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Edgeworth Expansions When the Parameter Dimension Increases with Sample Size

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Abstract

Suppose that we have a statistical model with q unknown parameters w, and an estimate $\hat{w}$, based on a sample of size n. A basic question is: what is the covariance of the estimate? The covariance is needed for the Central Limit Theorem (CLT). This gives a first approximation for the distribution of $\hat{w}$. But what if $q_n = n$ increases with n? How fast can it increase and the CLT still hold? An answer has so far only been given for the sample mean. The same is true for the Edgeworth expansions. These are expansions in powers of $n^{-1/2}$ for the density and distribution of $\hat{w}$. For fixed q, these expansions are important, as they show how small n can be for the CLT to apply. When it does, they can greatly improve the accuracy of the CLT. I give conditions that allow for the Edgeworth expansions to remain valid when $q_n = q$ increases with n. Earlier Edgeworth expansions when $q_n = q$ increases, have only been done for a sample mean, and only for a 2nd order Edgeworth expansion. In contrast, I consider a very large class of estimates, the class of non-lattice standard estimates. An estimate is said to be a standard estimate if its mean converges to its true value as n increases, and for $r \geq 1$, its rth order cumulants have magnitude n^{1-r} and can be expanded in powers of n^{-1}. For this class of estimates, I show that the Edgeworth expansions hold if q_n grows as a power of n less than $1/6$. That is, I give these expansions in powers of $n^{-1/2}q_n^3$. This large class of estimates has a huge range of potential applications, as estimates of high dimension are common in nearly all areas of applied statistics. The most important type of standard estimate is when $\hat{w}$ is a smooth function of a sample mean, of dimension p say. When either or both $q_n = q$ and $p_n = p$ increase with n, I give conditions on their growth for the Edgeworth expansions for $\hat{w}$ to remain valid: the eighth power of p times the sixth power of q cannot grow as fast as n. This holds for fixed $q = q_n$ if p_n grows less than a power of n less than $1/8$. This appears to be the first time when Edgeworth expansions have been given when not one, but two dimensions, are allowed to increase to ∞ with n. This gives two different pathways for allowing an increase in dimensionality. When $q = 1$, I give 5th order Edgeworth-Cornish-Fisher expansions for the standardized distribution and its quantiles of any smooth function of a sample mean of dimension p_n, when p_n is a power of n less than $1/2$. However for the special case when this function is linear, there is no restriction whatever on how fast p_n can increase! If also the components of the sample mean are independent, then these expansions are in powers of $(np)^{-1/2}$. I also give a method that greatly reduces the number of terms needed for the 2nd and 3rd order terms in the Edgeworth expansions, that is, for the 1st and 2nd order corrections to the CLTs. I also extend these results to the case where $\hat{w} \in R^q$ is a function of several independent sample means, each of dimension increasing with n, with total dimension p.

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1. Introduction

A basic need with any statistical model is, firstly, the estimation of its unknown parameters, say $w = (w_1, \ldots, w_q)'$, by an estimate $\hat{w}$ say, based on a sample of size n; and secondly, how to approximate the distribution of $\hat{w}$, or certain functions of $\hat{w}$. (Only for exponential families are exact distributions sometimes available.)

For a great many applications in various branches of applied statistics, the number of unknown parameters, q, can be large. So it is important to know if the standard results for fixed q remain valid if $q_n = q \to \infty$ as $n \to \infty$. A number of articles have given conditions on the growth of q_n for the Central Limit Theorem (CLT) for $\hat{w}$ to remain valid, when $\hat{w}$ is a sample mean. Kuelbs and Vidyashankar (2010) gave a highly technical CLT for the sample mean, without specifying how large q_n could be.

To specify growth rates, and to make comparisons, I use the notation

$$a_n = o(b_n) \text{ if } a_n/b_n \to 0 \text{ as } n \to \infty, \text{ and } a_n = O(b_n) \text{ if } a_n/b_n \text{ is bounded as } n \to \infty.$$

Theorem 2.1 of Koike (2026) gave conditions for a uniform CLT for the sample mean for *bounded* observations. His strange condition above his (2.4), $\log^3 q_n \le n$, holds when $q_n \le [n^{1/3}]$. Remarkably, (8) of Chernozhukov et al. (2017) gave a uniform CLT for $\hat{w}$ the sample mean of *bounded* random vectors that holds for $q_n = O(\exp(an^c))$ for $a > 0$ if $c < 1/7$. (Substitute into their condition to confirm.) For the sample mean, when $X_1, \ldots, X_n$ are log-concave, Fang and Koike (2021) gave a uniform CLT when $q_n = O(\exp(an^c))$, $a > 0$ and $c < 1/3$: substitute into RHS (1.4) of Koike (2026). Currently, this is the best rate available for q_n to grow. It will be interesting to see if this bound can be extended to a broader class of estimates than sample means, and if the log-concave condition can be removed.

Moving on to *second* order uniform Edgeworth-type expansions for a sample mean, the bound (2.4) of Koike (2026) appears to allow for $q_n = O(n^{1/3})$, if his undefined σ_* is bounded below.

Beyond sample means, Chernozhukov et al. (2013) considered the validity and accuracy of uniform Edgeworth expansions of $\max_{i=1}^{q} X_{ni}$ for large q when X_n is a standardized sample mean in R^q. Their condition on q_n holds if $\log q_n = o(n^{1/7})$: see (1.1) of Koike (2026).

Portnoy (1984, 1985) showed asymptotic normality for M-estimators with q_n regression parameters when $q_n/n^{1/2}$ is large. Donoho (2016) gave a CLT for M-estimates in a linear regression model of dimension q_n when $q_n/n \to a \in (0, 1)$.

In contrast, I give CLTs and 4th order Edgeworth expansions for *any* non-lattice standard estimate, though with stronger conditions on q_n than those noted above for a sample mean.

I now explain these terms. A *lattice* random variable is one which takes only integer values. For a CLT for the sample mean, when sampling from a lattice distribution, see (23.3) and chapter 5 of Bhattacharya and Rao (1976): extra terms are needed. See also Lazakovich (1982) and section 5 of Daniels (1987).

By *a CLT* for $\hat{w}$, I mean that for some covariance matrix V,

$$\text{as } n \to \infty, \; X_n = n^{1/2}(\hat{w} - w) \xrightarrow{\mathcal{L}} X \sim \mathcal{N}_q(0, V), \tag{1}$$

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a q-variate normal random vector, with density and distribution

$$\phi_V(x) = (2\pi)^{-q/2}(det\ V)^{-1/2}\exp(-x'V^{-1}x/2),\ \text{and}\ \Phi_V(x) = \int_{-\infty}^{x}\phi_V(x)dx. \quad (2)$$

By $\hat{w}$ *is a standard estimate*, I mean that $E[\hat{w}] \to w$ as $n \to \infty$, and for $r \geq 1$, its rth order cumulants have magnitude n^{1-r} and can be expanded in powers of n^{-1}. This condition arises naturally, as it holds if $\hat{w}$ is a function of sample means or empirical distributions, based on one or more random samples. Estimates based on a sample of independent but not identically distributed random vectors, are also generally standard estimates. For example, for a univariate sample mean $\bar{w} = n^{-1}\sum_{j=1}^{n} X_{jn}$, where X_{jn} has an rth cumulant κ_{rjn}, $\kappa_r(\bar{w}) = n^{1-r}\kappa_r$, where $\kappa_r = n^{-1}\sum_{j=1}^{n}\kappa_{rjn}$ is the average rth cumulant. For some examples, see Skovgaard (1981a, 1981b); Withers and Nadarajah (2020). The latter is for a function of a weighted mean of complex random matrices. It gives an application in electrical engineering of channel capacity for multiple arrays.

Estimates based on a stationary time series can also be standard estimates. See Withers (2025b).

So standard estimates have a huge range of potential applications. (Estimates of quantiles are not standard estimates. Nor are kernel estimates which use non-integer powers of n.)

For $\hat{w}$ non-lattice, the density and distribution of X_n of (1) can be expanded in powers of $n^{-1/2}$ about the density and distribution of X of (1). When an expansion contains s terms and a remainder, these are what I mean by *an Edgeworth expansion of order s*.

In summary:

- I extend both the CLT and Edgeworth expansions up to order 4, for estimates of large dimension $q = q_n$, from the case of a sample mean, to a broad class of estimates.
- I give for the first time these expansions for any q-valued smooth function of a p-valued sample mean, when p and q can both increase with sample size n.
- I give for the first time Edgeworth-Cornish-Fisher expansions for any univariate smooth function of a p-valued sample mean, when p increases with sample size n. For the special case when this function is linear, there is no restriction whatever on how fast p_n can increase!
- I extend the last two results to smooth functions of more than one vector mean.
- I give a way of greatly reducing the number of terms needed in 2nd and 3rd order Edgeworth expansions.
- I extend the last two results to smooth functions of more than one independent vector means.

Practical Implications

In many applications, models include a large number of parameters. Results such as this paper, extend the legitimacy of using standard inference procedures for high-dimensional parameters.

High dimensional inference now often relies on the bootstrap, Koike (2026), Gaussian approximation via concentration, and de-biased estimators. However the bootstrap at best only gives 2nd order expansions. Gaussian approximation relies on higher order cross-cumulants being zero, which is generally not the case. And bias-reduction alone is not enough to give even 2nd order expansions, as the effect of skewness is equally important. The approach here gives Edgeworth expansions of order $s \leq 4$.

I end this section with some results in the literature that I do not consider here. Kosorok and Ma (2007) considered sampling from vectors, and investigated the simultaneous estimation of the marginal distributions for large q_n. Fujikoshi (2009) considered the

asymptotic distributions of the canonical correlations between $X_1 \in R^p$ and $X_2 \in R^q$ with $p \leq q$. They derived asymptotic distributions of the canonical correlations when p is fixed, $q = q_n \to \infty$, and $q_n/n \to a \in [0,1)$ as $n \to \infty$. They assume that X_1, and X_2 have a joint *normal* distribution. Fujikoshi (2011) gave a number of results for large dimensions.

I use $E[Z]$ for the expected value of any random Z.

2. Overview

To be self-contained, Section 3 summarises the three Edgeworth expansions that I consider here for an estimate $\hat{w} \in R^q$ for the case when q does *not* depend on n.

How fast can $q = q_n$ increase with n, for these expansions to remain valid? Section 4 shows that they still hold true if $q_n = o(n^{1/6})$. This condition is much stronger than the condition $q_n = O(\exp(an^c))$ where $c < 1/3$ given for a CLT for a sample mean by Fang and Koike (2021) when $X_1, \ldots, X_n$ are log-concave. Also my condition $q_n = o(n^{1/6})$ is stronger than the condition $q_n = O(n^{1/3})$ needed by Koike (2026) for a 2nd order Edgeworth-type expansions for a sample mean. On the other hand, my result also holds for higher Edgeworth-type expansions, and for a far broader range of estimates than sample means.

Section 5 gives a theorem that reduces the number of terms needed for 2nd and 3rd order Edgeworth expansions. For example, if $q = 3$, it reduces the number of terms needed for 2nd and 3rd order Edgeworth expansions by 57% and 94%. This % reduction increases with q. If $q = 4$ or 5, it reduces the number of terms by 65% or 69% for the 2nd order Edgeworth expansions, and by 98% for the 3rd order Edgeworth expansions.

Sections 6–8 give results quite different from any in the literature. Section 6 considers the case when $\hat{w} = t(\bar{X})$ for $t : R^p \to R^q$ a smooth function of a sample mean $\bar{X}$ from a distribution on R^p with finite moments. When $q_n p_n = qp \to \infty$ as $n \to \infty$, it shows that the Edgeworth expansions for $\hat{w}$ remain valid if $p_n^8 q_n^6 = o(n)$. For example, this holds for *fixed* $q_n = q$ if $p_n = o(n^{1/8})$. These may be the 1st CLTs or Edgeworth expansions when, not one, but two parameters are allowed to increase to infinity with n.

When $q = 1$, Section 7 gives 5th order Edgeworth-Cornish-Fisher expansions for the standardized form of any univariate smooth function of a p-valued sample mean, when $p_n = p$ increases with sample size n. When $n \operatorname{var}(\hat{w})/p^2$ is bounded away from 0, these expansions still hold if $p_n = o(n^{1/2})$; but if the function is linear, there is no restriction whatever on how fast p_n can increase! When $n \operatorname{var}(\hat{w})/p$ is bounded away from 0, these expansions require the stronger condition, $p_n = o(n^{1/5})$, but this can be weakened to $p = o(n^{1/3})$ if the function is linear.

Section 8 extends Sections 6 and 7 to $\hat{w} = t(\bar{X}_1, \ldots, \bar{X}_I) : R^{p_1} \times \cdots \times R^{p_I} \to R^q$ where $\bar{X}_1, \ldots, \bar{X}_I$ are independent means of dimensions $p_1, \ldots, p_I$ from samples of sizes $n_1, \ldots, n_I$.

Section 9 shows how the pointwise Edgeworth expansions can be extended to uniform convergence, and briefly discusses confidence intervals. Section 10 gives a discussion including suggestions for future research. Section 11 gives some conclusions.

Main Results

Theorem 1 gives three 4th order pointwise Edgeworth expansions in powers of $n^{-1/2}q_n^3$, for any standard estimate of dimension $q_n = q$. So these require that $q_n = o(n^{1/6})$. As noted, this condition is stronger than the condition $q_n = O(n^{1/3})$ needed by Koike (2026) for a 2nd order Edgeworth-type expansions for a sample mean.

Theorem 2 gives a way to dramatically reduce the number of terms needed for the 2nd and 3rd order Edgeworth expansions.

Theorem 4 gives three 4th order pointwise Edgeworth expansions in powers of $n^{-1/2}p_n^4 q_n^3$, for estimates of dimension q_n that are functions of a sample mean, say $\bar{X}$, of dimension p_n, when *either or both* of p_n and q_n increase with n. So these require that

$p_n^8 q_n^6 = o(n)$. This appears to be the first time when an Edgeworth expansion has been given that allows for *two* dimensions to increase simultaneously with sample size.

If $\hat{w}$ is a univariate function of $\bar{X}$, so that $q = 1$, then 5th order Edgeworth-Cornish-Fisher expansions in powers of $n^{-1/2}p_n$, are given by Corollary 1 for the usual case when $n \text{ var}(\hat{w})/p^2$ is bounded away from 0. These need $p_n = o(n^{1/2})$. Corollary 2 gives these expansions in powers of $n^{-1/2}p_n^{5/2}$, for the case when $n \text{ var}(\hat{w})/p$ is bounded away from 0. These need $p_n = o(n^{1/5})$. However for a linear function, by Corollary 4, the last restriction can be weakened to $p_n = o(n^{1/3})$. Perhaps the most surprising results are given by Corollary 3 and Corollary 4: for $\hat{w} = c'\bar{X}$, these expansions are in powers of $n^{-1/2}$ with no restriction on how fast p_n increases, when $n \text{ var}(\hat{w})/p^2$ is bounded away from 0, or in powers of $(np^3)^{-1/2}$ when $n \text{ var}(\hat{w})/p$ is bounded away from 0. If also the components of $\bar{X}$ are independent, as is the case when its p components come from p independent random samples, then by Corollary 5, these expansions are in powers of $(np)^{-1/2}$: so remarkably, p_n now amplifies the effect of large n, whereas earlier it reduced the effect of large n.

Theorem 8 shows that Theorem 4 and Corollaries 1–5 still hold for $\hat{w}$ a function of not 1, but say I independent sample means of total dimension p. In this case, Theorems 9 and 10 give conditions for $n \text{ var}(\hat{w})/p^2$ and $n \text{ var}(\hat{w})/p$ to be bounded away from zero, as required by the I sample extensions of Corollaries 1 and 2.

3. Multivariate Edgeworth Expansions

Suppose that $\hat{w}$ is a *standard estimate* of $w \in R^q$ with respect to n. (n is typically the sample size.) That is, $E[\hat{w}] \to w$ as $n \to \infty$, where for $r \geq 1$ and $1 \leq i_1, \ldots, i_r \leq q$, the rth order cumulants of $\hat{w} = (\hat{w}_1, \ldots, \hat{w}_q)'$ can be expanded as

$$\bar{k}^{1\ldots r} = k^{i_1\ldots i_r} = \kappa(\hat{w}_{i_1}, \ldots, \hat{w}_{i_r}) \approx \sum_{d=r-1}^{\infty} n^{-d}\, \bar{k}_d^{1\ldots r}, \text{ where } \bar{k}_d^{1\ldots r} = k_d^{i_1\ldots i_r}, \tag{3}$$

and $\approx$ indicates an asymptotic expansion. That is,

$$\text{for } 1 \leq r \leq s,\ \bar{k}^{1\ldots r} = \sum_{d=r-1}^{s-1} n^{-d}\, \bar{k}_d^{1\ldots r} + O(n^{-s}).$$

For example, this condition holds if $\hat{w}$ is a smooth function of one or more sample means: see Sections 6 and 8. The *cumulant coefficients* $\bar{k}_d^{1\ldots r}$ may depend on n, but are bounded as $n \to \infty$. So *the bar replaces each* i_k *by* k. For example, $\bar{k}_0^1 = \bar{w}_1 = w_{i_1}$ and $\bar{k}_1^{12} = k_1^{i_1 i_2}$. I reserve i_k for this bar notation to avoid double subscripts and unwieldy equations So, (1) holds with $V = (k_1^{i_1 i_2}) = (\bar{k}_1^{12})$, $q \times q$. V may depend on n, but I assume that *det* V is bounded away from 0 as $n \to \infty$. We shall see that the cumulant coefficients needed for an Edgeworth expansion to s terms are as follows:

for $s = 1$, (the CLT), $\bar{k}_1^{12} = k_1^{i_1 i_2}$;
for $s = 2$, $\bar{k}_1^1 = k_1^{i_1}$, $\bar{k}_2^{123} = k_2^{i_1 i_2 i_3}$;
for $s = 3$, $\bar{k}_2^{12} = k_2^{i_1 i_2}$, $\bar{k}_3^{1234} = k_3^{i_1 i_2 i_3 i_4}$;
for $s = 4$, $\bar{k}_2^1 = k_2^{i_1}$, $\bar{k}_3^{123} = k_3^{i_1 i_2 i_3}$, $k_4^{1\ldots 5} = k_4^{i_1\ldots i_5}$.

Example 1. *If* $\hat{w} = \bar{X}$ *is the mean of a random sample with rth order cumulants* $\bar{\kappa}^{1\ldots r} = \kappa(X_{i_1}, \ldots, X_{i_r})$, *then* $\bar{k}_d^{1\ldots r} = I(d = r-1)\, \bar{\kappa}^{1\ldots r}$. *So these must be finite for an Edgeworth expansion to r terms.*

Example 2. *Given a sample of size n from a bivariate distribution with finite cumulants* κ_{rs}, *the empirical estimate of the covariance* $w = \kappa_{12}$ *is* $\hat{w} = (n-1)\tilde{w}/n$ *where* $\tilde{w}$ *is the usual*

unbiased estimate of w. So using the expressions for the cumulants of $\tilde{w}$ *in Stuart and Ord (1991),* $\kappa_r(\hat{w}) = (1-n^{-1})^r\,\kappa_r(\tilde{w})$ *can easily be expanded in powers of* n^{-1}.

$$\text{Let } \bar{P}_r^{1\ldots k} = P_r^{i_1\ldots i_k} \text{ be } \textit{the kth Edgeworth coefficient} \text{ of order } r \text{ for } \hat{w}.$$

These Edgeworth coefficients are needed for the Edgeworth expansions of order $r+1$. They can be defined by (8) below, and the requirement that they are symmetric in $i_1,\ldots,i_k$. So, conveniently, the components of the Edgeworth expansions are linear combinations of the Edgeworth coefficients. The Edgeworth coefficients are Bell polynomials in the cumulant coefficients, $\bar{k}_d^{1\ldots r}$ of (3), defined and given in Withers (2025a) for $1 \le r \le 3$. Their importance lies in their central role in the Edgeworth expansions of X_n of (1) in (8), (9) and (14) below. The Edgeworth coefficients $\bar{P}_r^{1\ldots k}$ needed for $r = 1,2,3$ are given in (19)–(21) of Withers (2025a):

$$\begin{aligned}
&\bar{P}_1^1 = \bar{k}_1^1,\ \bar{P}_1^{123} = \bar{k}_2^{123}/6,\ \bar{P}_2^{12} = \bar{k}_2^{12}/2 + \bar{k}_1^1\bar{k}_1^2/2, \qquad (4)\\
&\bar{P}_2^{1234} = \bar{k}_3^{1234}/4! + \mathcal{S}\,\bar{k}_1^4\bar{k}_2^{123}/6,\ \bar{P}_2^{1\ldots6} = \mathcal{S}\,\bar{k}_2^{123}\bar{k}_2^{456}/72,\\
&\bar{P}_3^1 = \bar{k}_2^1,\ \bar{P}_3^{123} = \bar{k}_3^{123}/6 + \mathcal{S}\,\bar{k}_2^{12}\bar{k}_1^3/2 + \bar{k}_1^1\bar{k}_1^2\bar{k}_1^3/6,\\
&\bar{P}_3^{1\ldots5} = \bar{k}_4^{1\ldots5}/5! + S_1/24 + S_2/12 + S_3/12\\
&\text{where } S_1 = \mathcal{S}\,\bar{k}_3^{1234}\bar{k}_1^5,\ S_2 = \mathcal{S}\,\bar{k}_2^{12}\bar{k}_2^{345},\ S_3 = \mathcal{S}\,\bar{k}_1^1\bar{k}_1^2\bar{k}_2^{345},\\
&\bar{P}_3^{1\ldots7} = \mathcal{S}\,\bar{k}_2^{123}\bar{k}_3^{4567}/144 + \mathcal{S}\,\bar{k}_2^{123}\bar{k}_2^{456}\bar{k}_1^7/72,\\
&\bar{P}_3^{1\ldots9} = \mathcal{S}\,\bar{k}_2^{123}\bar{k}_2^{456}\bar{k}_2^{789}/6^4, \qquad (5)
\end{aligned}$$

where $\mathcal{S}$ is the operator $\mathcal{S}$ that symmetrizes $\bar{f}^{1\ldots k}$ over $i_1,\ldots,i_k$. So these give the 4th order Edgeworth expansions, counting the CLT as the 1st order Edgeworth expansion.

Set $P(A) =$ Probability that A is true. By Withers (2025a) or Withers (1984/1986), for $\hat{w}$ non-lattice, the density and distribution of X_n can be expanded as

$$p_{X_n}(x) \approx \sum_{r=0}^{\infty} n^{-r/2} p_r(x), P(X_n \le x) \approx \sum_{r=0}^{\infty} n^{-r/2} P_r(x),\ x \in R^q, \qquad (6)$$

that is, for $s \ge 1$, $p_{X_n}(x)/\phi_V(x) = \sum_{r=0}^{s-1} n^{-r/2}\tilde{p}_r(x) + O(n^{-s/2})$,

$$\begin{aligned}
P(X_n \le x) &= \sum_{r=0}^{s-1} n^{-r/2} P_r(x) + O(n^{-s/2}), \qquad (7)\\
P(X_n \in C) &= \sum_{r=0}^{s-1} n^{-r/2} P_r(C) + O(n^{-s/2}),
\end{aligned}$$

where $p_0(x) = \phi_V(x)$, $P_0(x) = \Phi_V(x)$, of (2), and for $r \ge 1$,

$$\begin{aligned}
p_r(x)/\phi_V(x) &= \sum_{k=1}^{3r} [\tilde{p}_{rk} :\ k - r \text{ even}] = \tilde{p}_r(x) say,\\
\tilde{p}_{rk} &= \bar{P}_r^{1\ldots k}\bar{H}^{1\ldots k} = \sum_{i_1,\ldots,i_k=1}^{q} P_r^{i_1,\ldots,i_k}\, H^{i_1,\ldots,i_k}, \qquad (8)\\
P_r(x) &= \sum_{k=1}^{3r} [P_{rk}(x) :\ k - r \text{ even}],\\
P_{rk}(x) &= \bar{P}_r^{1\ldots k}\,\bar{H}_*^{1\ldots k} = \sum_{i_1,\ldots,i_k=1}^{q} P_r^{i_1,\ldots,i_k}\, H_*^{i_1,\ldots,i_k}, \qquad (9)
\end{aligned}$$

and $\bar{H}^{1\ldots k} = \bar{H}^{1\ldots k}(x,V) = H^{i_1\ldots i_k} = \phi_V(x)^{-1}\bar{O}^{1\ldots k}\,\phi_V(x)$,

$$\bar{H}^{1\ldots k}_{*} = \bar{H}^{1\ldots k}_{*}(x,V) = \bar{O}^{1\ldots k}\,\Phi_V(x) = \int_{-\infty}^{x} \bar{H}^{1\ldots k}\phi_V(x)dx, \tag{10}$$
$$\bar{O}^{1\ldots k} = (-\bar{\partial}_1)\ldots(-\bar{\partial}_k),\ \bar{\partial}_k = \partial_{i_k},\ \partial_i = \partial/\partial x_i.$$

Here, $\bar{H}^{1\ldots k}(x,V) = \bar{H}^{1\ldots k}$ and $\bar{H}^{1\ldots k}_{*}(x,V) = \bar{H}^{1\ldots k}_{*}$ are *the multivariate Hermite polynomial*, and *the integrated multivariate Hermite polynomial*. By Withers (2000),

$$\bar{H}^{1\ldots k} = E\left[(\bar{y}_1 + \sqrt{-1}\,\bar{Y}_1)\ldots(\bar{y}_k + \sqrt{-1}\,\bar{Y}_k)\right]$$

where $y = V^{-1}x$, $Y = V^{-1}X \sim \mathcal{N}_q(0, V^{-1})$, $\bar{y}_k = y_{i_k}$, and $\bar{Y}_k = Y_{i_k}$.

Note how (8) and (9) use the tensor summation convention: repetition of $i_1,\ldots,i_k$ in $\tilde{p}_{rk}$ of (8) and $P_{rk}(x)$ of (9) implies their implicit summation over their range, $1,\ldots,q$. Withers (2025a) gave $\bar{H}^{1\ldots k}$ explicitly for $k \le 6$, and for $k \le 9$ when $q = 2$.

So for $r \ge 0$, $\tilde{p}_r(x) = p_r(x)/\phi_V(x)$, $P_r(x)$ and $P_r(C)$ are the rth terms in the Edgeworth expansion for $p_{X_n}(x)/\phi_V(x)$, the density of X_n relative to its asymptotic normal density, and similarly, $P_r(x)$ and $P_r(C)$ are the rth terms in the Edgeworth expansions for $P(X_n \le x)$ and $P(X_n \in C)$. Also, for $r \ge 1$, $\tilde{p}_r(x), P_r(x)$ and $P_r(C)$ have $3r$ components, $\tilde{p}_{rk}, P_{rk}(x)$, and $P_{rk}(C)$, for $1 \le k \le 3r$, every 2nd component being zero.

$$\text{Set } \bar{\mu}^{1-2k} = E\left[\bar{Y}_1\ldots\bar{Y}_{2k}\right] = \sum^{1.3\ldots(2k-1)} \bar{V}^{12}\ldots\bar{V}^{2k-1,2k},$$

where $\sum^N \bar{f}^{1\ldots k}$ sums $\bar{f}^{1\ldots k}$ over all N permutations of $i_1,\ldots,i_k$ giving distinct values. For example, by section 2 of Withers (2025a),

$$\bar{\mu}^{1234} = \bar{V}^{12}\bar{V}^{34} + \bar{V}^{13}\bar{V}^{24} + \bar{V}^{14}\bar{V}^{23},\ \bar{H}^{1} = \bar{y}_1,\ \bar{H}^{12} = \bar{y}_1\bar{y}_2 - \bar{V}^{12},$$
$$\bar{H}^{123} = \bar{y}_1\bar{y}_2\bar{y}_3 - \sum^{3}\bar{y}_1\bar{V}^{23} = \bar{y}_1\bar{y}_2\bar{y}_3 - \bar{y}_1\bar{V}^{23} - \bar{y}_2\bar{V}^{13} - \bar{y}_3\bar{V}^{12},$$
$$\bar{H}^{1}_{*} = \bar{J}^{1},\ \bar{H}^{12}_{*} = \bar{J}^{12} - \bar{V}^{12}\Phi_V(x),\ \bar{H}^{123}_{*} = \bar{J}^{123} - \sum^{3}\bar{J}^{1}\bar{V}^{23}, \text{ where}$$
$$\bar{J}^{1-k} = \bar{J}^{1-k}(x,V) = E\left[\bar{Y}_1\ldots\bar{Y}_k\,I(X \le x)\right] = \bar{V}^{1,k+1}\ldots\bar{V}^{k,2k}\bar{M}^{k+1-2k}, \tag{11}$$

and $\bar{M}^{a-b} = \bar{M}^{a-b}(x,V) = E\left[\bar{X}_a\ldots\bar{X}_b\,I(X \le x)\right]$.

So the repeated $i_{k+1},\ldots,i_{2k}$ in (11) implies their repeated summation over $1,\ldots,q$. As x lies in R^q, $\int_{-\infty}^{x} dx$ in (10) stands for $\int_{-\infty}^{x_1} dx_1 \cdots \int_{-\infty}^{x_q} dx_q$.

Equations (4)–(7) give the Edgeworth expansions for the density and distribution of X_n of (1) to $O(n^{-2})$. $\tilde{p}_{rk}$ and P_{rk} each have q^k terms, but many are duplicates as $\bar{P}^{1\ldots k}_r$ is symmetric in $i_1,\ldots,i_k$. This is exploited in Section 5 to greatly reduce the number of terms in (9) and (14) below. So for example, by (6) and (7), the 2nd order Edgeworth expansions and the CLTs for the density and distribution of X_n are

$$p_{X_n}(x)/\phi_V(x) = 1 + n^{-1/2}\tilde{p}_1(x) + O(n^{-1}) = 1 + +O(n^{-1/2}),$$
$$P(X_n \le x) = \Phi_V(x) + n^{-1/2}P_1(x) + O(n^{-1}) = \Phi_V(x) + +O(n^{-1/2}).$$

for $x \in R^q$. For measurable $C \subset R^q$,

$$P(X_n \in C) \approx \sum_{r=0}^{\infty} n^{-r/2} P_r(C), \text{ where } P_0(C) = \Phi_V(C), \text{ and for } r \geq 1, \tag{12}$$

$$P_r(C) = E\left[p_r(X)\ I(X \in C)\right] = \sum_{k=1}^{3r} [P_{rk}(C) : k - r \text{ even}], \tag{13}$$

$$P_{rk}(C) = E\left[\tilde{p}_{rk}(X) I(X \in C)\right] = \int_C \tilde{p}_{rk}(x)\phi_V(x)dx$$

$$= \bar{P}_r^{1\ldots k} \bar{H}_*^{1\ldots k}(C) = \sum_{i_1,\ldots,i_k=1}^{q} P_r^{i_1,\ldots,i_k}\ H_*^{i_1,\ldots,i_k}(C), \tag{14}$$

$$\text{and } \bar{H}_*^{1\ldots k}(C) = E\left[\bar{H}^{1\ldots k}(X,V)\ I(X \in C)\right] = \int_C \bar{H}^{1\ldots k}\phi_V(x)dx.$$

Equation (14) is a 3rd use of the tensor sum convention.

If $-C = C$, then for r odd, $P_{rk}(C) = P_r(C) = 0$, so that

$$P(X_n \in C) \approx \sum_{r=0}^{\infty} n^{-r} P_{2r}(C) = \Phi_V(C) + n^{-1}P_2(C) + O(n^{-2}).$$

Examples 3 and 4 of Withers (2025a) gave $P_2(C)$ for

$$C = \{x : x'V^{-1}x \leq u\}, and\ C = \{x : |(V^{-1/2}x)_j| \leq u_j,\ j = 1,\ldots,q\}.$$

The main take-away here is that for $x \in R^q$, $C \subset R^q$, $\tilde{p}_r(x)$, $P_r(x)$, $P_r(C)$ of (8), (9) and (12), and $s \geq 1$,

$$p_{X_n}(x)/\phi_V(x) = \sum_{r=0}^{s-1} n^{-r/2}\tilde{p}_r(x) + O(n^{-s/2}), \tag{15}$$

$$P(X_n \leq x) = \sum_{r=0}^{s-1} n^{-r/2}P_r(x) + O(n^{-s/2}), \tag{16}$$

$$P(X_n \in C) = \sum_{r=0}^{s-1} n^{-r/2}P_r(C) + O(n^{-s/2}), \tag{17}$$

where, for example, by (4), using the tensor sum convention,

$$\tilde{p}_1(x) = p_1(x)/\phi_V(x) = \sum_{k=1,3} \tilde{p}_{1k},\ \tilde{p}_{11} = \bar{k}_1^1\ \bar{H}^1,\ \tilde{p}_{13} = \bar{k}_2^{123}\ \bar{H}^{123}/6,$$

$$P_1(x) = \sum_{k=1,3} P_{1k}(x),\ P_{11}(x) = \bar{k}_1^1\ \bar{H}_*^1,\ P_{13}(x) = \bar{k}_2^{123}\ \bar{H}_*^{123}/6,$$

$$P_1(C) = \sum_{k=1,3} P_{1k}(C),\ P_{11}(C) = \bar{k}_1^1\ \bar{H}_*^1(C),\ P_{13}(C) = \bar{k}_2^{123}\ \bar{H}_*^{123}(C)/6.$$

These asymptotic expansions generally diverge, as normal moments and Hermite polynomials increase very rapidly with their degree. Equations (15) and (16) are pointwise results. See Section 8 for uniform convergence.

We shall see that when $q_n = q \to \infty$, (15)–(17) remain true with $O(n^{-s/2})$ replaced by $O(\nu_n^{s/2})$ where ν_n depends on q_n, and also on p_n in the case when $\hat{w}$ is a smooth function of a sample mean of dimension p_n.

Under weaker conditions, $O(n^{-s/2})$ can be replaced by $o(n^{-(s-1)/2})$ for fixed q. So when $q_n = q \to \infty$, under weaker conditions, $O(\nu_n^{s/2})$ can be replaced by $o(\nu_n^{(s-1)/2})$. See for example Bhattacharya and Rao (1976).

4. The Case $q = q_n \to \infty$ as $n \to \infty$

Theorem 1. *Let $\hat{w}$ be a non-lattice estimate of $w \in R^q$, satisfying $E[\hat{w}] \to w$, and (3). Set $X_n = n^{1/2}(\hat{w} - w)$. Take $1 \le s \le 4$. Suppose that as $n \to \infty$,*

$$q = q_n \to \infty, \text{ and } \nu_n = n^{-1/2} q_n^3 \to 0, \text{ that is, } q_n = o(n^{1/6}).$$

Then, for $x \in R^q$, measurable $C \subset R^q$, $\tilde{p}_r(x)$ of (8), $P_r(x)$ of (9), and $P_r(C)$ of (13),

$$p_{X_n}(x)/\phi_V(x) = \sum_{r=0}^{s-1} n^{-r/2} \tilde{p}_r(x) + O(\nu_n^s), \tag{18}$$

$$P(X_n \le x) = \sum_{r=0}^{s-1} n^{-r/2} P_r(x) + O(\nu_n^s), \tag{19}$$

$$P(X_n \in C) = \sum_{r=0}^{s-1} n^{-r/2} P_r(C) + O(\nu_n^s). \tag{20}$$

Proof. Set $Q = q^2$. $\tilde{p}_{rk}(x)$ of (8), $P_{rk}(x)$ of (9), and $P_{rk}(C)$ of (13) and (14), each have q^k summations for $q = q_n$. So each has magnitude $O(q_n^k)$ for a given k. (As the values of k are determined by $r < s$, there is no need to bound $\bar{H}^{1\ldots k}$, $\bar{H}_*^{1\ldots k}$, $\bar{H}_*^{1\ldots k}(C)$, even though these increase rapidly with k for any given x or C.) So $p_r(x)$ of (8), $P_r(x)$ of (9), and $P_r(C)$ of (12), each have N_r terms, where

$$\begin{aligned} &\text{for } r \text{ odd, } N_r = q + q^3 + \cdots + q^{3r} = q(1 + Q + \cdots + Q^s) \\ &= q(Q^{s+1} - 1)/(Q - 1) \text{ for } s = (3r - 1)/2, \end{aligned} \tag{21}$$

$$\begin{aligned} &\text{and for } r \text{ even, } N_r = q^2 + q^4 + \cdots + q^{3r} = Q(1 + Q + \cdots + Q^s) \\ &= Q(Q^{s+1} - 1)/(Q - 1) \text{ for } s = 3r/2 - 1. \end{aligned} \tag{22}$$

So, $N_r = O(q^{3r})$ as $n \to \infty$.
So $p_r(x)$, $P_r(x)$ and $P_r(C)$ each have magnitude q_n^{3r} as $n \to \infty$.
So $n^{-s/2}$ $(p_s(x), P_s(x), P_s(C))$ has magnitude $n^{-s/2} q_n^{3s} = \nu_n^s$. The theorem follows. □

Note that the $r+1$th terms in (18)–(20), that is, $n^{-r/2}\tilde{p}_r(x)$, $n^{-r/2}P_r(x)$, $n^{-r/2}P_r(C)$, each have magnitude ν_n^r. This (19) is a pointwise result, but it implies that the remainder is uniform over $x \in C$ a bounded set, since the Hermite polynomials are continuous functions.

For applications of (20) to confidence regions and simultaneous inference, see Examples 3–5 of Withers (2025a).

Example 3. *Let $\hat{w}$ be the sample mean from a distribution on R^q with finite cross cumulants $\bar{\kappa}^{1\ldots r}, r \ge 1$. Then $E[\hat{w}] = w$, and only the leading coefficient in (3) is non-zero. As in Example 2 of Withers (2025a), by (4) and (5), the non-zero Edgeworth coefficients $P_r^{1\ldots k}$ needed for the three 4th order Edgeworth expansions of $\hat{w}$ with $V = (\bar{\kappa}^{12}) = (\kappa^{i_1 i_2})$, are as follows.*

$$\begin{aligned} &\bar{P}_1^{123} = \bar{\kappa}^{123}/3!,\ \bar{P}_2^{1234} = \bar{\kappa}^{1234}/4!,\ \bar{P}_2^{1\ldots 6} = \mathcal{S}\, \bar{\kappa}^{123}\bar{\kappa}^{456}/72,, \\ &\bar{P}_3^{1\ldots 5} = \bar{\kappa}^{1\ldots 5}/5!,\ \bar{P}_3^{1\ldots 7} = \mathcal{S}\, \bar{\kappa}^{123}\bar{\kappa}^{4\ldots 7}/144,\ \bar{P}_3^{1\ldots 9} = \mathcal{S}\, \bar{\kappa}^{123}\bar{\kappa}^{456}\bar{\kappa}^{789}/6^4. \end{aligned}$$

Substitution gives $(p_s(x), P_s(x), P_s(C))$ for $s = 1, 2, 3$. For example, as $q_n = q \to \infty$, the coefficients of $n^{-1/2}$ in the 2nd terms in the Edgeworth expansions for $p_{X_n}(x)/\phi_V(x)$, $P(X_n \le x)$, and $P(X_n \in C)$, are

$$\tilde{p}_1(x) = \tilde{p}_{13} = \bar{\kappa}^{123}\bar{H}^{123}/6 = O(q_n^3),$$
$$P_1(x) = P_{13}(x) = \bar{\kappa}^{123}\bar{H}_*^{123}/6 = O(q_n^3),$$
$$\text{and } P_1(C) = P_{13}(C) = \bar{\kappa}^{123}\bar{H}_*^{123}(C)/6 = O(q_n^3).$$

Example 4. *Suppose that $X_1, X_2, \ldots, X_n$ are independent random vectors in R^q, with mean $\hat{w} = \bar{X}$. Suppose that X_j has finite cross-cumulants*

$$\bar{\kappa}_{(j)}^{1\ldots r} = \kappa(X_{ji_1}, \ldots, X_{ji_r}) \text{ for } r \geq 1 \text{ and } i_1, \ldots, i_r \text{ in } 1, \ldots, q.$$

Then for $r \geq 1$,

$$\kappa(\hat{w}_{i_1}, \ldots, \hat{w}_{i_r}) = n^{1-r}\bar{\kappa}^{1\ldots r} \text{ where } \bar{\kappa}^{1\ldots r} = n^{-1}\sum_{j=1}^{n}\bar{\kappa}_{(j)}^{1\ldots r}.$$

Suppose also, that $\{\bar{\kappa}^{1\ldots r}\}$ are bounded in n, and that for $V = (\bar{\kappa}^{12})$, $\det V$ is bounded away from 0, as n increases. Then the Edgeworth coefficients needed for the three 4th order Edgeworth expansions for $\hat{w}$, are given by Example 1.

5. Reduction of Terms

My next theorem gives a way to reduce the number of terms in $\tilde{p}_r(x)$ of (8), $P_r(x)$ of (9), and $P_r(C)$ of (14), from q^k, to $L_k = q_k\,[1 + O(q^{-1})]$ as $q \to \infty$, where

$$q_k = \binom{q}{k} = q(q-1)\ldots(q-k+1)/k!.$$

(As I do not use q_n for q in this section, there is no ambiguity with this different use of q_k.)

$\tilde{p}_{rk}(x)$, $P_{rk}(x)$ and $P_{rk}(C)$ each have k summations over $1, \ldots, q$. So each have q^k terms. But many are duplicates as $\bar{P}_r^{1\ldots k}$ is symmetric in $i_1, \ldots, i_k$. Set $\binom{k}{a\ldots b} = k!/a!\ldots b!$, the multinomial coefficient. For example $\binom{3}{111} = 6$.

Theorem 2. *Let $T^{i_1\ldots i_k} \in R$ be symmetric in $i_1, \ldots, i_k$.*

Set $T^{i_1^2 i_2} = T^{i_1 i_1 i_2}$ and so on, and $u_k = \sum_{i_1,\ldots,i_k=1}^{q} T^{i_1\ldots i_k}$.

$$\text{Then, } u_1 = \sum_{i_1=1}^{q} T^{i_1},\ u_2 = \sum_{i_1=1}^{q} T^{i_1^2} + 2!\sum_{i_1>i_2}^{q_2} T^{i_1 i_2},$$
$$u_3 = \sum_{i_1=1}^{q} T^{i_1^3} + \binom{3}{1}\sum^{q(q-1)} T^{i_1^2 i_2} + 3!\sum_{i_1>i_2>i_3}^{q_3} T^{i_1 i_2 i_3},$$
$$u_4 = \sum_{i_1=1}^{q} T^{i_1^4} + \binom{4}{1}\sum^{q(q-1)} T^{i_1^3 i_2} + \binom{4}{2}\sum_{i_1>i_2}^{q_2} T^{i_1^2 i_2^2} + \binom{4}{211}\sum_{i_1>i_2,\, i_1>i_3}^{3q_3} T^{i_1^2 i_2 i_3}$$
$$+ 4!\sum_{i_1>i_2>i_3>i_4}^{q_4} T^{i_1 i_2 i_3 i_4},$$

$$u_5 = \sum_{i_1=1}^{q} T^{i_1^5} + \binom{5}{1} \sum^{q(q-1)} T^{i_1^4 i_2} + \binom{5}{2} \sum^{q(q-1)} T^{i_1^3 i_2^2} + \binom{5}{221} \sum_{i_1>i_2}^{5q_3} T^{i_1^2 i_2^2 i_3}$$
$$+ \binom{5}{2111} \sum_{i_2>i_3>i_4}^{4q_4} T^{i_1^2 i_2 i_3 i_4} + 5! \sum_{i_1>i_2>i_3>i_4>i_5}^{q_5} T^{i_1 i_2 i_3 i_4 i_5},$$
$$u_6 = \sum_{i_1=1}^{q} T^{i_1^6} + \binom{6}{1} \sum^{q(q-1)} T^{i_1^5 i_2} + \binom{6}{2} \sum^{q(q-1)} T^{i_1^4 i_2^2} + \binom{6}{3} \sum_{i_1>i_2}^{q_2} T^{i_1^3 i_2^3}$$
$$+ \binom{6}{411} \sum_{i_2>i_3}^{3q_3} T^{i_1^4 i_2 i_3} + \binom{6}{321} \sum^{6q_3} T^{i_1^3 i_2^2 i_3} + \binom{6}{222} \sum_{i_1>i_2>i_3}^{q_3} T^{i_1^2 i_2^2 i_3^2}$$
$$+ \binom{6}{3111} \sum_{i_2>i_3>i_4}^{4q_4} T^{i_1^3 i_2 i_3 i_4} + \binom{6}{2211} \sum_{i_1>i_2,i_3>i_4}^{6q_4} T^{i_1^2 i_2^2 i_3 i_4}$$
$$+ \binom{6}{21111} \sum_{i_2>i_3>i_4>i_5}^{5q_5} T^{i_1^2 i_2 i_3 i_4 i_5} + 6! \sum_{i_1>\cdots>i_6}^{q_6} T^{i_1 \ldots i_6},$$

where the sums are for distinct i_j. This reduces the number of terms in u_k from q^k to L_{qk} where

$$L_{q1} = q,\ L_{q2} = q + q_2,\ L_{q3} = q + 2q_2 + q_3,$$
$$L_{q4} = q + 3q_2 + 3q_3 + q_4,\ L_{q5} = q + 4q_2 + 5q_3 + 4q_4 + q_5,$$
$$L_{q6} = q + 5q_2 + 10q_3 + 14q_4 + 5q_5 + q_6.$$

For example, $(q,k) = (5,3) \Rightarrow L_{qk}/q^k = 0.28$; that is, Theorem 2 reduces the number of terms to 28%.

As a check, $q^k = \sum_{j=1}^{k} S(k,j)\ (q)_j$ where $(q)_j = j! q_j = q(q-1)\ldots(q-j+1)$, and $S(k,j)$ is the Stirling number of the 2nd kind. These are tabled on p. 310 of Comtet (1974). For example, the number of terms to calculate in u_4 is $q^4 = q + (4+3)(q)_2 + 6(q)_3 + (q)_4$.

To apply Theorem 2 to $u_k = \tilde{p}_{rk}$ of (8), or $u_k = P_{rk}(x)$ of (9), or $u_k = P_{rk}(C)$ of (14), take

$$T^{i_1 \ldots i_k} = \bar{P}_r^{1\ldots k}\ \bar{f}^{1\ldots k}, \text{ where } \bar{f}^{1\ldots k} = \bar{H}^{1\ldots k}, \text{ or } \bar{H}_*^{1\ldots k}, \text{ or } \bar{H}_*^{1\ldots k}(C), \tag{23}$$

and tensor summation is *not* used in (23). Then $u_k = \tilde{p}_{rk}$ of (8), or $P_{rk}(x)$ of (9), or $P_{rk}(C)$ of (14).

$$\tilde{p}_{r1} = \sum_{i_1=1}^{q} P_r^{i_1} H^{i_1},\ P_{r1}(x) = \sum_{i_1=1}^{q} P_r^{i_1} H_*^{i_1},\ P_{r1}(C) = \sum_{i_1=1}^{q} P_r^{i_1} H_*^{i_1}(C),$$
$$\tilde{p}_{r2} = \sum_{i_1=1}^{q} P_r^{i_1 i_1} H^{i_1 i_1} + 2 \sum_{i_1>i_2}^{q_2} P_r^{i_1 i_2} H^{i_1 i_2},\ P_{r2}(x) = \sum_{i_1=1}^{q} P_r^{i_1 i_1} H_*^{i_1 i_1} + 2 \sum_{i_1>i_2}^{q_2} P_r^{i_1 i_2} H_*^{i_1 i_2},$$
$$P_{r2}(C) = \sum_{i_1=1}^{q} P_r^{i_1 i_1} H_*^{i_1 i_1}(C) + 2 \sum_{i_1>i_2}^{q_2} P_r^{i_1 i_2} H_*^{i_1 i_2}(C),$$
$$\tilde{p}_{r3} = \sum_{i_1=1}^{q} P_r^{i_1 i_1 i_1} H^{i_1 i_1 i_1} + 3 \sum^{q(q-1)}_{i_1 \neq i_2} P_r^{i_1 i_1 i_2} H^{i_1 i_1 i_2} + 6 \sum_{i_1>i_2>i_3}^{q_3} P_r^{i_1 i_2 i_3} H^{i_1 i_2 i_3},$$

$$P_{r3}(x) = \sum_{i_1=1}^{q} P_r^{i_1 i_1 i_1} H_*^{i_1 i_1 i_1} + 3 \sum_{i_1 \neq i_2}^{q(q-1)} P_r^{i_1 i_1 i_2} H_*^{i_1 i_1 i_2} + 6 \sum_{i_1 > i_2 > i_3}^{q_3} P_r^{i_1 i_2 i_3} H_*^{i_1 i_2 i_3},$$

$$P_{r3}(C) = \sum_{i_1=1}^{q} P_r^{i_1 i_1 i_1} H_*^{i_1 i_1 i_1}(C) + 3 \sum_{i_1 \neq i_2}^{q(q-1)} P_r^{i_1 i_1 i_2} H_*^{i_1 i_1 i_2}(C) + 6 \sum_{i_1 > i_2 > i_3}^{q_3} P_r^{i_1 i_2 i_3} H_*^{i_1 i_2 i_3}(C).$$

This shows that we can replace q^k in the derivation of Theorem 1 by L_{qk} of Theorem 2. So for $r = 1, 2$, the number of terms in $p_r(x)$ of (8), $P_r(x)$ of (9), and $P_r(C)$ of (12), that is, N_r of (21) and (22), can be reduced to N'_r where

$$N'_1 = L_{q1} + L_{q3} = 2q + 2q_2 + q_3, \tag{24}$$

$$N'_2 = L_{q2} + L_{q4} + L_{q6} = 3q + 10q_2 + 10q_3 + 15q_4 + 5q_5 + q_6. \tag{25}$$

So,

$$q = 1 \Rightarrow N_1 = 2,\ N'_1 = 2,\ N_2 = 3,\ N'_2 = 3,$$

$$q = 2 \Rightarrow N_1 = 10,\ N'_1 = 6 = 0.6\ N_1,\ N_2 = 84,\ N'_2 = 16 = 0.19\ N_2,$$

$$q = 3 \Rightarrow N_1 = 30,\ N'_1 = 13 = 0.43\ N_1,\ N_2 = 819,\ N'_2 = 49 = 0.06\ N_2,$$

$$q = 4 \Rightarrow N_1 = 68,\ N'_1 = 24 = 0.35\ N_1,\ N_2 = 4368,\ N'_2 = 127 = 0.02\ N_2,$$

$$q = 5 \Rightarrow N_1 = 130,\ N'_1 = 40 = 0.31\ N_1,\ N_2 = 16,275,\ N'_2 = 295 = 0.02\ N_2,$$

where the values of N'_1/N_1 and N'_2/N_2 are approximate.

For example, if $q = 3$, Theorem 2 reduces the number of terms needed for 2nd and 3rd order Edgeworth expansions by 57% and 94%. If $q = 4$ or 5, Theorem 2 reduces the number of terms to calculate by 65% or 69% for the 2nd order Edgeworth expansions, and by 98% for the 3rd order Edgeworth expansions.

When $q = 2$, this reduction of terms was used in section 4 of Withers (2025a).

Example 5. *In Example 3, using the expression for u_3 in Theorem 2,*

$$6\tilde{p}_1(x) = \sum_{i_1=1}^{q} \kappa^{i_1^3} H^{i_1^3} + 3 \sum^{q(q-1)} \kappa^{i_1^2 i_2} H^{i_1^2 i_2} + 6 \sum_{i_1 > i_2 > i_3}^{q_3} \kappa^{i_1 i_2 i_3} H^{i_1 i_2 i_3},$$

$$6P_1(x) = \sum_{i_1=1}^{q} \kappa^{i_1^3} H_*^{i_1^3} + 3 \sum^{q(q-1)} \kappa^{i_1^2 i_2} H_*^{i_1^2 i_2} + 6 \sum_{i_1 > i_2 > i_3}^{q_3} \kappa^{i_1 i_2 i_3} H_*^{i_1 i_2 i_3},$$

$$6P_1(C) = \sum_{i_1=1}^{q} \kappa^{i_1^3} H_*^{i_1^3}(C) + 3 \sum^{q(q-1)} \kappa^{i_1^2 i_2} H_*^{i_1^2 i_2}(C) + 6 \sum_{i_1 > i_2 > i_3}^{q_3} \kappa^{i_1 i_2 i_3} H_*^{i_1 i_2 i_3}(C).$$

As $\bar{P}_1^1 = \bar{P}_2^{12} = 0$, L_{q1} in (24), and L_{q2} in (25) need to be deleted.

6. Functions of a Vector Sample Mean

Let $\bar{X}$ be the sample mean from a non-lattice distribution on R^p with mean μ, and finite cross cumulants $\bar{\kappa}^{1\ldots r} = \kappa^{j_1 \ldots j_r}, r \geq 1$. This expands the bar convention used earlier, as it now includes $j_1, j_2, \ldots$ in $1, \ldots, p$ as well as $i_1, i_2, \ldots$ in $1, \ldots, q$ as before. Let $t(.) : R^p \to R^q$ be a function with i_kth component $\bar{t}^k(.) = t^{i_k}(.)$ having finite derivatives at μ,

$$\bar{t}^k_{r-s} = \partial_{j_r} \ldots \partial_{j_s} t^{i_k}(\mu), \text{ where } \partial_j = \partial/\partial_{\mu_j},\ s \leq r,\ 1 \leq j \leq p. \tag{26}$$

Given k and $f^{i_1 \ldots i_k}$, I now write

$$\sum f^{i_1 \ldots i_k} \sim p^k, \text{ for } \sum f^{i_1 \ldots i_k} = O(p^k).$$

This follows by summing $i_1, \ldots, i_k$ over $1, \ldots, p$.

$$\text{Set } \sum^{2} \bar{t}^{1}_{a-b} \bar{t}^{2}_{c-d} = \bar{t}^{1}_{a-b} \bar{t}^{2}_{c-d} + \bar{t}^{2}_{a-b} \bar{t}^{1}_{c-d}.$$

More generally, for $\pi_1, \ldots, \pi_r$ any partition of $1, \ldots, k$, let $\sum^{N} \bar{t}^{1}_{\pi_1} \ldots \bar{t}^{r}_{\pi_r}$ denote the sum of $t^{b_1}_{\pi_1} \ldots t^{b_r}_{\pi_r}$ over all N permutations $b_1, \ldots, b_r$ of $j_1, \ldots, j_r$, giving distinct values. For example,

$$\sum^{3} \bar{t}^{1}_{13} \bar{t}^{2}_{2} \bar{t}^{3}_{4} = \bar{t}^{1}_{13} \bar{t}^{2}_{2} \bar{t}^{3}_{4} + \bar{t}^{2}_{13} \bar{t}^{1}_{2} \bar{t}^{3}_{4} + \bar{t}^{3}_{13} \bar{t}^{2}_{2} \bar{t}^{1}_{4}.$$

The next theorem gives the cumulant coefficients, $\bar{k}^{1\ldots r}_{d} = k^{i_1 \ldots i_r}_{d}$ of (3), for $\hat{w} = t(\bar{X})$, and track their magnitude in p from summing $j_1, \ldots, j_k$ over $1, \ldots, p$.

I now use the tensor sum convention for summing $j_1, \ldots, j_k$ for given $i_1, \ldots, i_k$. For example, given i_1 and i_2,

$$\bar{t}^{1}_{1} \bar{t}^{2}_{2} \, \bar{\kappa}^{12} = \sum_{j_1, j_2 = 1}^{p} t^{i_1}_{j_1} t^{i_2}_{j_2} \, \kappa^{j_1 j_2}, \quad \bar{t}^{1}_{12} \bar{\kappa}^{12} = \sum_{j_1, j_2 = 1}^{p} t^{i_1}_{j_1 j_2} \, \kappa^{j_1 j_2},$$

as the subscripts of t and the superscripts of κ are both js.

Theorem 3. *For $\hat{w} = t(\bar{X}) : R^p \to R^q$, (3) holds, where the cumulant coefficients, the $\bar{k}^{1\ldots r}_{d} = k^{i_1 \ldots i_r}_{d}$, needed for the 3rd order Edgeworth expansions of order $s = 1, 2, 3, 4$, are given as follows.*

$$\textit{For the CLT}, s = 1 : \; \bar{k}^{12}_{1} = k^{i_1 i_2}_{1} = V_{i_1 i_2} = \bar{t}^{1}_{1} \bar{t}^{2}_{2} \, \bar{\kappa}^{12} = \sum_{j_1, j_2 = 1}^{p} t^{i_1}_{j_1} t^{i_2}_{j_2} \, \kappa^{j_1 j_2} \sim p^2.$$

$$\textit{For } s = 2 : \; \bar{k}^{1}_{1} = k^{i_1}_{1} = \bar{t}^{1}_{12} \bar{\kappa}^{12} \; /2 = \sum_{j_1, j_2 = 1}^{p} t^{i_1}_{j_1 j_2} \, \kappa^{j_1 j_2} /2, \sim p^2,$$

$$\bar{k}^{123}_{2} = \bar{t}^{1}_{1} \bar{t}^{2}_{2} \bar{t}^{3}_{3} \, \bar{\kappa}^{123} + T^{123}_{1234} \, \bar{\kappa}^{12} \bar{\kappa}^{34} \sim p^3 + p^4 \sim p^4, \textit{ where } T^{123}_{1234} = \sum^{3} \bar{t}^{1}_{13} \bar{t}^{2}_{2} \bar{t}^{3}_{4}.$$

$$\textit{For } s = 3 : \; \bar{k}^{12}_{2} = T^{12}_{123} \, \bar{\kappa}^{123} /2 + T^{12}_{1234} \, \bar{\kappa}^{12} \bar{\kappa}^{34} /2 \sim p^3 + p^4 \sim p^4, \textit{ where}$$

$$T^{12}_{123} = \sum^{2} \bar{t}^{1}_{12} \bar{t}^{2}_{3}, \; T^{12}_{1234} = \sum^{2} \bar{t}^{1}_{123} \bar{t}^{2}_{4} + \bar{t}^{1}_{13} \bar{t}^{2}_{24},$$

$$\bar{k}^{1234}_{3} = \bar{t}^{1}_{1} \bar{t}^{2}_{2} \bar{t}^{3}_{3} \bar{t}^{4}_{4} \, \bar{\kappa}^{1234} + T^{1234}_{1\ldots5} \, \bar{\kappa}^{123} \bar{\kappa}^{45} + T^{1234}_{1\ldots6} \, \bar{\kappa}^{12} \bar{\kappa}^{34} \bar{\kappa}^{56} \sim p^4 + p^5 + p^6 \sim p^6,$$

$$\textit{where } T^{1234}_{1\ldots5} = \sum^{12} \bar{t}^{1}_{14} \bar{t}^{2}_{2} \bar{t}^{3}_{3} \bar{t}^{4}_{5}, \; T^{1234}_{1\ldots6} = \sum^{4} \bar{t}^{1}_{135} \bar{t}^{2}_{2} \bar{t}^{3}_{4} \bar{t}^{4}_{6} + \sum^{12} \bar{t}^{1}_{13} \bar{t}^{2}_{25} \bar{t}^{3}_{4} \bar{t}^{4}_{6}.$$

$$\textit{For } s = 4 : \; \bar{k}^{1}_{2} = \bar{t}^{1}_{123} \bar{\kappa}^{123} /6 + \bar{t}^{1}_{1234} \, \bar{\kappa}^{12} \bar{\kappa}^{34} /8 \sim p^3 + p^4 \sim p^4,$$

$$\bar{k}^{123}_{3} = T^{123}_{1234} \, \bar{\kappa}^{1234} /2 + T^{123}_{1\ldots5} \, \bar{\kappa}^{123} \bar{\kappa}^{45} + T^{123}_{1\ldots6} \, \bar{\kappa}^{12} \bar{\kappa}^{34} \bar{\kappa}^{56} \sim p^4 + p^5 + p^6 \sim p^6,$$

$$\textit{where } T^{123}_{1234} = \sum^{3} \bar{t}^{1}_{13} \bar{t}^{2}_{2} \bar{t}^{3}_{4},$$

$$T^{123}_{1\ldots5} = \sum^{6} \bar{t}^{1}_{124} \bar{t}^{2}_{3} \bar{t}^{3}_{5} /2 + \sum^{3} \bar{t}^{1}_{145} \bar{t}^{2}_{2} \bar{t}^{3}_{3} /2 + \sum^{6} \bar{t}^{1}_{12} \bar{t}^{2}_{34} \bar{t}^{3}_{5} /2 + \sum^{3} \bar{t}^{1}_{14} \bar{t}^{2}_{25} \bar{t}^{3}_{3},$$

$$T^{123}_{1\ldots6} = \sum^{3} \bar{t}^{1}_{1235} \bar{t}^{2}_{4} \bar{t}^{3}_{6} /2 + \sum^{6} \bar{t}^{1}_{123} \bar{t}^{2}_{45} \bar{t}^{3}_{6} + \sum^{6} \bar{t}^{1}_{135} \bar{t}^{2}_{24} \bar{t}^{3}_{6} /2 + \bar{t}^{1}_{13} \bar{t}^{2}_{25} \bar{t}^{3}_{46}$$

$$\bar{k}^{1\ldots5}_{4} = \bar{t}^{1}_{1} \cdots \bar{t}^{5}_{5} \, \bar{\kappa}^{1\ldots5} + T^{1\ldots5}_{1\ldots6} \, \bar{\kappa}^{1234} \bar{\kappa}^{56} + U^{1\ldots5}_{1\ldots6} \, \bar{\kappa}^{123} \bar{\kappa}^{456}$$

$$+ T^{1\ldots5}_{1\ldots7} \, \bar{\kappa}^{123} \bar{\kappa}^{45} \bar{\kappa}^{67} + T^{1\ldots5}_{1\ldots8} \, \bar{\kappa}^{12} \bar{\kappa}^{34} \bar{\kappa}^{56} \bar{\kappa}^{78} \sim p^5 + \cdots + p^8 \sim p^8,$$

$$\textit{where } T^{1\ldots5}_{1\ldots6} = \sum^{20} \bar{t}^{1}_{15} \bar{t}^{2}_{2} \bar{t}^{3}_{3} \bar{t}^{4}_{4} \bar{t}^{5}_{6}, \; U^{1\ldots5}_{1\ldots6} = \sum^{15} \bar{t}^{1}_{14} \bar{t}^{2}_{2} \bar{t}^{3}_{3} \bar{t}^{4}_{5} \bar{t}^{5}_{6},$$

$$T^{1\ldots5}_{1\ldots7} = \sum^{30} \bar{t}^{1}_{146} \bar{t}^{2}_{2} \bar{t}^{3}_{3} \bar{t}^{4}_{5} \bar{t}^{5}_{7} + \sum^{60} \bar{t}^{1}_{14} \bar{t}^{2}_{26} \bar{t}^{3}_{3} \bar{t}^{4}_{5} \bar{t}^{5}_{7} + \sum^{60} \bar{t}^{1}_{14} \bar{t}^{2}_{56} \bar{t}^{3}_{2} \bar{t}^{4}_{3} \bar{t}^{5}_{7},$$

$$T^{1\ldots5}_{1\ldots8} = \sum^{5} \bar{t}^{1}_{1357} \bar{t}^{2}_{2} \bar{t}^{3}_{4} \bar{t}^{4}_{6} \bar{t}^{5}_{8} /5 + \sum^{60} \bar{t}^{1}_{135} \bar{t}^{2}_{27} \bar{t}^{3}_{4} \bar{t}^{4}_{6} \bar{t}^{5}_{8} + \sum^{60} \bar{t}^{1}_{13} \bar{t}^{2}_{25} \bar{t}^{3}_{47} \bar{t}^{4}_{6} \bar{t}^{5}_{8}.$$

Proof. This is a special case of Theorem 2 of Withers (2024) with $\bar{k}_d^{1\dots r}$ replaced by $I(d = r-1)\,\bar{\kappa}^{1\dots r}$. □

The heavy algebra here need not concern us, unless we are making an application. We need only note that different terms have different magnitudes as powers of p. What is important here is the dominant magnitude in these sums.

Example 6. *Suppose that we have a random sample $Y_1, \dots, Y_n$ in R. Let $\bar{X}_i$ be the mean of the ith powers of the sample. Then $t(\hat{w}) = \bar{X}_2 - \bar{X}_1^2$ is the sample variance where $t(\mu) = \mu_2 - \mu_1^2$. In this case $p = 2$, $\hat{w}_i = \bar{X}_i$, and $q = 1$.*

Lemma 1. *For $k \geq 1$, and $r = 1, 2, 3$, $\bar{P}_r^{1\dots k}$ of (4)–(5) satisfies $\bar{P}_r^{1\dots k} \sim p^{r+k}$.*
So for $\bar{f}^{1\dots k} = f^{i_1\dots i_k} \in R$, $\bar{P}_r^{1\dots k}\,\bar{f}^{1\dots k} = \sum_{i_1,\dots,i_k=1}^{q} \bar{P}_r^{1\dots k}\,\bar{f}^{1\dots k} \sim p^{r+k}q^k$.

$$\text{Set } P_r = \sum_{k=1}^{3r} [\bar{P}_r^{1\dots k}\,\bar{f}^{1\dots k} :\ k - r \text{ even}]. \text{ Then } P_r \sim p^{4r}q^{3r}. \tag{27}$$

Proof. Use Theorem 3 to check that $\bar{P}_r^{1\dots k} \sim p^{r+k}$. The dominant term in P_r is for $k = 3r$. So, (27) holds. □

Theorem 4. *Set $\hat{w} = t(\bar{X}) : R^p \to R^q$. Suppose that $\hat{w}$ is non-lattice, that $E[\hat{w}] \to w$, and that (3) holds. Suppose that as $n \to \infty$, $p_n q_n = pq \to \infty$, and that*

$$\nu_n = n^{-1/2}\,p_n^4\,q_n^3 \to 0. \text{ That is, } p_n^8 q_n^6 = o(n). \tag{28}$$

Then (18)–(20) hold for $s = 1, 2, 3, 4$.

Proof. For P_r of (27), $n^{-r/2}P_r \sim \nu_n^r$. Now take $\bar{f}^{1\dots k}$ of (23). □

For example, (28) holds for fixed $q = q_n$ if $p_n = o(n^{1/8})$, and for fixed $p = p_n$ if $q_n = o(n^{1/6})$. I now apply Theorem 2 to the components of Theorem 3.

Theorem 5. *Take L_{pr} of Theorem 2.*

$$\begin{aligned}
&\text{For } s = 1 :\ \bar{k}_1^{12} = V_{i_1 i_2} \sim L_{p2}.\\
&\text{For } s = 2 :\ \bar{k}_1^{1} \sim L_{p2}, \text{ reduced from } p^2,\\
&\bar{k}_2^{123} \sim L_{p3} + 3L_{p4}, \text{ reduced from } p^3 + 3p^4.\\
&\text{For } s = 3 :\ \bar{k}_2^{12} \sim 2L_{p3} + 3L_{p4}, \text{ reduced from } 2p^3 + 3p^4,\\
&\bar{k}_3^{1234} \sim L_{p4} + 12L_{p5} + 16L_{p6}, \text{ reduced from } p^4 + 12p^5 + 16p^6.\\
&\text{For } s = 4 :\ \bar{k}_2^{1} \sim L_{p3} + L_{p4}, \text{ reduced from } p^3 + p^4,\\
&\bar{k}_3^{123} \sim 3L_{p4} + 18L_{p5} + 16L_{p6}, \text{ reduced from } 3p^4 + 18p^5 + 16p^6,\\
&\bar{k}_4^{1\dots5} \sim L_{p5} + (20+15)L_{p6} + 150L_{p7} + 125L_{p8},\\
&\text{reduced from } p^5 + (20+15)p^6 + 150p^7 + 125p^8.
\end{aligned}$$

Let us write these as

$$\bar{k}_d^{1\dots r} \sim N_{rd}.$$

For example, $L_{p2}/p^2 = (p+1)/2p \to 1/2$ as $p \to \infty$. For $p = 4$, the number of calculations needed for $\bar{k}_2^{123}$ is reduced by the factor

$$N_{32}/(p^3 + 3p^4) = (L_{p3} + 3L_{p4})/(p^3 + 3p^4) = 3/26.$$

One can now work out similar results for $T^{i_1 \dots i_k}$ of (23), and so for the terms of the Edgeworth expansions, $\tilde{p}_r(x)$, $P_r(x)$ and $P_r(C)$. For example,

$$\begin{aligned}
&\tilde{p}_{11} = \bar{P}_1^1 \bar{H}^1 = \bar{k}_1^1 \bar{H}^1 \sim qN_{11} = qL_{p2}, \\
&\tilde{p}_{13} = \bar{P}_1^{123} \bar{H}^{123} \sim k_2^{123} \bar{H}^{123} \sim L_{q3} N_{32}, \\
&\Rightarrow \tilde{p}_1(x) \sim qN_{11} + L_{q3} N_{32} = qL_{p2} + L_{q3}(L_{p3} + 3L_{p4}).
\end{aligned}$$

Similarly , taking $T^{i_1 i_2} = P_2^{i_1 i_2} H^{i_1 i_2}$ without implicit summation,

$$\tilde{p}_{22} = \bar{P}_2^{12} \bar{H}^{12} = \sum_{i_1=1}^{q} T^{i_1^2} + 2 \sum_{i_1 > i_2}^{q_2} T^{i_1 i_2} \sim qL_{p2} + q_2 L_{p0}.$$

7. The Case $q = 1$ and $\bar{X}$ of Large Dimension p_n

This appears to be the first time that Edgeworth expansions for this important case have been considered. I give 5th order Edgeworth-Cornish-Fisher expansions.

As in Section 5, let $\bar{X}$ be the sample mean from a non-lattice distribution on R^p with mean μ, and finite cross cumulants $\bar{\kappa}^{1 \dots r} = \kappa^{j_1 \dots j_r}, r \geq 1$. Let $t(.) : R^p \to R$ be a function with finite derivatives at μ,

$$\bar{t}_{r-s} = \partial_{j_r} \dots \partial_{j_s} t(\mu), \text{ where } \partial_j = \partial / \partial_{\mu_j}, \; s \leq r, \; 1 \leq j \leq p. \tag{29}$$

Theorem 6. *Take $\hat{w} = t(\bar{X}) : R^p \to R$. Its cumulants can be expanded as*

$$\kappa_r(\hat{w}) \approx \sum_{j=r-1}^{\infty} n^{-j} \, a_{rj} \text{ for } r \geq 1, \tag{30}$$

where the cumulant coefficients, a_{rj}, needed for the 3rd order Edgeworth expansions of order $s = 1, 2, 3, 4$, are given as follows.

$$\begin{aligned}
&\text{For the CLT}, s = 1 : \; a_{21} = \bar{t}_1 \bar{t}_2 \, \bar{\kappa}^{12} = \sum_{j_1, j_2 = 1}^{p} t_{j_1} t_{j_2} \, \kappa^{j_1 j_2} \sim p^2. \\
&\text{For } s = 2 : \; a_{11} = \bar{t}_{12} \bar{\kappa}^{12} \, /2 = \sum_{j_1, j_2 = 1}^{p} t_{j_1 j_2} \, \kappa^{j_1 j_2} \, /2 \sim p^2, \\
&a_{32} = \bar{t}_1 \bar{t}_2 \bar{t}_3 \, \bar{\kappa}^{123} + T_{1234}^{123} \, \bar{\kappa}^{12} \bar{\kappa}^{34} \sim p^3 + p^4 \sim p^4, \text{ where } T_{1234}^{123} = 3 \bar{t}_{13} \bar{t}_2 \bar{t}_4. \\
&\text{For } s = 3 : \; a_{22} = T_{123} \, \bar{\kappa}^{123} / 2 + T_{1234} \, \bar{\kappa}^{12} \bar{\kappa}^{34} / 2 \sim p^3 + p^4 \sim p^4, \text{ where} \\
&T_{123} = 2 \bar{t}_{12} \bar{t}_3, \; T_{1234} = 2 \bar{t}_{123} \bar{t}_4 + \bar{t}_{13} \bar{t}_{24}, \\
&a_{43} = \bar{t}_1 \bar{t}_2 \bar{t}_3 \bar{t}_4 \, \bar{\kappa}^{1234} + T_{1 \dots 5} \, \bar{\kappa}^{123} \bar{\kappa}^{45} + T_{1 \dots 6} \, \bar{\kappa}^{12} \bar{\kappa}^{34} \bar{\kappa}^{56} \sim p^4 + p^5 + p^6 \sim p^6, \\
&\text{where } T_{1 \dots 5} = 12 \bar{t}_{14} \bar{t}_2 \bar{t}_3 \bar{t}_5, \; T_{1 \dots 6} = 4 \bar{t}_{135} \bar{t}_2 \bar{t}_4 \bar{t}_6 + 12 \bar{t}_{13} \bar{t}_{25} \bar{t}_4 \bar{t}_6. \\
&\text{For } s = 4 : \; a_{12} = \bar{t}_{123} \bar{\kappa}^{123} / 6 + \bar{t}_{1234} \, \bar{\kappa}^{12} \bar{\kappa}^{34} / 8 \sim p^3 + p^4 \sim p^4, \\
&a_{33} = T_{1234} \, \bar{\kappa}^{1234} / 2 + T_{1 \dots 5} \, \bar{\kappa}^{123} \bar{\kappa}^{45} + T_{1 \dots 6} \, \bar{\kappa}^{12} \bar{\kappa}^{34} \bar{\kappa}^{56} \sim p^4 + p^5 + p^6 \sim p^6, \\
&\text{where } T_{1234} = 3 \bar{t}_{13} \bar{t}_2 \bar{t}_4, \\
&T_{1 \dots 5} = 6 \bar{t}_{124} \bar{t}_3 \bar{t}_5 / 2 + 3 \bar{t}_{145} \bar{t}_2 \bar{t}_3 / 2 + 6 \bar{t}_{12} \bar{t}_{34} \bar{t}_5 / 2 + 3 \bar{t}_{14} \bar{t}_{25} \bar{t}_3, \\
&T_{1 \dots 6} = 3 \bar{t}_{1235} \bar{t}_4 \bar{t}_6 / 2 + 6 \bar{t}_{123} \bar{t}_{45} \bar{t}_6 + 6 \bar{t}_{135} \bar{t}_{24} \bar{t}_6 / 2 + \bar{t}_{13} \bar{t}_{25} \bar{t}_{46}, \\
&a_{54} = \bar{t}_1 \cdots \bar{t}_5 \, \bar{\kappa}^{1 \dots 5} + T_{1 \dots 6} \, \bar{\kappa}^{1234} \bar{\kappa}^{56} + U_{1 \dots 6} \, \bar{\kappa}^{123} \bar{\kappa}^{456} \\
&+ T_{1 \dots 7} \, \bar{\kappa}^{123} \bar{\kappa}^{45} \bar{\kappa}^{67} + T_{1 \dots 8} \, \bar{\kappa}^{12} \bar{\kappa}^{34} \bar{\kappa}^{56} \bar{\kappa}^{78} \sim p^5 + \cdots + p^8 \sim p^8, \\
&\text{where } T_{1 \dots 6} = 20 \bar{t}_{15} \bar{t}_2 \bar{t}_3 \bar{t}_4 \bar{t}_6, \; U_{1 \dots 6} = 15 \bar{t}_{14} \bar{t}_2 \bar{t}_3 \bar{t}_5 \bar{t}_6,
\end{aligned}$$

$$T_{1...7} = 30\bar{t}_{146}\bar{t}_2\bar{t}_3\bar{t}_5\bar{t}_7 + 60\bar{t}_{14}\bar{t}_{26}\bar{t}_3\bar{t}_5\bar{t}_7 + 60\bar{t}_{14}\bar{t}_{56}\bar{t}_2\bar{t}_3\bar{t}_7,$$
$$T_{1...8} = 5\bar{t}_{1357}\bar{t}_2\bar{t}_4\bar{t}_6\bar{t}_8/5 + 60\bar{t}_{135}\bar{t}_{27}\bar{t}_4\bar{t}_6\bar{t}_8 + 60\bar{t}_{13}\bar{t}_{25}\bar{t}_{47}\bar{t}_6\bar{t}_8.$$

Proof. Putting $i_1 = \cdots = i_k = 1$ in (3) gives (30). The rest follows from Theorem 3. □

Example 7. *Suppose that* $\hat{w} = \bar{X}'B\bar{X}/2$ *where* $B' = B \in R^{p\times p}$. *So* $w = t(\mu) = \mu'B\mu/2$ *has 1st and 2nd derivatives*

$$\bar{t}_r = (B\mu)_{j_r},\ \bar{t}_{rs} = B_{j_r j_s}.$$

The higher derivatives of $t(\mu)$ *are zero. So* $\{a_{rj}\}$ *of (30) simpliify as follows.*

$a_{12} = 0$, *and the following are zero: the 1st term of* T_{1234} *needed for* a_{22}, *the 1st term of* $T_{1...6}$ *needed for* a_{43}, *the 1st 2 terms of* $T_{1...5}$ *needed for* a_{33}, *the 1st 3 terms of* $T_{1...6}$ *needed for* a_{33}, *the 1st 2 terms of* $T_{1...5}$ *needed for* a_{33}, *the 1st term of* $T_{1...7}$ *needed for* a_{54}, *and the 1st 2 terms of* $T_{1...8}$ *needed for* a_{54}. *For example, if* $B = I_p$, *then*

$$\hat{w} = |\bar{X}|^2/2,\ t(\mu) = |\mu|^2/2,\ \bar{t}_r = \mu_{j_r},\ \bar{t}_{rs} = I(j_r = j_s).$$

By Theorem 4, the Edgeworth expansions (18)–(20) hold for $s = 1,2,3,4$ if $\nu_n = n^{-1/2}\ p_n^4 \to 0$, that is, $p_n = o(n^{1/8})$. However the usual Edgeworth expansion when $q = 1$ is for its *standardized* estimate,

$$Y_n = n^{1/2}(\hat{w} - w)/a_{21}^{1/2} \text{ and } A_{ri} = a_{ri}/a_{21}^{r/2}, \tag{31}$$

the standardized cumulant coefficients. Suppose that $N \sim \mathcal{N}(0,1)$ is a unit normal random variable with density and distribution

$$\phi(y) = (2\pi)^{-1/2}e^{-y^2/2},\ \Phi(y) = Prob.(N \le y) = \int_{-\infty}^{y}\phi(y)dy.$$

Consider first *the case of fixed* p. Then Y_n of (31) has the Edgeworth-Cornish-Fisher expansions in powers of $n^{-1/2}$,

$$P_n(y) = Prob.(Y_n \le y) \approx \Phi(y) - \phi(y)\sum_{r=1}^{\infty} n^{-r/2}\ h_r(y), \tag{32}$$

$$p_n(y) = dP_n(y)/dy \approx \phi(y)\ [1 + \sum_{r=1}^{\infty} n^{-r/2}\ \bar{h}_r(y)], \tag{33}$$

$$\Phi^{-1}(P_n(y)) \approx y - \sum_{r=1}^{\infty} n^{-r/2}\ f_r(y),\ P_n^{-1}(\Phi(y)) \approx y + \sum_{r=1}^{\infty} n^{-r/2}\ g_r(y), \tag{34}$$

where $Prob.(A)$ is the probability that A is true. $h_r(y), \bar{h}_r(y), f_r(y)$ and $g_r(y)$ are polynomials in y. They are also polynomials in *the standardized cumulant coefficients* A_{ri} of (31). The leading polynomials are

$$h_1(y) = f_1(y) = g_1(y) = A_{11} + A_{32}H_2/6,\ \bar{h}_1(y) = A_{11}H_1 + A_{32}H_3/6, \tag{35}$$
$$h_2(y) = (A_{11}^2 + A_{22})H_1/2 + (A_{11}A_{32} + A_{43}/4)H_3/6 + A_{32}^2H_5/72,$$
$$f_2(y) = (A_{22}/2 - A_{11}A_{32}/3)H_1 + A_{43}H_3/24 - A_{32}^2(4y^3 - 7y)/36,$$
$$g_2(y) = A_{22}H_1/2 + A_{43}H_3/24 - A_{32}^2(2y^3 - 5y)/36,$$
$$\bar{h}_2(y) = (A_{11}^2 + A_{22})H_2/2 + (A_{11}A_{32} + A_{43}/4)H_4/6 + A_{32}^2H_6/72,$$

where H_k is the kth Hermite polynomial. By Withers (2000),

$$H_k = H_k(y) = \phi(y)^{-1}(-d/dy)^k\phi(y) = E\,(y+\sqrt{-1}N)^k \text{ for } k \geq 0: \tag{36}$$
$$H_0 = 1,\ H_1 = y,\ H_2 = y^2-1,\ H_3 = y^3-3y,\ H_4 = y^4-6y^2+3.$$
$$H_5 = y^5-10y^3+15y,\ H_6 = y^6-15y^4+45y^2-15.$$

See for example, section 1 of Withers (2025a). (A8) there gives H_k for $k \leq 9$. For $h_k(y), f_k(y), g_k(y)$ when $k = 3, 4$, see Withers (1984/1986).

Since $h_r(y)$ is even/odd for r odd/even,

$$Prob.(|Y_n| \leq y) \approx \Phi(y) - 2\phi(y)\sum_{r=1}^{\infty} n^{-r}\, h_{2r}(y) \text{ for } y > 0. \tag{37}$$

Now consider *the case when $p_n = p$ increases with n.*

Corollary 1. *Suppose that*

$$a_{21}/p^2 \text{ is bounded away from 0 as } p \to \infty \text{ so that } a_{21}^{-r/2} \sim p^{-r}, \tag{38}$$

for $r \geq 1$. Then

$$A_{11} \sim p,\ A_{32} \sim p,\ A_{22} \sim p^2,\ A_{43} \sim p^2,\ A_{12} \sim p^3,\ A_{33} \sim p^3,\ A_{54} \sim p^3.$$

So for $1 \leq r \leq 4$, $h_r(y)$, $f_r(y)$, $g_r(y)$, $\overline{h}_r(y)$ are all $O(p^r)$. Also,

$$\text{if } p = o(n^{1/2}), \text{ then } \nu_n = n^{-1/2}p \to 0,$$

so that for $1 \leq s \leq 5$,

$$P_n(y) = Prob.(Y_n \leq y) = \Phi(y) - \phi(y)\sum_{r=1}^{s-1} n^{-r/2}\, h_r(y) + O(\nu_n^r),$$
$$p_n(y) = dP_n(y)/dy = \phi(y)\,[1 + \sum_{r=1}^{s-1} n^{-r/2}\,\overline{h}_r(y)] + O(\nu_n^r),$$
$$\Phi^{-1}(P_n(y)) = y - \sum_{r=1}^{s-1} n^{-r/2}\, f_r(y) + O(\nu_n^r),$$
$$P_n^{-1}(\Phi(y)) = y + \sum_{r=1}^{s-1} n^{-r/2}\, g_r(y) + O(\nu_n^r),$$

where $\sum_{r=1}^{s-1} = 0$ if $s = 1$.

Proof. A_{ri} is a_{ri} multiplied by ($a_{21}^{-r/2} \sim p^{-r}$). The magnitude of a_{ri} is given by Theorem 6. So for example,

$$A_{11} \sim p^2 a_{21}^{-1/2} \sim p,\ A_{32} \sim p^4 a_{21}^{-3/2} \sim p,\ A_{22} \sim p^4 a_{21}^{-2} \sim p^2.$$

Now check the magnitudes of h_r and the others. For example by (35), $h_1(y) \sim A_{11} + A_{32} \sim p$. □

While the condition (38) generally holds, it fails if the components of $\bar{X}$ are independent. In this case, $a_{21} = \sum_{j=1}^{p} t_j^2\,\kappa^{jj} \sim p$, not p^2.

Corollary 2. *Suppose that*

$$a_{21}/p \text{ is bounded away from } 0 \text{ as } p \to \infty. \tag{39}$$

So, $a_{21}^{-r/2} \sim p^{-r/2}$ *for* $r \geq 1$. *Then*

$$A_{11} \sim p^{3/2}, A_{32} \sim p^{5/2}, A_{22} \sim p^3, A_{43} \sim p^4, A_{12} \sim p^{7/2}, A_{33} \sim p^{9/2}, A_{54} \sim p^{13/2}.$$

So for $1 \leq r \leq 4$, $h_r(y)$, $f_r(y)$, $g_r(y)$, $\overline{h}_r(y)$ *are* $O(p^{5r/2})$. *So for* $1 \leq s \leq 5$, $\hat{w}$ *satisfies the Edgeworth-Cornish-Fisher expansions of Corollary 1 if*

$$v_n = n^{-1/2} p^{5/2} \to 0 \text{ as } n \to \infty, \text{ that is, if } p = o(n^{1/5}).$$

Proof. $A_{ri} \sim p^{r/2} A'_{ri}$ where A'_{ri} is A_{ri} of Corollary 1. □

There is one very important case that is an exception. I now suppose that

$$\hat{w} = c' \bar{X}, \text{ so that } w = c' \mu, \text{ where } c \in R^p \text{ is a given constant.} \tag{40}$$

Then for $r \geq 1$, $a_{ri} = 0$ for $i \geq r$ and

$$a_{r,r-1} = \bar{c}_1 \cdots \bar{c}_r \, \bar{\kappa}^{1 \ldots r} \sim p^r. \tag{41}$$

Corollary 3. *Suppose that (40) holds. Then for* $r \geq 3$, $A_{r,r-1} \sim 1$. *So, for* $1 \leq r \leq 4$, $h_r(y)$, $f_r(y)$, $g_r(y)$, $\overline{h}_r(y)$ *are all bounded in* p. *So for* $1 \leq s \leq 5$, $\hat{w}$ *satisfies the Edgeworth-Cornish-Fisher expansions of Corollary 1 with* $v_n = n^{-1/2}$, no matter how p_n increases with n!

Proof. By (38) and (41), $A_{r,r-1} = a_{r,r-1} \, a_{21}^{-r/2} \sim p^r \, p^{-r} = 1$. □

Corollary 4. *Suppose that (39) holds. Then for* $r \geq 3$, $A_{r,r-1} \sim p^{r/2}$, *and for* $1 \leq r \leq 4$, $h_r(y)$, $f_r(y)$, $g_r(y)$, $\overline{h}_r(y)$ *are* $O(p^{3r/2})$. *So for* $1 \leq s \leq 5$, $\hat{w}$ *satisfies the Edgeworth-Cornish-Fisher expansions of Corollary 1 if*

$$v_n = n^{-1/2} p^{3/2} \to 0 \text{ as } n \to \infty, \text{ that is, if } p = o(n^{1/3}).$$

Proof. By (41), $a_{r,r-1} \sim p^r$. So $A_{r,r-1} \sim p^r \, p^{-r} = p^{r/2}$. Now check the magnitude of h_r and the others. For example, $h_1(y) \sim A_{32} \sim p^{3/2}$. □

Corollary 5. *Suppose that the components of* $\bar{X}$ *are independent and that (39) holds. Then* $a_{21} = \sum_{j=1}^{]} c_j \kappa^{jj}$. *Also* $A_{ri} = 0$ *for* $i \neq r-1$, *and for* $r \geq 3$, $A_{r,r-1} \sim p^{1-r/2}$. *Also for* $1 \leq r \leq 4$, $h_r(y)$, $f_r(y)$, $g_r(y)$, $\overline{h}_r(y)$ *are* $O(p^{-r/2})$. *So for* $1 \leq s \leq 5$, $\hat{w}$ *satisfies the Edgeworth-Cornish-Fisher expansions of Corollary 1 with*

$$v_n = N^{-1/2} \text{ where } N = np.$$

Proof. Check that $a_{r,r-1} \sim p$. Then check the magnitudes of $h_r(y)$, $f_r(y)$, $g_r(y)$, $\overline{h}_r(y)$. □

So remarkably, in this case p_n amplifies the effect of large n, whereas earlier it had reduced the effect of large n. However in retrospect, Corollary 5 is not surprising, since one can write $\hat{w}/p$ as the mean of N independent (but not identically distributed) random variables, and then apply Example 4.

8. Functions of I Vector Sample Means

Given $I \geq 1$, for $1 \leq i \leq I$, let $\bar{X}_i$ be the mean of a sample of size n_i from a non-lattice distribution on R^{p_i} with mean μ_i, and finite cross cumulants $\bar{\kappa}_i^{1\ldots r} = \kappa_i^{j_1\ldots j_r}, r \geq 1$. Set

$$p = p_1 + \cdots + p_I,\ \mu' = (\mu_1', \ldots, \mu_I') \in R^p,\ n = \min_{i=1}^{I} n_i,\ \nu_i = n/n_i \leq 1, \tag{42}$$

$$p_k' = \sum_{I_1=1}^{I} p_{I_1}^k \nu_{I_1}^{k-1} \leq p_k'' = \sum_{I_1=1}^{I} p_{I_1}^k \leq p^k. \tag{43}$$

So p is the total dimension of μ' and $(\bar{X}_1', \ldots, \bar{X}_I')$. For $1 \leq j \leq p_i$, let μ_i^j be the jth component of μ_i. Set $\partial_i^j = \partial/\partial\mu_i^j$.

Let $t(.) : R^{p_1} \times \cdots \times R^{p_I} \to R^q$ be a function with i_kth component $\bar{t}^k(.) = t^{i_k}(.)$ having finite derivatives at μ,

$$\bar{t}_{r-s}^k = \partial_{j_r} \ldots \partial_{j_s} t^{i_k}(\mu), \text{ where } \partial_j = \partial/\partial_{\mu_j},\ s \leq r,\ 1 \leq j \leq p.$$

The next theorem uses the T and U notation of Theorem 3.

Theorem 7.

$$\textit{Suppose that } \bar{X}_1, \ldots, \bar{X}_I \textit{ above are independent.} \tag{44}$$

$$\textit{Set } \hat{w} = t(\bar{X}_1, \ldots, \bar{X}_I)) : R^{p_1} \times \cdots \times R^{p_I} \to R^q. \tag{45}$$

Then (3) holds, where the cumulant coefficients, the $\bar{k}_d^{1\ldots r} = k_d^{i_1\ldots i_r}$, *needed for the 3rd order Edgeworth expansions of order* $s = 1, 2, 3, 4$, *are given as follows.*

$$\textit{For the CLT, } s = 1 :\ \bar{k}_1^{12} = k_1^{i_1 i_2} = V_{i_1 i_2} = \bar{t}_1^1 \bar{t}_2^2 \sum_{I_1=1}^{I} \bar{\kappa}_{I_1}^{12} \nu_{I_1} = \sum_{j_1, j_2=1}^{p} t_{j_1}^{i_1} t_{j_2}^{i_2} \sum_{I_1=1}^{I} \kappa_{I_1}^{j_1 j_2} \nu_{I_1} \sim p_2'.$$

$$\textit{For } s = 2 :\ \bar{k}_1^1 = k_1^{i_1} = \bar{t}_{12}^1 \sum_{I_1=1}^{I} \bar{\kappa}_{I_1}^{12} \nu_{I_1}/2 = \sum_{j_1, j_2=1}^{p} t_{j_1 j_2}^{i_1} \sum_{I_1=1}^{I} \kappa_{I_1}^{j_1 j_2} \nu_{I_1}/2 \sim p_2',$$

$$\bar{k}_2^{123} = \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \sum_{I_1=1}^{I} \bar{\kappa}_{I_1}^{123} \nu_{I_1}^2 + T_{1234}^{123} \sum_{I_1, I_2=1}^{I} \bar{\kappa}_{I_1}^{12} \nu_{I_1} \bar{\kappa}_{I_2}^{34} \nu_{I_2} \sim p_3' + (p_2')^2.$$

$$\textit{For } s = 3 :\ \bar{k}_2^{12} = T_{123}^{12} \sum_{I_1=1}^{I} \bar{\kappa}_{I_1}^{123} \nu_{I_1}^2/2 + T_{1234}^{12} \sum_{I_1, I_2=1}^{I} \bar{\kappa}_{I_1}^{12} \nu_{I_1} \bar{\kappa}_{I_2}^{34} \nu_{I_2}/2 \sim p_3' + (p_2')^2,$$

$$\bar{k}_3^{1234} = \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \sum_{I_1=1}^{I} \bar{\kappa}_{I_1}^{1234} \nu_{I_1}^3 + T_{1\ldots5}^{1234} \sum_{I_1, I_2=1}^{I} \bar{\kappa}_{I_1}^{123} \nu_{I_1}^2 \bar{\kappa}_{I_2}^{45} \nu_{I_2} + T_{1\ldots6}^{1234} \sum_{I_1, I_2, I_3=1}^{I} \bar{\kappa}_{I_1}^{12} \nu_{I_1} \bar{\kappa}_{I_2}^{34} \nu_{I_2} \bar{\kappa}_{I_3}^{56} \nu_{I_3}$$
$$\sim p_4' + p_2' p_3' + (p_2')^3,$$

$$\textit{For } s = 4 :\ \bar{k}_2^1 = \bar{t}_{123}^1 \sum_{I_1=1}^{I} \bar{\kappa}_{I_1}^{123} \nu_{I_1}^2/6 + \bar{t}_{1234}^1 \sum_{I_1, I_2=1}^{I} \bar{\kappa}_{I_1}^{12} \nu_{I_1} \bar{\kappa}_{I_2}^{34} \nu_{I_2}/8 \sim p_3' + (p_2')^2,$$

$$\bar{k}_3^{123} = T_{1234}^{123} \sum_{I_1=1}^{I} \bar{\kappa}_{I_1}^{1234} \nu_{I_1}^3/2 + T_{1\ldots5}^{123} \sum_{I_1, I_2=1}^{I} \bar{\kappa}_{I_1}^{123} \nu_{I_1}^2 \bar{\kappa}_{I_2}^{45} \nu_{I_2} + T_{1\ldots6}^{123} \sum_{I_1, I_2, I_3=1}^{I} \bar{\kappa}_{I_1}^{12} \nu_{I_1} \bar{\kappa}_{I_2}^{34} \nu_{I_2} \bar{\kappa}_{I_3}^{56} \nu_{I_3}$$
$$\sim p_4' + p_2' p_3' + (p_2')^3,$$

$$\bar{k}_4^{1\dots5} = \bar{t}_1^1 \cdots \bar{t}_5^5 \sum_{I_1=1}^{I} \bar{\kappa}_{I_1}^{1\dots5} \nu_{I_1}^4 + T_{1\dots6}^{1\dots5} \sum_{I_1,I_2=1}^{I} \bar{\kappa}_{I_1}^{1234} \nu_{I_1}^3 \bar{\kappa}_{I_2}^{56} \nu_{I_2} + U_{1\dots6}^{1\dots5} \sum_{I_1,I_2=1}^{I} \bar{\kappa}_{I_1}^{123} \nu_{I_1}^2 \bar{\kappa}^{456} \nu_{I_2}^2$$
$$+ T_{1\dots7}^{1\dots5} \sum_{I_1,I_2,I_3=1}^{I} \bar{\kappa}_{I_1}^{123} \nu_{I_1}^2 \bar{\kappa}_{I_2}^{45} \nu_{I_2} \bar{\kappa}_{I_3}^{67} \nu_{I_3} + T_{1\dots8}^{1\dots5} \sum_{I_1,I_2,I_3=1}^{I} \bar{\kappa}_{I_1}^{12} \nu_{I_1} \bar{\kappa}_{I_2}^{34} \nu_{I_2} \bar{\kappa}_{I_3}^{56} \nu_{I_3} \bar{\kappa}_{I_4}^{78} \nu_{I_4}$$
$$\sim p_5' + p_2' p_4' + (p_3')^2 + (p_2')^2 p_3' + (p_2')^4.$$

Proof. In Theorem 3, replace $\bar{\kappa}^{1\dots k}/n^{k-1}$ by $\sum_{I_1=1}^{I} \bar{\kappa}_{I_1}^{1\dots k}/n_{I_1}^{k-1}$. That is, replace $\bar{\kappa}^{1\dots k}$ by $\sum_{I_1=1}^{I} \bar{\kappa}_{I_1}^{1\dots k} \nu_{I_1}^{k-1}$. This form holds by (44). □

Theorem 8. *Lemma 1, Theorems 3–6, and Corollaries 1–5 hold with p the total dimension of (42) for $\hat{w}$ of (45).*

Proof. By (43), we can replace the bounds $\sim p_1', p_2', p_3' + (p_2')^2, \dots$ of Theorem 7 by the bounds $\sim p^2, p^2, p^4, \dots, p^8$ of Theorem 3. The rest follows. □

I now give conditions for $n \operatorname{var}(\hat{w})/p^2$ and $n \operatorname{var}(\hat{w})/p$ to be bounded away from zero, as required by the I sample extensions of Corollaries 1 and 2. Note that

$$a_{21} = \sum_{j=1}^{I} \nu_j a_{21}^j \text{ where } a_{21}^j = \sum_{j_1,j_2=1}^{p_j} t_{j_1}^j t_{j_2}^j \kappa_j^{j_1 j_2} \tag{46}$$

is the asymptotic variance of $n^{1/2}[t(\hat{\mu}_{(j)}) - t(\mu)]$ and $\hat{\mu}_{(j)}$ is μ with μ_j replaced by $\bar{X}_j$.

Theorem 9. *The condition in Corollary 1 that $n\ var(\hat{w})/p^2$ is bounded away from zero, holds for $\hat{w}$ of (45), if for $1 \le j \le I$, a_{21}^j/p_j^2 is bounded away from zero.*

Proof. Suppose that for $1 \le j \le I$, $a_{21}^j/p_j^2 > c_j > 0$ where c_j does not depend on p. Set $c_0 = \min_{j=1}^{I} \nu_j c_j$. By (46),

$$a_{21} > \sum_{j=1}^{I} \nu_j c_j p_j^2 \ge c_0 \sum_{j=1}^{I} p_j^2 \ge I^{-1} c_0 (\sum_{j=1}^{I} p_j)^2.$$

For, if $Z = p_j$ with probability I^{-1}, $j = 1, \dots, I$, then

$$I^{-1}(\sum_{j=1}^{I} p_j)^2 = E\ Z^2 \ge (E\ Z)^2 = (I^{-1} \sum_{j=1}^{I} p_j)^2.$$

□

Theorem 10. *The condition in Corollary 2 that $n\ var(\hat{w})/p$ is bounded away from zero, holds for $\hat{w}$ of (45), if for $1 \le j \le I$, a_{21}^j/p_j is bounded away from zero.*

Proof. Suppose that for $1 \le j \le I$, $a_{21}^j/p_j > c_j > 0$ where c_j does not depend on p. Set $c_0 = \min_{j=1}^{I} \nu_j c_j$. By (46), $a_{21} > \sum_{j=1}^{I} \nu_j c_j p_j \ge c_0 \sum_{j=1}^{I} p_j = c_0 p$. □

9. Uniform Convergence

Koike (2026), Chernozhukov et al. (2017), and Fang and Koike (2021) gave uniform convergence for the CLT and the 2nd order Edgeworth expansion for the sample mean with increasing q_n.

But for the general standard estimate, I have so far only given pointwise convergence results. By Corollary A3 of Withers (2025a) with s replaced by $s+1$, we have

Theorem 11. *For $r \geq 1$ and $1 \leq i_1, \ldots, i_r \leq q$, suppose that the rth order cumulants of $\hat{w} = (\hat{w}_1, \ldots, \hat{w}_q)'$ can be expanded as*

$$\kappa(\hat{w}_{i_1}, \ldots, \hat{w}_{i_r}) = \sum_{d=r-1}^{R_r} n^{-d}\, k_d^{i_1 \ldots i_r} + o(n^{-R_r}), \tag{47}$$

where each $k_d^{i_1 \ldots i_r}$ is bounded in n. Then under the Cramer-type condition (A9) of Corollary A3 of Withers (2025a), X_n of (1) satisfies

$$\sup_{x \in R^q} |P(X_n \leq x) - \sum_{r=0}^{s-1} n^{-r/2} P_r(x)| = o(n^{-(s-1)/2}), \tag{48}$$

for $s = 2$, (47) holds with $R_1 = R_2 = 1$, $R_3 = 2$;

for $s = 3$, (47) holds with $R_1 = 1$, $R_2 = R_3 = 2$, $R_4 = 3$;

and for $s = 4$, (47) holds with $R_1 = R_2 = 2$, $R_3 = R_4 = 3$.

So (47) is a weak form of (3), and (48) is a uniform form of (7). So for example, if these conditions hold for $s = 4$, then the remainder in (48) is $o(n^{-3/2})$.

For the sample mean with $q = 1$ and $s = 2$, this result and an analogous result for the density $p_{X_n}(x)$, and the uniform Berry-Esseen bound can be found in Theorems 2.1, 4.1 and 5.1 of Feller (1966). It would be interesting to know whether Berry-Esseen type bounds can be derived.

Using this theorem, one can now obtain uniform versions of the Edgeworth expansions for $P(X_n \leq x$ in (7), (19) of Theorem 4, and in Corollaries 1–5. Presumably similar uniform results are obtainable for the density of X_n and for $P(X_n \in C)$ for a suitable class of sets C in R^q.

However I do not know of any uniform convergence results for Cornish-Fisher expansions for quantiles, even for fixed q.

10. Conclusions

Let $\hat{w}$ be a standard estimate of an unknown $w \in R^q$. That is, $\hat{w}$ is a consistent estimate, and for $r \geq 1$, its rth order cumulants have magnitude n^{1-r}, and can be expanded in powers of n^{-1}. This is a very large class of estimates. It includes functions of sample moments and empirical distributions, samples of independent but not identically distributed random vectors, and samples from stationary series.

Then by Section 2, for fixed q, and non-lattice estimates, Edgeworth-type expansions hold for the density and distribution of $X_n = n^{1/2}(\hat{w} - w)$, in terms of the Edgeworth coefficients given in Withers (2025a) for $s \leq 4$. For $s \geq 1$, the first s terms of the three Edgeworth expansions, give the density and distribution of X_n, and also $P(X_n \in C)$, to $O(n^{-s/2})$ as $n \to \infty$. Theorem 1 shows that this remains true when $q_n = q \to \infty$, if the remainder, $O(n^{-s/2})$, is replaced by $O(\nu_n^s)$, where $\nu_n = n^{-1/2}\, q_n^3$. So the three Edgeworth expansions hold when $\nu_n \to \infty$, that is, when $q_n = o(n^{1/6})$.

Theorem 2 gives formulas that dramatically reduce the number of terms needed by 2nd and 3rd order Edgeworth-type expansions, that is, for 1st and 2nd order corrections to the CLT.

When $\hat{w} = t(\bar{X}) : R^p \to R^q$, and $p_n q_n = pq \to \infty$, Theorem 4 shows that the three 4th order Edgeworth expansions considered here hold if $p_n^8 q_n^6 = o(n)$. For example, this holds for fixed $q = q_n$ if $p_n = o(n^{1/8})$.

If also, $q = 1$, Corollaries 1 and 2 give Edgeworth-Cornish-Fisher expansions for the standardized estimate. The results were unexpected. For the usual case when the asymptotic variance is $O(p^2/n)$, I only require that $p = o(n^{1/2})$. When the asymptotic variance is $O(p/n)$, as is the case when the components of $\bar{X}$ are independent, I require that $p = o(n^{1/5})$, a stronger condition. However for a *linear* function of a mean, by Corollaries 3 and 4, for the usual case when the asymptotic variance is $O(p^2/n)$, there is *no restriction* on the growth of p, while if the asymptotic variance is $O(p/n)$, I require that $p = o(n^{1/3})$. If also the components of $\bar{X}$ are independent, these Edgeworth-Cornish-Fisher expansions are in powers if $N^{-1/2}$ where $N = np$, so that p amplifies the effect of n.

11. Discussion and Future Directions

It should now be possible to extend the confidence intervals and confidence regions of error $O(n^{-s/2})$ given in Withers (1982, 1989), to the situation where q and p are allowed to increase with n, with $O(n^{-s/2})$ replaced by $O(v_n^{s/2})$.

Theorem 1 showed that the three Edgeworth expansions considered here, hold for standard estimates when $q_n = o(n^{1/6})$. Is this result optimal? It is certainly not optimal for a sample mean. For, as noted in Section 1, Theorem 2.1 of the recent paper Koike (2026) gives conditions for a 2nd order Edgeworth expansion for the distribution of a sample mean, when $q_n = O(n^c)$ for *any* c, or even when $q_n = O(\exp(an^c))$ for $a > 0$ and $c < 1/3$. It will be interesting to see if such weak conditions on q_n can be extended to a wide class of estimates, such as standard estimates.

My Theorem 4 showed that the three 4th order Edgeworth expansions hold for $\hat{w} = t(\bar{X}) : R^p \to R^q$, when $p_n^8 q_n^6 = o(n)$. It remains to be seen how much this condition can be weakened.

The inequalities in (43) with the stronger bounds in Theorem 7, suggest that the Edgeworth expansions of order s can be strengthened in terms of the p'_k of (43).

When $q = 1$, Corollaries 1–5 give 5th order Edgeworth-Cornish-Fisher expansions for the standardized estimate under various conditions. It should not be hard to extend the results of Sections 6 and 7 to $\hat{w} = T(F_n) \in R^q$, where $F_n(x)$ is the empirical distribution of a random sample of size n from a distribution $F(x)$ on R^p, or, using Section 8, to I samples.

The method employed here may also be able to extend some of the results for large q_n in the references from a sample mean to a standard estimate. It may also be able to extend results for large numbers of regression parameters beyond those noted in Portnoy (1984, 1985) and Donoho (2016).

Another worthwhile extension would be to strengthen these results to tilted Edgeworth expansions.

Extreme estimates, such as a sample minimum or maximum are a special class of standard estimates with the leading cumulant coefficients of (3) zero:

$$\bar{k}_{r-1}^{1\ldots r} = 0 \text{ for } r \geq 2.$$

So $V = 0_{q\times q}$ in (1), and the Edgeworth expansions do not apply. Instead, $\hat{w}$ has a multivariate extreme value limit when standardized, and its distribution and density have expansions in powers of certain functions of n. See for example Withers and Nadarajah (2019) if $q = 1$ and $\hat{w}$ the minimum of a positive sample, or Withers and Nadarajah (2018) if $q = 2$ and $\hat{w}$ is the minimum and maximum of a sample from a distribution on $[w_1, w_2]$. or more generally, see Withers and Nadarajah (2023). Nevertheless, analogues of the results here should be obtainable.

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