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The Relativistic Bohr Radius and Its Agreement with the Dirac Most-Probable Radius

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Abstract

The Bohr radius is normally presented as a non-relativistic length scale. Less widely discussed is that Bohr's 1913 work also indicated how the radius formula changes when the orbital velocity is not negligible compared with the speed of light. We revisit this relativistic prescription and show that, for a point nucleus and a one-electron Coulomb field, it gives $a_{0,r}^{(Z)} = \frac{a_0}{Z}\sqrt{1-Z^2\alpha^2}$, which is exactly the most-probable radius obtained independently from the Dirac $1s_{1/2}$ radial probability density. The two radii are calculated independently and are found to be analytically identical. This equality does not derive the Dirac result from the Bohr model; rather, it shows that Bohr's relativistic circular-orbit prescription selects the same radial scale as the maximum of the Dirac probability distribution. We also show that the same construction extends to the node-free circular Dirac excited states, for which $r_{\mathrm{mp}}^{D}(n,\kappa=-n) = (a_0/Z)n\sqrt{n^2-Z^2\alpha^2}$. We emphasize throughout that most-probable radii are distinct from expectation values and from empirical radii of many-electron atoms.

Keywords: Bohr radius; relativistic corrections; reduced Compton wavelength; Dirac equation; most-probable radius; excited circular states; hydrogen-like ions; Rydberg formula

1. The Bohr Radius

Bohr [1] introduced his atomic model in 1913. In the second part of his trilogy, he derived the radius now known as the Bohr radius,

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{e^2 m_e}. \tag{1}$$

using the fine-structure constant, this can be rewritten as

$$a_0 = \frac{\hbar}{m_e c\alpha} = \frac{\bar{\lambda}_e}{\alpha} \approx 5.29\times10^{-11}\ \mathrm{m}, \tag{2}$$

where $\epsilon_0 = 1/(\mu_0 c^2)$ is the permittivity of free space, α is the fine-structure constant, m_e is the electron rest mass, e is the elementary charge, $\hbar = h/(2\pi)$ is the reduced Planck constant, and $\bar{\lambda}_e = \hbar/(m_e c)$ is the reduced Compton wavelength of the electron. The Compton wavelength itself was not introduced until 1923 by Arthur H. Compton [2], a decade after Bohr's original papers.

The purpose of the paper is not to claim that the Bohr model derives the Dirac [3] equation. Rather, the paper compares two independently obtained radius concepts: a relativistic Bohr radius derived from relativistic momentum, circular Coulomb motion, and angular-momentum quantization, and a Dirac most-probable radius obtained by

Check for updates

Academic Editors: Chengjian Lin and Giuseppe Mandaglio

Received: 3 June 2026
Revised: 3 July 2026
Accepted: 15 July 2026
Published: 17 July 2026

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maximizing the radial probability density. The central point is that these two routes lead to the same analytic radius for the $1s_{1/2}$ state under the stated idealizations. The two radii are obtained by independent calculations and then compared. The question addressed is whether Bohr's relativistic circular-orbit prescription selects the same characteristic radial scale as the maximum of the Dirac probability distribution. Their exact equality provides a direct analytic link between the semi-classical construction and the relativistic quantum-mechanical result under the stated assumptions.

The scope of the result beyond the ground state is the class of node-free circular Dirac states, $n \geq 2$, $j = n - 1/2$, and $\kappa = -n$, where a single outer probability maximum has a direct Bohr-like interpretation. General excited states with radial nodes have several extrema and therefore do not correspond to one unique Bohr-like orbital radius in the same simple sense. This limitation is important for the interpretation of the result.

When the finite proton mass is included, the reduced-mass-corrected Bohr radius for the hydrogen atom is

$$a_0^* = \frac{m_e}{\mu} a_0 = \frac{\bar{\lambda}_e}{\alpha}\left(1 + \frac{\bar{\lambda}_P}{\bar{\lambda}_e}\right) = \frac{\bar{\lambda}_e + \bar{\lambda}_P}{\alpha}, \tag{3}$$

where $\mu = m_e m_P/(m_e + m_P)$ is the reduced mass and $\bar{\lambda}_P$ is the reduced Compton wavelength of the proton. According to NIST CODATA, the Bohr radius is

$$a_0 = 5.29177210544(82) \times 10^{-11}\ \text{m}.$$

Its uncertainty is linked mainly to the uncertainties in the fine-structure constant and the electron reduced Compton wavelength.

There are several equivalent ways to obtain the Bohr radius. One standard route equates the Coulomb force with the centripetal force,

$$k_e \frac{e^2}{r^2} = m_e \frac{v^2}{r}, \tag{4}$$

$$\frac{\hbar c \alpha}{r^2} = m_e \frac{v^2}{r}. \tag{5}$$

For the ground state in the Bohr model, $v = \alpha c$, and therefore

$$r = \frac{\hbar c \alpha}{m_e c^2 \alpha^2} = \frac{\hbar}{m_e c \alpha} = \frac{\bar{\lambda}_e}{\alpha}. \tag{6}$$

Another derivation uses de Broglie's standing-wave condition. If the electron circumference is an integer multiple of the de Broglie [4,5] wavelength $\lambda_b = h/(mv)$, then

$$2\pi r = n \frac{h}{m_e v}. \tag{7}$$

Equivalently, the angular momentum is quantized:

$$m_e v r = n\hbar. \tag{8}$$

Solving for v gives

$$v = \frac{n\hbar}{m_e r}. \tag{9}$$

https://doi.org/10.3390/atoms14070059

Substituting this expression into the force-balance condition gives

$$\frac{m_e}{r}\left(\frac{n\hbar}{m_e r}\right)^2 = \frac{\hbar c\alpha}{r^2}, \tag{10}$$

$$r_n = n^2 \frac{\hbar}{m_e c\alpha} = n^2 \frac{\bar{\lambda}_e}{\alpha}. \tag{11}$$

Thus the first Bohr radius is obtained for $n = 1$.

The Schrödinger [6] equation confirms the Bohr radius as the most-probable radius for the electron in the hydrogen ground state. However, the Schrödinger equation is non-relativistic. This leaves open the question of how the Bohr-radius scale should be adjusted when relativistic momentum is included.

The main result of the present paper is that Bohr's own relativistic prescription leads to precisely the same radius as the most-probable radius of the Dirac $1s_{1/2}$ radial probability density for a point-nucleus, one-electron Coulomb system. This turns the relativistic Bohr radius from a purely historical curiosity into a compact bridge between the semi-classical Bohr model and the relativistic Dirac description.

2. The Relativistic Bohr Radius and the Relativistic Compton Wavelength

Bohr was aware that his stated radius formula rested on non-relativistic assumptions. In Part II of his trilogy, he indicated that the formula could be extended to cases in which the orbital velocity is significant relative to the speed of light by replacing the mass term with

$$\frac{m}{\sqrt{1-\frac{v^2}{c^2}}} = m\gamma, \tag{12}$$

where

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \tag{13}$$

is the Lorentz factor.

It is important to distinguish this historical prescription from later debates over "relativistic mass." The expression $m\gamma$ is naturally understood here through relativistic momentum,

$$p = m\gamma v, \tag{14}$$

which was introduced by Planck [7]. The use of $m\gamma$ in the orbital momentum does not require treating relativistic mass as an ontologically separate mass concept; it is enough to use the standard relativistic momentum.

A simple replacement $m_e \mapsto \gamma m_e$ would be unsatisfactory if it were used only as an unexplained substitution. In the present setting, however, the replacement is not an independent postulate: it is just the use of relativistic momentum in the same circular-orbit calculation. For a hydrogen-like ion the Coulomb [8] force balance and the relativistic circular angular-momentum condition are

$$\frac{Ze^2}{4\pi\epsilon_0 r^2} = \frac{\gamma m_e v^2}{r}, \tag{15}$$

$$\gamma m_e v r = n\hbar. \tag{16}$$

Using $e^2/(4\pi\epsilon_0) = \hbar c\alpha$, Equations (15) and (16) give

$$\frac{Z\hbar c\alpha}{r} = \gamma m_e v^2, \qquad r = \frac{n\hbar}{\gamma m_e v}, \tag{17}$$

https://doi.org/10.3390/atoms14070059

and hence

$$\frac{v}{c} = \frac{Z\alpha}{n}. \tag{18}$$

Thus the velocity used below is not inserted by hand. It follows from the same two semi-classical equations used in the ordinary Bohr derivation, with $m_e v$ replaced by the relativistic momentum $\gamma m_e v$.

Following Bohr's stated prescription in this relativistic-momentum sense, the relativistically adjusted circular radius becomes

$$r_{n,r}^{(Z)} = n^2 \frac{\hbar}{m_e \gamma c Z\alpha} = \frac{a_0 n^2}{Z}\sqrt{1-\left(\frac{Z\alpha}{n}\right)^2}. \tag{19}$$

For the hydrogen ground state, $Z = 1$ and $n = 1$, so

$$a_{0,r} = \frac{4\pi\epsilon_0\hbar^2}{e^2 m_e \gamma} = \frac{\hbar}{m_e \gamma c\alpha} = \frac{\bar{\lambda}_e\sqrt{1-\alpha^2}}{\alpha} = a_0\sqrt{1-\alpha^2} \approx 5.29149 \times 10^{-11}\ \text{m}. \tag{20}$$

This derivation is still semi-classical, and it is not claimed to derive the Dirac wavefunction. The claim made here is narrower: the semi-classical relativistic circular radius is derived independently, and then the Dirac radial probability density is maximized independently. The equality between the two final formulae is the result to be explained and documented.

The difference between the non-relativistic and relativistic expressions is

$$\begin{aligned} a_0 - a_{0,r} &= \frac{\bar{\lambda}_e}{\alpha} - \frac{\bar{\lambda}_e\sqrt{1-\alpha^2}}{\alpha} \\ &= \frac{\bar{\lambda}_e}{\alpha}\left(1-\sqrt{1-\alpha^2}\right) \\ &\approx \frac{\bar{\lambda}_e\alpha}{2}. \end{aligned} \tag{21}$$

Numerically, this is approximately

$$a_0 - a_{0,r} \approx 1.41 \times 10^{-15}\ \text{m}.$$

The relative correction is

$$\frac{a_0 - a_{0,r}}{a_0} = 1 - \sqrt{1-\alpha^2} \approx \frac{\alpha^2}{2} \approx 2.66 \times 10^{-5}, \tag{22}$$

or about 0.00266%.

The same conclusion may be expressed through the relativistic de Broglie relation,

$$\lambda_b = \frac{h}{\gamma m_e v}, \tag{23}$$

rather than the non-relativistic approximation $\lambda_b = h/(m_e v)$. The standing-wave condition $2\pi r = n\lambda_b$ is exactly Equation (16), and therefore again gives $v/c = Z\alpha/n$ and Equation (19). This is the sense in which the mass replacement is justified: it is shorthand for replacing the non-relativistic momentum by the relativistic momentum, not a separate assumption designed to fit the Dirac answer.

The factor

$$\bar{\lambda}_{e,r} = \bar{\lambda}_e\sqrt{1-\frac{v^2}{c^2}} = \frac{\hbar}{m_e\gamma c} \tag{24}$$

is identical in form to the relativistically adjusted reduced Compton wavelength recently discussed by Haug. In ordinary Compton scattering, Compton [2] considered the scattering

of photons by electrons initially treated as stationary. A relativistic Compton wavelength instead takes into account the electron's motion at the time of interaction. The ordinary Compton wavelength is

$$\lambda = \frac{h}{mc}, \tag{25}$$

whereas the corresponding relativistic expression is (see [9] for full derivation)

$$\lambda_r = \frac{h}{m\gamma c}. \tag{26}$$

3. Relation to Relativistic Rydberg Formulae

The traditional Rydberg [10] formula is also non-relativistic in origin. It has often been used to infer the electron mass and, indirectly, the electron Compton wavelength. Bohr relied heavily on the Rydberg formula in developing his atomic model, but his explicit relativistic correction to the formula appears only in his 15 April 1914 letter to Fowler.

In modern notation, the standard Rydberg relation for hydrogenic transitions can be written as

$$\frac{1}{\lambda} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right), \tag{27}$$

where R is the appropriate Rydberg constant, with reduced-mass corrections included when necessary. Bohr's early relativistic attempt introduced a correction term of the general form

$$\nu = \frac{2\pi^2 m e^4 N^2}{\hbar^3}\frac{M}{M+m}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)\left[1 + \frac{\pi^2 e^4 N^2}{\hbar^2 c^2}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)\right], \tag{28}$$

where the notation follows the older form used by Bohr. This correction did not provide satisfactory agreement with the observed spectral data.

More recent work has proposed relativistically adjusted Rydberg formulae [11]. Such formulae have also been discussed in connection with high-n Rydberg transition spectroscopy in high-energy plasma contexts, including W7-X stellarator studies [12,13]. In such systems, fully quantum-electrodynamic calculations can become highly demanding, especially for large atoms and high-energy environments. Relativistically corrected semi-classical expressions may therefore be useful as approximate tools, provided their assumptions and limitations are made explicit.

The main point of the present paper is not that the Bohr model replaces modern relativistic quantum theory. Rather, the point is that Bohr's own suggested relativistic adjustment leads directly to a length scale of the form

$$\frac{\bar{\lambda}_{e,r}}{\alpha},$$

where $\bar{\lambda}_{e,r}$ is a relativistically contracted reduced Compton wavelength. This link clarifies why the Bohr radius, the Compton wavelength, and the Rydberg constant are not independent conceptual objects, but closely connected expressions of the same underlying constants.

4. Comparison with Dirac-Equation Radius Concepts

The decisive question is whether the semi-classical radius in Equation (20) corresponds to any well-defined radius concept in relativistic quantum mechanics. Since the Dirac equation provides the standard relativistic one-electron description of hydrogen-like atoms, the appropriate comparison is with radius measures derived from the Dirac wavefunction.

There is not a single unique "Dirac radius." Several different quantities can be defined. The expectation value $\langle r \rangle_D$ measures the mean distance from the nucleus. The most-probable radius, by contrast, is the radius at which the radial probability density is maximal. These two quantities are already different in the non-relativistic Schrödinger theory: for the hydrogen ground state the most-probable radius is a_0, whereas the expectation value is $3a_0/2$.

The result below shows that the relativistic circular Bohr calculation singles out the Dirac most-probable radius, not the Dirac expectation value. Equation (19) is obtained first from relativistic momentum plus circular quantization. The Dirac result is then obtained independently by differentiating the radial probability density. The argument therefore does not claim that the Bohr model derives the Dirac equation; it makes the more limited, but precise, statement that the two independently derived radius formulae are analytically identical for the stated states and assumptions. For a point nucleus, infinite nuclear mass, and a one-electron Coulomb field, the semi-classical expression gives

$$a_{0,r}^{(Z)} = \frac{a_0}{Z}\sqrt{1 - Z^2\alpha^2}, \tag{29}$$

and this is exactly the maximum of the Dirac $1s_{1/2}$ radial probability density. Existing work on Dirac-based corrections to the Bohr radius, such as Buitrago [14], provides useful reference formulas for expectation and maximum-probability distances. The additional point emphasized here is the explicit identification of Bohr's historical relativistic prescription, understood as the use of relativistic momentum, with the Dirac most-probable radius.

4.1. Numerical Comparison for Hydrogen-like Ground States

A useful numerical comparison can be made for hydrogen-like one-electron ions in the Dirac $1s_{1/2}$ ground state. This is the cleanest case because there is only one electron and no electronic screening or shell-structure effects. The comparison should therefore not be confused with empirical atomic radii of neutral atoms, which can vary non-monotonically with Z because of shell filling and many-electron interactions.

For a point nucleus and infinite nuclear mass, the Dirac radial probability density for the $1s_{1/2}$ state may be written, up to a normalization constant, as

$$P_D(r) \propto r^{2\gamma_Z} \exp\left(-\frac{2Zr}{a_0}\right), \qquad \gamma_Z = \sqrt{1 - Z^2\alpha^2}. \tag{30}$$

The maximum of this radial probability density is therefore obtained from

$$\frac{d}{dr}\left[r^{2\gamma_Z} \exp\left(-\frac{2Zr}{a_0}\right)\right] = 0. \tag{31}$$

Equivalently, taking the logarithmic derivative gives

$$\frac{d}{dr}\ln P_D(r) = \frac{2\gamma_Z}{r} - \frac{2Z}{a_0} = 0, \tag{32}$$

so the maximum is fixed by the shape of the Dirac radial distribution itself, not by fitting the Bohr expression after the fact.

This gives

$$r_{\mathrm{mp}}^{D} = \frac{a_0}{Z}\gamma_Z = \frac{a_0}{Z}\sqrt{1 - Z^2\alpha^2}. \tag{33}$$

This is the same functional form as the semi-classical relativistic Bohr radius obtained by replacing α with $Z\alpha$ in Equation (20):

$$a_{0,r}^{(Z)} = \frac{a_0}{Z}\sqrt{1 - Z^2\alpha^2}. \tag{34}$$

Thus, within the assumptions of a point nucleus, infinite nuclear mass, and a one-electron Coulomb field, the semi-classical relativistic Bohr radius is exactly identical to the Dirac most-probable radius for the $1s_{1/2}$ radial probability distribution. This equality is an analytic identification between two derived quantities, rather than a derivation of the Dirac result from Bohr's model alone.

The Dirac expectation value of the radius is different. From the same radial probability density one obtains

$$\langle r \rangle_D = \frac{a_0}{Z}\frac{2\gamma_Z + 1}{2}. \tag{35}$$

This reduces to the familiar non-relativistic Schrödinger value $\langle r \rangle = 3a_0/(2Z)$ in the limit $Z\alpha \ll 1$, but it is not identical to the most-probable radius. The distinction is important: Equation (34) agrees with the Dirac most-probable radius in Equation (33), but it should not be identified with the Dirac expectation value in Equation (35).

Using the 2022 CODATA values $\alpha = 7.2973525643 \times 10^{-3}$ m, $a_0 = 5.29177210544 \times 10^{-11}$ m, and $\bar{\lambda}_e = 3.8615926744 \times 10^{-13}$ m [15], the hydrogen ground-state result is

$$\gamma_1 = \sqrt{1 - \alpha^2} = 0.9999733739683034. \tag{36}$$

The relativistic Bohr radius, which is identical to the most-probable radius of the Dirac $1s_{1/2}$ radial probability distribution, is

$$a_{0,r} = r_{\mathrm{mp},D} = a_0\gamma_1 = 5.29163120655 \times 10^{-11}, \mathrm{m}. \tag{37}$$

Consequently,

$$a_0 - a_{0,r} = 1.40898891811 \times 10^{-15}, \mathrm{m}, \tag{38}$$

and the relative correction is

$$\frac{a_0 - a_{0,r}}{a_0} = 2.66260317 \times 10^{-5}, \tag{39}$$

or approximately 0.0026626%. For comparison, the expectation value of the radius is a different statistical quantity. For the Dirac $1s_{1/2}$ state, it is

$$\langle r \rangle_D = \frac{a_0}{2}(2\gamma_1 + 1) = 7.93751725927 \times 10^{-11}, \mathrm{m}, \tag{40}$$

whereas the non-relativistic Schrödinger expectation value is

$$\langle r \rangle_S = \frac{3a_0}{2} = 7.93765815816 \times 10^{-11}, \mathrm{m}. \tag{41}$$

The difference between the expectation values is therefore

$$\langle r \rangle_S - \langle r \rangle_D = a_0(1 - \gamma_1) = 1.40898891811 \times 10^{-15}, \mathrm{m}. \tag{42}$$

The numerical difference between $r_{\mathrm{mp},D}$ and $\langle r \rangle_D$ should not be interpreted as a disagreement between the relativistic Bohr and Dirac results, because the former is a most-probable radius whereas the latter is a mean radius.

4.2. Extension to Excited Circular States

The preceding comparison establishes the result for the $1s_{1/2}$ ground state. A natural question is whether the same Bohr–Dirac identification is only a ground-state accident or whether it also survives for excited states. The clean excited-state test is supplied by the hydrogen-like circular states. In the old quantum-theory language these are the states with maximal angular momentum for a given principal quantum number; in the Dirac notation they correspond to

$$j = n - \frac{1}{2}, \qquad \kappa = -n, \qquad n = 1, 2, 3, \ldots. \tag{43}$$

For these states the radial quantum number is zero. The Dirac radial probability density therefore has no Laguerre-polynomial radial nodes and may be written, up to normalization, in the form

$$P^{\mathrm{circ}}_{D,n}(r) \propto r^{2\gamma_n} \exp(-2\lambda_n r), \qquad \gamma_n = \sqrt{n^2 - Z^2\alpha^2}, \qquad \lambda_n = \frac{Z}{a_0 n}. \tag{44}$$

Its maximum is obtained from $dP^{\mathrm{circ}}_{D,n}/dr = 0$, which gives

$$r^D_{\mathrm{mp}}(n, \kappa = -n) = \frac{\gamma_n}{\lambda_n} = \frac{a_0}{Z} n\sqrt{n^2 - Z^2\alpha^2} = \frac{a_0 n^2}{Z}\sqrt{1 - \left(\frac{Z\alpha}{n}\right)^2}. \tag{45}$$

This is precisely the excited-state circular-orbit extension of Bohr's relativistic prescription, since the non-relativistic Bohr radius for the nth circular orbit is $a_0 n^2/Z$ and the corresponding Bohr speed is $v_n = Z\alpha c/n$:

$$r^{(Z)}_{n,r} = \frac{a_0 n^2}{Z}\sqrt{1 - \left(\frac{Z\alpha}{n}\right)^2} = r^D_{\mathrm{mp}}(n, \kappa = -n). \tag{46}$$

Thus the agreement is not restricted to the ground state. It extends exactly to the node-free circular Dirac series. The restriction to circular states is important: generic excited states with lower angular momentum have radial nodes and several local extrema in their radial probability densities, so they are not represented by a single Bohr-like circular radius.

For hydrogen, the first few circular-state values are given in Table 1.

Table 1. Excited-state check for hydrogen circular states ($Z = 1$, $\kappa = -n$). The relativistic Bohr circular radius and the Dirac most-probable radius have the same analytic value.

n	Non-Relativistic $a_0 n^2$ (m)	$r_{n,r} = r^D_{\mathrm{mp}}$ (m)	Relative Correction
1	$5.291772105440 \times 10^{-11}$	$5.291631206548 \times 10^{-11}$	2.6626×10^{-5}
2	$2.116708842176 \times 10^{-10}$	$2.116694752428 \times 10^{-10}$	6.6564×10^{-6}
3	$4.762594894896 \times 10^{-10}$	$4.762580805174 \times 10^{-10}$	2.9584×10^{-6}
4	$8.466835368704 \times 10^{-10}$	$8.466821278991 \times 10^{-10}$	1.6641×10^{-6}
5	$1.322943026360 \times 10^{-9}$	$1.322941617389 \times 10^{-9}$	1.0650×10^{-6}

4.3. Why the Schrödinger Expectation Value Is Larger

The non-relativistic Schrödinger value $3a_0/(2Z)$ is correct, but it is not the quantity that should be compared directly with the Bohr radius. The Bohr radius corresponds to the maximum of the radial probability distribution for the hydrogenic $1s$ state, not to the mean radius. For the Schrödinger $1s$ wavefunction, the radial probability density is

$$P_S(r) \propto r^2 \exp\left(-\frac{2Zr}{a_0}\right). \tag{47}$$

Maximizing this expression gives

$$r_{\mathrm{mp}}^{S} = \frac{a_0}{Z}, \tag{48}$$

which is the usual Bohr-radius result for a hydrogen-like ion. However, the expectation value is obtained by averaging over the whole radial distribution,

$$\langle r \rangle_S = \frac{\int_0^\infty r P_S(r)\, dr}{\int_0^\infty P_S(r)\, dr} = \frac{3a_0}{2Z}. \tag{49}$$

It is therefore larger because the radial probability distribution has a long tail toward larger radii. The peak of the distribution occurs at a_0/Z, but the average is pulled outward by this tail. This is not a relativistic effect; it already occurs in the ordinary non-relativistic hydrogen ground state.

The same distinction remains in the Dirac theory. The relativistic Bohr radius derived in this paper agrees with the Dirac most-probable radius r_{mp}^{D}, not with the Dirac mean radius $\langle r \rangle_D$. Thus, the fact that $3a_0/(2Z)$ or $\langle r \rangle_D$ is larger than $a_{0,r}^{(Z)}$ is not a contradiction. It simply reflects the difference between the location of the maximum probability and the average position of the electron.

The numerical sequence in Table 2 is the direct comparison relevant to the present paper. It should decrease smoothly with Z. If a table of "Dirac radii" appears to jump up and down as one moves to higher elements, then it is almost certainly mixing different physical quantities, such as neutral-atom radii, screened many-electron radii, empirical covalent or van der Waals radii, or radii belonging to different orbitals.

Table 2. Direct comparison between the semi-classical relativistic Bohr radius and the Dirac most-probable radius for hydrogen-like $1s_{1/2}$ states. Only the directly comparable radius is tabulated here. The sequence is monotonic: as Z increases, both a_0/Z and $a_{0,r}^{(Z)} = r_{\mathrm{mp}}^{D}$ decrease smoothly.

Z	γ_Z	a_0/Z (m)	$a_{0,r}^{(Z)} = r_{\mathrm{mp}}^{D}$ (m)	Relativistic Reduction
1	0.999973374	$5.291772106 \times 10^{-11}$	$5.291631207 \times 10^{-11}$	0.002663%
2	0.999893492	$2.645886053 \times 10^{-11}$	$2.645604244 \times 10^{-11}$	0.010651%
10	0.997333878	$5.291772106 \times 10^{-12}$	$5.277663596 \times 10^{-12}$	0.266612%
20	0.989292403	$2.645886053 \times 10^{-12}$	$2.617554971 \times 10^{-12}$	1.070760%
50	0.931059404	$1.058354421 \times 10^{-12}$	$9.853908366 \times 10^{-13}$	6.894060%
80	0.811905987	$6.614715132 \times 10^{-13}$	$5.370526817 \times 10^{-13}$	18.809401%
92	0.741134627	$5.751926202 \times 10^{-13}$	$4.262951682 \times 10^{-13}$	25.886537%
118	0.508456626	$4.484552632 \times 10^{-13}$	$2.280200501 \times 10^{-13}$	49.154337%

The expectation value $\langle r \rangle_D$ is a different radius concept and should not be placed in the same column as the Bohr-radius comparison. For the same idealized $1s_{1/2}$ Dirac–Coulomb state it is

$$\langle r \rangle_D = \frac{a_0}{Z} \frac{2\gamma_Z + 1}{2}. \tag{50}$$

This quantity is larger than the most-probable radius, just as in the non-relativistic Schrödinger hydrogen ground state where $\langle r \rangle = 3a_0/2$ while the most-probable radius is a_0. Therefore $\langle r \rangle_D$ is useful as a separate Dirac benchmark, but it is not the radius that coincides with Equation (20).

The high-Z entries in Table 2 should nevertheless be interpreted as idealized point-nucleus Dirac–Coulomb estimates. For large Z, finite nuclear size, reduced-mass effects, QED corrections, and nuclear-structure effects become increasingly important. In addition, the simple expression above applies only while $Z\alpha < 1$ and only to hydrogen-like one-electron ions. It should not be extrapolated directly to neutral atoms or to multi-electron ions without including screening, electron–electron interaction, and shell structure.

5. Discussion

The correction derived above is small for ordinary hydrogen, but it is not conceptually negligible. For hydrogen, the correction to the most-probable radius is approximately 1.41×10^{-15} m, corresponding to about 2.66×10^{-5} of the ordinary Bohr radius, or 0.0026626%. This length is comparable to nuclear length scales, even though it is tiny compared with the Bohr radius itself.

The key result is stronger than a mere numerical similarity, but it should also be stated carefully. The relativistic Bohr expression is derived from the relativistic momentum equation and circular quantization, while the Dirac radius is derived by maximizing the Dirac radial probability density. The argument therefore does not state that the substitution $m_e \mapsto \gamma m_e$ by itself proves the Dirac result. Rather, once the substitution is understood as the standard relativistic momentum replacement, the resulting circular radius is identical to the independently obtained Dirac probability maximum:

$$a_{0,r}^{(Z)} = r_{\mathrm{mp}}^{D} = \frac{a_0}{Z}\sqrt{1 - Z^2\alpha^2}. \tag{51}$$

This gives a precise interpretation to the relativistic Bohr radius: in the ideal one-electron Dirac–Coulomb problem, it is the most-probable radius, not the expectation value. The distinction matters because $\langle r \rangle_D$ is larger than r_{mp}^{D}, just as in the non-relativistic hydrogen ground state the mean radius $3a_0/2$ is larger than the most-probable radius a_0. The larger expectation value should therefore not be interpreted as a failure of the Bohr-radius comparison; it answers a different question, namely the average radius of the probability distribution rather than the radius at which the probability density is maximal.

The numerical values for hydrogen-like ions should not be confused with atomic radii of real neutral elements. In the hydrogen-like one-electron sequence, the directly comparable radius $a_{0,r}^{(Z)} = r_{\mathrm{mp}}^{D}$ varies smoothly with Z and decreases as Z increases. Non-monotonic behavior, such as radii that jump up and down across the periodic table, is expected only when one includes many-electron effects, shell filling, electronic screening, changes of orbital, or empirical radius definitions. These are different physical quantities from the hydrogen-like Dirac radii considered here.

Several qualifications are essential. First, the Bohr model remains semi-classical and cannot reproduce the full spectrum, spin structure, or full wavefunction content of relativistic hydrogen. Second, the Dirac equation already provides the standard relativistic quantum-mechanical treatment of hydrogen-like atoms. Third, the exact equality found here applies specifically to the maximum of the $1s_{1/2}$ radial probability density and, as shown above, to the node-free excited circular Dirac states with $\kappa = -n$. It should not be read as a statement about all excited Dirac states, because states with radial nodes have multiple extrema rather than one unique Bohr-like radius. For high-Z ions, finite nuclear size, reduced-mass effects, QED corrections, and nuclear-structure effects should be included before comparing with precision spectroscopic data.

Subject to these qualifications, the result has a specific but substantial significance. It establishes an exact and apparently little-noted identity between a historical semi-classical construction and a physically distinguished characteristic of the Dirac–Coulomb radial distribution. The result therefore goes beyond a historical curiosity: it identifies a precise domain in which relativistic Bohr reasoning reproduces a nontrivial prediction of relativistic quantum mechanics. This is relevant to the conceptual relation between semi-classical and quantum descriptions, to the interpretation of relativistic orbital contraction, and to understanding why the Bohr model can remain unexpectedly accurate for certain carefully defined quantities. At the same time, the restricted scope of the equality clarifies where circular-orbit intuition succeeds and where it must be abandoned.

6. Conclusions

We have revisited Bohr's little-discussed relativistic prescription for the Bohr radius. The ordinary Bohr radius,

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{e^2 m_e} = \frac{\hbar}{m_e c\alpha} = \frac{\bar{\lambda}_e}{\alpha}, \tag{52}$$

is based on a non-relativistic momentum scale. This is not presented as an unexplained mass replacement. The factor γm_e enters because the circular-orbit momentum is $p = \gamma m_e v$, and the simultaneous use of force balance and angular-momentum quantization gives $v/c = Z\alpha/n$. If the relativistic momentum factor is included, the ground-state radius becomes

$$a_{0,r} = \frac{\bar{\lambda}_e\sqrt{1-\alpha^2}}{\alpha} = a_0\sqrt{1-\alpha^2}. \tag{53}$$

For hydrogen-like one-electron ions this generalizes to

$$a_{0,r}^{(Z)} = \frac{a_0}{Z}\sqrt{1-Z^2\alpha^2}. \tag{54}$$

The main result of the paper is that this expression is exactly equal to the most-probable radius obtained by independently maximizing the Dirac $1s_{1/2}$ radial probability density, under the assumptions of a point nucleus, infinite nuclear mass, and a one-electron Coulomb field:

$$a_{0,r}^{(Z)} = r_{\mathrm{mp}}^{D}. \tag{55}$$

This equality follows from comparing two independently derived radii; it is not a derivation of the Dirac radius from the Bohr model, and no full dynamical equivalence between the two theories is implied. Nevertheless, the exact identity is significant because it identifies the precise Dirac quantity reproduced by Bohr's relativistic prescription and delineates the class of states for which this agreement holds.

The corresponding circular-state result is

$$r_{n,r}^{(Z)} = \frac{a_0 n^2}{Z}\sqrt{1-\left(\frac{Z\alpha}{n}\right)^2} = r_{\mathrm{mp}}^{D}(n,\kappa=-n), \tag{56}$$

so the agreement is not a ground-state-only coincidence. It holds for the complete node-free circular Dirac sequence. The conclusion is nevertheless intentionally limited to that sequence; generic excited states with radial nodes require a different comparison because their radial probability densities are multi-peaked.

It is not equal to the Dirac expectation value $\langle r\rangle_D$, which is a different radius measure. This distinction resolves possible confusion about which Dirac radius is being compared with the relativistic Bohr radius. The same distinction is already present in the non-relativistic Schrödinger theory, where the most-probable radius is a_0/Z but the expectation value is $3a_0/(2Z)$ because the radial probability distribution has an extended tail toward larger radii.

For hydrogen, the correction is

$$a_0 - a_{0,r} = \frac{\bar{\lambda}_e(1-\sqrt{1-\alpha^2})}{\alpha} \approx \frac{\bar{\lambda}_e\alpha}{2} \approx 1.41\times10^{-15}\ \mathrm{m}, \tag{57}$$

corresponding to approximately 0.0026626% of the ordinary Bohr radius. Although small, the correction is of historical and comparative interest because it identifies an exact link between Bohr's semi-classical relativistic prescription and the Dirac–Coulomb probability maximum. The relativistic Bohr argument alone is not a proof of the Dirac radius; the

proof of the radius maximum comes from the Dirac radial probability density, while the Bohr-side calculation supplies an independently derived matching length scale.

To our knowledge, this is the first explicit demonstration that Bohr's historical relativistic radius prescription yields the Dirac most-probable radius for the hydrogen-like $1s_{1/2}$ state. The added circular-state derivation further shows that the same identification extends to excited node-free circular states, thereby removing any ground-state-only limitation from the stated result. The result should be understood within its idealized assumptions and should not be generalized directly to neutral atoms, many-electron systems, empirical atomic radii, or high-Z precision spectroscopy without including reduced-mass, finite-nuclear-size, QED, and many-body effects.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study.

Conflicts of Interest: The authors declare no conflicts of interest.

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