

Article

On Fast Jerk-Continuous Motion Functions with Higher-Order Kinematic Restrictions for Online Trajectory Generation II: Arbitrary Initial and Final Jerks

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Abstract

Finding smooth and fast motion functions within the kinematic limits of a robot or automated system is an important task for reducing vibration and increasing efficiency. Much research is concerned with point-to-point motion as needed in pick-and-place operations but prescribing general boundary conditions is also interesting for setting the state at intermediate waypoints in multipoint tasks or for reacting to unforeseen events having an arbitrary state. This paper addresses the following trajectory problem: jerk-continuous, fast online trajectory generation under kinematic restrictions with arbitrary initial/final velocity and jerk, while initial and final accelerations are constrained to zero. Fast means that at any time at least one restriction is reached. This extends previous work to additionally prescribing initial and final jerks and can hence be seen as an important intermediate step towards the full solution where the accelerations can also be arbitrarily chosen. We further develop the method of going systematically through the velocity–acceleration plane applied in previous research by introducing new concepts and techniques regarding how points in this plane can be connected and additional rounds can be inserted. The algorithms we develop find an admissible solution for any given distance within the timeframe suitable for online trajectory generation although a faster isolated solution might be missed. We tested the algorithm in a huge number of random configurations and show that the motion times are shorter than those the only other available algorithm provides.

Keywords: online trajectory generation; fast motion function; jerk continuity; arbitrary initial and final jerk; snap restriction



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1. Introduction

It is one of the most basic tasks in mechanical engineering to design the motion of an object from a given initial state to a desired final state such that the respective function can be given as input to a controller. Correspondingly, this task has been addressed in the literature very often (cf. [1,2]), be it in a special application scenario, for a specific class of tasks (e.g., point-to-point motion), or for satisfying certain task requirements (e.g., vibration reduction, quick reaction in online trajectory generation). In order to provide a “smooth” motion, requirements on the continuity level of the motion function are stated. Particularly in the more recent literature ([2–14], see also the references in [2]), the advantage of continuous jerk functions has been shown. Therefore, the current article only considers motion functions having this property. Besides smoothness, time efficiency is another important requirement. To achieve this, one should obviously use the limits of the device

under consideration as far as possible, i.e., the maximum velocity, acceleration, jerk, and snap (derivative of jerk). For this, methods applying optimization algorithms have been used (see overview in [1]) but they are too slow for online trajectory generation where the function should be computed within a controller cycle. For the latter to be possible, motion functions are usually directly constructed using a piecewise approach. This leads to the task already partially tackled in [2]: find a fast jerk-continuous motion function within the kinematic limits of the system in a computationally efficient way where fast means that at any instant in time at least one restriction is reached.

This task can be classified regarding the kind of initial and final state conditions. In point-to-point motion (also called rest-to-rest, for short: RR, see [2]) all derivatives up to the third are zero. This is normally used in pick-and-place tasks. If the object has initially and finally a constant velocity (i.e., higher derivatives are zero) as in certain manufacturing processes (cf. [15]), the task is abbreviated as GG (as in [2], the abbreviations are adopted from [16]). When initial and final velocities, accelerations, and jerks can be set arbitrarily, the task is abbreviated as BB+ (the + stands for additionally given jerks). This occurs for example in the following application scenarios:

- When one has to change the motion at an arbitrary stage due to unforeseen events (see [17]).
- In a multipoint task (cf. [18]), the waypoints have to be interconnected and one wants to be able to prescribe arbitrary conditions for these instead of assuming that they are rest points with zero velocity and acceleration.
- When parts of the motion function are determined by technological requirements and an intermediate piece between these already given motion parts must be designed and inserted (cf. [19]).

Table 1 of [2] gives an overview of the literature up to 2022 and Table 1 below adds more recent work. Most of the recent articles deal with the RR situation ([3–12]) and have, as an additional restriction, the condition that when the velocity function reaches its maximum value (not necessarily the maximum velocity of the system) the jerk goes back to zero. In our own previous work [2], we covered the GG task by going systematically through the velocity–acceleration plane (introduced by Broquère [20]) without making that assumption and showed the gain in time efficiency (for RR as a special sub-case of GG see also the more detailed investigation in [21]). The work by Boryga et al. [13] also covers the GG task but uses special polynomials. This leads to relatively simple computations and infinite continuity of derivatives but since no phases with constant velocity, acceleration, and/or jerk are possible, time efficiency is sub-optimal. The article by Zhao [14] allows for arbitrary initial velocity and acceleration (the final state is rest) but also goes back to zero jerk at the maximum of the velocity function and does not always find a solution. If the latter is the case, the motion function first goes back to rest. For the general BB+ task, there is still only the algorithm by Ezair et al. [22], which also makes the assumption regarding the velocity mentioned above. Therefore, for the general BB+ task, an approach dropping this assumption is still pending. Given the complexity in our tackling of the GG task in [2], it is unrealistic to generalize the approach in one step. Therefore, as an intermediate task, we allow for arbitrary initial velocities and jerks but let the respective accelerations be zero. We abbreviate this task as GG+. This task is solved in the current article. In doing so, we provide new techniques and concepts developed in Section 2 that are likely to prove useful for tackling the general BB+ task. We also set up the piecewise linear jerk models to be used in the overall construction of the motion function. In Section 3, depending on the signs of initial and final jerk, we systematically go through the velocity–acceleration plane, making sure that all possible given distances are covered using the techniques and concepts developed in Section 2. Section 4 deals with implementation aspects and

provides performance results corroborating the computational efficiency. In Section 5, we compare our approach with the only other available algorithm by Ezair et al. [22] in example situations and in a test run with randomly generated data.

Table 1. Jerk-continuous motion functions with restrictions 2022–2026 (addendum to [2]).

Author/Year	Boundary Conditions	Type of Jerk Function	Continuity Level	Kinematic Restrictions	Constant Phases	Additional Assumptions
Stretti et al. 2023, Bruzzone et al. 2026 [3,4]	RR	Piecewise elliptic, 7 segments	3	v, a, j	v, a	j = 0 at max./min. velocity
Yu et al. 2024 [5]	RR	Piecewise linear, 15 segments	3	v, a, j, sn	v, a, j, sn	j = 0 at max./min. velocity
Fang et al. 2023 [6]	RR	Piecewise trigon. and zero, 7 segments	4	v, a, j	v, a	j = 0 at max./min. velocity
Fang et al. 2024 [7]	RR+	Piecewise linear, 15 segments or higher order	3+	v, a, j, sn+	v, a, j, sn+	j = 0 at max./min. velocity
Han et al. 2024 [8]	RR	Piecewise sigmoid and constant, 15 segments	∞	v, a, j, sn	v, a, j	j = 0 at max./min. velocity; sn = 0 at max./min. acceleration
Wu et al. 2023 [9]	RR	Piecewise linear, 15 segments	3	v, a, j, sn	v, a, j, sn	All restrictions are reached
Hu et al. 2025 [10]	RR	0 or polynomial of degree 2	3	v, a, j	v, a	j = 0 at max./min. velocity
Liu et al. 2026 [11]	RR	Piecewise trigon. and constant, 15 segments	3	v, a, j	v, a, j	j = 0 at max./min. velocity
Zöllner 2026 [12]	RR	Piecewise linear, 15 segments or higher order	3+	v, a, j, sn+	v, a, j, sn+	j = 0 at max./min. velocity
Boryga et al. 2022 [13]	GG	Polynomials with degree ≥ 4	∞	v, a, j	None	Only special functions considered
Alpers 2022 [2]	GG	Piecewise linear, 15 segments	3	v, a, j, sn	v, a, j, sn	None
Zhao et al. 2022 [14]	BR	Piecewise linear and trig., 15 segments	3	v, a, j	v, a, j	j = 0 at max./min. velocity; does not always provide a solution
Own 2026	GG+	Piecewise linear, 15 segments	3	v, a, j, sn	v, a, j, sn	None

The main contributions of this research can be summarized as follows:

- The contribution provides an algorithm for computing a jerk-continuous, fast motion function under kinematic restrictions (velocity, acceleration, jerk, and snap) with

arbitrary initial and final velocities and jerks and zero acceleration, which because of its computational efficiency, is suitable for online trajectory generation as defined in the seminal book by Kröger [17]. It is the first contribution that avoids the assumption that the jerk has to go back to zero at an intermediate velocity, which enhances time efficiency.

- New concepts like connectibility and continuous connectibility are developed that show how the procedure of going systematically through the velocity–acceleration plane used in [2] can be extended when additional initial and final jerks are prescribed. These concepts are likely to be of use when tackling the more general task where initial and final accelerations can be also prescribed.
- A comparison with the only other algorithm by Ezair et al. [22], which is suitable for online trajectory generation and allows for arbitrary initial and final velocities and jerks, shows that our approach improves the time efficiency often considerably, although at the cost of a much higher level of complexity.

2. Concepts, Models, and Techniques

The task to be solved in this contribution can be stated as follows: Find a jerk-continuous motion function when the following values are given:

- $v_{max}, a_{max}, j_{max}, sn_{max}$ (each > 0 ; we assume symmetry, i.e., $v_{min} = -v_{max}$ etc.);
- Initial velocity and jerk: v_A and j_A ; final velocity and jerk: v_E and j_E ;
- Distance: Δs .

Initial and final accelerations are still assumed to be zero as in [2]. For constructing such a function, we again use the 15-segment profile with piecewise constant snap (cf. Figure 1 in [2]). Moreover, we work in the velocity–acceleration plane introduced in [20] and also described in [2] (cf. Figures 2 and 3 in [2]). We do not distinguish between the two cases I and II, as done in [2], depending on whether the maximal jerk is reached since the initial and final jerks can now be prescribed arbitrarily. Since the situation considered in the current paper is much more complex than the GG task in [2], we work coarser with less case distinctions to retain manageability. As in [2], we assume $\Delta v \geq 0$ since one can simply reverse the direction of the distance otherwise.

Consider two points in the velocity–acceleration plane lying on the velocity axis (i.e., $a = 0$) having jerks $j_A \geq 0$ and $j_E \leq 0$ named $(v_A, 0, j_A)$ and $(v_E, 0, j_E)$, with $v_A \leq v_E$. In the sequel, we simply write (v_A, j_A) when the acceleration is zero. In the case of zero jerk, there is always an arc in the upper halfplane connecting both points (see [2]); however, this is no longer the case with arbitrary jerks. The leftmost point on the velocity axis right of v_A that can be reached is the one where the jerk function is linear with $-sn_{max}$ as the slope (see Figure 1a). We denote this velocity by $\overline{v}_A$ (see Figure 1b).

An easy computation yields

$$\overline{v}_A = v_A + \frac{2}{3} \cdot \frac{j_A^3}{sn_{max}^2} \quad (1)$$

In the same manner, we define $\overline{v}_E$ for (v_E, j_E) with $j_E \leq 0$, which is the rightmost velocity that can be reached directly from (v_E, j_E) going through the lower halfplane.

$$\overline{v}_E = v_E + \frac{2}{3} \cdot \frac{j_E^3}{sn_{max}^2} \quad (2)$$

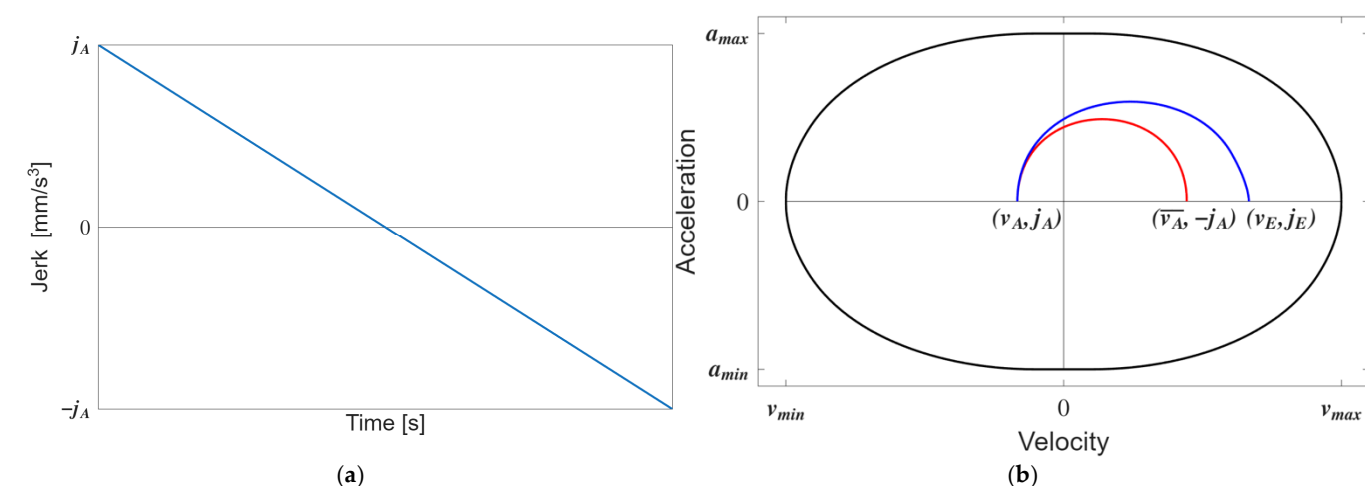


Figure 1. (a) Jerk function for going from j_A to $-j_A$. (b) Going from (v_A, j_A) to the leftmost point $(\bar{v}_A, -j_A)$ and to a point (v_E, j_E) further right.

In order to connect (v_A, j_A) to (v, j_E) with jerk $j_E \neq -j_A$, one needs a jerk function consisting of two pieces as shown in Figure 2, called a two-connection in the sequel. For $j_A \leq |j_E|$, the snap is first positive and then negative, which we denote as pm-connection (pm standing for plus-minus); for $j_A > |j_E|$, it is the other way around, denoted as mp-connection. Another easy computation provides the velocity difference for these jerk functions as

$$\Delta v = \frac{4j_A^3 - 4j_E^3 + 3\sqrt{2j_A^2 + 2j_E^2}|j_E^2 - j_A^2|}{12sn_{max}^2} \quad (3)$$

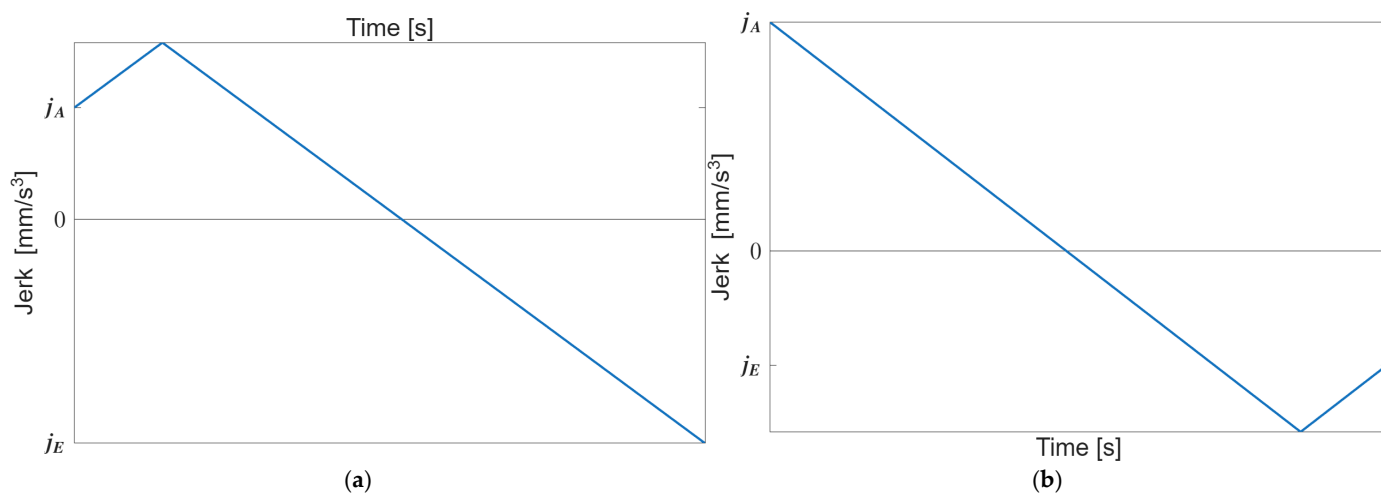


Figure 2. (a) Jerk function for a pm-connection. (b) Jerk function for an mp-connection.

This is the minimum difference between v_A and v_E for a direct connection to exist between (v_A, j_A) and (v, j_E) in the upper halfplane, and it is denoted by $\min \Delta v(j_A, j_E)$ in the sequel. If $\Delta v \geq \min \Delta v(j_A, j_E)$ holds, then (v_A, j_A) and (v, j_E) are called connectible, otherwise non-connectible. An analogous consideration can be made for $j_A \leq 0$ and $j_E \geq 0$ and $\Delta v \leq 0$, where the connection is going through the lower halfplane of the velocity–acceleration plane from right to left. There, we get the same formula for $\min \Delta v(j_A, j_E)$ but the result is negative, so the term “minimum” here relates to the absolute value.

Assume that (v_A, j_A) and (v_E, j_E) are connectible. When we fix v_A , j_A , and v_E and we change j_E such that $|j_E|$ is closer to j_A than before, then connectibility is retained (a proof is given in Appendix B.1). In the sequel, we not only need to compute the minimum

velocity difference from j_A and j_E but also to determine limits (wrt. Connectibility) for j_A or j_E when Δv and j_E or j_A are given. For example, if $j_A \geq 0$ and $\Delta v \geq \min \Delta v(j_A, j_E)$ are given, there are values j_{El} (l indicating “lower” referring to the absolute value!) with $|j_{El}| \leq j_A$ and j_{Eu} (u indicating “upper”) with $|j_{Eu}| \geq j_A$ such that $\Delta v = \min \Delta v(j_A, j_{El})$ and $\Delta v = \min \Delta v(j_A, j_{Eu})$. Then, (v_A, j_A) and $(v_A + \Delta v, j_E)$ are connectible if and only if $|j_{El}| \leq |j_E| \leq |j_{Eu}|$. These values can be determined by solving the equation $\Delta v = \min \Delta v(j_A, j_E)$ for j_E . A computer algebra system (CAS) like Maple® turns this into a problem of finding the zeroes of a polynomial (see Appendix A.1) for which standard procedures are available. One then has to pick those zeroes that fulfil the original equation and $j_{max} \geq |j_E|$. Analogously, one can compute j_{Al} and j_{Au} when Δv and j_E are given.

We now assume $j_A \geq 0$, $j_E \leq 0$, and $\Delta v \geq 0$. If $\Delta v > \min \Delta v(j_A, j_E)$, one needs at least three pieces in the jerk function and constant pieces must be inserted when j_{max} , a_{max} , and/or v_{max} are exceeded (see [2] Figures 5 and 11). We call them pmp-connections (plus–minus–plus for the snap potentially with intermediate zero snap pieces). One gets 8 function types, which are shown in Figure 3. Since some pieces might have a length of zero, we have less distinctions than in [2], which helps to retain manageability.

For $j_A = j_E = 0$, there are examples for the resulting curve in the velocity–acceleration plane presented in [2]; the time intervals, etc., can also be computed as in [2], either directly or by reducing the problem to finding the zeros of a polynomial. Which of the eight models is applicable depends on Δv . The critical values of Δv where a model change takes place can be determined in the same way as was done for the zero jerk case in [2]. The corresponding curves run in the upper halfplane of the velocity–acceleration plane. Note that the same types partially also work when $j_A < 0$ holds. If $(\overline{v_A}, -j_A)$ and (v_E, j_E) are connectible, then simply start the jerk function with the negative j_A and run linearly with positive maximum snap to $-j_A$ and then continue as in Figure 3. When $(\overline{v_A}, -j_A)$ and (v_E, j_E) are not connectible, then run as in Figure 4a, i.e., the minus part (negative snap) begins before $-j_A$ is reached. This is only possible when $|j_E| \leq |j_A|$, i.e., otherwise a pmp-connection does not exist. In this case, one has to use an mpm-connection (snap has the signs minus–plus–minus), which is shown in Figure 4b. By making the jerk smaller first, and then larger and again smaller, one can achieve $|j_E| > |j_A|$. Note that the mpm-connections are just the pmp-connections multiplied by -1 , so one gets the respective function graphs by reflecting those of Figure 3 at the time axis.

An mpm-connection is also necessary when $j_A \geq 0$, $j_E \leq 0$, and $\Delta v \geq 0$ holds but (v_A, j_A) and (v_E, j_E) are not connectible. Then, the curve in the velocity–acceleration plane runs through $(\overline{v_A}, -j_A)$ or $(\overline{v_E}, -j_E)$ depending on whether $|j_A|$ is smaller or larger than $|j_E|$, which is shown in Figure 5 (in the case of $|j_A| = |j_E|$, it runs through both points since then $(\overline{v_A}, -j_A)$ and $(\overline{v_E}, -j_E)$ must be connectible).

A similar consideration yields for $j_A \geq 0$, $j_E \geq 0$ that a pmp-connection is possible when (v_A, j_A) and $(\overline{v_E}, -j_E)$ are connectible or when this is not the case but $|j_A| \leq |j_E|$ holds. Otherwise, one has to use an mpm-connection. For the remaining case, $j_A \leq 0$, $j_E \geq 0$, one can always use a pmp-connection, which might run through $(\overline{v_A}, -j_A)$ (Figure 6a) or $(\overline{v_E}, -j_E)$ (Figure 6b) or both.

In order to provide an overview of the type of connection to be used, we first define the following quantities in Formulae (4) and (5):

$$\Delta v_{mod} = \begin{cases} v_E - v_A, & j_A \geq 0, j_E \leq 0 \\ v_E - \overline{v_A}, & j_A \leq 0, j_E \leq 0 \\ \overline{v_E} - v_A, & j_A \geq 0, j_E \geq 0 \\ \overline{v_E} - \overline{v_A}, & j_A \leq 0, j_E \geq 0 \end{cases} \quad (4)$$

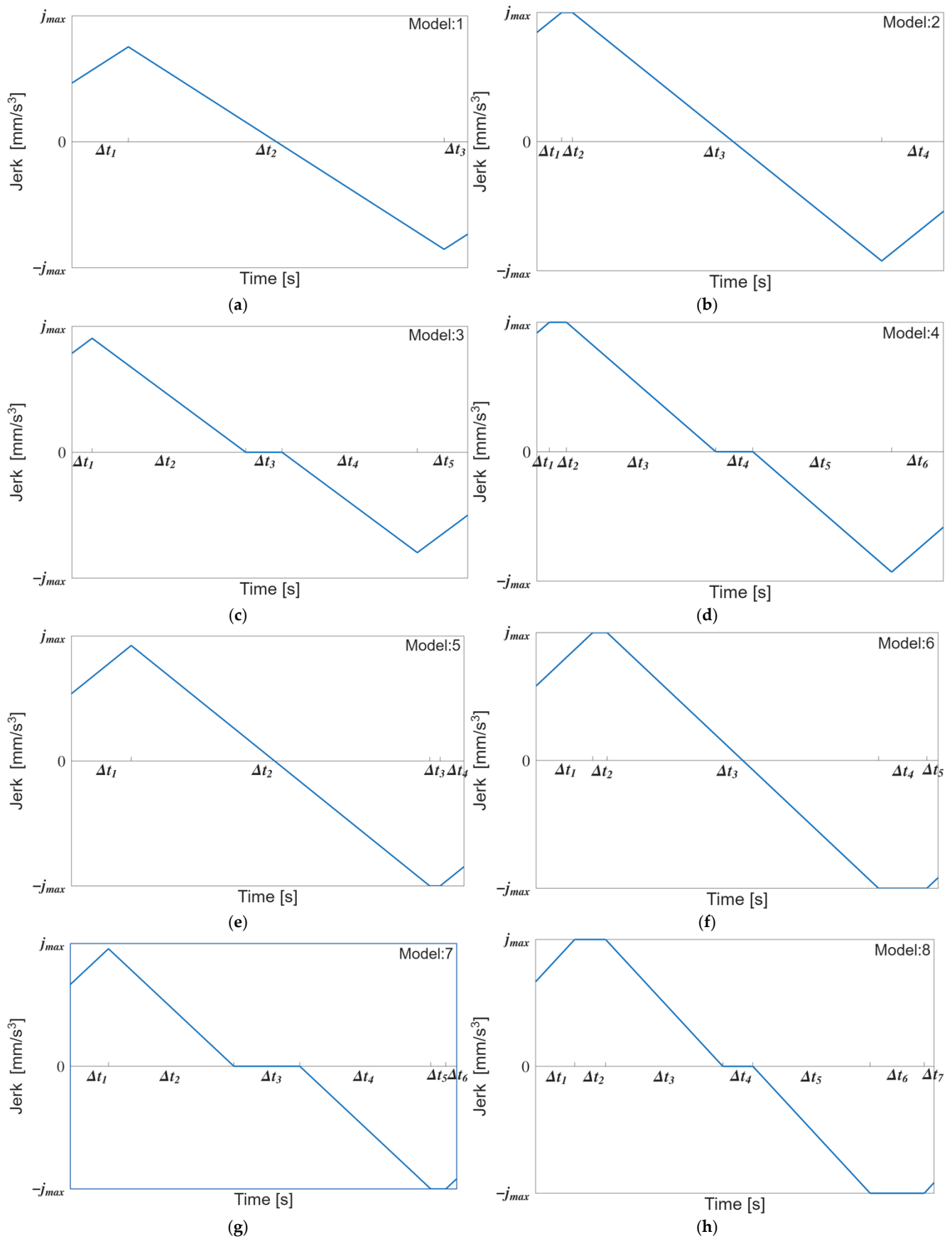


Figure 3. Jerk models for pmp-connection. (a) j_{min} , j_{max} , a_{max} not reached. (b) j_{max} reached. (c) a_{max} reached. (d) j_{max} , a_{max} reached. (e) j_{min} reached. (f) j_{max} , j_{min} reached. (g) j_{min} , a_{max} reached. (h) j_{min} , j_{max} , a_{max} reached.

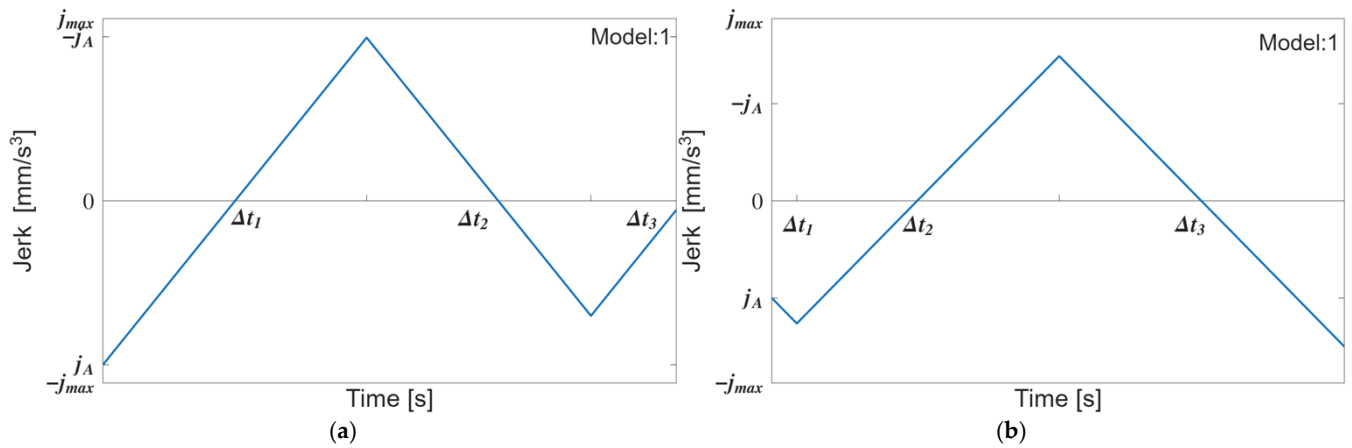


Figure 4. (a) pmp-connection for $|j_E| \leq |j_A|$. (b) mpm-connection for $|j_E| > |j_A|$.

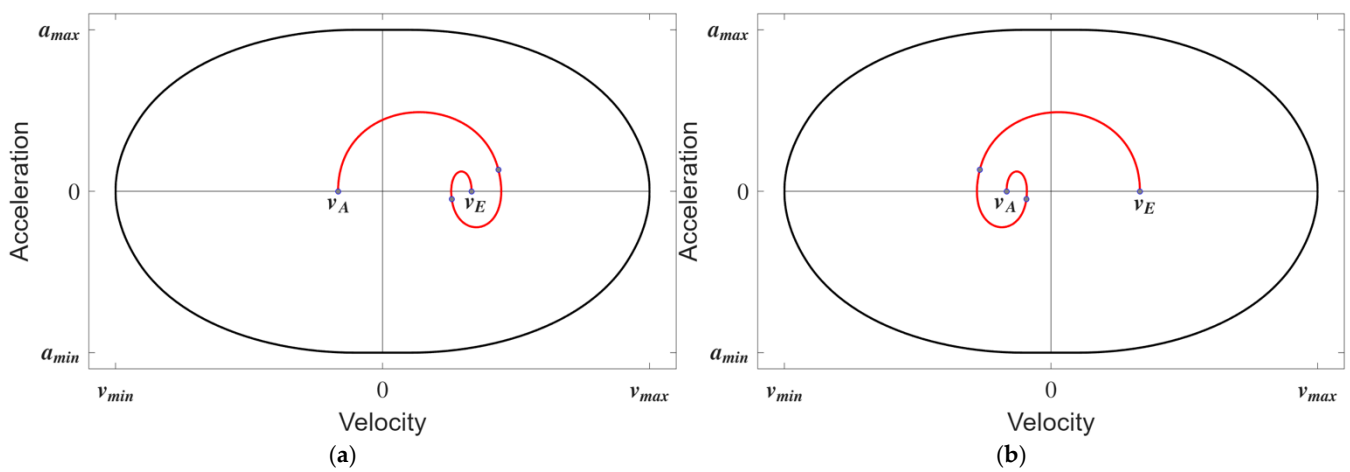


Figure 5. (v_A, j_A) and (v_E, j_E) are not connectable. (a) $|j_A| > |j_E|$. (b) $|j_A| < |j_E|$.

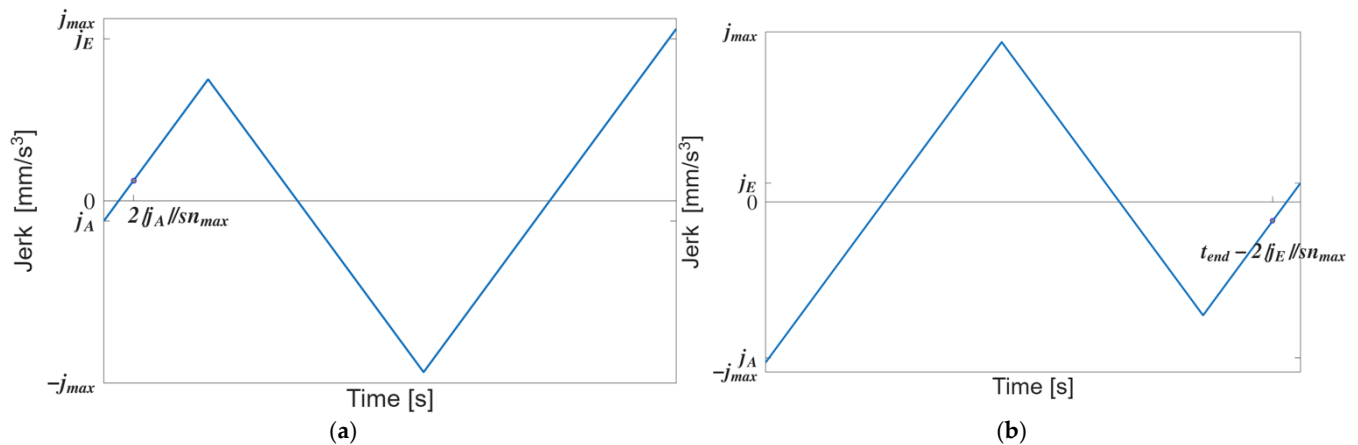


Figure 6. Jerk function for $j_A \leq 0$ and $j_E \geq 0$. (a) $|j_A| < |j_E|$. (b) $|j_A| > |j_E|$.

$$\min \Delta v_{mod} = \begin{cases} \min \Delta v(j_A, j_E), & j_A \geq 0, j_E \leq 0 \\ \min \Delta v(-j_A, j_E), & j_A \leq 0, j_E \leq 0 \\ \min \Delta v(j_A, -j_E), & j_A \geq 0, j_E \geq 0 \\ \min \Delta v(-j_A, -j_E), & j_A \leq 0, j_E \geq 0 \end{cases} \quad (5)$$

Note that for $\Delta v_{mod} \geq \min \Delta v_{mod}$ and $\Delta v \geq 0$, one can always use a pmp-connection since for $j_A \leq 0$ one can start with a p-piece going to $-j_A$ and for $j_E \geq 0$ one can finish with a p-piece going from $-j_E$ to j_E . Table 2 gives an overview of when to use pmp or mpm for

$\Delta v_{mod} < \min \Delta v_{mod}$, depending on the signs of j_A and j_E and the relationship between $|j_A|$ and $|j_E|$.

Table 2. Connection types for $\Delta v_{mod} < \min \Delta v_{mod}$ and $v_A \leq v_E$.

j_A	j_E	Relation j	Connection
≥ 0	≤ 0	n.a.	mpm
< 0	≤ 0	$ j_A \leq j_E $	mpm
< 0	≤ 0	$ j_A > j_E $	pmp
≥ 0	> 0	$ j_A \leq j_E $	pmp
≥ 0	> 0	$ j_A > j_E $	mpm
< 0	> 0	n.a.	pmp

An analogous deliberation can be made for $\Delta v < 0$ (i.e., $v_E < v_A$) when going from (v_A, j_A) to (v_E, j_E) through the lower halfplane. The negative quantities Δv_{mod} and $\min \Delta v_{mod}$ can be defined as in Formulae (4) and (5) by substituting $\geq$ or $>$ with $\leq$ or $<$ and vice versa. For $\Delta v_{mod} \leq \min \Delta v_{mod}$, one can always use an mpm-connection; for $\Delta v_{mod} > \min \Delta v_{mod}$, one has to exchange pmp and mpm in Table 2 and substitute $\geq$ or $>$ with $\leq$ or $<$ and vice versa in columns j_A and j_E . For later reference in Section 3, Table 3 provides the respective results.

Table 3. Connection types for $\Delta v_{mod} > \min \Delta v_{mod}$ and $v_A > v_E$.

j_A	j_E	Relation j	Connection
≤ 0	≥ 0	n.a.	pmp
> 0	≥ 0	$ j_A \leq j_E $	pmp
> 0	≥ 0	$ j_A > j_E $	mpm
≤ 0	< 0	$ j_A \leq j_E $	mpm
≤ 0	< 0	$ j_A > j_E $	pmp
> 0	< 0	n.a.	mpm

We again assume $j_A \geq 0$, $j_E \leq 0$, and $\Delta v \geq 0$ and that (v_A, j_A) and (v_E, j_E) are connectible, which implies that there is a pmp-connection in the upper halfplane (Figure 7a). This connection realizes one specific distance, Δs . When the distance to be travelled is larger, it was the strategy in [2] to go from (v_A, j_A) (through the upper halfplane) to an intermediate velocity and jerk (v_m, j_m) on the velocity axis (i.e., $a = 0$), and from there through the lower halfplane to (v_E, j_E) (cf. Figure 3 in [2]). Because there $j_E = 0$ holds, it was possible to cover all distances larger than Δs this way since j_m runs continuously from 0 to $-j_{max}$ and retains that value while v_m runs further to v_{max} . If $j_E < 0$, this is no longer feasible since when going to the right there is a discontinuity in the distance covered because of the additional loop to be made (cf. Figure 7b). A similar problem occurred in [23] where we considered motion functions that are only acceleration-continuous (see Figure 7 in [23]). There, the solution consisted of making a continuous transition from the direct connection from (v_A, a_A) to (v_E, a_E) to the connection adding a loop at the end by working with four pieces. Here, we do the same by adding another m-piece to the pmp models, i.e., by having so-called pmpm-connections. Figure 8 shows the corresponding jerk function for the first pmpm model (without zero-snap intervals).

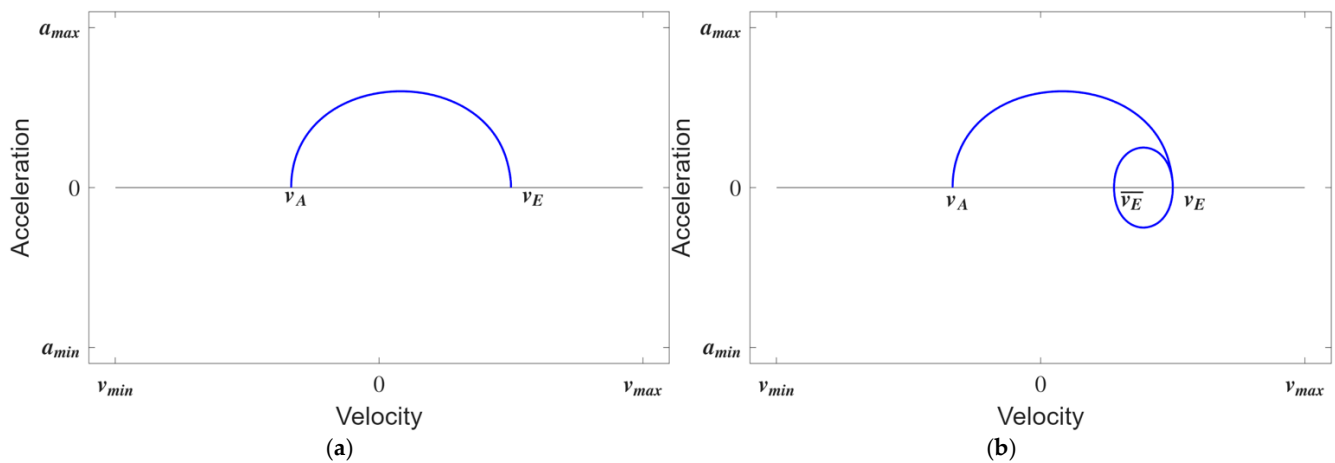


Figure 7. (a) Direct pmp-connection from (v_A, j_A) to (v_E, j_E) . (b) Adding a loop at (v_E, j_E) .

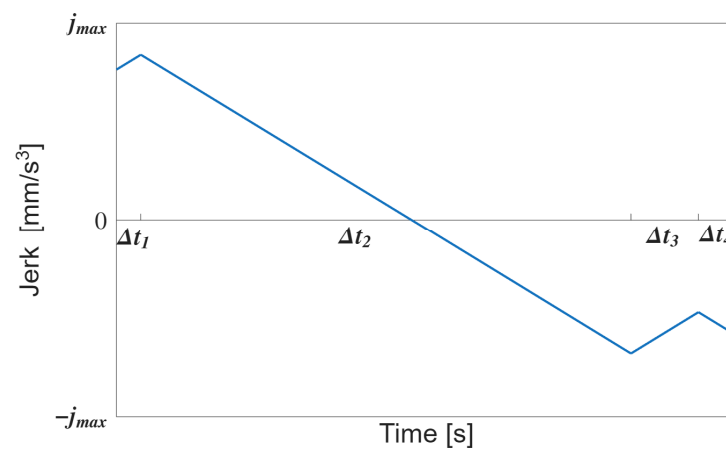


Figure 8. Jerk function for pmpm, model 1.

If one lets Δt_4 , i.e., the length of the fourth interval, run from 0 to $2 \frac{|j_E|}{sn_{max}}$, then the direct connection is transformed into the curve with the additional loop at the end shown in Figure 7b. Figure 9 depicts curves for different values of Δt_4 , thus showing the transition. In a similar way, one can get a continuous transition from the direct pmp-connection to an mpmp-connection with an additional loop at (v_A, j_A) . Here, at the starting point of the jerk function, a piece with negative snap (minus) is added.

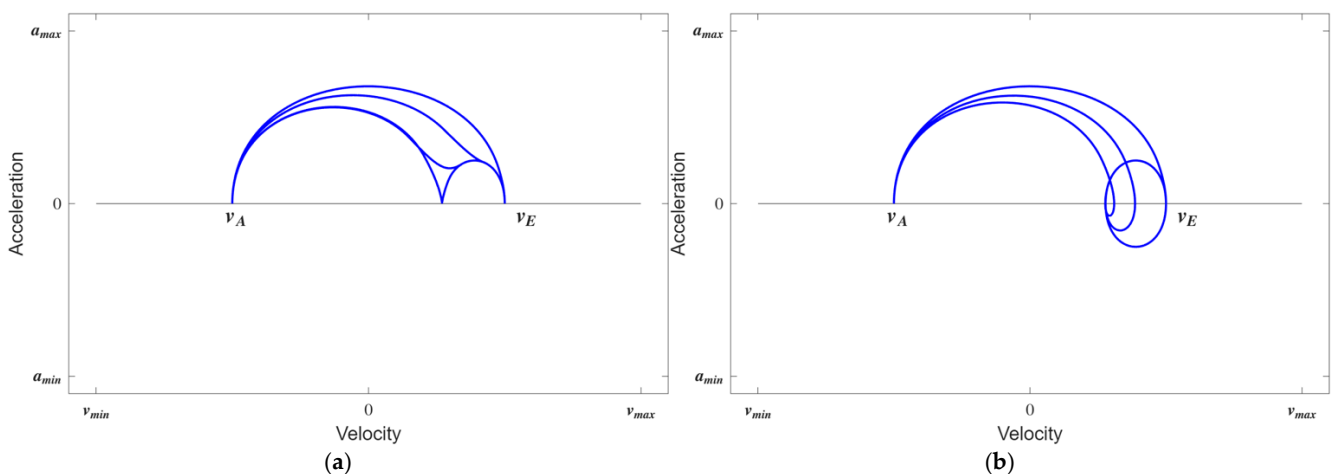


Figure 9. Curves for a growing value of Δt_4 . (a) From 0 until a peak at the velocity axis is reached. (b) Further growth until the loop at (v_E, j_E) is added.

Since there are 8 model variations of pmp, there are also 8 variations of mpmp and pmpm and a model switch might take place when Δt_1 (or Δt_4) runs from 0 to $2 \frac{|j_A|}{sn_{max}}$ (or $2 \frac{|j_E|}{sn_{max}}$). The Δt_1 (or Δt_4) values at which the changes occur can be found in a similar way as shown for the model change in pmp when Δv grows (see [2]). One ends up with symbolic solutions (formulae) or polynomial equations for which quick and reliable zero finders are available. Note that when, for example, Δt_1 goes from 0 to $2 \frac{|j_A|}{sn_{max}}$, the model change will switch direction. One might start with a model with zero-snap intervals and go down to the pure four-section mpmp-connection (without zero-snap intervals) and then go back to the starting model. Actually, the going up part is not simply the going down part in reverse order. In the example shown in Figure 10a, one has the model sequence 6->5->1->2->6 (the numbers correspond to the pmp model numbers in Figure 3). Moreover, the process can switch direction more than once. In the example depicted in Figure 10b, one has the sequence 8->6->8->7->8. This means that at each step one has to investigate whether the next Δt_1 value for going down or that for going up comes first and choose the respective model.

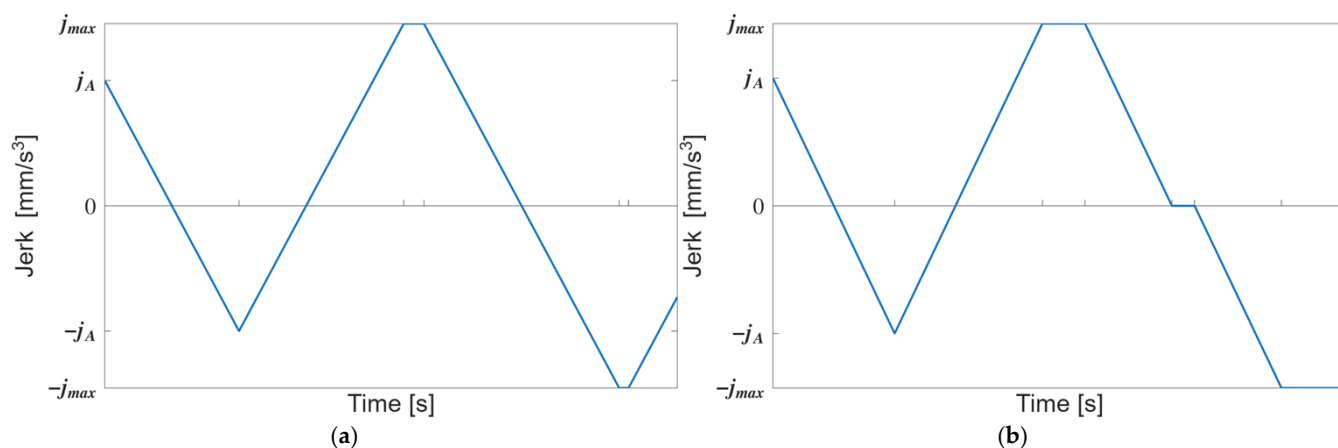


Figure 10. Starting models with model change for mpmp. (a) 6->5->1->2->6. (b) 8->6->8->7->8.

The approach using pmpm- or mpmp-connections described above does not always provide solutions for $\Delta t_4 \in [0, 2 \frac{|j_E|}{sn_{max}}]$ or for $\Delta t_1 \in [0, 2 \frac{|j_A|}{sn_{max}}]$ when Δv is too small. Connectivity is not sufficient for solutions to exist. Figure 11a shows that when increasing Δt_4 starting with 0, the curve begins to “buckle” and this needs sufficient distance between v_A and v_E . Later, when the curve has a double point (i.e., crosses itself), the necessary distance decreases again, as shown in Figure 11b. But in-between there might be an interval of Δt_4 (or Δt_1) values for which no solution exists (the equations provide negative values for the length of time intervals). Figure 11 provides an example where the critical Δt_4 values and the corresponding curves in the velocity–acceleration plane are given. To allow for the space needed by the additional m-section at the end or the beginning the p-section at the beginning or the end has to shrink. At the critical Δt_4 (or Δt_1) values, the Δt_1 (or Δt_4) values are 0 and then become negative (i.e., lead to infeasible solutions), so the critical curves are mpm-connections. In fact, one gets the same mpm-connection in both cases (pmpm and mpmp) since the mpm-connection between (v_A, j_A) and (v_E, j_E) is uniquely determined. The critical Δt_4 values are computed by setting Δt_1 to zero in the first pmpm-connection model and the corresponding critical Δt_1 values by setting Δt_4 to zero in the first mpmp-connection model. By using a CAS programme like Maple® 2025, this is again transformed into a problem of finding zeros of a polynomial. If there are two feasible solutions (i.e., lying in the respective interval and providing non-negative values for the time intervals), then the open interval between them consists of the Δt_4 or

Δt_1 values providing no feasible solution. When there is a zero with multiplicity 2 or there are none, for all values of Δt_4 or Δt_1 there are solutions. Using this, one can even get a polynomial equation for determining the critical velocity difference $\min \Delta v_{ct}$ (where the index ct stands for “continuous transition”) between v_A and v_E (with given values for j_A, j_E) from which no feasible zeros can be found, which means that the procedure described above provides a continuous transition to the connection with a loop. This is described in more detail in Appendix A.3. We provided an example here simply for illustration. Assume $j_A = 15, j_E = -7$, then the minimal velocity difference for connectivity can be computed with Formula (3) as $\min \Delta v = 22.61$. The minimal velocity difference for continuous transition (computed according to Appendix A.3) is 26.20. So, for values of $\Delta v = v_E - v_A$ with $22.61 \leq \Delta v < 26.20$, the points (v_A, j_A) and (v_E, j_E) are connectible but not with continuous transition. For the sake of brevity, we call (v_A, j_A) and (v_E, j_E) continuously connectible when they are connectible with continuous transition.

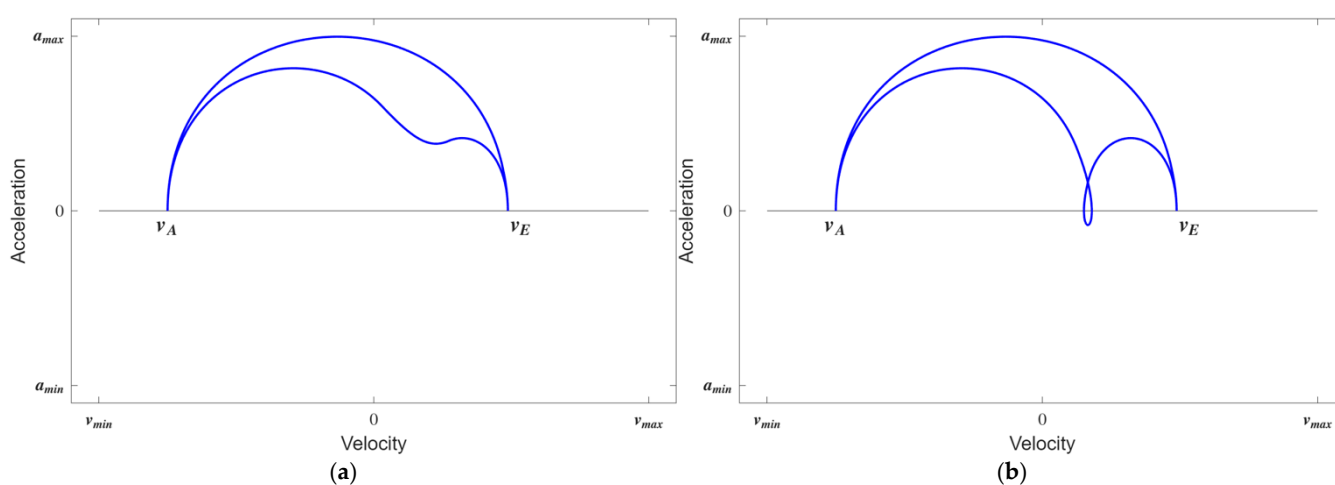


Figure 11. Boundary curves for solutions to exist. Data: $sn_{max} = 10$, $j_{max} = 15$, $a_{max} = 12$, $\Delta v = 24.7523$, $j_A = 14$, $j_E = -10$. (a) $\Delta t_4 = 1.1968$. (b) $\Delta t_4 = 1.7730$.

We now consider the cases where (v_A, j_A) and (v_E, j_E) have a connection consisting of only 1 or 2 pieces of non-zero snap, i.e., a pm-connection in the case of $|j_A| < |j_E|$, an mp-connection for $|j_A| > |j_E|$, and an m-connection for $|j_A| = |j_E|$ (which also can be considered as pmp or mpm with one or two intervals having the length 0). If $|j_A| < |j_E|$, then only for $\Delta t_1 = 0$ and $\Delta t_1 = 2 \frac{|j_A|}{sn_{max}}$ there are mpm-connections, which for $\Delta t_1 = 0$ is the original pm-connection and for $\Delta t_1 = 2 \frac{|j_A|}{sn_{max}}$ has a loop at (v_A, j_A) (Figure 12a). From there, one can get continuously to the function having a loop at the right side (i.e., at (v_E, j_E)) by working with pmpm (i.e., adding a p-piece at the front), where Δt_4 runs from the final Δt of the mpm-connection to $2 \frac{|j_E|}{sn_{max}}$ (see Figure 12b). Correspondingly, it is possible to go from the loop at the right side to the loop at the left side. An analogous argument applies in the case of $|j_A| > |j_E|$, and for $|j_A| = |j_E|$ one goes from (v_A, j_A) to (v_E, j_E) and then adds a loop going from (v_E, j_E) to itself. From there, one can go to the left or to the right.

Deliberations similar to those in the previous paragraphs for inserting loops and for getting from a loop at one end to a loop at the other end can be made when $j_A \leq 0$ and $j_E \leq 0$ or when $j_A \geq 0$ and $j_E \geq 0$. We just consider the first case because an analogous argumentation holds for the other one. There are two possibilities according to Table 2/Table 3. If $(\overline{v_A}, -j_A)$ and (v_E, j_E) are connectible, the connection between (v_A, j_A) and (v_E, j_E) is pmp, otherwise it is mpm. If it is pmp, one can construct a continuous transition to the version having an additional loop at (v_E, j_E) by adding an m-piece, i.e., working with pmpm. Note that it might be necessary to intersect the v-axis left of $(\overline{v_A}, -j_A)$,

which will happen if $(\overline{v_A}, -j_A)$ and (v_E, j_E) are not continuously connectible. One then has to make sure that the point of intersection will not be left of v_{min} in order to yield a feasible solution. Because within the pmpm-connection there is a 2-connection in the upper or in the lower halfplane, this is guaranteed if the following inequation holds:

$$v_{min} \leq \min\{v_A - \min\Delta v(0, j_A), v_E - \min\Delta v(0, j_E)\} \quad (6)$$

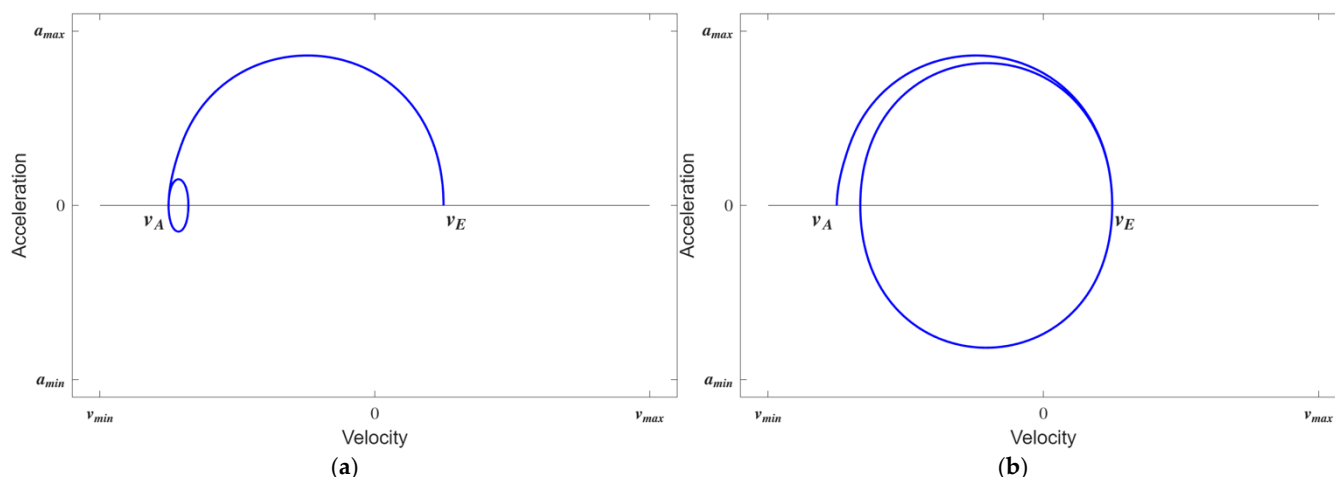


Figure 12. pm-connection between (v_A, j_A) and (v_E, j_E) . (a) With an additional loop at (v_A, j_A) (mpm). (b) With an additional loop at (v_E, j_E) (pmpm).

We will take that into account in our considerations below. Let the connection between (v_A, j_A) and (v_E, j_E) still be pmp. Then, it is not possible to add a loop at (v_A, j_A) by using mpmp in a continuous transition since for this the first piece (starting at (v_A, j_A)) must be a p-piece. Otherwise, one goes more and more to the left. But one can get a continuous transition from the curve with an additional loop at (v_E, j_E) to the curve with the loop at (v_A, j_A) by using the pmpm-connection and letting Δt_1 run from its initial value (where the loop was at (v_E, j_E)) to $2 \frac{|j_A|}{sn_{max}}$. Now assume that the connection between (v_A, j_A) and (v_E, j_E) is mpmp. Then, one has a continuous transition to the connection with an additional loop at (v_A, j_A) by working with a pmpm-connection, i.e., by adding a p-piece at the beginning. Again, one might overshoot v_{min} to the left, so this has to be taken into account by making sure that the inequality (6) holds. One cannot generate a loop at the end by using mpmp because for such a loop the final piece must be m. But again, one can get a continuous transition from the curve with an additional loop at (v_A, j_A) to the curve with the loop at (v_E, j_E) by using the pmpm-connection and letting Δt_4 run from its initial value (where the loop was at (v_A, j_A)) to $2 \frac{|j_E|}{sn_{max}}$.

In the sequel, it will sometimes be necessary to generate another duplicate loop from one loop, say at (v_A, j_A) . For $j_A \geq 0$, the loop at (v_A, j_A) is mp. From this one can add a loop by working with an mpmp-connection where Δt_2 (length of the time interval of the second piece, which is p) runs from 0 to $2 \frac{|j_A|}{sn_{max}}$. Figure 13 shows the intermediate curves. When doing this transition, the intersection with the velocity axis might overshoot v_{max} . We will take this into account below. An analogous consideration holds for $j_A < 0$ (loop to the left). An easy computation yields the following distance covered and time needed for a loop at (v_A, j_A) :

$$\Delta s = \text{sign}(j_A) \cdot \frac{4j_A(j_A^3 + 3sn_{max}^2 \cdot v_A)}{3sn_{max}^3} \quad (7)$$

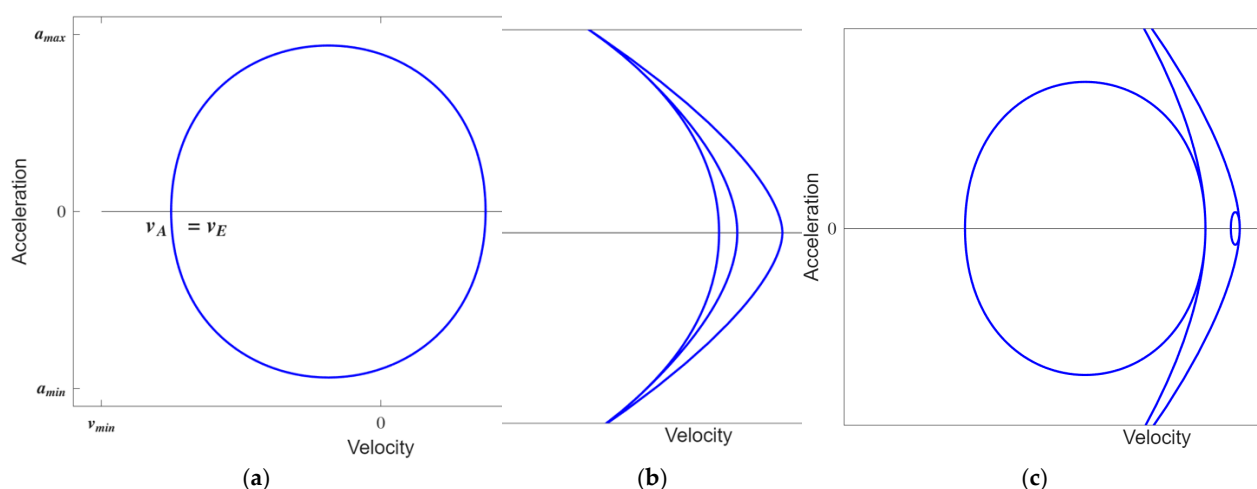


Figure 13. Duplication of a loop. (a) Original loop. (b) Zoom into right part for jerk at $a = 0$ approaching 0. (c) Zoom into right part for jerk at $a = 0$ going back to $-j_E$.

$$\Delta t = 4 \frac{|j_A|}{sn_{max}} \quad (8)$$

If we have a situation, as shown in Figure 7b, of realizing a certain given distance Δs , and we want to cover larger distances, we proceed similarly as in the zero jerk case (cf. [2]) via intermediate velocity–jerk pairs (v_m, j_m) right of (v_E, j_E) . Since the part from $(\bar{v}_E, -j_E)$ to (v_E, j_E) remains fixed, we omit this for the moment and only consider the part going from (v_A, j_A) via (v_m, j_m) to $(\bar{v}_E, -j_E)$. For notational simplicity, we call the new end-point again (v_E, j_E) , i.e., we now have $\Delta v \geq 0$ and $j_A, j_E \geq 0$ (for $\Delta v < 0$ and $j_A, j_E \leq 0$, the line of argumentation is completely analogous). Compared to the zero jerk case, it is not clear which jerk j_m should be attached to an intermediate velocity v_m . With $v_A \leq v_E$ and zero jerk, it was obvious that the curve section lying in the lower halfplane is the critical one and one lets $|j_m|$ grow such that (v_m, j_m) and (v_E, j_E) can be connected. The respective formula is stated in [2] (Equation (8): $vm2jm$ function; once $-j_{max}$ is reached, j_m is constantly $-j_{max}$). In the case of arbitrary jerk, the situation is more complicated. We consider the case $j_A, j_E \geq 0$ and $v_m \geq v_A, v_E$, i.e., going right (going left works analogously). Let $j_m \leq j_{m_start}$ be given (j_{m_start} might, e.g., be $-j_A$ or $-j_E$). We work reversely and compute the corresponding v_m . Compute $\min \Delta v(j_A, j_m)$ and $\min \Delta v(j_E, j_m) = -\min \Delta v(j_m, j_E)$. If $v_A + \min \Delta v(j_A, j_m) > v_E + \min \Delta v(j_E, j_m)$, then the upper section from (v_A, j_A) to (v_m, j_m) is the limiting part and we set $v_m = v_A + \min \Delta v(j_A, j_m)$. Otherwise, the lower part is the limiting one (in the case of equality, both parts are equally limiting) and we set $v_m = v_E + \min \Delta v(j_m, j_E)$. The roles might change when j_m runs from j_{m_start} down to $-j_{max}$. We consider the following cases:

Let $v_A < v_E$ and $j_A < j_E$. Assume that (v_A, j_A) and $(\bar{v}_E, -j_E)$ are not connectible, i.e., $v_A + \min \Delta v(j_A, -j_E) > v_E + \min \Delta v(j_E, -j_E)$, where $\min \Delta v(j_E, -j_E) = \frac{2}{3} \cdot \frac{j_E^3}{sn_{max}^2}$ according to Formula (2), i.e., the upper part is the limiting section. This inequality is equivalent to $\min \Delta v(j_A, -j_E) - \min \Delta v(j_E, -j_E) > v_E - v_A$. In Appendix B.2, we prove that for any $j_m \leq -j_E$, the following inequality holds:

$$\min \Delta v(j_A, j_m) - \min \Delta v(j_E, j_m) \geq \min \Delta v(j_A, -j_E) - \min \Delta v(j_E, -j_E) \quad (9)$$

Therefore, for any $j_m \leq -j_E$, we have $\min \Delta v(j_A, j_m) - \min \Delta v(j_E, j_m) > v_E - v_A$, and hence the upper part is always the limiting section, which serves to compute v_m for a given j_m or vice versa.

Now assume that $v_A + \min\Delta v(j_A, -j_E) \leq v_E + \min\Delta v(j_E, -j_E)$, i.e., (v_A, j_A) and $(\bar{v}_E, -j_E)$ are connectible. The limiting part for $j_m = -j_E$ is the lower part and we have $\min\Delta v(j_A, -j_E) - \min\Delta v(j_E, -j_E) \leq v_E - v_A$. We will show in Appendix B.3 that when $|j_m|$ becomes larger, the expression $\min\Delta v(j_A, j_m) - \min\Delta v(j_E, j_m)$ also grows and it is not bounded, i.e., can become arbitrarily large. Therefore, for some j_{m_limit} we have

$$\min\Delta v(j_A, j_{m_limit}) - \min\Delta v(j_E, j_{m_limit}) = v_E - v_A \quad (10)$$

and from then on, the upper part is the limiting part. In order to find j_{m_limit} , one has to solve Equation (10) for j_{m_limit} which, using a CAS, leads to a polynomial equation of degree 8. Note that if $j_{m_limit} < -j_{max}$, then there will be no change from the lower to the upper part. The following configuration provides an example for a change from the lower to the upper part: $v_A = 17$; $v_E = 40$; $j_{max} = 25$; $sn_{max} = 10$; $j_A = 7$; $j_E = 15$. Figure 14 shows that starting from $j_m = -15$ and going left, first $v_A + \min\Delta v(j_A, j_m)$ (blue curve) lies below $v_E + \min\Delta v(j_E, j_m)$ (red curve); starting from left of about $j_m = -18.5$ and going left, the result is the other way round. The figure might suggest that the differences are only small, but using the wrong half for determining v_m leads to negative values for time intervals during the computation, so the distinction is crucial.

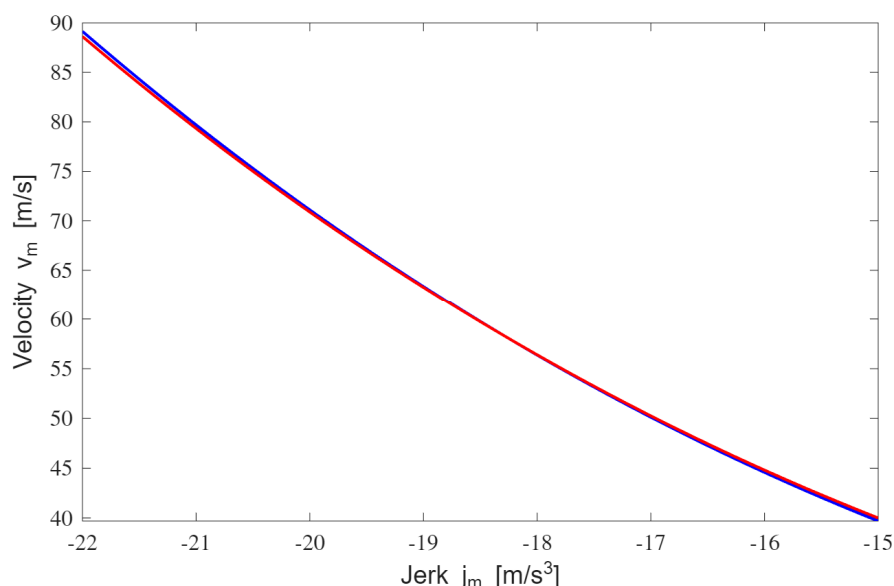


Figure 14. Computation of v_m for $j_m = -15 \dots -22$ (going right to left!) using the lower part ending at v_E (red curve) first and then the upper part starting at v_A (blue curve).

Now, let $v_A < v_E$ and $j_A \geq j_E$. Then, $|j_m| \leq j_A$, i.e., $|j_m|$ is closer to j_A than to j_E (or equally close for $j_A = j_E$), and hence $\min\Delta v(j_A, j_m) \leq \min\Delta v(j_E, j_m)$. Because of $v_A < v_E$, we have $v_A + \min\Delta v(j_A, j_m) < v_E + \min\Delta v(j_E, j_m)$ such that the lower part is always the limiting one. For $v_A \geq v_E$, analogous considerations apply.

All cases are summarized in Table 4:

If one starts with j_{m_start} and goes down to $-j_{max}$, then $j_{m2vm}(j_m) = v_A + \min\Delta v(j_A, j_m)$ if, according to Table 3, the upper half is limiting; the result is $j_{m2vm}(j_m) = v_E + \min\Delta v(j_E, j_m)$ if the curve in the lower halfplane is limiting. Vice versa, the formulae can also be used to compute j_m from v_m , where a CAS turns the equation into a polynomial one such that the computation takes more time.

Table 4. Change in limiting part.

Relation v	Relation j	Connectivity	Change	First Limiting Part
$v_A < v_E$	$j_A \geq j_E$	n.a.	No	Lower
$v_A < v_E$	$j_A < j_E$	$(v_A, j_A), (\overline{v_E}, -j_E)$ connectible	Yes	Lower
$v_A < v_E$	$j_A < j_E$	$(v_A, j_A), (\overline{v_E}, -j_E)$ not connectible	No	Upper
$v_A \geq v_E$	$j_A \geq j_E$	$(\overline{v_A}, -j_A), (v_E, j_E)$ connectible	Yes	Upper
$v_A \geq v_E$	$j_A \geq j_E$	$(\overline{v_A}, -j_A), (v_E, j_E)$ not connectible	No	Lower
$v_A \geq v_E$	$j_A < j_E$	n.a.	No	Upper

Finally, we make some useful restrictions regarding the values for v_{max} , a_{max} , j_{max} , sn_{max} . As already stated in the previous paper, we assume that $j_{max} \leq \sqrt{2a_{max}sn_{max}}$, since otherwise j_{max} cannot be reached starting at the velocity axis (i.e., $a = 0$) without exceeding a_{max} . Moreover, we assume that starting from rest ($v = 0$, $a = 0$, $j = 0$), it is possible to reach $-j_{max}$ at v_{max} , i.e.,

$$\min \Delta v(0, -j_{max}) \leq v_{max} \quad (11)$$

This enables us to have only the absolutely necessary number of extra “rounds” as we will see in Section 3 when reporting the results. One can reduce j_{max} in order to guarantee the condition.

3. Results

In this section, we describe how one can traverse the velocity–acceleration plane in a continuous way such that all possible distances are covered. For this, we make case distinctions according to the signs of j_A and j_E :

- Case I: $j_A \geq 0$ and $j_E \leq 0$
- Case II: $j_A < 0$ and $j_E \leq 0$
- Case III: $j_A \geq 0$ and $j_E > 0$
- Case IV: $j_A < 0$ and $j_E > 0$

In each case, we proceed similarly: We distinguish further according to the connectibility property of (v_A, j_A) and (v_E, j_E) (Case I), $(\overline{v_A}, -j_A)$ and (v_E, j_E) (Case II), (v_A, j_A) and $(\overline{v_E}, -j_E)$ (Case III), and $(\overline{v_A}, -j_A)$ and $(\overline{v_E}, -j_E)$ (Case IV). In each sub-case, we have a special connection between (v_A, j_A) and (v_E, j_E) as a starting configuration, which covers a special distance Δs . From this, we proceed to the left or to the right in order to cover all real distances. In this process, we identify intermediate configurations where a change in the model type takes place (e.g., from pmp to pmpm). We specify with which type of model one can get from one intermediate configuration to the next one and which running parameter is used to do this in a continuous way. Once $(v_{max}, 0)$ (or $(v_{min}, 0)$) is reached, when going to the right (or left) one can stop since one can include pieces with constant velocity v_{max} (or v_{min}) for covering any further distance. The algorithm for finding a configuration to realize a given distance then consists of computing the distances for the intermediate configurations, recognizing between which intermediate ones the given distance is covered, and then applying the bisection method in order to find the parameter value that provides the configuration to realize the given distance. This is similar to [2], but we work coarser in that we do not investigate model changes within a certain model type (like the eight pmp models).

3.1. Case I: $j_A \geq 0$ and $j_E \leq 0$

3.1.1. Continuous Connectivity of (v_A, j_A) and (v_E, j_E)

Figure 15 shows the configurations where a model type change occurs. The starting configuration 1 (Figure 15a) is of type pmp since we assume connectivity. Since going from there to the left is completely analogous to going to the right, we only consider the latter sub-case. After the starting configuration, one uses the model type pmpm to get to configuration 2 according to Section 2. From there, we let the jerk j_m of the intermediate velocity v_m run from j_E to $j_{min} = -j_{max}$ (v_m runs from v_E to v_{mjmin}). The first part from (v_A, j_A) to (v_m, j_m) has type pmp, the second part from (v_m, j_m) to (v_E, j_E) has type mpm. If j_m reaches $-j_{max}$ before v_m is equal to v_{max} , we have configuration 3.1; otherwise, the next intermediate configuration is 3.2. If one has configuration 3.1, then going further right to $(v_{max}, -j_{max})$ also leads to configuration 3.2 with $j_{vmax} = -j_{max}$. At v_{max} , we let the jerk run from j_{vmax} to 0 leading to configuration 4. Note that the connection between (v_A, j_A) and (v_E, j_E) in the upper halfplane might also reach a_{max} , which is possible with pmp-connections (see Figure 3).

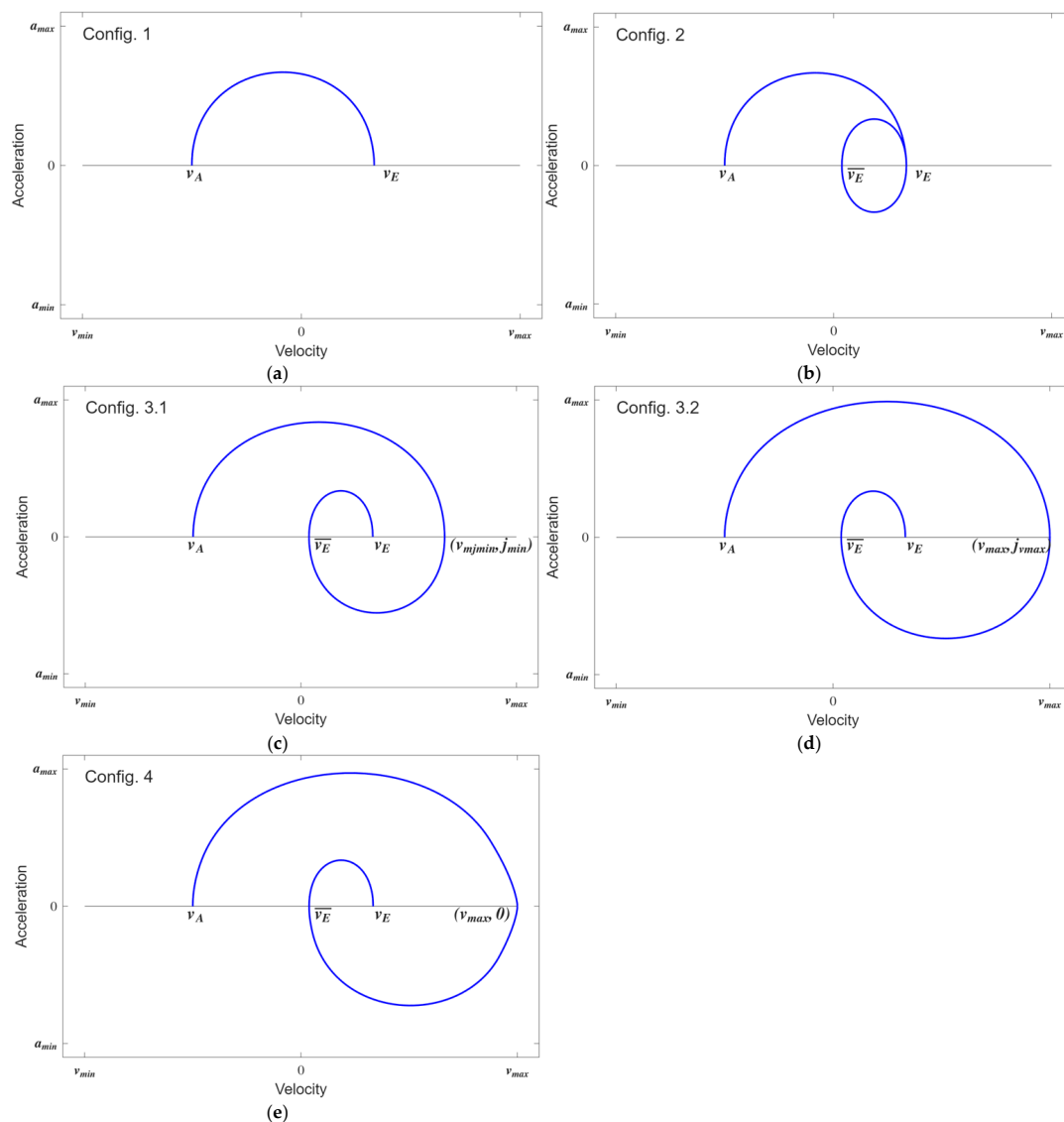


Figure 15. Switching configurations: Case I, continuous connectivity. (a) Starting configuration. (b) Adding a round at v_E . (c) Going down to j_{min} . (d) Going further to v_{max} . (e) Going down to $j = 0$ at v_{max} .

Figure 16 provides an overview of possible sequences of intermediate configurations.

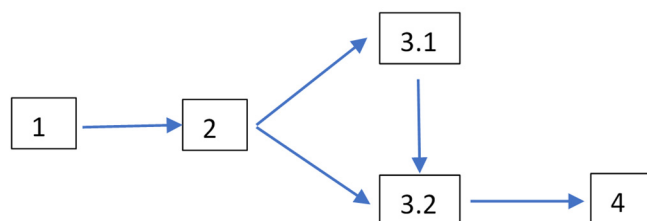


Figure 16. Sequence of intermediate configurations: Case I, continuous connectivity.

Table 5 lists the transitions, the model type used in a transition, and the running parameters as well as their ranges. Note that in the transition 1->2, the parameter Δt_{end} equals Δt_4 for the pmpm model with four segments, i.e., without the interval of constant jerk (Figure 8). Otherwise, Δt_{end} is the length of the final segment. In the transitions 2->3.1 and 2->3.2, one might use j_m or v_m as the parameter. The model type for the lower part in the transition 3.2->4 can switch from mpm to pmp when j_m runs from j_{vmax} to 0 according to Table 3. In this case, the curve in Figure 15e does not intersect the velocity axis at $\bar{v}_E$ but at a smaller velocity value (left of $\bar{v}_E$).

Table 5. Model types used for transition and parameter ranges: Case I, continuous connectivity.

Transition	Model Type	Parameter and Range
1->2	pmpm	$\Delta t_{end} = 0..2 \frac{ j_E }{sn_{max}}$
2->3.1	Upper part: pmp Lower part: mpm	$j_m = j_E..j_{max}$ or $v_m = v_E..v_{mjmin}$
3.1->3.2	Upper part: pmp Lower part: mpm	$v_m = v_{mjmin}..v_{max}$
2->3.2	Upper part: pmp Lower part: mpm	$j_m = j_E..j_{vmax}$ or $v_m = v_E..v_{max}$
3.2->4	Upper part: pmp Lower part: mpm or pmp (cf. Table 2/Table 3)	$j_m = j_{vmax}..0$

3.1.2. Non-Continuous Connectivity of (v_A, j_A) and (v_E, j_E)

We now assume that (v_A, j_A) and (v_E, j_E) are connectible but not continuously connectible. Figure 17 shows the configurations where a model type change occurs. The starting configuration 1 (Figure 17a) is of type pmp since we assume connectibility. Since going from there to the left is completely analogous to going to the right, we only consider the latter sub-case. After the starting configuration, one uses the model type pmpm to get to configuration 2 according to Section 2, where Δt_4 runs from 0 to the first critical value of Δt_4 , which we denote by Δt_{4c1} . The solutions between configurations 1 and 2 form an isolated solution area realizing a certain range of distances. The next feasible configuration, 3 (see Figure 17c), starts with the second critical value of Δt_4 , which is denoted by Δt_{4c2} . This is the starting point of the connected solution area realizing all real distances. From here, we go to the right by letting Δt_4 run from Δt_{4c2} to $2 \frac{|j_E|}{sn_{max}}$, reaching configuration 4. Proceeding to configurations 5.1, 5.2, and 6.1 is completely analogous to going to 3.1, 3.2, and 4 in the previous section, Section 3.1.1. But now, it might be the case that (v_A, j_A) and $(v_{max}, 0)$ are not connectible when connectibility is only possible going down to some j_{lim} , i.e., the next configuration after 5.2 is 6.2. Then, one needs an additional round at (v_{max}, j_{lim}) , as shown in Figure 17i in configuration 7. One can get to this configuration as described in Section 2 by going from (v_{max}, j_{lim}) to (v_E, j_E) using pmpm and letting Δt_1 run from 0 to $2 \frac{|j_{lim}|}{sn_{max}}$. The upper part of configuration 7 is the same as in configuration 6.2, and

also does not change in configuration 8. To reach the latter, one lets the jerk j_m at the second traversal of v_{max} run from j_{lim} to 0. The connection from (v_{max}, j_{lim}) to (v_{max}, j_m) is pmp and the connection from (v_{max}, j_m) to (v_E, j_E) is mpm or pmp, according to Table 2/Table 3.

Figure 18 provides an overview of possible sequences of intermediate configurations. Table 6 lists the transitions, the model type used in a transition, and the running parameters as well as their ranges.

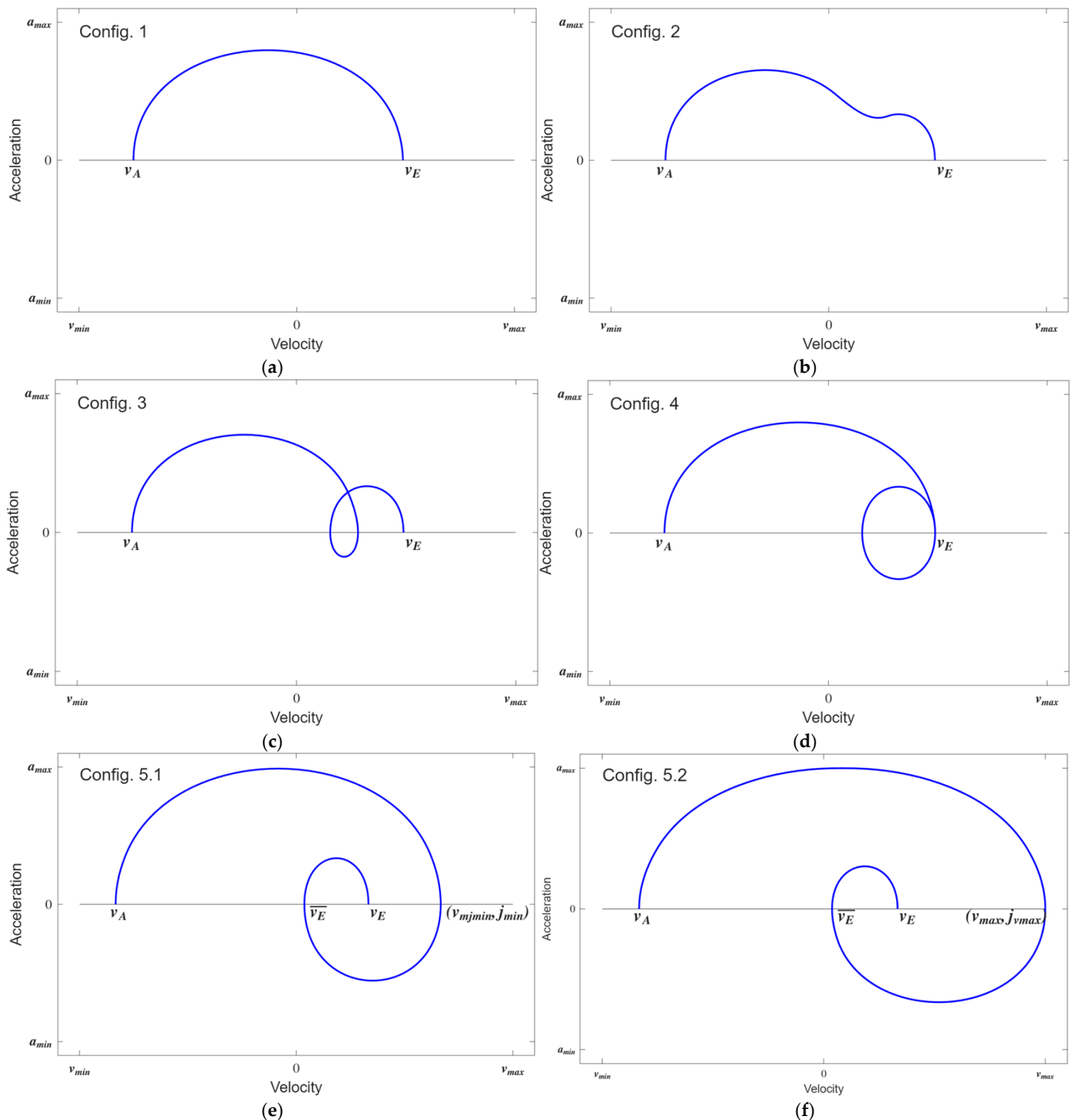


Figure 17. Cont.

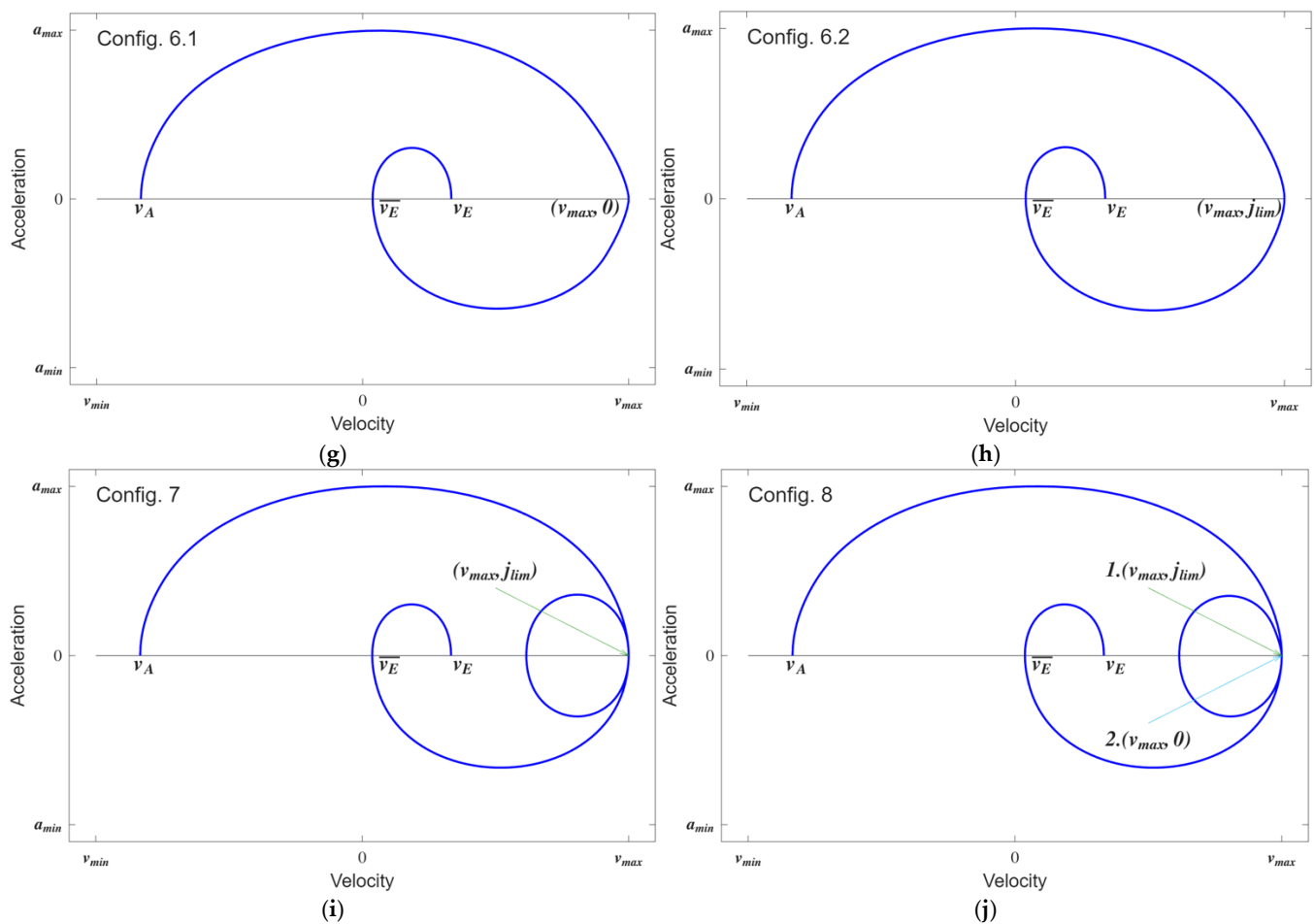


Figure 17. Switching configurations: Case I, connectible. (a) Starting configuration. (b) Going with Δt_4 up to the first critical value. (c) Starting again at the second critical value of Δt_4 . (d) Adding a round at v_E . (e) Going down to j_{min} . (f) Going further to v_{max} . (g) Going down to $j = 0$ at v_{max} . (h) Going down to j_{lim} at v_{max} . (i) Adding a round at v_{max} . (j) Going down to $j = 0$ at v_{max} .

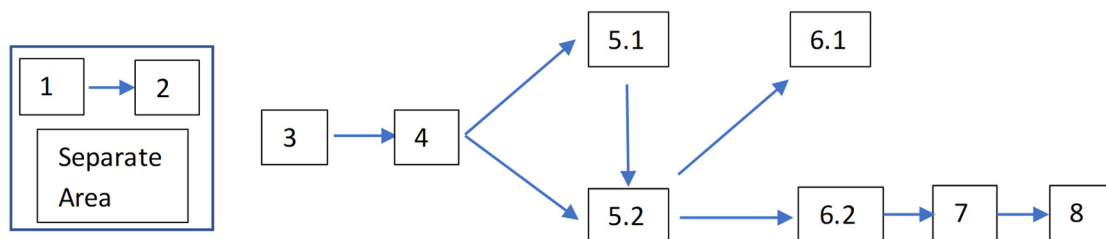


Figure 18. Sequence of intermediate configurations: Case I, connectible.

Table 6. Model types used for transition and parameter ranges: Case I, connectible.

Transition	Model Type	Parameter and Range
1->2	pmpm or mpmp	$\Delta t_4 = 0.. \Delta t_{4c1}$
3->4	pmpm	$\Delta t_4 = \Delta t_{4c2} \cdot 2 \frac{ j_E }{sn_{max}}$
4->5.1	Upper part: pmp Lower part: mpmp	$j_m = j_E.. -j_{max}$ or $v_m = v_E.. v_{mjmin}$
5.1->5.2	Upper part: pmp Lower part: mpmp	$v_m = v_{mjmin}.. v_{max}$

Table 6. Cont.

Transition	Model Type	Parameter and Range
4->5.2	Upper part: pmp Lower part: mpm	$j_m = j_E \cdot j_{vmax}$ or $v_m = v_E \cdot v_{max}$
5.2->6.1	Upper part: pmp Lower part: mpm or pmp (cf. Table 2/Table 3)	$j_m = j_{vmax} \cdot 0$
5.2->6.2	Upper part: pmp Lower part: mpm or pmp (cf. Table 2/Table 3)	$j_m = j_{vmax} \cdot j_{lim}$
6.2->7	Upper part: pmp (as in 6.2); from (v_{max}, j_m) to (v_E, j_E) : pmpm	$\Delta t_1 = 0.2 \frac{ j_{lim} }{sn_{max}}$
7->8	Upper part: pmp (as in 6.2); from (v_{max}, j_{lim}) to (v_{max}, j_m) : pmp from (v_{max}, j_m) to (v_E, j_E) : mpm or pmp (cf. Table 2/Table 3)	$j_m = j_{lim} \cdot 0$

3.1.3. No Connectivity of (v_A, j_A) and (v_E, j_E)

We now assume that (v_A, j_A) and (v_E, j_E) are not connectible and $|j_A| \geq |j_E|$, since for $|j_A| < |j_E|$ we can proceed analogously by switching going left and going right. According to Table 2, there exists an mpm-connection running partially through the lower halfplane, as shown in Figure 19a. It intersects the velocity axis at v_{mr} (with jerk j_{mr}) and at $\overline{v_E}$. We first assume $v_{mr} \leq v_{max}$, i.e., the connection is feasible and the configuration 1.1r is our starting one. From there, we proceed to the right as in Section 3.1.1. We let the jerk j_m of the intermediate velocity v_m run from j_{mr} to $j_{min} = -j_{max}$ (v_m runs from v_{mr} to v_{mjmin}). The first part from (v_A, j_A) to (v_m, j_m) has type pmp, the second part from (v_m, j_m) to (v_E, j_E) has type mpm. If j_m reaches $-j_{max}$ before v_m is equal to v_{max} , we have configuration 2.1r; otherwise, the next intermediate configuration is 2.2r (Figure 19b,c). If one has configuration 2.1r, then going further right to $(v_{max}, -j_{max})$ also leads to configuration 2.2r with $j_{vmax} = -j_{max}$. At v_{max} , we let the jerk run from j_{vmax} to 0 leading to configuration 3.1r if this is possible (i.e., (v_A, j_A) and $(v_{max}, 0)$ are connectible). Otherwise, we stop at jerk j_{lim} with the minimal possible absolute value and get to configuration 3.2r (Figure 19e). We leave the first part from (v_A, j_A) to (v_{max}, j_{lim}) as is and consider the mpm-connection from (v_{max}, j_{lim}) to (v_E, j_E) . We get to the next configuration 4.1r (Figure 19f) by adding a p-piece at the beginning of the mpm-connection, letting Δt_1 run from 0 to $2 \frac{|j_{lim}|}{sn_{max}}$. The pmpm-connection might intersect the velocity axis left of $\overline{v_E}$ but we do not overshoot at v_{min} because of inequality (11). Then, we let the jerk at the second intersection of the velocity axis at v_{max} decrease to 0, i.e., we have a part connecting (v_{max}, j_{lim}) to $(v_{max}, 0)$ and another one from $(v_{max}, 0)$ to (v_E, j_E) , as depicted in Figure 19g. By including a phase of constant velocity v_{max} at $(v_{max}, 0)$, one can cover arbitrarily large distances.

Figure 20 provides an overview of possible sequences of intermediate configurations. Table 7 lists the transitions, the model type used in a transition, and the running parameters as well as their ranges.

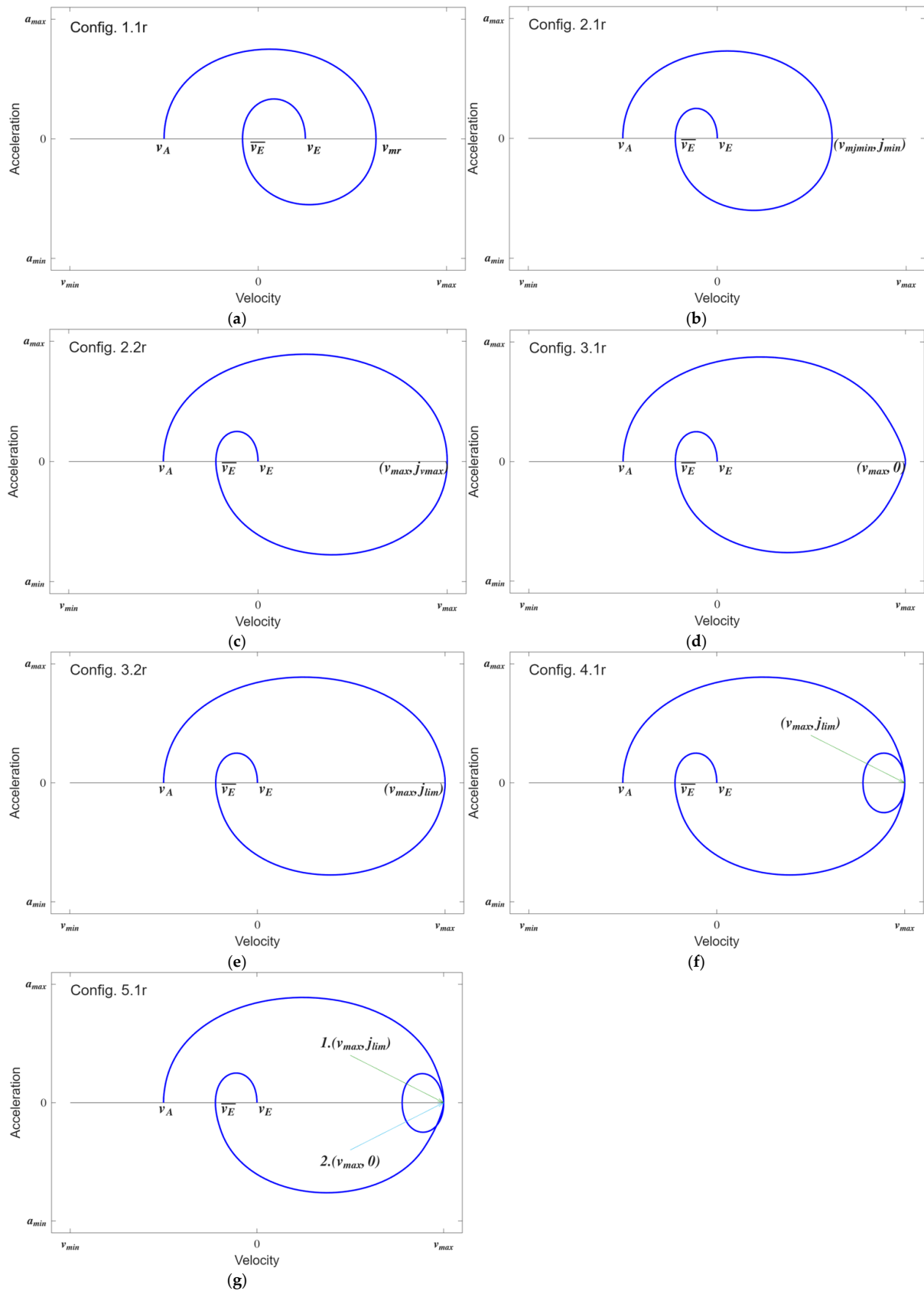


Figure 19. Switching configurations: Case I, non-connectible, going right, part I. (a) Starting configuration. (b) Going down to j_{min} . (c) Going further to v_{max} . (d) Going down to $j = 0$ at v_{max} . (e) Going down to j_{lim} at v_{max} . (f) Adding a round at v_{max} . (g) Going down to $j = 0$ at v_{max} .

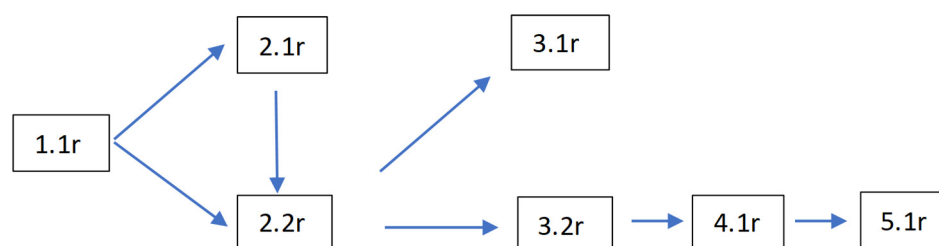


Figure 20. Sequence of intermediate configurations: Case I, non-connectible, going right, part I.

Table 7. Model types used for transition and parameter ranges: Case I, non-connectible, going right, part I.

Transition	Model Type	Parameter and Range
1r->2.1r	Upper part: pmp Lower part: mpm	$j_m = j_{mr} - j_{max}$ or $v_m = v_{mr} \cdot v_{mjmin}$
1r->2.2r	Upper part: pmp Lower part: mpm	$j_m = j_{mr} \cdot j_{vmax}$ or $v_m = v_{mr} \cdot v_{max}$
2.1r->2.2r	Upper part: pmp Lower part: mpm	$v_m = v_{mjmin} \cdot v_{max}$
2.2r->3.1r	Upper part: pmp Lower part: mpm or pmp (cf. Table 2/Table 3)	$j_m = j_{vmax} \cdot 0$
2.2r->3.2r	Upper part: pmp Lower part: mpm or pmp (cf. Table 2/Table 3)	$j_m = j_{vmax} \cdot j_{lim}$
3.2r->4.1r	Upper part: pmp (as in 3.2r); from (v_{max}, j_{lim}) to (v_E, j_E) : pmpm	$\Delta t_1 = 0.2 \frac{ j_{lim} }{a_{max}}$
4.1r->5.1r	Upper part: pmp (as in 3.2r); from (v_{max}, j_{lim}) to (v_{max}, j_m) : pmp from (v_{max}, j_m) to (v_E, j_E) : mpm or pmp (cf. Table 2/Table 3)	$j_m = j_{lim} \cdot 0$

We now assume that in Figure 19a the inequality $v_{mr} > v_{max}$ holds, i.e., the connection is not feasible. In this case, we first use an mp-connection to go from (v_A, j_A) to (v_{max}, j_{vmax}) , and then with a pmp-connection (according to Table 3) we go further on to (v_E, j_E) , such that we have an mpmp-connection for the whole motion depicted in Figure 21a. This intersects the velocity axis left of $\bar{v}_E$. One might interpret the situation also by stating that adding a p-piece at the end allows to go further left and avoid the overshoot at v_{max} . From this starting configuration, 1r, we again proceed by going to the right covering arbitrarily large distances in a continuous way. We get to the next configuration, 2r, by leaving the first part from (v_A, j_A) to (v_{max}, j_{vmax}) unchanged and using a pmpm-connection from (v_{max}, j_{vmax}) to (v_E, j_E) , thus adding an extra round at (v_E, j_E) , as shown in Figure 21b. When doing this transition, we do not overshoot at v_{min} because of inequality (11), as explained above. From configuration 2r, we get to configuration 3r (Figure 21c) by increasing Δt_{end} in the pmpm-connection up to some Δt_{lim} until the second intersection with the velocity axis reaches v_{max} having a jerk value of j_{lim} . Then, we let the jerk at this point decrease to 0, i.e., we have a part connecting (v_{max}, j_{vmax}) to $(v_{max}, 0)$ and another one from $(v_{max}, 0)$ to (v_E, j_E) , as depicted in Figure 21d. This again enables us to include a phase of constant velocity v_{max} .

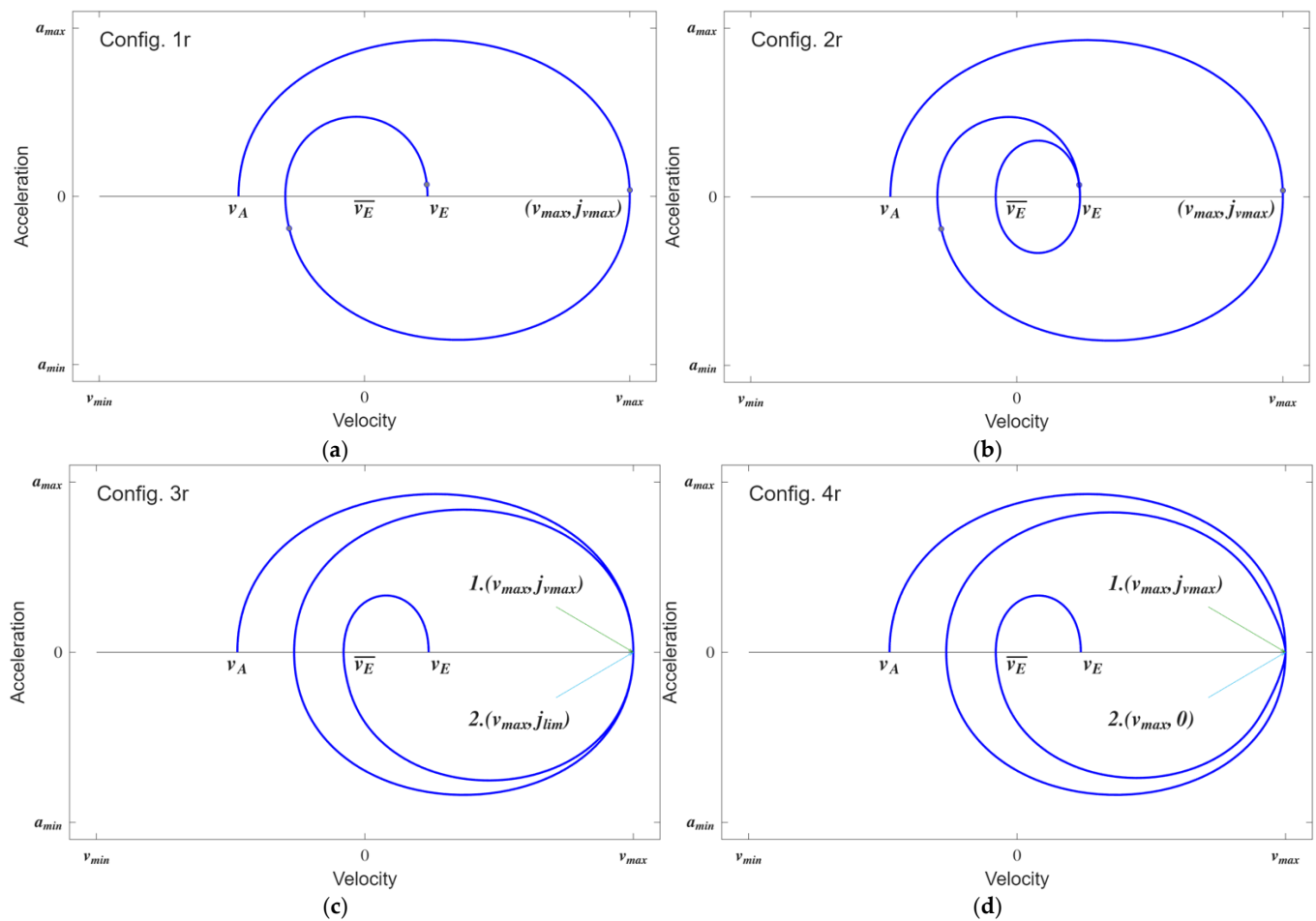


Figure 21. Switching configurations: Case I, non-connectible, going right, part II. (a) Starting configuration. (b) Adding a round at v_E . (c) Moving a round to v_{max} . (d) Going down to $j = 0$ at v_{max} .

Figure 22 provides an overview of possible sequences between the intermediate configurations. Table 8 lists the transitions, the model type used in a transition, and the running parameters as well as their ranges.



Figure 22. Sequence of intermediate configurations: Case I, non-connectible, going right, part II.

Table 8. Model types used for transition and parameter ranges: Case I, non-connectible, going right, part II.

Transition	Model Type	Parameter and Range
1r->2r	Upper part (fix): mp Lower part: pmpm	$\Delta t_{end} = 0.2 \frac{ j_E }{sn_{max}}$
2r->3r	Upper part (fix): mp Lower part: pmpm	$\Delta t_{end} = 2 \frac{ j_E }{sn_{max}} \cdot \Delta t_{lim}$
3r->4r	Upper part (fix): mp Lower part: from (v_{max}, j_{vmax}) to (v_{max}, j_m) : pmp; from (v_{max}, j_m) to (v_E, j_E) : mpm or pmp (cf. Table 2/Table 3)	$j_m = j_{lim} \cdot 0$

We now proceed from the starting configuration 1.1r (for $v_{mr} \leq v_{max}$) or 1r (for $v_{mr} > v_{max}$) to the left, so 1.1r = 1.1l and 1r = 1l, respectively. If in configuration 1.1l, the inequality $\Delta t_1 < 2 \frac{|j_A|}{sn_{max}}$ holds (i.e., $v_{mr} > \bar{v}_A$), we let Δt_1 run from its starting value to $2 \frac{|j_A|}{sn_{max}}$, such that the curve runs through $(\bar{v}_A, -j_A)$, leading to configuration 2.1l (Figure 23a). If in configuration 1.1l the inequality $\Delta t_1 \geq 2 \frac{|j_A|}{sn_{max}}$ holds already, we have 2.1l = 1.1l. It is possible that in configuration 2.1l we have $v_{ml} < v_{min}$, where the second intersection with the velocity axis is denoted by v_{ml} and then this configuration is infeasible. Then, we let Δt_1 run only from its starting value to the value Δt_{1vmin} , where $v_{ml} = v_{min}$ holds, leading to configuration 2.2l (Figure 24a). When our starting configuration is 1l, we have $\Delta t_1 < 2 \frac{|j_A|}{sn_{max}}$ and we always go to configuration 2.1l by letting Δt_1 run from its starting value to $2 \frac{|j_A|}{sn_{max}}$. Because of Equation (11), $v_{ml} < v_{min}$ is not possible.

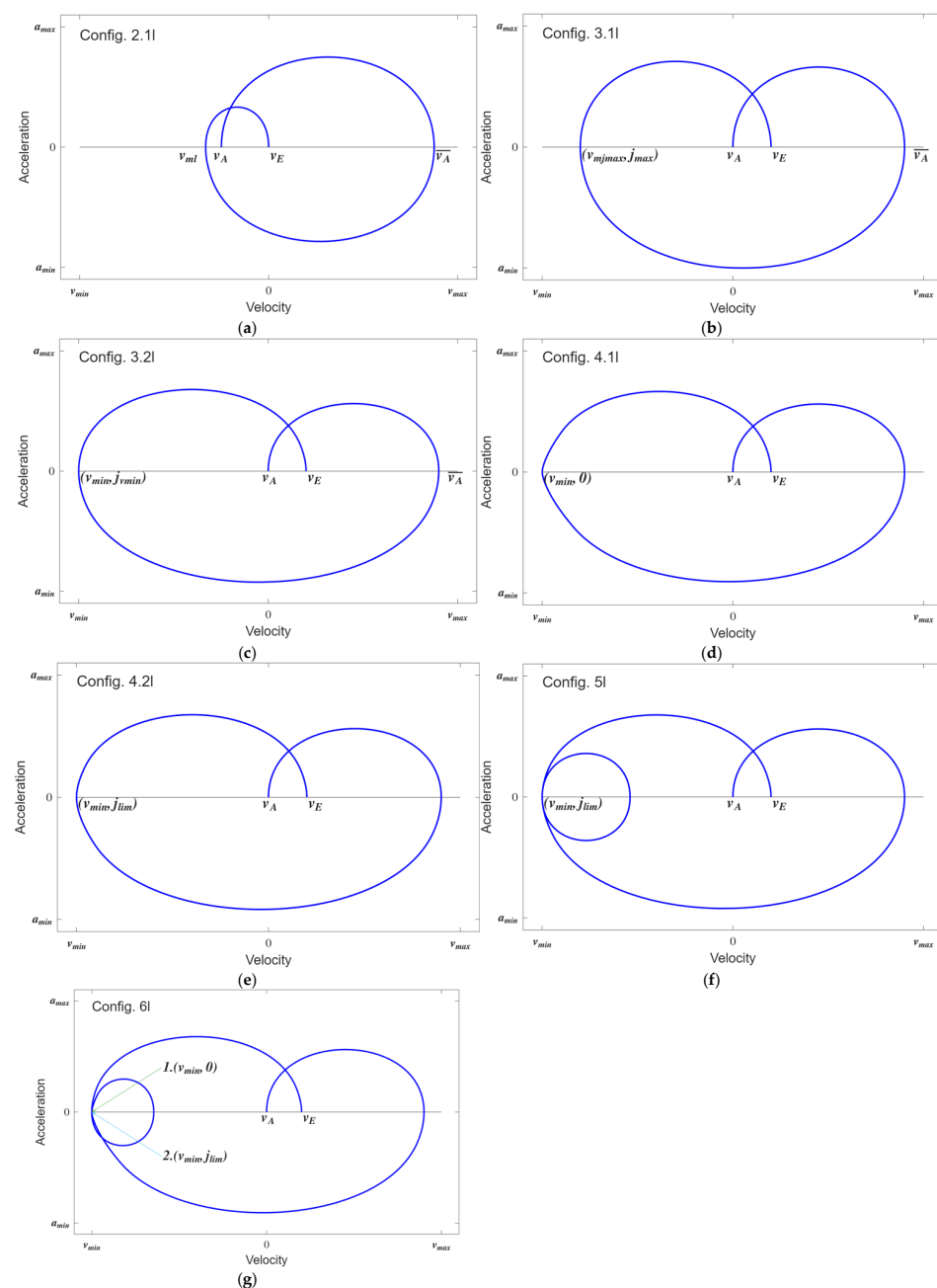


Figure 23. Switching configurations: Case I, non-connectible, going left, part I. (a) Increasing Δt_1 to have a half-round to $\bar{v}_A$. (b) Going up to j_{max} . (c) Going further to v_{min} . (d) Going down to $j = 0$ at v_{min} . (e) Going down to j_{lim} at v_{min} . (f) Adding a round at v_{min} . (g) Going down to $j = 0$ at v_{min} .

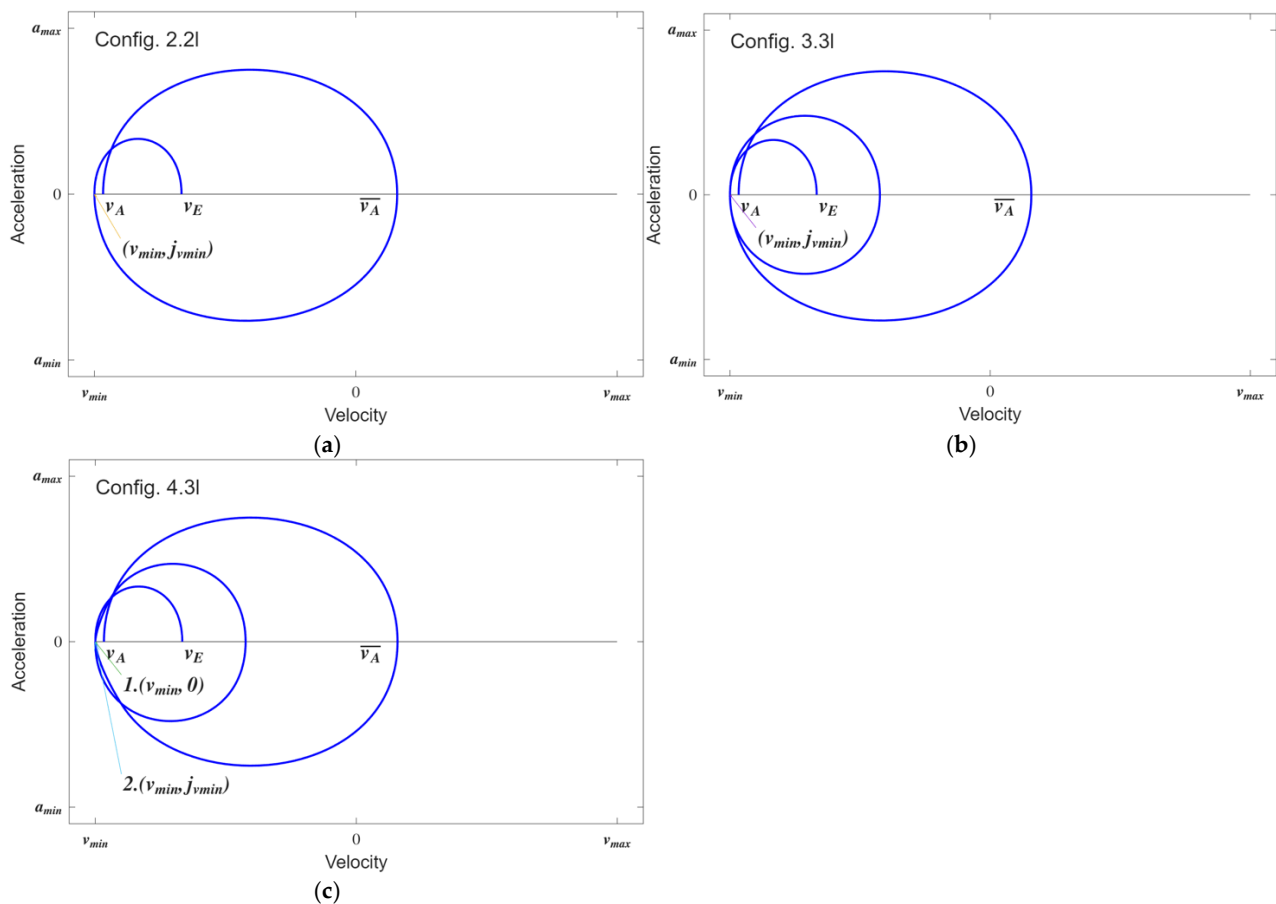


Figure 24. Switching configurations: Case I, non-connectible, going left, part II. (a) Increasing Δt_1 until reaching v_{min} . (b) Adding a round at v_{min} . (c) Going down to $j = 0$ at v_{min} .

Assume we are at configuration 2.1l (Figure 23a) and the curve intersects the velocity axis at the intermediate velocity v_{ml} with a jerk value of j_{ml} . We let the jerk j_m of the intermediate velocity v_m run from j_{ml} to j_{max} (v_m runs from v_{ml} to v_{mjmax}). The first part from (v_A, j_A) to (v_m, j_m) has type mpm, the second part from (v_m, j_m) to (v_E, j_E) has type pmp. If j_m reaches j_{max} before v_m is equal to v_{min} , we have configuration 3.1l; otherwise, the next intermediate configuration is 3.2l (Figure 23b,c). If one has configuration 3.1l, then going further left to (v_{min}, j_{max}) also leads to configuration 3.2l, with $j_{vmin} = j_{max}$. At v_{min} , we let the jerk run from j_{vmin} to 0, leading to configuration 4.1l if possible (i.e., $(v_{min}, 0)$ and (v_E, j_E) are connectible, Figure 23d). Otherwise, we stop at the minimal possible jerk j_{lim} and get to configuration 4.2l (Figure 23e). In 4.2l, we leave the last part from (v_{min}, j_{lim}) to (v_E, j_E) as is and consider the mpm-connection from (v_A, j_A) to (v_{min}, j_{lim}) . We get to the next configuration, 5l (Figure 23f), by adding a p-piece at the end of the mpm-connection, letting Δt_4 run from 0 to $2 \frac{|j_{lim}|}{s_{lim}^{max}}$. The mpmp-connection might intersect the velocity axis right of $\bar{v}_A$ but we do not overshoot at v_{max} because of inequality (11). Then, we let the jerk at the first intersection of the velocity axis at v_{min} decrease to 0 in order to get to configuration 6l, i.e., we have a part connecting (v_A, j_A) to $(v_{min}, 0)$ and another one from $(v_{min}, 0)$ to (v_{min}, j_{lim}) , as depicted in Figure 23g. By including a phase of constant velocity v_{min} at $(v_{min}, 0)$ in configuration 4.1l or 6l, one can cover arbitrarily small distances.

If one does not get to configuration 2.1l but to 2.2l (Figure 24a), one proceeds as described above for configuration 4.2l, which now includes a round at (v_{min}, j_{vmin}) (configuration 3.3l, Figure 24b), and then goes from (v_A, j_A) to $(v_{min}, 0)$ and from $(v_{min}, 0)$ to (v_{min}, j_{vmin}) (configuration 4.3l, Figure 24c).

Figure 25 provides an overview of possible sequences between the intermediate configurations. Table 9 lists the transitions, the model type used in a transition, and the running parameters as well as their ranges.

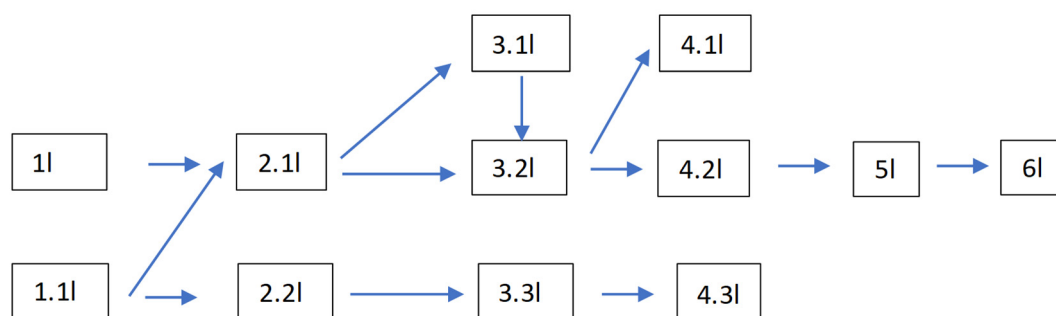


Figure 25. Sequence of intermediate configurations: Case I, non-connectible, going left.

Table 9. Model types used for transition and parameter ranges: Case I, non-connectible, going left.

Transition	Model Type	Parameter and Range
1.1l->2.1l	mpmp	$\Delta t_1 = \Delta t_{1start} \cdot 2 \frac{ j_A }{sn_{max}}$
1l->2.1l	mpmp	$\Delta t_1 = \Delta t_{1start} \cdot 2 \frac{ j_A }{sn_{max}}$
1.1l->2.2l	mpmp	$\Delta t_1 = \Delta t_{1start} \cdot \Delta t_{1vmin}$
2.1l->3.1l	First part: half-round from (v_A, j_A) to $(\overline{v_A}, -j_A)$; Second part: mpm; Third part: pmp	$j_m = j_{ml} \cdot j_{max}$ or $v_m = v_{ml} \cdot v_{mjmax}$
2.1l->3.2l	First part: half-round from (v_A, j_A) to $(\overline{v_A}, -j_A)$; Second part: mpm; Third part: pmp	$j_m = j_{ml} \cdot j_{vmin}$ or $v_m = v_{ml} \cdot v_{min}$
2.2l->3.3l	First part: mpmp from (v_A, j_A) to (v_{min}, j_{vmin}) Second part: as in 2.2l	$\Delta t_{end} = 0.2 \frac{ j_{vmin} }{sn_{max}}$
3.1l->3.2l	First part: half-round from (v_A, j_A) to $(\overline{v_A}, -j_A)$; Second part: mpm; Third part: pmp	$v_m = v_{mjmax} \cdot v_{min}$
3.2l->4.1l	First part: from (v_A, j_A) to (v_{min}, j_m) : mpm or pmp (cf. Table 2/Table 3); Second part: from (v_{min}, j_m) to (v_E, j_E) : pmp	$j_m = j_{vmin} \cdot 0$
3.2l->4.2l	First part: from (v_A, j_A) to (v_{min}, j_m) : mpm or pmp (cf. Table 2/Table 3); Second part: from (v_{min}, j_m) to (v_E, j_E) : pmp	$j_m = j_{vmin} \cdot j_{lim}$
3.3l->4.3l	First part: from (v_A, j_A) to (v_{min}, j_m) : mpm or pmp (cf. Table 2/Table 3) Second part: from (v_{min}, j_m) to (v_{min}, j_{vmin}) : pmp Third part: from (v_{min}, j_{vmin}) to (v_E, j_E) : as in 3.3l	$j_m = j_{vmin} \cdot 0$
4.2l->5l	First part: from (v_A, j_A) to (v_{min}, j_{lim}) : mpmp; Second part: from (v_{min}, j_{lim}) to (v_E, j_E) : pmp (as in 4.2l); First part: from (v_A, j_A) to (v_{min}, j_m) : mpm or pmp (cf. Table 2/Table 3)	$\Delta t_{end} = 0.2 \frac{ j_{lim} }{sn_{max}}$
5l->6l	Second part: from (v_{min}, j_m) to (v_{min}, j_{lim}) : mpm or pmp (cf. Table 2/Table 3) Third part: from (v_{min}, j_{lim}) to (v_E, j_E) : as in 5l	$j_m = j_{lim} \cdot 0$

3.2. Case II: $j_A < 0$ and $j_E \leq 0$

Besides using the techniques already applied in Case I, new challenges can come up once the intermediate velocity v_m has reached the restriction v_{min} or v_{max} . We just consider v_{min} here. Assume that the difference between v_A and v_{min} is so small that (v_A, j_A) and $(v_{min}, 0)$ are not connectible, i.e., physically it is not possible to get from velocity v_A having a jerk value of j_A to v_{min} and stay there (i.e., $a = 0, j = 0$) due to the combination of Δv and j_A . Then, starting at (v_A, j_A) , the jerk can only go down to some value $j_{lim} > 0$. If

now (v_{min}, j_{lim}) and (v_E, j_E) are not continuously connectible, one cannot insert a round at v_{min} in a continuous way as was done in Case I (neither in the lower part from (v_A, j_A) to (v_{min}, j_{lim}) nor in the upper part from (v_{min}, j_{lim}) to (v_E, j_E)). The discontinuity creates a gap in covered distances. So, this is a triggering condition for a gap and the physical reason for its occurrence is that, given the jerks j_A and j_E , both v_A and v_E are too close to v_{min} for going directly down to jerk zero and for continuously inserting a round at v_{min} . One can argue analogously when $(v_{min}, 0)$ and (v_E, j_E) are not connectible, leading to the corresponding second triggering condition for a gap. If the distance to be covered falls into this gap, the handling strategy consists of inserting the additional round right at the intermediate velocity of the starting configuration and retaining this round when going left and right. Then, reaching v_{min} , one already has the round available, thus avoiding a gap. Note that for the distances already covered without the extra round inserted from start onward, this approach provides longer times but it will only be used when the distance to be covered lies in the gap.

Moreover, if neither (v_A, j_A) and $(v_{min}, 0)$ nor $(v_{min}, 0)$ and (v_E, j_E) are connectible, one needs two extra-rounds at v_{min} : one for getting the jerk from the limit value j_{lim} down to 0 and another one for then getting the jerk up to the limit value of the upper part. But as described in the previous section, it is possible to duplicate a round, so this can be handled.

3.3. Case III: $j_A \geq 0$ and $j_E > 0$

This case can be handled completely analogously to the previous one by exchanging left and right.

3.4. Case IV: $j_A < 0$ and $j_E > 0$

In this case, it is sufficient to distinguish between the connectible and non-connectible situations. No new challenges occur, so one can use the techniques already applied in the previous cases. For details, see the Supplementary Materials.

4. Implementation and Validation

We assume that the data on kinematic restrictions, initial and final velocities and jerks, and the distance to be covered are given as stated at the beginning of Section 2. Then, the general main algorithm is quite simple (special cases are dealt with subsequently): For all sub-cases stated in Section 3, first the sequence of intermediate configurations is determined beginning with the starting model and then going to the right and to the left as described above. This leads to the actually valid sub-sequence of the possible sequences of models listed in the respective figures in Section 3 (and in the Supplementary Materials, respectively). For the starting model and each subsequent configuration, the covered distance is computed. Then, depending on whether the given distance is smaller or larger than the distance covered by the starting model, one proceeds to the left or to the right until the distance covered at the intermediate configuration is larger or smaller than the distance to be covered. This way, one obtains the transition to work with. The tables listing the model types used for transitioning from one configuration to the next provide the model type and the running parameter including its range. Then, a simple bisection method is applied to find the model and the time intervals for covering the given distance within a certain error margin.

The situation becomes more complicated when there is an area of isolated solutions. This occurs in the case of non-continuous connectibility right at the starting configuration (see Figure 11a) when Δt_1 or Δt_4 runs from 0 to the first critical value, Δt_{1c1} or Δt_{4c1} (in the sequel, we just consider Δt_1 since the same considerations apply to Δt_4). If one wants to check whether the given distance is covered when running Δt_1 , it is not sufficient to inspect

the distances at the boundaries since the covered distance need not develop monotonously. Therefore, we sub-divide the interval $[0, \Delta t_{1c1}]$ into sub-intervals and compute the covered distances at the boundaries of the sub-intervals. If the given distance lies between the distances covered at the boundaries, we can apply the bisection method and get a solution. The solutions obtained in this way are later compared with solutions obtained in the non-isolated area. In our implementation, we chose 10 as the number of sub-intervals. A low number enhances computational efficiency maybe at the cost of missing a better solution. In order to better control the trade-off between the security of finding an existing solution and computational efficiency, the following procedure could be applied:

- Sub-divide the time interval into a chosen number of intervals.
- If the distance to be covered does not lie between two distances covered at sub-interval boundaries, compute the quadratic approximation of the boundary points (covered distance depending on Δt_1 or Δt_4), as well as its maximum/minimum.
- If the distance to be covered lies beyond the extremum (possibly including a security factor), the search is stopped; otherwise, the sub-dividing is refined around the extremum and the process is repeated until a given maximum number of iterations is reached.

By setting the number of sub-intervals and iterations as well as the value of a security factor, one can control the process.

The situation becomes even more complicated when proceeding as stated in the first paragraph leads to ending up in a gap, as described in Section 3, Case II, for the sub-case of non-continuous connectivity. Then, one has to work with an additional round from the start.

As in [2], we use Matlab[®] 2025 as the implementation environment from which we essentially need to use the programming language and the polynomial root solver. For each sub-case, we set up the following test scenario in order to check whether viable solutions are found:

- $v_{max} \in [0.1, 20]$, $a_{max} \in [5, 30]$, $j_{max} \in [10, 100]$, $sn_{max} \in [50, 500]$ are chosen randomly (units: $\frac{mm}{s}$, $\frac{mm}{s^2}$, $\frac{mm}{s^3}$, and $\frac{mm}{s^4}$, respectively; in the sequel we will omit the units). sn_{max} and a_{max} are chosen first, and then j_{max} , such that condition $j_{max} \leq \sqrt{2a_{max}sn_{max}}$ holds, and then v_{max} , such that inequality (11) holds (cf. end of Section 2).
- j_A and j_E are chosen randomly between j_{min} and j_{max} , such that the sign conditions for the sub-case under consideration are fulfilled.
- v_A is chosen randomly between v_{min} and v_{max} , such that the sub-case is possible at all; v_E is chosen randomly between v_A and v_{max} , such that the sub-case occurs.
- Δs is chosen randomly between the smallest and largest covered distance at the end configurations going left and right.

For each sub-case listed in Section 3 (except for III, which works like II), we made 10^5 runs, repeating each one 10 times to determine the smallest time needed for computation. All runs succeeded in providing a solution within the given error margin of 10^{-3} for Δs . Regarding numerical problems, we used the same strategies as already listed in [2].

As in [2], we used the time-taking facilities provided by Matlab[®] to measure the time needed to find a solution. The underlying hardware configuration was a Dell Pro 16 Plus notebook running Windows 11 with an Intel Ultra 7 processor (8 kernels) working at up to 4.8 GHz and 32 GB RAM. The results of the runs are listed in Table 10. As in [2], the values for mean time, standard deviation, and maximum time suggest that, in general, the times are well below 1 ms but, occasionally, they might exceed this value. Again, using a profiling tool shows that about 60% of the computation time is spent using the root solver. So, using a faster polynomial root solver could be one improvement. Moreover,

in our trials, we first computed all intermediate configurations going left and right from the starting configuration in order to check whether these are found. One could improve on performance by going only in one direction and only as far as necessary for the given distance to be exceeded.

Table 10. Computational efficiency: Time needed for finding the solution, 100.000 runs each sub-case.

Case	Sub-Case	Mean	St.-Deviation	Maximum
I: +/—	Continuous connectibility	257 μ s	62 μ s	515 μ s
	Non-continuous connectibility	479 μ s	123 μ s	1500 μ s
	No connectibility	195 μ s	54 μ s	432 μ s
II: —/—	Continuous connectibility	263 μ s	83 μ s	639 μ s
	Non-continuous connectibility	506 μ s	154 μ s	878 μ s
	No connectibility	193 μ s	54 μ s	466 μ s
IV: —/+	Connectibility	194 μ s	51 μ s	607 μ s
	No connectibility	193 μ s	54 μ s	482 μ s

Note that besides running the algorithm in a PC environment as we did, a test in a real controller environment as is performed in [17] for the acceleration-continuous case is still pending.

5. Discussion and Comparison with Related Work

In the literature, there are many trajectory generation algorithms that not only take into account kinematic restrictions but also address more generally dynamic aspects, disturbances, or application-specific problems. Since our work is located in the field of online trajectory generation under kinematic restrictions, we include in our comparison only contributions from this field. In [2], we already compared our algorithm for the rest-in-rest (RR) situation as well as for the GG situation (zero initial and final jerk and acceleration), and showed the advantages of our approach over a range of given distances. The only algorithm that allows for prescribing arbitrary initial and final jerks is the one by Ezair et al. [22]. In this section, we first consider two examples to demonstrate and explain the time efficiency advantage of our approach. We use an example from Case I with continuous connectibility to show some principle differences. We also use an example from Case I with no connectibility to shed light on the problems that occur when it is not possible to go down to jerk 0 at v_{max} directly, i.e., when an extra round is needed. In order to take into account a much wider range of conditions, we then provide statistical results of a test run with randomly generated input data (restrictions, initial/final conditions, and distance).

Example 1. The given data in this example are: $v_{max} = 70$, $a_{max} = 30$, $j_{max} = 20$, $sn_{max} = 10$ resp. 30, $v_A = 0$, $v_E = 30$, $j_A = 15$, $j_E = -7$. Figure 26a shows this example with $sn_{max} = 10$, and Figure 26b shows this example with $sn_{max} = 30$. In the outer parts (left of about 50 and right of about 150) of Figure 26a, the advantage of our approach comes from not going back to zero jerk at the maximally achieved velocity. As can be seen in Figure 26b, the differences become smaller when the snap is higher. Then, one can go down from maximal jerk to zero very quickly, such that this restriction of the algorithm by Ezair et al. [22] becomes less important quantitatively. More interesting is the inner part. In Figure 26a, our algorithm changes from going

left to going right at a distance near 50 and one has a discontinuity in used time (for the same effect for acceleration-continuous motion functions cf. [23] Figure 8a). The algorithm of Ezair et al. [22] obviously changes later to the right and in this range of distances (from about 50 to 150) provides solutions that are considerably worse. This is due to the usage of a kind of bisection method that converges but not necessarily to the best solution, which is a principle restriction of the method. In our approach, it is completely clear which solution is taken when going systematically through the velocity–acceleration plane. One might start the algorithm of Ezair et al. [22] several times with randomly chosen starting intervals; however, in our hardware configuration (see Section 4), the algorithm needed 3.5 ms for $sn_{max} = 10$ (our approach: 248 μ s) for a single run, such that several tries would make it unsuitable for online trajectory generation.

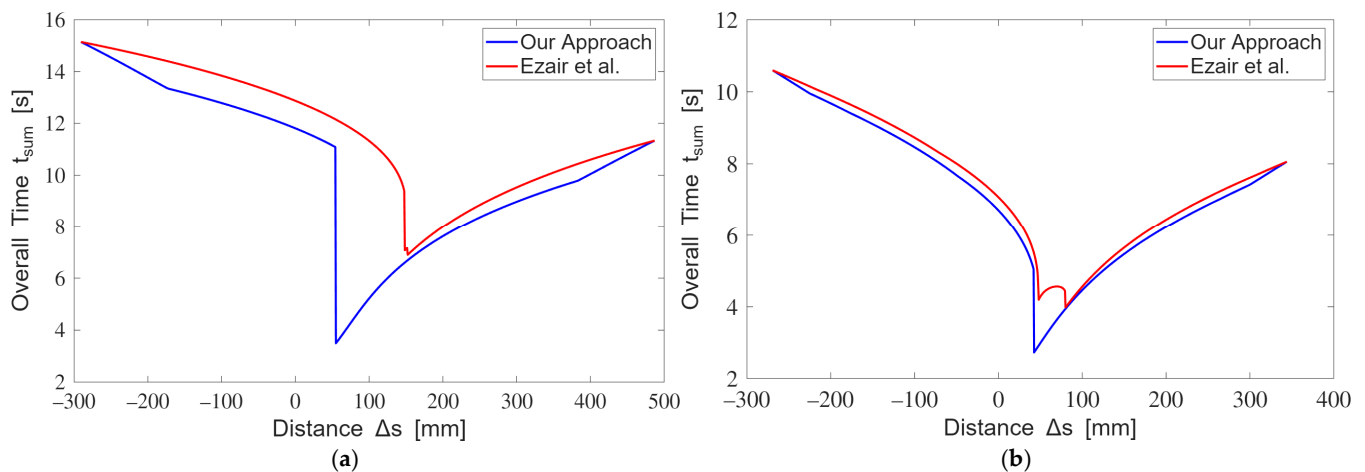


Figure 26. Example 1: Time needed over the range of given distances: (a) $sn_{max} = 10$. (b) $sn_{max} = 30$.

Table 11 provides data on the maximum and mean absolute and relative differences of the time advantage of our approach compared to the one by Ezair et al., based on the 1000 sample points used for generating the graph. One can recognize that enlarging the maximum snap reduces the advantage of our approach.

Table 11. Maximum and mean absolute and relative differences for time in Example 1.

Max. Snap [mm/s ⁴]	Max. Abs. Difference [s]	Mean Abs. Difference [s]	Max. Rel. Difference [%]	Mean Rel. Difference [%]
10	8.65	1.30	71.22	11.23
30	2.75	0.28	50.23	4.41

Example 2. The given data in this example are: $v_{max} = 70$, $a_{max} = 30$, $j_{max} = 20$, $sn_{max} = 10$, $v_A = 47$, $v_E = 57$, $j_A = 15$, $j_E = -7$. These data are chosen such that we have no connectivity of (v_A, j_A) and (v_E, j_E) , and (v_A, j_A) and $(v_{max}, 0)$ are also non-connectible, such that one needs an additional round at v_{max} for going down to zero jerk to be able to include an additional segment with constant velocity v_{max} . The sequence of models when going to the right is 1.1r->2.2r->3.2r->4.1r->5.1r (shown as a possible sequence in Figure 20). Figure 27a shows the needed time over the range of possible distances going from $(v_{min}, 0)$ to $(v_{max}, 0)$ for our approach and the approach of Ezair et al. In the left part, and in a considerable section within the right part, we observe the same behaviour as described in Example 1; however, zooming into the right end of Figure 27a, where both curves look rather identical, reveals that the approach of Ezair et al. [22] has a small advantage regarding needed time. Figure 27b shows the corresponding velocity function produced by the algorithm of Ezair et al. for a distance in this part ($\Delta s = 574.65$). One recognizes that the function slightly overshoots v_{max} , and then goes down to v_{max} again. There is no part in

the algorithm that prevents such an overshooting phenomenon, so the algorithm might also produce non-admissible solutions violating the kinematic restrictions. This shows, again, that it is hardly possible to circumvent the inherent complexity of the problem in a simple algorithm.

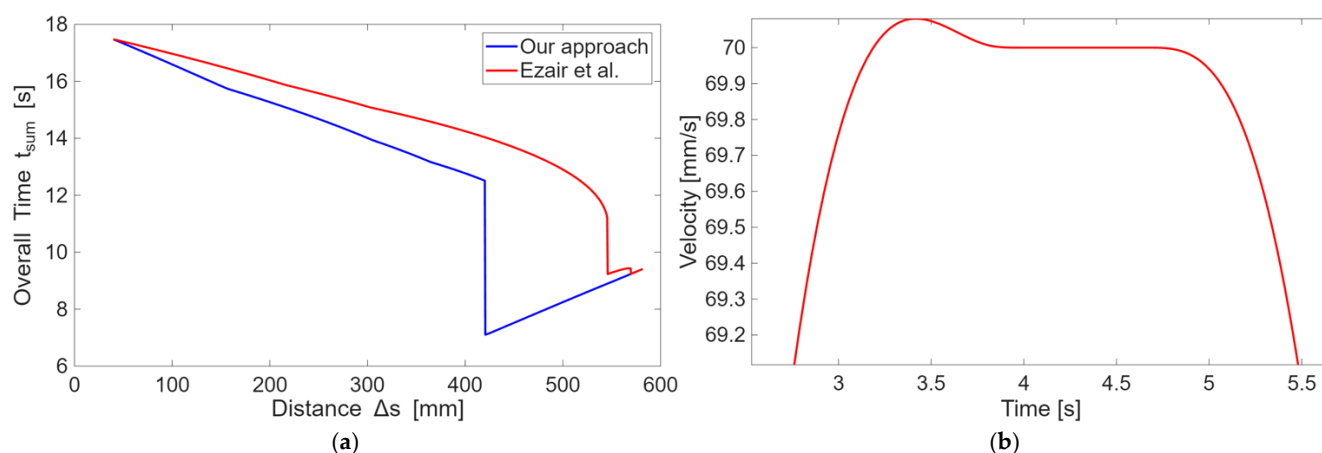


Figure 27. Example 2: (a) Time needed over the range of given distances. (b) Zoom into the velocity function of Ezair et al. [22] for $\Delta s = 574.65$.

Table 12 provides the same data for Example 2 as reported for Example 1 in Table 11. The results are similar to those shown in Table 11.

Table 12. Maximum and mean absolute and relative differences for time in Example 2.

Max. Snap [mm/s ⁴]	Max. Abs. Difference [s]	Mean Abs. Difference [s]	Max. Rel. Difference [%]	Mean Rel. Difference [%]
10	6.92	1.79	49.36	12.98

In order to compare both approaches for a much wider range of conditions, we performed test runs with randomly chosen input data on restrictions, initial and final velocities and jerks, and the distance to be covered. We used the same setting as reported in Section 4 for testing the implementation (boundaries for kinematic restrictions, etc.) and conducted 10^5 test runs for Case I with continuous connectivity, where we do not assume that the algorithm by Ezair et al. leads to an overshooting phenomenon, as observed in example 2. Table 13 shows the mean, median, and standard deviation values of the relative and absolute differences. The considerable difference between the mean and median indicates that the values are biased by very large differences (see, e.g., the medium region of distances in Figure 26a), such that a closer look at the frequency distributions is required.

Table 13. Mean, median, and standard deviation values for absolute and relative differences.

Difference	Mean	Median	Standard Deviation
Absolute	0.181 s	0.064 s	0.389 s
Relative	12.98%	5.81%	19.03%

Table 14 contains the frequency distribution of the relative differences, i.e., the difference between the time needed in the algorithm by Ezair et al. [22] and the time needed in our algorithm relative to the time needed by Ezair et al. [22]. The data shows that for about 80% of the test runs, the time advantage is smaller or equal to 10%; however, there is also a considerable amount of cases with much higher percentages as was also recognised in Example 1. But in absolute values, a high percentage might mean just a few milliseconds,

which is irrelevant. Therefore, in Table 15, we provide a frequency distribution for the absolute differences. This shows that for approximately 41% of the runs, the advantage is below 0.05s, which depending on the application scenario, is likely to be irrelevant. This result is expectable since, in our test runs, we allowed the snap to be up to 500 (mm/s⁴) (see Section 4) and we already recognised in Example 1 that increasing the snap value decreases the advantage, since going back to zero jerk at the intermediate velocity can be realised within a short timeframe.

Table 14. Frequency distribution of relative differences.

Interval [%]	Frequency
[0, 1]	5544
(1, 5]	37,321
(5, 10]	36,202
(10, 20]	4063
(20, 50]	7502
>50	9368

Table 15. Frequency distribution of absolute differences.

Interval [s]	Frequency
[0, 0.01]	6443
(0.01, 0.05]	35,031
(0.05, 0.1]	24,820
(0.1, 0.5]	24,848
(0.5, 1]	4774
(1, 2]	3105
(2, 5]	951
>5	28

One can conclude from the data that the advantage of our approach very much depends on the conditions, such as the system parameters. One needs more information on the application scenario to decide whether the time advantage is worth the additional complexity of our approach.

6. Conclusions

In this article, we solve the problem of finding a fast motion function for online trajectory generation when kinematic restrictions on velocity, acceleration, jerk, and snap are given, as well as the initial and final velocities and jerks and the distance to be covered. As in [2], fast means that at any instant in time, at least one restriction is reached. Note that we do not claim time optimality because we cannot prove that, and the complexity encountered in this contribution indicates to avoid such claims without formal proof. By introducing new concepts like connectivity and new techniques like inserting extra-rounds, it was possible to extend the approach of systematically going through the velocity–acceleration plane to the task allowing for arbitrary initial and final jerks. The exposition shows that, again, a considerable amount of inherent complexity comes to the fore (particularly when reaching v_{max}, v_{min}), which is manageable by making suitable case distinctions. This also justifies consideration of the problem as an important intermediate step to solve the general problem, where arbitrary initial and final accelerations can also be prescribed, which is the ultimate goal of this line of research. The article also gives insight into the limitations of the only currently available approach by Ezair et al. for tackling this general problem, which might give a solution that violates a restriction by overshooting or lead to solutions quite different from the optimal one.

Our approach has clear limitations: It models the environment only by prescribing values for kinematic restrictions; other aspects like the system dynamics, external disturbances, or application-specific issues are not considered. Moreover, the computational efficiency still needs testing in a real controller environment.

In future research, we will drop the assumption that the initial and final accelerations must be zero in order to find a solution to the general problem by going systematically through the velocity–acceleration plane. To which extent reducing the problem to the GG+ situation is possible (analogous to the procedure for acceleration-continuous functions used in [23]) needs to be investigated. Moreover, the problem of multi-axis synchronization in robotics needs to be addressed once the one-dimensional case is solved, as described in the seminal book by Kröger [17] for acceleration-continuous functions.

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Abbreviations

The following abbreviations are used in this manuscript:

CAS	Computer Algebra System
p	Plus
m	Minus
pm	Plus-minus
mp	Minus-plus
pmp	Plus-minus-plus
mpm	Minus-plus-minus
pmpm	Plus-minus-plus-minus
mpmp	Minus-plus-minus-plus

Symbols

The following abbreviations are used in this manuscript:

$v_{max}, a_{max}, j_{max}, sn_{max}$	Maximum velocity, acceleration, jerk, and snap
v_A, v_E	Initial and final velocity
j_A, j_E	Initial and final jerk
j_m, j_{ml}, j_{mr}	Jerk at intermediate velocity v_m , going left, going right
j_{lim}	Limit for jerk when going down
j_{m_limit}	Limit for jerk when a change in the limiting part (upper or lower) occurs
j_{vmin}, j_{vmax}	Jerk at v_{min} and v_{max}
j_{Al}, j_{Au}	Lower and upper absolute value of j_A for having connectibility when Δv and j_E are given
j_{El}, j_{Eu}	Lower and upper absolute value of j_E for having connectibility when Δv and j_A are given
$\overline{v_A}, \overline{v_E}$	Velocities reached when going a half-round from (v_A, j_A) or (v_E, j_E)
v_m, v_{ml}, v_{mr}	Intermediate velocity v_m , going left, going right

v_{mjmin}, v_{mjmax}	Velocity when reaching j_{min} or j_{max}
$min\Delta v(j_A, j_E)$	Minimum velocity difference for connectivity of (v_A, j_A) and (v_E, j_E) for $j_A \geq 0$ and $j_E \leq 0$
$min\Delta v_{mod}(j_A, j_E)$	Modified minimum velocity difference for arbitrary j_A, j_E
$min\Delta v_{ct}(j_A, j_E)$	Minimum velocity difference required for continuous connectivity
Δv_{mod}	Modified velocity difference
$\Delta t_{1c1}, \Delta t_{1c2}$	First or second critical value for Δt_1 in pmpm- or mpmp-connection
$\Delta t_{4c1}, \Delta t_{4c2}$	First or second critical value for Δt_4 in pmpm- or mpmp-connection
Δt_{1start}	Starting value for Δt_1
Δt_{1vmin}	Value for Δt_1 when reaching v_{min}
Δt_{lim}	Value for final time interval in pmpm-connection when reaching v_{max}

Appendix A. Formulae and Polynomials

Appendix A.1. Computing j_{El}, j_{Eu} from j_A and Δv , and j_{Al}, j_{Au} from j_E and Δv

If $j_A \geq 0$ and $\Delta v \geq min\Delta v(j_A, -j_A) \geq 0$ are given, j_{El} or j_{Eu} can be computed from Formula (3), such that (v_A, j_A) and (v_E, j_E) are connectible for all j_E with $j_{Eu} \leq j_E \leq -j_A$ or $-j_A \leq j_E \leq j_{El}$ (so l or u stands for lower or upper bound, referring to the absolute value!). Solving this equation for j_E using the CAS Maple® leads to finding the zeros of the following polynomial of degree 6:

$$Z^6 - 9j_A^2 Z^4 + (-48\Delta v sn_{max}^2 + 16j_A^3) Z^3 - 9j_A^4 Z^2 - 72\Delta v^2 sn_{max}^4 + 48\Delta v j_A^3 sn_{max}^2 + j_A^6 \quad (A1)$$

If there is no real zero in $[-j_A, 0]$, (v_A, j_A) and (v_E, j_E) are connectible for any $j_E \in [-j_A, 0]$, i.e., $j_{El} = 0$. Otherwise, j_{El} is the real zero for which Equation (3) holds. There must be a unique real zero in $[-\infty, -j_A]$ for which Equation (3) holds. This is j_{Eu} .

If $j_E \leq 0$ and $\Delta v \geq min\Delta v(-j_E, j_E) \geq 0$ are given, j_A can be computed from Formula (3), such that (v_A, j_A) and (v_E, j_E) are connectible. Solving this equation for j_A using the CAS Maple® leads to finding the zeros of the following polynomial of degree 6:

$$Z^6 - 9j_E^2 Z^4 + (48\Delta v sn_{max}^2 + 16j_E^3) Z^3 - 9j_E^4 Z^2 - 72\Delta v^2 sn_{max}^4 - 48\Delta v j_E^3 sn_{max}^2 + j_E^6 \quad (A2)$$

If there is no real zero in $[0, -j_E]$, (v_A, j_A) and (v_E, j_E) are connectible for any $j_A \in [0, -j_E]$, i.e., $j_{Al} = 0$. Otherwise, j_{Al} is the real zero for which Equation (3) holds. There must be a unique real zero in $[-j_E, \infty)$ for which Equation (3) holds. This is j_{Au} .

For $j_A < 0$ and $\Delta v < min\Delta v(j_A, -j_A) \leq 0$ (traversing through the lower halfplane), one can proceed analogously.

Appendix A.2. Computing Time Intervals for Pmpm-Connections (Model 1) for a Given Δt_4

If (v_A, j_A) and (v_E, j_E) are given and connectible, then the model must fulfill the following three equations, where $t_4 = \Delta t_1 + \Delta t_2 + \Delta t_3 + \Delta t_4$:

$$v(t_4) = v_E, a(t_4) = 0, j(t_4) = j_E \quad (A3)$$

Using the CAS programme Maple® transforms this into a problem of finding Δt_2 as sn_{max}^{-1} times a zero of the following polynomial of degree 4:

$$12Z^4 + (-48\Delta t_4^2 sn_{max}^2 - 96\Delta t_4 j_E sn_{max} - 24j_A^2 - 24j_E^2) Z^2 + (-48\Delta t_4^3 sn_{max}^3 - 144\Delta t_4^2 j_E sn_{max}^2 - 96\Delta t_4 j_E^2 sn_{max} - 48\Delta v sn_{max}^2 + 16j_A^3 - 16j_E^3) Z - 12\Delta t_4^4 sn_{max}^4 - 48\Delta t_4^3 j_E sn_{max}^3 + 12\Delta t_4^2 j_A^2 sn_{max}^2 - 60\Delta t_4^2 j_E^2 sn_{max}^2 + 24\Delta t_4 j_A^2 j_E sn_{max} - 24\Delta t_4 j_E^3 sn_{max} - 3j_A^4 + 6j_A^2 j_E^2 - 3j_E^4 \quad (A4)$$

One has to pick the zero where the resulting values for Δt_1 and Δt_3 are non-negative real numbers and the original Equations (A3) are fulfilled. Then, Δt_1 and Δt_3 are computed according to Equations (A5) and (A6).

$$\Delta t_3 = \frac{2\Delta t_2^2 sn_{max}^2 + 4\Delta t_2 \Delta t_4 sn_{max}^2 + 2\Delta t_4^2 sn_{max}^2 + 4\Delta t_2 j_E sn_{max} + 4\Delta t_4 j_E sn_{max} - j_A^2 + j_E^2}{4sn_{max}^2 \Delta t_2} \quad (A5)$$

$$\Delta t_1 = \frac{(\Delta t_2 - \Delta t_3 + \Delta t_4) sn_{max} - j_A + j_E}{sn_{max}} \quad (A6)$$

For the other 7 models, for a pmpm-connection one can proceed analogously.

Appendix A.3. Computing $\min \Delta v_{ct}$

We consider the pmpm-connection resulting from adding an m-piece to pmp. There, Δt_4 is a free variable that runs from 0 to $2 \frac{|j_E|}{sn_{max}}$ when (v_A, j_A) and (v_E, j_E) are continuously connectible. If the latter is not the case, there are two values for Δt_4 for which Δt_1 becomes 0; that is, the connection is mpm, and these define an interval such that Δt_4 values lying in-between do not yield a feasible pmpm-connection since they result in negative lengths of time intervals. The values of Δt_4 , for which Δt_1 becomes 0, can be computed as sn_{max}^{-1} times a zero of the following polynomial (simply set $\Delta t_1 = 0$ and solve for Δt_4 instead of setting a value for Δt_4 , see Appendix A.2):

$$(36j_A^2 - 36j_E^2)Z^4 + (144\Delta v sn_{max}^2 - 48j_A^3 + 144j_A^2 j_E - 96j_E^3)Z^3 + (432\Delta v j_E sn_{max}^2 + 18j_A^4 - 144j_A^3 j_E + 180j_A^2 j_E^2 - 54j_E^4)Z^2 + (288\Delta v j_E^2 sn_{max}^2 + 36j_A^4 j_E - 96j_A^3 j_E^2 + 72j_A^2 j_E^3 - 12j_E^5)Z + 72\Delta v^2 sn_{max}^4 - 48\Delta v j_A^3 sn_{max}^2 + 48\Delta v j_E^3 sn_{max}^2 - j_A^6 + 9j_A^4 j_E^2 - 16j_A^3 j_E^3 + 9j_A^2 j_E^4 - j_E^6 \quad (A7)$$

In order to find $\min \Delta v_{ct}$, one considers Δv as unknown and states as additional requirement that the polynomial (A7) has a double zero; that is, the interval for Δt_4 providing an infeasible solution has a length of zero and therefore one has continuous connectibility. This requirement implies that the value of Δt_4 is also a zero of the derivative of the polynomial. This gives two equations with two unknowns, Δv and Z . The CAS Maple® turns this again into another problem of polynomial zero finding: Z is a zero of the following polynomial of degree 4:

$$6Z^4 + 24j_E Z^3 + (-j_A^2 + 29j_E^2)Z^2 + (-2j_A^2 j_E + 10j_E^3)Z - j_A^2 j_E^2 + j_E^4 \quad (A8)$$

Then, Δv can be computed as:

$$\Delta v = -\left((12j_A^2 - 12j_E^2)Z^3 + (-12j_A^3 + 36j_A^2 j_E - 24j_E^3)Z^2 + (3j_A^4 - 24j_A^3 j_E + 30j_A^2 j_E^2 - 9j_E^4)Z + 3j_A^4 j_E - 8j_A^3 j_E^2 + 6j_A^2 j_E^3 - j_E^5 \right) / (12sn_{max}^2 (3Z^2 + 6Zj_E + 2j_E^2)) \quad (A9)$$

One has to pick that zero of (A8) as a value for Z , which gives a value for Δv according to (A9), which is not smaller than $\min \Delta v(j_A, j_E)$.

If $j_E = 0$, one gets $\min \Delta v(j_A, j_E) = \min \Delta v_{ct}(j_A, j_E)$ since the loop to the right is reduced to a point and one can continuously get to the loop to the left by letting Δt_1 run from its initial value to $2 \frac{|j_A|}{sn_{max}}$.

Appendix B. Proofs

Appendix B.1

Proof. Let (v_A, j_A) and (v_E, j_E) be connectible. If j_E is modified such that for the new value j_E^* the absolute value $|j_E^*|$ is closer to $|j_A|$ than $|j_E|$. Then (v_A, j_A) and (v_E, j_E^*) are also connectible.

We assume $j_A \geq 0$ and $j_E \leq 0$ (otherwise proceed analogously). First, let $|j_A| \leq |j_E|$. Then, $|j_E^*|$ has to decrease for getting closer to $|j_A|$ and from (3) we get immediately that $\min \Delta v(j_A, j_E^*) < \min \Delta v(j_A, j_E)$ and hence the assertion. Now, let $|j_A| > |j_E|$. Then, $|j_E^*|$ has to increase for getting closer to $|j_A|$, and so j_E^* has to decrease. We consider in (3) $\min \Delta v(j_A, j_E)$ as function of j_E holding all other quantities fixed. We show that the function is increasing, when j_E goes from $-j_A$ to 0, which proves the assertion. For this, we differentiate (3) wrt. j_E and we have to show that the values are non-negative. We can neglect the denominator since it is always positive, i.e., we must show

$$-12j_E^2 + \frac{6(j_A^2 - j_E^2)j_E}{\sqrt{2j_A^2 + 2j_E^2}} - 6j_E\sqrt{2j_A^2 + 2j_E^2} \geq 0. \quad (\text{A10})$$

Multiplying with $\sqrt{2j_A^2 + 2j_E^2}$ and rearranging leads to:

$$6j_E(j_A^2 + 3j_E^2) \leq -12j_E^2\sqrt{2j_A^2 + 2j_E^2}. \quad (\text{A11})$$

Since both sides are negative, squaring leads to:

$$36j_E^2(j_A^2 + 3j_E^2)^2 \geq 144j_E^4(2j_A^2 + 2j_E^2). \quad (\text{A12})$$

Simplifying this using the CAS Maple[®] yields

$$36j_E^2(j_A - j_E)^2(j_A + j_E)^2 \geq 0 \quad (\text{A13})$$

which is obviously true. $\square$

Appendix B.2

Proof. Let $0 \leq j_A < j_E$. Then, $\min \Delta v(j_A, j_m) - \min \Delta v(j_E, j_m) \geq \min \Delta v(j_A, -j_E) - \min \Delta v(j_E, -j_E)$ holds for any $j_m \leq -j_E$.

Substituting the terms with $\min \Delta v$ according to Formula (3), multiplying with $12sn_{\max}^2$, and simplification using the CAS Maple[®] yields the following equivalent inequality:

$$(j_A^2 - j_E^2)\sqrt{j_A^2 + j_E^2} + (-j_A^2 + j_m^2)\sqrt{j_A^2 + j_m^2} + (j_E^2 - j_m^2)\sqrt{j_E^2 + j_m^2} \geq 0. \quad (\text{A14})$$

This can be better handled when setting $a = j_A^2$, $b = j_E^2$, $c = j_m^2$, such that a , b , and c are non-negative and $a < b \leq c$

$$(a - b)\sqrt{a + b} + (-a + c)\sqrt{a + c} + (b - c)\sqrt{b + c} \geq 0. \quad (\text{A15})$$

This is equivalent to:

$$(a - b)\sqrt{a + b} + (-a + c)\sqrt{a + c} \geq (c - b)\sqrt{b + c}. \quad (\text{A16})$$

Since both sides are positive, squaring yields equivalently

$$(a - b)^2(a + b) + (-a + c)^2(a + c) + 2(a - b)(-a + c)\sqrt{a + b}\sqrt{a + c} \geq (c - b)^2(b + c). \quad (\text{A17})$$

and hence

$$2(a - b)(-a + c)\sqrt{a + b}\sqrt{a + c} \geq (c - b)^2(b + c) - (a - b)^2(a + b) - (-a + c)^2(a + c). \quad (\text{A18})$$

Simplification using the CAS Maple[®] leads to:

$$2(a-b)(-a+c)\sqrt{a+b}\sqrt{a+c} \geq -(2a+b+c)(a-c)(a-b). \quad (\text{A19})$$

Division by the negative expression $2(a-b)(-a+c)$ yields

$$\sqrt{a+b}\sqrt{a+c} \leq \frac{a}{2} + b + c. \quad (\text{A20})$$

Squaring both sides (which are positive) and simplifying again leads to $\frac{(b-c)^2}{4} \geq 0$, which is obviously true. $\square$

Appendix B.3

Proof. $\min \Delta v(j_A, j_m) - \min \Delta v(j_E, j_m)$ grows monotonously without upper bound for $0 < j_A < j_E$, $j_m \leq -j_E$ and growing $|j_m|$.

Compute $\min \Delta v(j_A, j_m) - \min \Delta v(j_E, j_m)$ according to Formula (3) and simplify by multiplying with $12sn_{max}^2$. Differentiating the expression with respect to j_m yields:

$$6j_m \sqrt{2j_A^2 + 2j_m^2} + 2 \frac{(-3j_A^2 + 3j_m^2)j_m}{\sqrt{2j_A^2 + 2j_m^2}} - 6j_m \sqrt{2j_E^2 + 2j_m^2} + 2 \frac{(3j_E^2 - 3j_m^2)j_m}{\sqrt{2j_E^2 + 2j_m^2}}. \quad (\text{A21})$$

We have to show that this is negative since for growing $|j_m|$ the value of j_m decreases. Division by j_m , which is < 0 , and simplification by the CAS Maple[®] yields

$$6 \frac{(-j_E^2 - 3j_m^2) \sqrt{2j_A^2 + 2j_m^2} + (j_A^2 + 3j_m^2) \sqrt{2j_E^2 + 2j_m^2}}{\sqrt{2j_A^2 + 2j_m^2} \sqrt{2j_E^2 + 2j_m^2}} > 0. \quad (\text{A22})$$

Since the denominator is positive, the numerator must also be positive, i.e.,

$$(j_A^2 + 3j_m^2) \sqrt{2j_E^2 + 2j_m^2} > (j_E^2 + 3j_m^2) \sqrt{2j_A^2 + 2j_m^2}. \quad (\text{A23})$$

Since both sides are positive, this is equivalent to

$$(j_A^2 + 3j_m^2)^2 (2j_E^2 + 2j_m^2) > (j_E^2 + 3j_m^2)^2 (2j_A^2 + 2j_m^2). \quad (\text{A24})$$

Setting $a = j_A^2$, $b = j_E^2$, $c = j_m^2$ as in Appendix B.2 yields:

$$(a + 3c)^2 (b + c) > (b + 3c)^2 (a + c). \quad (\text{A25})$$

Simplification by the CAS Maple[®] yields equivalently

$$(ab + ac + cb - 3c^2)(a - b) > 0. \quad (\text{A26})$$

Because of $0 < a < b \leq c$, both factors are negative, i.e., the first assertion is true. Applying L'Hospital's rule for computing the limit of the difference $\min \Delta v(j_A, j_m) - \min \Delta v(j_E, j_m)$ for $j_m \rightarrow -\infty$ yields ∞ , so there is no upper bound for the difference proving the second assertion. $\square$

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