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A Theory-Consistent CVAR Scenario for a Monetary Model with Forward-Looking Expectations [†]

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Abstract: A theory-consistent CVAR scenario describes a set of testable regularities capturing basic assumptions of the theoretical model. Using this concept, the paper considers a standard model for exchange rate determination with forward-looking expectations and shows that all assumptions about the model's shock structure and steady-state behavior can be formulated as testable hypotheses on common stochastic trends and cointegration. The basic stationarity assumptions of the monetary model failed to obtain empirical support. They were too restrictive to explain the observed long persistent swings in the real exchange rate, the real interest rates, and the inflation and interest rate differentials.

Keywords: theory-consistent CVAR; expectations; international puzzles; long swings; persistence; imperfect knowledge



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1. Introduction

How to take forward-looking expectations to the data has been much debated in economics. Most economists seem to prefer model consistent rational expectations to ensure that agents' expectations are consistent with the assumed theoretical model. These models are mostly taken to the data using calibration and Bayesian priors often focussing on a few key parameters of the theoretical model. But, as forcefully argued by among others [Hendry and Mizon \(2000\)](#) and [Spanos \(2009\)](#), this is scientifically valid only if the assumed structure of the economic model is empirically correct.

Numerous econometric studies have shown that basic assumptions underlying economic models tend to be rejected against the data. See, for example, the articles of the special issue "Using Econometrics for Assessing Economic Models" ([Juselius 2009](#)). [Hendry and Mizon \(2014\)](#) shows that the law of iterated expectations fails to hold intertemporally if there are unpredictable breaks in the economic relationships. Such breaks are endemic in economic relations and points to a major problem for the rational expectations assumption. This is confirmed in David Hendry's important work on forecasting which has documented that the predictive performance of these models are miserably poor (see [Clements and Hendry 2001, 2011](#) and [Hendry and Ericsson 2003](#)). In stark contrast to the rational expectations hypothesis, David Hendry and coauthors have shown that, in our nonstationary economies with complex feedback dynamics and structural breaks, the forecasts from these models are completely outperformed by those of simple time-series models devoid of economic content.

The idea of this paper is complementary to David Hendry's work on forecasting performance by focussing on how to formulate testable hypotheses on the pulling and pushing behavior of economic models with forward looking expectations without compromising scientific principles. We pay particular attention to the case when the data-generating process contains unit roots and breaks and show that empirical consistency often entails

theoretical inconsistency. This is largely in line with David Hendry's wide-ranging research, the ultimate goal of which has always been to understand how the economy works, rather than how it is assumed to.

As David Hendry has tirelessly emphasized, a reality check of the assumptions of an economic model has to be performed in an empirical model that correctly describes the data-generating process. This entails to adequately account for a large number of issues, such as unit roots and breaks, complex feed-back dynamics, regime changes, and extreme events. For this purpose, we propose to translate all basic assumptions of the economic model into a set of testable hypotheses on the pulling and pushing forces of a well-specified Cointegrated VAR (CVAR) model, in short a theory-consistent CVAR scenario (Juselius 2006; Juselius 2017; Juselius and Franchi 2007; Hoover et al. 2008). The advantage of the CVAR is that it offers a precise description of the data generating process and that its probability assumptions are testable, the disadvantage is that it is basically consistent with backward-looking expectations. To overcome this problem, we propose a simple procedure for how to translate forward-looking expectations into testable hypotheses on the common trends and the long-run equilibrium relationships in the framework of a CVAR model.

To illustrate the ideas, the paper derives a CVAR scenario for a standard monetary model for exchange rate determination with forward-looking expectations. Basic hypotheses on shock structure and steady-state behavior are formulated as a set of empirical regularities one would expect to see in the data provided the monetary model is empirically valid. Its basic long-run properties are tested using data on interest rates, prices, and the nominal exchange rate for Germany and USA in the post Bretton Woods—pre EMU period. The results show that the monetary model fails to explain the long and persistent swings typical of foreign currency markets. In particular, it is the stationarity assumption of the real exchange rate, the interest rate and inflation rate differential, and the real interest rates that is too restrictive.

The paper is organized as follows. Section 2 suggests a simple rule for associating expectations with observables. Section 3 discusses general principles for formulating a theory-consistent CVAR scenario with unobserved expectations. Section 4 summarizes the basic properties of the Dornbush overshooting model for exchange rate determination. Section 5 gives a practical rule for how to associate expectations with observables in the monetary model. Section 6 derives the theory-consistent CVAR scenario for the monetary model. Section 7 introduces the empirical CVAR model and Section 8 reports the rank determination tests. The results show that the number of stochastic trends of first and second order are not tenable with the basic assumptions underlying the theoretical model. Section 9 concludes.

2. Associating Expectations with Observables

The overriding idea of a theory-consistent CVAR scenario is to classify economic variables and relations according to their persistency property. The latter is measured by their order of integration, such as $I(0)$ for a stationary process, $I(1)$ for a first order persistent process, and $I(2)$ for a second order persistent process. Based on a simple rule for how to associate expected with observed entities, theoretically consistent persistency properties are derived and compared to observed magnitudes.

Because economic variables/relations do not move infinitely away from their equilibrium values, as true unit root processes do, one might argue that such a classification is theoretically flawed. While the former argument is obviously correct, the latter is not necessarily so. This is because economic variables/relations frequently exhibit a persistence that is empirically indistinguishable from a unit root or a double unit root process over a finite sample period. Thus, an empirical single/double unit root can be considered a measure of persistence that describes a *near* $I(1)$ or *near* $I(2)$ process. In this view, a unit root is not a structural parameter, but rather a convenient tool for classifying the data into homogeneous groups in terms of stationarity, and $I(1)$ and $I(2)$ type of nonstationarity. It implies that a variable can be classified as stationary in one period, but nonstationary

in another period—even when the latter overlaps with the former. As long as causally related variables are similarly classified, the procedure will work as intended. For example, unemployment and inflation can be classified as stationary in one sample, while nonstationary in another, and in both cases describe a plausible Phillips curve relationship. From an econometric point of view, nonstationarity is helpful as it gives statistically more precise estimates. For a detailed discussion, see Juselius (2013).

To formulate a theory-consistent CVAR scenario, we need a rule for how to associate the unobserved expectations in a theoretical model with the observed variables. The following three points explain:

1. Economic actors are assumed to form expectations which are broadly consistent with the order of integration of the variable in question. For example, if the theory predicts that $x_t \sim I(0)$, then its expectation $x_t^e \sim I(0)$, if $x_t \sim I(1)$ then $x_t^e \sim I(1)$ and if $x_t \sim I(2)$ then $x_t^e \sim I(2)$, where $x_t^e = x_{t+1|t}^e$ denotes the expected value of x_{t+1} formed at time t .
2. The *expectation shock*, $v_t = x_t - x_t^e$, is assumed to be $I(0)$ when $x_t \sim I(1)$, whereas $I(1)$ when $x_t \sim I(2)$ implying that $(\Delta x_t - \Delta x_t^e) \sim I(0)$.
3. The *forecast error*, $\varepsilon_{t+1} = x_{t+1} - x_t^e$ is assumed to be a non-systematic white noise process, unless an unanticipated structural break or an extreme event hit the variable at the forecasted time period $t + 1$. In the latter case, the forecast error is $x_{t+1} - x_t^e = \varepsilon_{t+1} + \psi Di_{t+1}$, where Di is a dummy variables that is 0 for t and 1 for $t + 1$. Since such extraordinary events are known *ex post*, they are assumed foremost to affect the forecast errors *ex ante*. The results below are derived assuming that forecast errors conditional on D_t are white noise.

We illustrate the idea for two cases, first assuming x_t is $I(1)$ and then that it is $I(2)$. We disregard $x_t \sim I(3)$, as it is considered to be empirically implausible and $x_t \sim I(0)$, as it defines a stationary process for which cointegration and stochastic trends have no additional informational value.

1. $x_t \sim I(1)$ is assumed to follow a random walk, $x_t = x_{t-1} + \varepsilon_t$, where ε_t is white noise. A consistent forecasting rule is $x_t^e = x_t$. In this simple case the *expectation shock* $v_t = x_t^e - x_t$ would be zero, but for a more general autoregressive model, v_t would be a stationary process. The *forecast error* is $x_{t+1} - x_t^e = \varepsilon_{t+1}$. Inserting $x_t^e = x_t$ gives $x_{t+1} - x_t = \varepsilon_{t+1}$. Thus, a white noise forecast error does not change the assumed persistency property of the economic variable.
2. $x_t \sim I(2)$ is assumed to follow a random walk in Δx_t , i.e., $x_t = x_{t-1} + \Delta x_{t-1} + \varepsilon_t$. A consistent forecasting rule is $x_t^e = x_t + \Delta x_t$. Hence, the difference between the observed value and the forecast, $v_t = x_t^e - x_t = \Delta x_t$, is an $I(1)$ process. The forecast error is again assumed to be a non-systematic white noise process, i.e., $x_{t+1} - x_t^e = \varepsilon_{t+1}$. Inserting $x_t^e = x_t + \Delta x_t$ in the above gives $\Delta^2 x_{t+1} = \varepsilon_{t+1}$. Thus, the process remains unchanged also in this case.

Furthermore, $x_t \sim I(1)$ implies $\Delta x_t \sim I(0)$, and $x_t \sim I(2)$ implies that $\Delta x_t \sim I(1)$ and $\Delta^2 x_t \sim I(0)$. Assumption 1 and the following two corollaries formalize the basic idea:

Assumption 1. When $x_t \sim I(1)$, $(x_t - x_t^e)$ is $I(0)$ or possibly zero. When $x_t \sim I(2)$, $(x_t - x_t^e)$ is $I(1)$.

Corollary 1. When $x_t \sim I(1)$, x_t, x_{t+1} and x_t^e have the same persistency property, i.e., they share a common stochastic $I(1)$ trend. When $x_t \sim I(2)$, $\Delta x_t, \Delta x_{t+1}$ and Δx_t^e share a common stochastic $I(1)$ trend.

Corollary 2. When $x_t \sim I(1)$, the cointegration relations $\{\beta' x_t^e, \beta' x_{t+1}, \beta' x_t\} \sim I(0)$, i.e., they share the same stationarity property. When $x_t \sim I(2)$, the disequilibrium relations $\{\beta' x_t^e, \beta' x_t, \beta' x_{t+1}, d' \Delta x_t^e, d' \Delta x_t, d' \Delta x_{t+1}\} \sim I(1)$, share a common stochastic $I(1)$ trend, the multicointegrated relations $\{(\beta' x_t^e + d' \Delta x_t^e), (\beta' x_t + d' \Delta x_t), (\beta' x_{t+1} + d' \Delta x_{t+1})\} \sim I(0)$ share the

same stationarity properties, and finally the medium-run relations $\{\tau' \Delta x_t^e, \tau' \Delta x_t, \tau' \Delta x_{t+1}\} \sim I(0)$ share the same stationarity properties.¹

Hence, Assumption 1 allows us to make valid inference in the CVAR model on theoretically assumed long-run relationships even though the postulated behavior is a function of expected rather than observed outcomes.

3. On the Formulation of a Theory-Consistent CVAR Scenario

The VAR(k) model is given by:

$$x_t = \Pi_1 x_{t-1} + \dots + \Pi_k x_{t-k} + \mu_0 + \mu_1 t + \Phi D_t + \varepsilon_t, \quad t = 1, \dots, T \quad (1)$$

$$\varepsilon_t \sim N(0, \Omega)$$

where x_t is a vector of economic variables, μ_0 is a constant, t is a linear trend, and D_t a vector of m dummy variables controlling for unanticipated effects of extraordinary events. The sample period, T , is assumed to describe a reasonably constant parameter regime, at least for the long-run parameters of interest. Depending on the purpose of the study, the time span of such a period can vary considerably. For example, it can be very short if the relevant expectations are of day-to-day or minute-to-minute changes, or it can be very long if the relevant expectations are of technology changes.

The specification, $\varepsilon_t \sim N(0, \Omega)$, implies that the difference between the process x_t and its conditional mean, $x_t - E(x_t \mid x_{t-1}, \dots, x_{t-k}, \mu_0, t, D_t)$, is a white noise process. If agents' expectations, $x_{t|t-1}^e$, were formed by the conditional mean of the VAR model, then expectations would not deviate systematically from realizations as long as $D_t = 0$, i.e., as long as no unanticipated extreme events happened at time t . If $D_t \neq 0$, then the forecast error, $\varepsilon_t + \Phi D_t$, will deviate from a white noise process. But in period $t + 1$, when the extraordinary effect is known, the forecast error would again be white noise. This is similar to the premise behind the rational expectations hypothesis that "all that could have been known about the structure of the economy was actually known at the time when plans were made". Thus, conditional on D_t , expectations and observables will have the same persistency property.

The white noise assumption in (1) together with Assumption 1 and the two corollaries ensure that a theory-consistent CVAR scenario can be adequately tested in the framework of an empirical CVAR model. Juselius (2017) formulates the following five steps for how to include forward-looking expectations into a testable theory-consistent CVAR scenario:

1. Express the expectations variable(s) as a function of observed variables. For example, according to Uncovered Interest Rate Parity (UIP), the expected change in the nominal exchange rate is equal to the interest rate differential. Hence, the persistency property of the latter is also a measure of the persistency property of the unobservable expected change in nominal exchange rate and can, therefore, be empirically tested.
2. Translate the postulated behavioral relations of the theoretical model into a set of hypothetical conditions on their persistency properties. The next section illustrates that the Dornbush standard monetary models is consistent with the purchasing power parity and the uncovered interest rate parity holding as stationary conditions (or at most near $I(1)$).
3. For a given order of integration of the unobserved expectations variable and the expectation shocks derive the theory-consistent order of integration for all remaining variables.
4. Translate the stochastically formulated theoretical model into a theory-consistent CVAR scenario by formulating the basic assumptions underlying the theoretical model as a set of testable hypotheses on cointegration relations and common trends.
5. Estimate a well-specified VAR model and check the empirical adequacy of the derived theory-consistent CVAR scenario.

The following notation will be used to discriminate between different types of shocks: $\varepsilon_t \sim \text{Niid}(0, \sigma_\varepsilon^2)$ is a white noise process; $v_t = x_{t+1|t}^e - x_t$; and $u_t = f(\varepsilon_t)$ is an unobserved 'structural' shock assumed to be a linear function of the shocks to the system.

4. A Standard Monetary Model for the Real Exchange Rate

The overshooting model by [Dornbusch \(1976\)](#) and [Dornbusch and Frankel \(1988\)](#) is widely considered to be the workhorse model in international macroeconomics. It explains the long swings in the real exchange rate by assuming that the nominal exchange rate is overshooting its equilibrium value as a result of price rigidities. Explicitly or implicitly it is also based on the assumption that (i) the rate of equilibrium adjustment to Purchasing Power Parity (PPP) is identical for relative prices and nominal exchange rates ([Frydman and Goldberg \(2007\)](#))², (ii) equilibrium in the goods market is characterized by PPP, (iii) Uncovered Interest Parity (UIP) is a market clearing mechanism, and (iv) the international Fisher parity holds as a stationary condition. These are basic features that will have to be formulated as restrictions on the pulling and pushing forces in a theory-consistent CVAR scenario for this model.

PPP states that $S_{ppp} = P_d / P_f$, where S_{ppp} stands for the equilibrium exchange rate, P_d stands for domestic prices and P_f for foreign prices. It implies that the nominal exchange rate, S_t , should reflect relative prices, $P_{d,t} / P_{f,t}$ in equilibrium. In a time-series context, logs are preferable to levels and the real exchange rate is defined as:

$$q_t = s_t - p_{d,t} + p_{f,t}, \quad (2)$$

where lower cases stand for logarithmic values and a subscript d for a domestic and f for a foreign economy. In equilibrium, the real exchange rate, q , is defined by relative prices being equal to the nominal exchange rate, i.e., $q_{ppp} = 0$. When prices are measured by a price index, the equilibrium value, q_{ppp} , is undefined and the observed average real exchange rate is generally different from zero. Figure 1, upper panel, in the Appendix plots the mean adjusted graphs of the nominal exchange rate, s_t , together with relative prices, $p_{d,t} - p_{f,t} = s_{ppp}$. The long and persistent swings of the nominal exchange rate around its equilibrium value are remarkably large.

Theoretically, the real exchange rate is assumed to deviate from its long-run equilibrium value by an equilibrium error ($q_t - \bar{q}$), which in the Dornbusch/Frankel type of models is assumed to be an AR(1) process:

$$\Delta q_t = -\alpha(q_{t-1} - \bar{q}) + \varepsilon_{q,t}, \quad (3)$$

where \bar{q} is the sample average, $0 < \alpha < 1$ measures the speed of adjustment and $\varepsilon_{q,t}$ is white noise. Even though q_t in (3) describes a stationary process some versions of the monetary model allow α to be very close to zero so that the real exchange rate is a near $I(1)$ process. The graph of the real exchange rate, q , in Figure 1, lower panel, shows that it has inherited the persistent swings from the nominal exchange rate and that the slowly increasing long-run trend visible both in the nominal exchange rate and the relative price has been annihilated in the real rate.

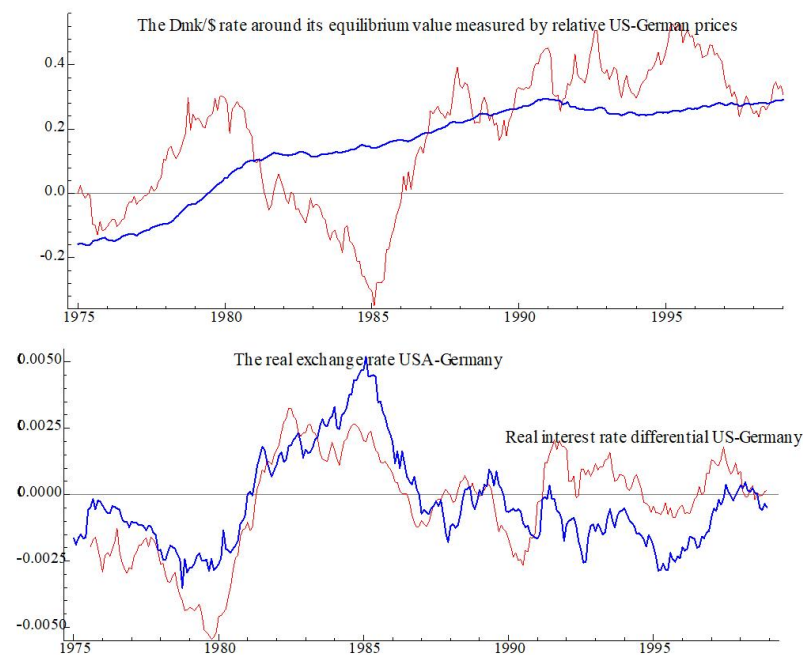


Figure 1. The graphs of nominal exchange rate and the price differential (**upper panel**) and the real exchange rate together with the bond rate differential (**lower panel**).

The Uncovered Interest Rate Parity (UIP) is defined as:

$$i_{d,t} - i_{f,t} = s_t^e - s_t, \quad (4)$$

where i stands for a nominal interest rate. Figure 2, upper panel, in the Appendix, plots the differenced nominal exchange rate, Δs_t , together with a 12 months moving average. The aim of the latter is to visualize any persistent movements in the differenced process that would be difficult to get sight of in the very volatile short-run changes. The graph of the bond rate differential in the lower panel shows that it has exhibited much more persistent swings than the differenced nominal exchange rate.

The Fisher Parity states that the nominal interest rate is equal to the expected inflation rate plus an independent real interest rate. The latter is assumed to reflect the ratio of average profit per capital in the economy, which is difficult to measure on the aggregate level. In practice, the real rate has often been approximated with the real GDP growth rate, typically assumed to be stationary with a non-zero mean³. Accordingly, the real interest rate is considered stationary with a constant mean.

The Fisher parity is defined as:

$$i_{j,t} = r_{j,t} + \Delta p_{j,t}^e + \varepsilon_{F,t}, \quad j = d, f \quad (5)$$

where $r_{j,t}$ is an (unobserved) real interest rate, $p_{j,t}^e$ is a shortcut for $p_{j,t+1|t}^e$, and $\varepsilon_{F,t}$ is a white noise error. Figure 3 in the Appendix shows the *ex post* real interest rate for USA in the upper panel and for Germany in the lower panel. Both fluctuate in long persistent swings around an approximate mean value of 5%.

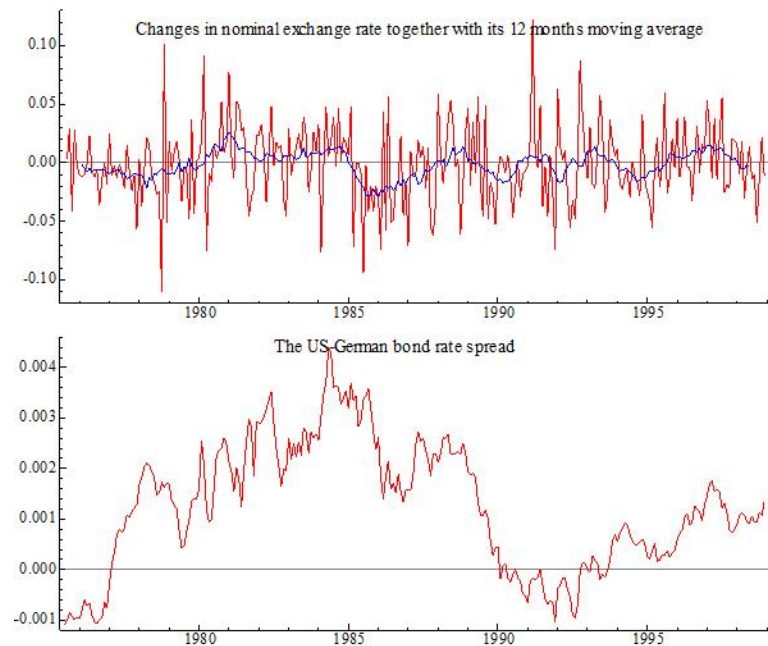


Figure 2. Graphs of the changes in the DK-\$ nominal exchange rate together with its 12 months moving average (**upper panel**) and the US-German bond rate spread (**lower panel**).

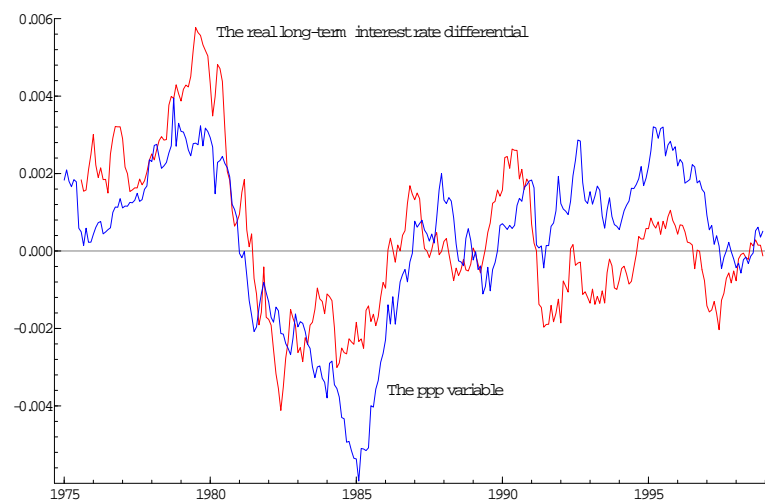


Figure 3. The graph of the real bond rate differential together with the negative of the real exchange rate.

Finally, (3) and (4) together with Assumption 1 define the international Fisher Parity:

$$(i_{d,t} - i_{f,t}) = (\Delta p_{d,t}^e - \Delta p_{f,t}^e) + (r_d - r_f), \quad (6)$$

implying equality between the real interest rates up to a real productivity differential. Figure 4, upper panel, in the Appendix, shows the bond rate differential together with a 12 months moving average and the lower panel the observed inflation rate differential together with a 12 months moving average. Both fluctuate in long persistent and dissimilar swings.

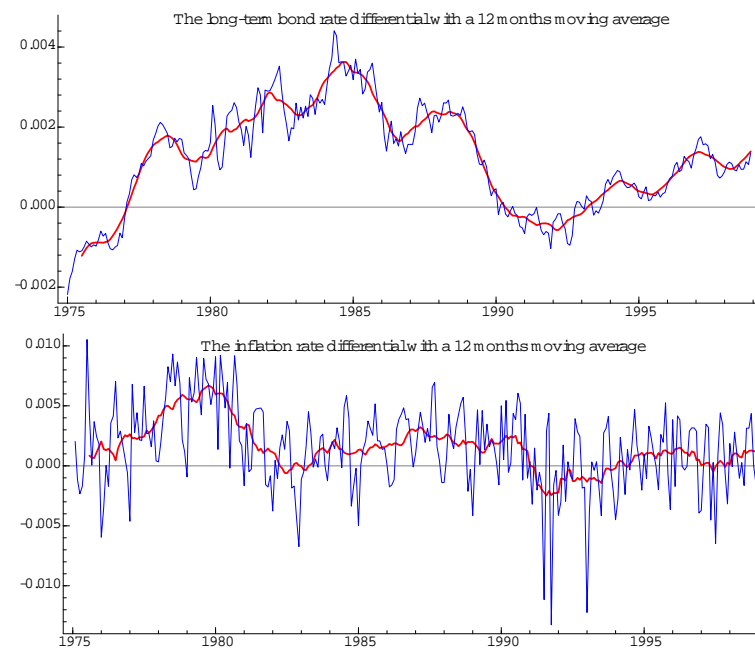


Figure 4. The graphs of the bond rate differential (**upper panel**) and the inflation rate differential (**lower panel**) together with their 12 months moving average.

5. Anchoring Expectations to Observables

The first step is now to derive theory-consistent time-series properties for the relevant variables and relations in the monetary model using Assumption 1 to handle unobserved expectations.

The UIP condition states that

$$s_t^e - s_t = i_{d,t} - i_{f,t}, \quad (7)$$

implying that the interest rate differential is a measure of the expected change in the nominal exchange rate. Thus, in accordance with step 1 above, we can use the order of integration of the observed interest rate differential as a persistency measure of the of the expected change in the nominal exchange rate.

The change in interest rates are assumed unpredictable and can be described by:

$$i_{j,t} = i_{j,t-1} + \varepsilon_{j,t}, \quad j = d, f \quad t = 1, \dots, T \quad (8)$$

where $\varepsilon_{j,t}$ is a stationary error term. Integrating (8) over the sample period gives:

$$i_{j,t} = i_{j,0} + \sum_{i=1}^t \varepsilon_{j,i}, \quad j = d, f \quad t = 1, \dots, T \quad (9)$$

where the cumulation of the interest rate shocks is a measure of the stochastic $I(1)$ trend in the interest rate.

Under Assumption 1, $s_t^e - s_t = v_{s,t}$ is stationary (when $s_t \sim I(1)$). Thus, for UIP to hold as a market clearing mechanism ($i_{d,t} - i_{f,t}$) must be stationary. This implies that the stochastic trend in the interest rates must be identical, so that $\sum_{i=1}^t (\varepsilon_{d,i}^p - \varepsilon_{f,i}^p) = 0$. The interest rate differential can then be expressed as:

$$i_{d,t} - i_{f,t} = i_{d,0} - i_{f,0}. \quad (10)$$

The Fisher parity (5) is equivalently expressed as

$$\Delta p_{j,t}^e = i_{j,t} - r_{j,t}, \quad j = d, f. \quad (11)$$

where the real interest rate is assumed to be stationary with a constant mean, $r_{j,t} = r_j + \varepsilon_{rj,t}$. Under Assumption 1, $\Delta p_{t+1}^e = \Delta p_t + v_{p_t}$ where $v_{p_t} \sim I(0)$ so the inflation rate can be expressed as:

$$\Delta p_{j,t} = i_{j,t} - r_{j,t} - v_{p,j,t}, \quad j = d, f. \quad (12)$$

Inserting (9) in (12) gives an expression for the stochastic properties of the inflation rates:

$$\Delta p_{j,t} = i_{j,0} - r_{j,t} + \sum_{s=1}^t \varepsilon_{j,s} - v_{p,j,t}, \quad j = d, f \quad (13)$$

Hence, inflation rate is $I(1)$ with the same stochastic trend as the interest rate.

An expression for the price level is obtained by integrating $\Delta p_{j,t}$ over t :

$$p_{j,t} = (i_{j,0} - r_j)t + \sum_{s=1}^t \sum_{i=1}^s \varepsilon_{j,i} + \sum_{i=1}^t \varepsilon_{rj,i} - \sum_{s=i}^t v_{p_{j,i}} + p_{j,0}, \quad j = d, f \quad (14)$$

Thus, the price level contains a linear time trend, a second order stochastic trend originating from twice cumulated shocks to the interest rate, and first order stochastic trends originating from cumulated inflation expectation shocks. The linear trend in prices derives from the initial value of the nominal interest rate corrected for the mean value of the real interest rate, so the slope of the linear trend is approximately equal to the initial value of the inflation rate.

Using (12), the international Fisher parity can be formulated as:

$$\begin{aligned} (\Delta p_{d,t} - \Delta p_{f,t}) &= (i_{d,t} - i_{f,t}) - (r_d - r_f) - (v_{p_{d,t}} - v_{p_{f,t}}) \\ &= (i_{d,0} - i_{f,0}) - (r_d - r_f) - (v_{p_{d,t}} - v_{p_{f,0}}). \end{aligned} \quad (15)$$

International parity conditions imply equalization of the real interest rates up to the real productivity growth differential ($r_d - r_f$). All components on the r.h.s. are stationary, hence the inflation differential is stationary. $(\Delta p_{d,t} - \Delta p_{f,t}) \sim I(0)$ implies $(p_{d,t} - p_{f,t}) \sim I(1)$, hence prices are cointegrated (1, -1) from $I(2)$ to $I(1)$ and, therefore, satisfy long-run price homogeneity. Integrating (15) over t gives an expression for relative prices

$$p_{d,t} - p_{f,t} = \left\{ (i_{d,0} - r_d) - (i_{f,0} - r_f) \right\} t - \sum_{i=1}^t (v_{p_{d,i}} - v_{p_{f,i}}) + (p_{d,0} - p_{f,0}). \quad (16)$$

Thus, the relative prices contain a linear time trend due to the initial value of relative interest rates and a stochastic $I(1)$ trend originating from cumulated inflation expectation shocks.⁴

An expression for the nominal exchange rate can be obtained from the uncovered interest rate parity in (7) using Assumption 1 to replace Δs_t^e with $\Delta s_t + v_{s,t}$:

$$\Delta s_t = i_{d,t} - i_{f,t} - v_{s,t}. \quad (17)$$

Inserting (10) into (17):

$$\Delta s_t = (i_{d,0} - i_{f,0}) - v_{s,t}$$

Integrating (17) over t gives an expression for the level of nominal exchange rate:

$$s_t = (i_{d,0} - i_{f,0})t - \sum_{i=1}^t v_{s,i} + s_0 \quad (18)$$

showing that the nominal exchange rate contains a local linear trend originating from the initial values of the interest rate differential and a stochastic $I(1)$ trend originating from cumulated expectation shocks to the nominal exchange rate.

An expression for the real exchange rate can be found by subtracting (18) from (16):

$$p_{d,t} - p_{f,t} - s_t = - \sum_{i=1}^t (v_{p_{d,i}} - v_{p_{f,i}} - v_{s,i}) - (r_d - r_f)t + (p_{d,0} - p_{f,0} - s_0).$$

The expectation shocks to relative prices are likely to approximately equal the expectation shocks to the nominal exchange rate under purchasing power parity, so that $v_{p_{d,i}} - v_{p_{f,i}} \simeq v_{s,i}$. Under this assumption, the real exchange rate is stationary around a trend that captures the relative productivity differential of the two countries.

6. A Theory-Consistent Scenario

The derived theory-consistent stochastic properties for prices, the nominal exchange rate, and the interest rates allow us to translate the behavioral equilibrium relationships underlying the monetary model to a set of testable hypotheses on cointegration and common trends in the CVAR model. The derivations of these properties started from the assumption that $s_t^e - s_t = i_{d,t} - i_{f,t}$, where $(i_d - i_f) \sim I(0)$, implying one common stochastic $I(1)$ trend in the interest rates. This is the only common stochastic trend in the system, as the stochastic properties of the remaining variables are derived from the stochastic properties of the interest rates.

According to (14), the $I(2)$ trend in prices originates from the twice cumulated interest rate shocks and the $I(1)$ trend from the once cumulated expectation shocks to inflation forecasts. Under Assumption 1, white noise expectations shocks do not change the long-run properties of the system. Therefore, provided the monetary model is an adequate description of the real economy, the CVAR system would be driven by one second order stochastic trend and be equilibrium error correcting to four cointegration relations. The common autonomous shock, $u_{1,t}$, would be a linear combination of the estimated interest rate residuals, $\hat{u}_{1,t} = \alpha'_{\perp} \hat{\varepsilon}_t$, where $\alpha'_{\perp} = [0, 0, 0, 1, 1]$.

As discussed above, the common shock, $u_{1,t}$, cumulates once in the interest rates and the nominal exchange rate, but twice in the price variables (see also Juselius 2006, Chapter 2.5). The theory-consistent CVAR scenario is consistent with $\{r = 4, s_1 = 0, s_2 = 1\}$, where s_1, s_2 is the number of autonomous $I(1)$ and $I(2)$ trends respectively. It is formulated as follows:

$$\begin{bmatrix} p_d \\ p_f \\ s \\ i_d \\ i_f \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} [\Sigma \Sigma u_1] + \begin{bmatrix} b_1 & 1 & 0 & 0 \\ b_2 & 0 & 1 & 0 \\ b_3 & 0 & 0 & 1 \\ c_1 & 0 & 0 & 0 \\ c_1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Sigma u_1 \\ \Sigma v_{d,p} \\ \Sigma v_{f,p} \\ \Sigma v_s \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ 0 \\ 0 \end{bmatrix} t + Z_t. \quad (19)$$

where $\Sigma \Sigma u_1$ is a shorthand for $\sum_{s=1}^t \sum_{j=1}^s u_{1,j}$, Σu_1 a shorthand for $\sum_{j=1}^t u_{1,j}$ and similarly for $\Sigma v_{p,d}$, $\Sigma v_{p,f}$, and $\Sigma v_{p,s}$. Z_t is a catch-all for all short-term effects in the vector process. If the common stochastic $I(2)$ trend affects the two prices with identical coefficients, then $(p_d - p_f) \sim I(1)$ is consistent with long-run price homogeneity.

The coefficients c_i , b_i , and d_i are not expressed as functions of the parameters of the theoretical model as this would require specification also of the short-run dynamics. Thus, the above CVAR scenario is only informative about the conditions under which the more important long-run behavior of the model is empirically valid. M. Juselius (2010) shows that the short-run implications of the economic model are of interest only if the model first is able to satisfactorily describe the long-run properties of the data.

If $b_1 - b_2 = b_3$, $d_1 - d_2 = d_3$ and $\Sigma v_{d,p} - \Sigma v_{f,p} - \Sigma v_s = 0$, then the scenario can be formulated for the real exchange rate as follows:

$$\begin{bmatrix} s - p_d + p_f \\ \Delta p_d \\ \Delta p_f \\ i_d \\ i_f \end{bmatrix} = \begin{bmatrix} 0 \\ c_1 \\ c_1 \\ c_1 \\ c_1 \end{bmatrix} [\Sigma u_1] + Z_t. \quad (20)$$

showing that the four irreducible cointegration relations correspond to the PPP, the UIP, and the Fisher parities:

1. $p_d - p_f - s \sim I(0)$
2. $i_d - i_f \sim I(0)$
3. $i_d - \Delta p_d \sim I(0)$,
4. $i_f - \Delta p_f \sim I(0)$,

Again, other irreducible relations can be found by linear combinations. For example, $(i_d - i_f) - (i_d - \Delta p_d) + (i_f - \Delta p_f) = (\Delta p_d - \Delta p_f) \sim I(0)$ is a linear combination of 2, 3, and 4.

7. The Specification of the CVAR Model

The empirical analysis is based on German-US data for the post Bretton Woods, pre-EMU period⁵. The sample starts in 1975:8 and ends in 1998:12. The VAR model defined in (1) is formulated in levels, but is now reformulated in levels, first and second order differences anticipating the finding that the process x_t is $I(2)$. The VAR specification has three lags, a broken linear trend, and four dummy variables:

$$\Delta^2 x_t = \Gamma \Delta x_{t-1} + \Pi x_{t-1} + \Gamma_1 \Delta^2 x_{t-1} + \mu_0 + \mu_{01} Ds_{91:1,t} + \mu_1 t + \mu_1 t_{91:1} + \phi_1 D_{tax,t} + \phi_2 Dp_{86:2} + \phi_3 Dp_{91:1} + \varepsilon_t, \quad (21)$$

where $x_t = [p_{d,t}, p_{f,t}, s_t, b_{d,t}, b_{f,t}]$ and p_t stands for CPI prices, s_t for the Dmk/dollar exchange rate, b_t for long-term bond rates, a subscript d for Germany and a subscript f for USA, $t_{91:1,t}$ is a linear trend starting in 1991:1 and $Ds_{91,t}$ is a step dummy, which is 1 from 1991:1–1998:12, otherwise 0. Both control for the reunification of East and West Germany. $D_{tax,t}$ is an impulse dummy accounting for three different excise taxes levied on prices to pay for the German reunification, $Dp_{86:2}$ is controlling for a large shock to the US price and bond rate in 1986:2, possibly a delayed consequence of the Plaza Accord in 1985, and $Dp_{91:1}$ accounts for a large shock to the German price in 1991:1 caused by the reunification.

A battery of mis-specification tests show no significant residual autocorrelation of order 1 and 2 and no significant residual ARCH of order 2—except for the nominal exchange rate with a p-value of 0.02. Due to excess kurtosis, residual normality is however rejected for all variables except the nominal exchange rate. This is mostly because the sample period includes the money targeting regime in the first half of the eighties with wild fluctuations of the interest rates. Also the inflation rates exhibit long-tailed residual distributions to some extent. Because residual normality was rejected due to excess kurtosis rather than skewness, implying distributions are still symmetrical, we consider the VAR model to be reasonably well-specified.

The hypothesis that x_t is $I(1)$ is formulated as a reduced rank hypothesis on $\Pi = \alpha\beta$, where α is $p \times r$ and β is $p_1 \times r$ with $p_1 = p + 2$. The hypothesis that x_t is $I(2)$ is formulated as an additional reduced rank hypothesis $\alpha'_\perp \Gamma \beta_\perp = \xi \eta'$, where $\Gamma = -(I - \Gamma_1)$, ξ, η are $(p - r) \times s_1$ and $\alpha_\perp, \beta_\perp$ are the orthogonal complements of α, β respectively. See Johansen (1996).

Because the $I(2)$ condition is defined as a complex restriction on the Γ matrix, the representation in (21) is not optimal for likelihood inference. To circumvent this problem, Johansen (1997) propose the following parameterization:⁶

$$\begin{aligned} \Delta^2 x_t = & \alpha \left[\begin{pmatrix} \beta \\ \tau_{01} \\ \tau_0 \end{pmatrix}' \begin{pmatrix} x_{t-1} \\ t_{91:1,t-1} \\ t-1 \end{pmatrix} + \begin{pmatrix} d \\ d_{01} \\ d_0 \end{pmatrix}' \begin{pmatrix} \Delta x_{t-1} \\ D_{s91:1,t-1} \\ 1 \end{pmatrix} \right] \\ & + \zeta \begin{pmatrix} \tau \\ \tau_{01} \\ \tau_0 \end{pmatrix}' \begin{pmatrix} \Delta x_{t-1} \\ D_{s91:1,t-1} \\ 1 \end{pmatrix} + \\ & + \Gamma_1 \Delta^2 x_{t-1} + \Phi_1 D_{tax,t} + \Phi_2 Dp_{86,2t} + \phi_3 Dp_{91:1,t} + \varepsilon_t, \end{aligned} \quad (22)$$

where $\tau = [\beta, \beta_{\perp}]$ and d is proportional to τ_{\perp} . In (21), an unrestricted constant (and a step dummy) will cumulate twice to a quadratic trend, and a linear (broken) trend to a cubic trend. By specifying the broken trend to be restricted to the β relations and the differenced broken trend to the d relations of model (22) these undesirable effects are avoided. For a more detailed discussion, see Doornik and Juselius (2017), (Juselius 2006, Chapter 17).

8. Rank Determination

The standard trace test procedure proposed by Nielsen and Rahbek (2007) starts with the most restricted model ($r = 0, s_1 = 0, s_2 = 5$), continues to the end of the row, and proceeds similarly row-wise from left to right until the first non-rejection. Table 1 report the tests statistics that have been calculated by bootstrapping. All trace tests in the row $r = 0$ were rejected and are not reported in the table.

As a robustness check the characteristic roots of the unrestricted and rank restricted versions of the model are also reported in Table 1. The unrestricted VAR has five large roots, four of which are almost on the unit circle (0.98) plus a fifth one which is large but not equally close to one (0.85). The choice of reduced rank indices should therefore be consistent with five, possibly four, unit roots in the characteristic polynomial. The first non-rejection is at $\{r = 1, s_1 = 3, s_2 = 1\}$ with a p-value of 0.14, implying that five of the characteristic roots are restricted to be on the unit circle. The largest unrestricted root is equal to 0.53, ensuring that all persistent movements in the data have been accounted for.

Table 1. Determination of the two rank indices.

$p - r$	r	Rank Test Statistics					
		$s_2 = 5$	$s_2 = 4$	$s_2 = 3$	$s_2 = 2$	$s_2 = 1$	$s_2 = 0$
4	1		354.2 [0.00]	236.7 [0.00]	144.2 [0.00]	70.5 [0.14]	65.3 [0.50]
3	2			139.8 [0.00]	59.9 [0.25]	41.5 [0.42]	38.2 [0.72]
1	4					10.3 [0.97]	4.4 [0.95]
Six Largest Characteristic Roots (Modulus of)							
Unrestricted VAR			0.98	0.98	0.98	0.98	0.85
$r = 1, s_1 = 3, s_2 = 1$			1.0	1.0	1.0	1.0	0.53
$r = 2, s_1 = 1, s_2 = 2$			1.0	1.0	1.0	1.0	0.55
$r = 4, s_1 = 0, s_2 = 1$			1.0	1.0	0.96	0.96	0.84
							0.58

The case ($r = 2, s_1 = 1, s_2 = 2$) with a p-value of 0.25 is also consistent with five unit roots in the characteristic polynomial. The difference between the two cases is that the former corresponds to one multicointegration relation (between levels and differences) and three medium-run cointegration relations among the differenced variables, whereas the

latter corresponds to two multicointegration relations and one medium-run relation among the differenced variables. In terms of likelihood the two cases are very similar.

Finally, the case ($r = 4, s_1 = 0, s_2 = 1$) corresponds to the theory-consistent choice of rank indices. The large p -value (0.97) of its test statistic tells us there is at least one $I(2)$ trend in the model, but this is generally uninformative about the correct choice of rank indices. This is because it does not say anything about whether the system has more than one trend. The estimated characteristic roots reported in the last row of Table 1 show that this choice will leave two unrestricted near unit roots (0.96) and one additional large root (0.84) in the model. Thus, any test of the stationarity hypotheses derived in Section 6 based on the theory-consistent CVAR scenario would be completely unreliable, a conclusion strongly supported by the persistent graphs in the Appendix.

Altogether, the results show that the standard Dornbush-Frankel monetary model cannot explain the long persistent movements in the data. Similar results have been found in (Juselius 2006, Chapter 21) for Denmark versus Germany, Juselius and MacDonald (2004) for Japan versus USA, Juselius and Macdonald (2006) for Germany versus USA, Juselius and Assenmacher (2017) for Switzerland versus USA, and Juselius and Stillwagon (2018) for UK versus USA. Common to the above papers is the finding that Purchasing Power Parity needs Uncovered Interest Parity to become a stationary parity relation and that this is the key to a variety of economic puzzles.

In a companion paper, Juselius (2017) derives a theory-consistent scenario for a monetary model in which rational expectations are replaced by imperfect knowledge based expectations (Frydman and Goldberg 2011; Frydman and Goldberg 2007). This change turns out to have strong implications for the theory-consistent persistency properties of the variables and the relations. In stark contrast to the stationarity assumptions of the rational expectations model, the imperfect knowledge based monetary model is consistent with a pronounced persistence of near $I(2)$ for the price and the interest rate differentials and similarly for the nominal and real exchange rate. The empirical testing in Juselius (2017) supports the derived theoretical results: the price and the interest rate differentials were found to be near $I(2)$, whereas the nominal and real exchange rate were borderline near $I(2)$ (the $I(1)$ hypothesis was only rejected with a small p -value of 0.07).

9. Conclusions

The paper demonstrates that structuring the data according its (near) unit root properties using the Cointegrated VAR (CVAR) model provides a powerful way of confronting a theory model with the data. This is formalized as a theory-consistent CVAR scenario which describes the empirical regularities we should expect to see if the theory model is empirically relevant. To overcome the problem of unobserved expectations, the paper demonstrates that basic hypotheses about the expectation's formation can be translated into testable hypotheses in the scenario.

To illustrate, the paper translates most of the basic hypotheses underlying a standard monetary model for nominal exchange rate determination into testable hypotheses of a well specified CVAR model. The empirical results show that the monetary model is not able to explain the persistent movements in the data, suggesting that informationally less demanding "imperfect knowledge" based models (Frydman and Goldberg 2011; Frydman and Goldberg 2007) is likely to be superior in this respect. For example, Johansen et al. (2010) finds for a similar data set that an imperfect knowledge based model is in line with the persistency properties of the data. In a companion paper Juselius 2017 derives a CVAR scenario for an "imperfect knowledge" based monetary model for exchange rate determination and finds overwhelmingly strong empirical support for this model.

The real exchange rate and the real interest rate are among the most important determinants for the real economy, which emphasizes the importance of understanding the causes underlying their long persistent movements away from fundamental values. Numerous CVAR reality checks of a variety of economic models show that the persistency properties of economic data are systematic and untenable with mainstream macroeconomic models

assuming rational expectations and stationary equilibrium relationships. Instead, these results are much more in line with disequilibrium models (Stiglitz 2018 and Guzman and Stiglitz 2020), emphasizing the need for economic models that are based on persistent disequilibria and complex feed-back dynamics.

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Notes

- ¹ Section 6 provides a definition of β , d and τ .
- ² Cheung et al. (2004) find that this feature is not supported by empirical evidence.
- ³ If the real GDP growth rates instead were $I(1)$, then the real GDP would be $I(2)$ and, hence, nominal GDP and GDP price deflator would have to be $I(3)$ to be cointegrated. This strongly supports the stationarity assumption.
- ⁴ Note that, the trend $\sum_{i=1}^t (v_{p_{d,i}} - v_{p_{f,i}})$ is assumed to be near $I(1)$ and, hence, is not assumed to move infinitely away from the long-run productivity differential trend.
- ⁵ All calculations are done using the software program CATS 3 in Oxmetrics (Doornik and Juselius (2017)).
- ⁶ It has been implemented in CATS 3 for OxMetrics. See Doornik and Juselius (2017).

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