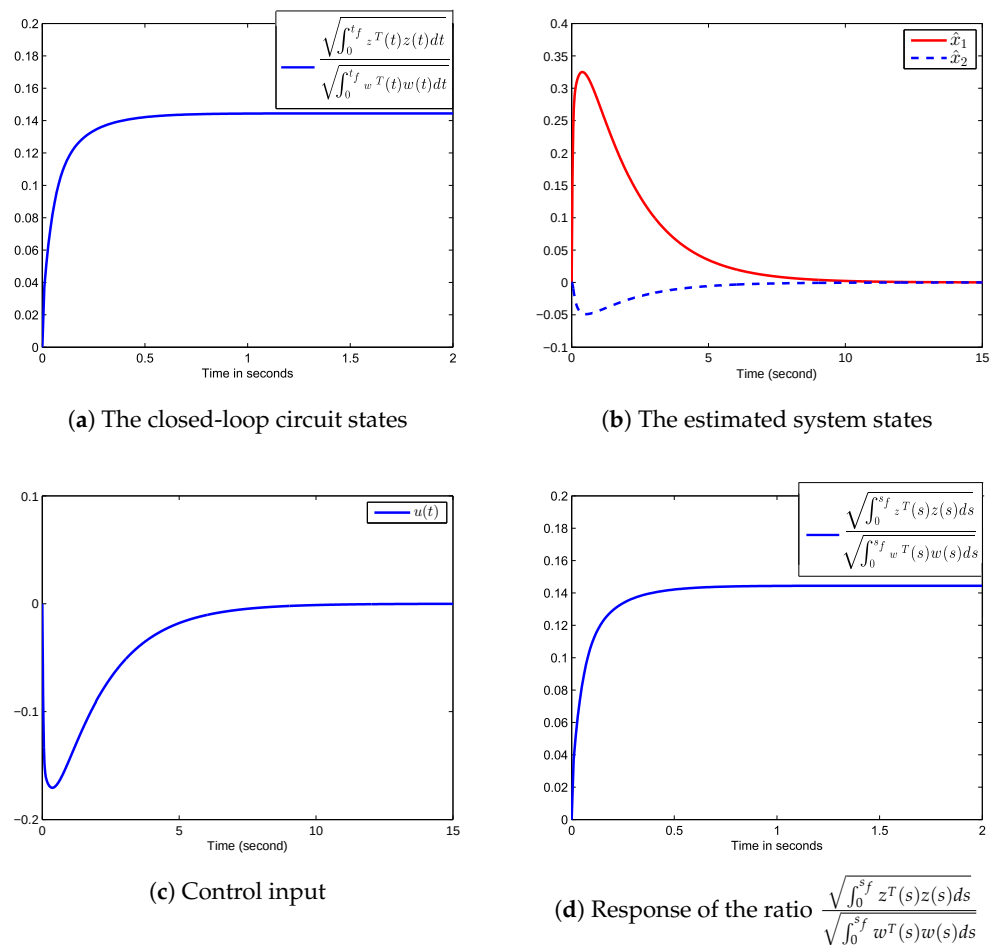


Figure 3. This is a wide figure. Schemes follow the same formatting. If there are multiple panels, then they should be listed as follows. (a) Description of what is contained in the first panel. (b) Description of what is contained in the second panel. (c) Description of what is contained in the third panel. (d) Description of what is contained in the fourth panel. Figures should be placed in the main text near the first time they are cited.

5. Conclusions

The observer-based OFC problem of nonlinear systems was studied via T-S fuzzy affine models in this article. A piecewise affine state observer was devised to estimate the unmeasurable states, and relaxed controller and observer co-design conditions were derived via a half-quantity bounding technique. The proposed scheme also removes the requirement that all control input matrices be common and uncertainty-free. Future research will be directed toward (1) further reducing the design conservatism, e.g., by generalizing the region-wise crisp controller (Equation (12)) to a piecewise fuzzy observer-based output-feedback controller and by introducing more sophisticated relaxation techniques for the cross-term bounding; (2) extending the proposed co-design scheme to fuzzy systems with time-varying delays and to sampled data or event-triggered output-feedback settings, where the triggering condition and the controller or observer gains can be solved jointly [34]; and (3) investigating the application of the proposed design to experimental platforms and conducting an extended numerical sensitivity study.

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