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Super Riemann Surfaces and Fatgraphs

Albert S. Schwarz [1,*] and Anton M. Zeitlin [2]

1 Department of Mathematics, University of California at Davis, Davis, CA 95616, USA
2 Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA; zeitlin@lsu.edu
* Correspondence: schwarz@math.ucdavis.edu

Abstract: Our goal is to describe superconformal structures on super Riemann surfaces (SRSs) based on data assigned to a fatgraph. We start from the complex structures on punctured $(1|1)$-supermanifolds, characterizing the corresponding moduli and the deformations using Strebel differentials and certain Čech cocycles for a specific covering, which we reproduce from fatgraph data, consisting of $U(1)$-graph connection and odd parameters at the vertices. Then, we consider dual $(1|1)$-supermanifolds and related superconformal structures for $N = 2$ super Riemann surfaces. The superconformal structures, $N = 1$ SRS, are computed as the fixed points of involution on the supermoduli space of $N = 2$ SRS.

Keywords: moduli space; supergeometry; string theory

1. Introduction

1.1. Some History and Earlier Results

The geometry of the moduli spaces of (punctured) Riemann surfaces has been a central topic in modern mathematics for many years. Since the 1980s, string theory served as a significant source of ideas in studying moduli spaces. For a proper description of string theory, one has to consider certain generalizations of moduli spaces related to the fact that strings, while propagating, should carry extra anticommutative parameters, thus generating what is known as a superconformal manifold as introduced by M.A. Baranov and A.S. Schwarz [1] or a super Riemann surface (SRS) as independently introduced by D. Friedan [2] (see also [3–8] for a review). It turned out that such spaces' geometry is quite involved; see, e.g., [9]. An important task is, of course, related to the parametrization of such supermoduli.

There are several ways of looking at the parametrization problem. For example, one could deal with supermoduli spaces of punctured Riemann surfaces with the negative Euler characteristic from the point of view of higher Teichmüller theory as a subset in the character variety for the corresponding supergroup. In the case of original moduli spaces using methods of hyperbolic geometry, R. Penner described coordinates in the universal cover of moduli space, the Teichmüller space, as the subspace of the character variety of $PSL(2,\mathbb{R})$, so that the corresponding Riemann surfaces appear here from the uniformization point of view as a factor of the upper half-plane via the element of the related character variety, i.e., the Fuchsian subgroup [10].

The action of the mapping class group in these coordinates is rational. It could be described combinatorially using decorated triangulations or dual objects, known as metric fatgraphs or ribbon graphs for the corresponding Riemann surfaces. Thus constructed coordinates were generalized to the case of reductive groups [11]. The supergroup case has remained a mystery until recently. In [12–14], such coordinates were constructed in the framework of the higher Teichmüller spaces associated with supergroups $OSp(1|2)$ and $OSp(2|2)$, which correspond to the Teichmüller spaces $N = 1$ and $N = 2$ SRSs. The desired $N = 1$ and $N = 2$ SRSs could be reconstructed using the elements of character variety via the appropriately modified uniformization approach [6,15].

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Citation: Schwarz, A.S.; Zeitlin, A.M. Super Riemann Surfaces and Fatgraphs. *Universe* **2023**, *9*, 384. https://doi.org/10.3390/universe9090384

Academic Editor: Gerald B. Cleaver

Received: 5 July 2023
Revised: 24 August 2023
Accepted: 24 August 2023
Published: 26 August 2023

There is a different, more "hands-on" approach to the moduli spaces of punctured Riemann surfaces, where one can directly see the transition functions for the corresponding complex structures, which we discuss in more detail below. One can start from the parameterization of moduli spaces via the so-called Strebel differentials, which again can be described using metric fatgraphs [16]. This approach allowed (see [17]) us to explicitly "glue" the Riemann surface by constructing transition functions.

In this paper, we want to generalize this construction in the case of super Riemann surfaces. We start by describing the moduli space of $(1|1)$-supermanifolds. This result also describes the moduli space of $N = 2$ super Riemann surfaces. Finally, we study the moduli space of $N = 1$ super Riemann surfaces using the fact that this space can be obtained as a set of fixed points of the involution of the space of $(1|1)$-supermanifolds constructed in [7].

We would also like to mention recent progress in studying supermoduli spaces from various perspectives. While our approach deals with the real parametrization of supermoduli in parallel to work on super-Teichmüller theory [12–14], a lot of exciting features of supermoduli are related to the algebro-geometric description. The main result of Donagi and Witten [18] that supermoduli space is not projected led to a renewed interest in the subject in the modern era and supergeometry in general. One can mention recent works by Felder, Kazhdan, and Polishchuk dealing with the Schottky approach for supermoduli [19], as well as the general treatment of supermoduli spaces as Deligne–Mumford stacks [20]. Some other recent results, which use both real and complex geometry points of view and are related to enumerative invariants related to supermoduli [21,22].

1.2. The Structure of the Paper and Main Results

In Section 2, we review basic notions related to $(1|1)$-supermanifolds, $N = 1$ and $N = 2$ super Riemann surfaces (SRS). We devote special attention to the punctured $N = 1$ SRS with two puncture classes corresponding to various spin structure choices: Ramond (R) and Neveu–Schwarz (NS).

In Section 3, we define two instrumental objects that come from geometric topology. The first object is a fatgraph (or ribbon graph). This graph is homotopically equivalent to the punctured surface with the cyclic ordering of half-edges at every vertex, which comes from the orientation of the surface so that each puncture is associated with a particular cycle on the graph. The second object is a spin structure on the fatgraph, making it a *spin fatgraph*. We describe spin structures as the classes of orientations on fatgraphs based on the works [12–14], where $N = 1$, $N = 2$ SRSs were studied from a uniformization perspective. This construction allows distinguishing boundary components of such spin fatgraphs, separating them into two sets based on comparing their orientation and the orientation induced by the surface. Those two sets correspond to NS and R punctures in the uniformization picture.

Section 4 is devoted to an important construction allowing us to relate the data assigned to the fatgraphs to the theory of moduli of Riemann surfaces following [16,17]. Specifically, we explicitly describe the moduli spaces of surfaces F^c with marked points, using special covering $\{U_v, V_p\}$, with one neighborhood U_v for every vertex v and V_p for every puncture p. The set of $\{U_v\}$ has only double overlaps $U_v \cap U_{v'}$, corresponding to edges $\{v, v'\}$, so $\cup_v U_v = F$ is a punctured surface. U_p overlaps with all U_v for all the vertices surrounding the puncture. To construct the corresponding transition functions $w' = f_{v',v}(w), y = f_{p,v}(w)$ on overlaps, we consider the fatgraph with one positive number per edge, producing the *metric fatgraph*. Then, we attach the infinite stripe to the edge, with the width being the corresponding parameter. The transition functions rise from gluing stripes corresponding to edges into neighborhoods U_v, with the width being a positive parameter assigned to the edge. The key ideas of this description, which is due to Kontsevich [16] and further elaborated by Mulase and Penkava [17], lie within the theory of Strebel differentials. These are holomorphic quadratic differentials on a punctured surface with certain extra conditions. One can reconstruct the metric fatgraph and the corresponding complex structure for every Strebel differential so that their zeroes define the vertices of the fatgraphs, and the order of zero determines the valence of the corresponding vertex. At the same time, the punctures correspond to their double poles. All this can be summarized in the fact that Strebel differentials parametrize the trivial $\mathbb{R}_+^s$-bundle over the moduli space of Riemann surfaces with s punctures.

In Section 5, we use this fatgraph description to characterize the moduli space of $(1|1)$ supermanifolds with punctures: we use the term "puncture" for marked points or $(0|1)$ divisors assigned to marked points on the underlying Riemann surface. At first, we consider the split $(1|1)$ supermanifolds, which can be viewed as Riemann surfaces with a line bundle $\mathcal{L}$ over them. The corresponding moduli space can be then described by the flat $U(1)$ connections on the corresponding metric fatgraphs with zero monodromies around the punctures, accompanied by a fixed divisor at punctures, one for every degree.

Next, we describe this construction's deformation by expressing the tangent bundle's odd parts to the corresponding moduli space as Čech cocycles on the Riemann surface F. These cocycles lead to the infinitesimal deformations of the transition functions, which could be continued beyond the infinitesimal level.

Parametrizing such Čech cocycles is a nontrivial problem, which, however, can be solved in the case when $\deg(\mathcal{L}) = 1 - g - n - r/2$, where n is the number of point punctures and r is the (even) number of $(0|1)$-divisor punctures and g is a genus. In this case, the corresponding cocycles can be characterized by the ordered sets of complex odd parameters for every vertex, where the number of parameters in each set depends on the valence of the vertex. This is roughly twice more parameters than needed, so there are equivalences between complex structures constructed in such a way. We characterize those equivalences explicitly using sections of the appropriate line bundles.

Thus, the fatgraph description of the split case, together with the parametrization of cocycles, immediately leads to the complete parametrization of the complex structures of $(1|1)$-supermanifolds with such a degree.

We note that on the level of uniformization, this is an important subclass of supermanifolds obtained in [13], corresponding to flat connections with zero monodromies around punctures.

In Section 6, we use the results of Dolgikh, Rosly, and Schwarz [7], who explicitly described the equivalence between $N = 2$ super Riemann surfaces and $(1|1)$-supermanifolds, expressing the transition functions of $N = 2$ SRSs using the transition functions for $(1|1)$-supermanifolds obtained in Section 5.

In Section 7, we first discuss the involution on the moduli space of $N = 2$ SRS, such that the fixed points of this involution are $N = 1$ SRS. We then describe the split case, characterizing various choices of the corresponding line bundle using spin structures on the fatgraph, thus looking at the corresponding supermoduli space with the given assignment of R and NS punctures as a 2^{2g} covering space over the moduli space of punctured Riemann surfaces. We then apply the involution to the deformations, first on the infinitesimal level and then continuing beyond, using the superconformal condition. This eventually leads to our main Theorem 12, which describes deformations of $N = 1$ SRS.

2. $(1|1)$ Supermanifolds, $N = 1$ and $N = 2$ Super Riemann Surfaces, and Superconformal Transformations

2.1. Super Riemann Surfaces and Superconformal Transformations

We remind the reader that a *complex supermanifold* of dimension $(1|1)$ (see, e.g., [23]) over some Grassmann algebra S is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of supercommutative S-algebras over X such that $(X, \mathcal{O}_X^{\mathrm{red}})$ can be identified with a Riemann surface (where $\mathcal{O}_X^{\mathrm{red}}$ is obtained from $\mathcal{O}_X$ by quoting out nilpotents), and for some open sets $U_\alpha \subset X$ and some linearly independent elements $\{\theta_\alpha\}$, we have $\mathcal{O}_{U_\alpha} = \mathcal{O}_{U_\alpha}^{\mathrm{red}} \otimes S[\theta_\alpha]$. We will also refer to $(X, \mathcal{O}_X^{\mathrm{red}})$ as a *base manifold*. These open sets U_α serve as coordinate neighborhoods for supermanifolds with coordinates $(z_\alpha, \theta_\alpha)$. The coordinate transformations on the overlaps $U_\alpha \cup U_\beta$ are given by the following formulas $z_\alpha = f_{\alpha\beta}(z_\beta, \theta_\beta)$, $\theta_\alpha = \psi_{\alpha\beta}(z_\beta, \theta_\beta)$, where $f_{\alpha\beta}$, $\psi_{\alpha\beta}$ are even and odd functions, respectively.

A super Riemann surface (SRS) Σ [6,8] over some Grassmann algebra S is a complex supermanifold of dimension $1|1$ over S, with one more extra structure: there is an odd subbundle $\mathcal{D}$ of $T\Sigma$ of dimension $0|1$, such that for any nonzero section D of $\mathcal{D}$ on an open subset U of Σ, D^2 is nowhere proportional to D, i.e.; we have the exact sequence:

$$0 \longrightarrow \mathcal{D} \longrightarrow T\Sigma \longrightarrow \mathcal{D}^2 \longrightarrow 0. \tag{1}$$

One can pick the holomorphic local coordinates in such a way that this odd vector field will have the form $f(z,\theta)D_\theta$, where $f(z,\theta)$ is a nonvanishing function and:

$$D_\theta = \partial_\theta + \theta\partial_z, \quad D_\theta^2 = \partial_z. \tag{2}$$

Such coordinates are called $superconformal$. The transformation between two superconformal coordinate systems (z,θ), (z',θ') is determined by the condition that $\mathcal{D}$ should be preserved, namely,

$$D_\theta = (D_\theta\theta')D_{\theta'}, \tag{3}$$

Locally, one obtains

$$z' = u(z) + \theta\eta(z)\sqrt{\partial_z u(z)}, \quad \theta' = \eta(z) + \theta\sqrt{\partial_z u(z) + \eta(z)\partial_z\eta(z)}, \tag{4}$$

so that the constraint on the transformation emerging from the local change in coordinates is $D_\theta z' - \theta' D_\theta \theta' = 0$.

2.2. $N = 2$ Super Riemann Surfaces

$N = 2$ super Riemann surfaces ($N = 2$ SRS) is a generalization of super Riemann surfaces, which is a supermanifold of dimension $(1|2)$ with extra structure. Its tangent bundle has two subbundles $\mathcal{D}_+$ and $\mathcal{D}_-$, so that each of them is integrable, meaning that if $D_\pm$ are nonvanishing sections of $\mathcal{D}_\pm$, we have

$$D_+^2 = aD_+, \quad D_-^2 = bD_- \tag{5}$$

for some functions a and b. At the same time, the direct sum $\mathcal{D}_+ \oplus \mathcal{D}_-$ is non-integrable, so $[D_+, D_-]$ is a basis for the tangent bundle. That is, for $N = 2$ super Riemann surfaces Σ, one has the following exact sequence:

$$0 \longrightarrow \mathcal{D}_+ \oplus \mathcal{D}_- \longrightarrow T\Sigma \longrightarrow \mathcal{D}_+ \otimes \mathcal{D}_- \longrightarrow 0. \tag{6}$$

As in the case of super Riemann surfaces, one can show that there exist superconformal coordinates in which $\mathcal{D}_+$ and $\mathcal{D}_-$ are locally generated by:

$$D_+ = \partial_{\theta_+} + \frac{1}{2}\theta_-\partial_z, \quad D_- = \partial_{\theta_-} + \frac{1}{2}\theta_+\partial_z, \tag{7}$$

so that $D_\pm^2 = 0$, $[D_+, D_-] = \partial_z$.

It turns out that there is an equivalence between $(1|1)$ supermanifolds and $N = 2$ SRSs, as was estabished by Dolgikh, Rosly, and Schwarz [7]. One can notice that there is an involution $\theta_+ \leftrightarrow \theta_-$. The corresponding complex $(1|1)$ supermanifold constructed from the $N = 2$ SRSs after the involution is, of course, generally a different one, and it is called *dual*. In fact, such a dual supermanifold turns out to be a supermanifold of $(0|1)$ divisors of the original one. The self-dual $(1|1)$ supermanifolds are, of course, $N = 1$ super Riemann surfaces.

We will discuss these questions in more detail later in the text.

2.3. Punctures: Ramond and Neveu–Schwarz

Let us now discuss the types of punctures on $N = 1$ super Riemann surface.

The *NS puncture* is a natural generalization of the puncture of ordinary Riemann surfaces and can be considered as any point (z_0, θ_0) on the super Riemann surface. Locally, one can associate with it a $(0|1)$-dimensional divisor of the form $z = z_0 - \theta_0\theta$, which is the orbit with respect to the action of the group generated by D, and this divisor uniquely determines the point (z_0, θ_0) due to the superconformal structure.

Let us consider the case in which the puncture is at $(0, 0)$ locally. In its neighborhood, let us pick a coordinate transformation

$$z = e^w, \qquad \theta = e^{w/2}\eta, \tag{8}$$

such that the neighborhood (without the puncture) is mapped to a *supertube* with w sitting on a cylinder $w \sim w + 2\pi i$, and D_θ becomes

$$D_\theta = e^{-w/2}(\partial_\eta + \eta\partial_w). \tag{9}$$

Hence, (w, η) are superconformal coordinates, and we have the full equivalence relation given by

$$w \sim w + 2\pi i, \qquad \eta \longrightarrow -\eta. \tag{10}$$

The case of a *Ramond puncture* is a whole different story. On the level of super Riemann surfaces, the associated divisor is determined as follows. In this case, we are looking at the case in which the condition that D^2 is linearly independent of D is violated along some $(0|1)$ divisor. That is, in some local coordinates (z, θ) near the Ramond puncture with coordinates $(0, 0)$, $\mathcal{D}$ has a section of the form

$$D^*_\theta = \partial_\theta + z\theta\partial_z.$$

We see that its square vanishes along the *Ramond divisor* $z = 0$. One can map the neighborhood patch to the supertube using a different coordinate transformation

$$z = e^w, \qquad \theta = \eta, \tag{11}$$

whose coordinates on the supertube will be superconformal, since

$$D_\eta = \partial_\eta + \eta\partial_w. \tag{12}$$

Notice that the identifications we have to impose on (w, η) now become

$$w \sim w + 2\pi i, \qquad \eta \longrightarrow +\eta. \tag{13}$$

To describe Ramond punctures globally, consider a subbundle $\mathcal{D}$ is generated by such operators D^*_η for the Ramond punctures $p_1, p_2, \ldots p_{n_R}$. We have the exact sequence

$$0 \longrightarrow \mathcal{D} \longrightarrow T\Sigma \longrightarrow \mathcal{D}^2 \otimes \mathcal{O}(\mathscr{P}) \longrightarrow 0, \tag{14}$$

where $\mathscr{P} = \sum_{i=1}^{n_R} \mathscr{P}_i$ is a divisor where $\mathcal{D}^2 = 0 \bmod \mathcal{D}$.

In the split case $T\Sigma|_X = TX \oplus \mathcal{G}$, we can divide the tangent space to $T\Sigma$ into even and odd parts, which we can identify with $\mathcal{D}^2 \otimes \mathcal{O}(\mathscr{P})$ and $\mathcal{D}$, respectively. Also, notice that after reducing it to the base manifold, $\mathcal{O}(\mathscr{P}) = \mathcal{O}(\sum_{i=1}^{n_R} p_i)$. Therefore,

$$\mathcal{G}^2 = TX \otimes \mathcal{O}\Big(-\sum_{i=1}^{n_R} p_i\Big).$$

Since $\deg(TX) = 2 - 2g$, this automatically implies that $\deg(\mathcal{G}) = 1 - g - n_R/2$, leading to the fact that there should be an even number of such punctures, known as Ramond or simply R punctures. We refer to the Section 4.2.2. of [8] for more details.

3. Fatgraphs and Spin Structures

From now on, we will consider Riemann surfaces of genus g with s punctures ($s > 0$) and a negative Euler characteristic, which we will denote as F_g^s or simply F. The corresponding closed version will be denoted as F^c.

Consider the *fatgraph* τ, corresponding to an s-punctured surface F. This is a graph that is homotopically equivalent to F, with cyclic orderings on half-edges for every vertex [10] induced by the orientation of the surface. Let t_0, t_1 denote the set of vertices and edges of t, respectively. Let w be an orientation on the edges $t_1 \subset t$. As in [12], we define a *fatgraph reflection* at a vertex v of (t, w) to reverse the orientations of w on every edge of t incident to v.

Definition 1. *We define $\mathcal{O}(\tau)$ as the equivalence classes of orientations on a trivalent fatgraph τ spine of F, where the equivalence relation is given by $\omega_1 \sim \omega_2$ iff ω_1 and ω_2 differ by a finite number of fatgraph reflections. It is an affine H^1-space where the cohomology group $H^1 := H^1(F; \mathbb{Z}_2)$ acts on $\mathcal{O}(\tau)$ by changing the orientation of the edges along cycles.*

In [12,13,24] various realizations of the spin structures on the surface F, characterized by a trivalent fatgraph τ, are described. These results can be easily generalized to a fatgraph with vertices of any valence.

In fact, following [25], a spin structure can be characterized by a quadratic form $q : H_1(F; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$ so that for any cycles a, b, one has $q(a + b) = q(a) + q(b) + a \cdot b$, where $a \cdot b$ denotes the intersection form.

The space of all orientation classes is an affine H^1-space. We fix a fatgraph t, where we denote by $o_\omega(e)$ the orientation of the edge $e \in t$ in the orientation ω. We define $\delta_{\omega_1,\omega_2} : \tau_1 \longrightarrow \mathbb{Z}_2$ by

$$\delta_{\omega_1,\omega_2}(e) := \left\{ \begin{array}{ll} +1 & o_{\omega_1}(e) = o_{\omega_2}(e), \\ -1 & o_{\omega_1}(e) \neq o_{\omega_2}(e), \end{array} \right. \tag{15}$$

which defines an element in $H^1(F; \mathbb{Z}_2)$.

Proposition 1 ([12,13])**.** *The set of spin structures is isomorphic to the space of quadratic forms $\mathcal{Q}(F)$ on $H_1(F; \mathbb{Z}_2)$, and also isomorphic to $\mathcal{O}(\tau)$, as affine H^1-spaces.*

This leads to the following important consequence.

Theorem 1 ([13])**.** *Given an oriented simple cycle $\phi \in \pi_1(F)$ homotopic to a path on the fatgraph τ with orientation class $[\omega]$, the corresponding quadratic form is given by*

$$q([\gamma]) = (-1)^{L_\gamma}(-1)^{N_\gamma} = (-1)^{R_\gamma}(-1)^{\overline{N}_\gamma}, \tag{16}$$

where L_ϕ (resp. R_ϕ) is the number of left (resp. right) turns of γ on the fatgraph τ, and N_ϕ (resp. $\overline{N}_\gamma$) is the number of edges of τ such that γ, and ω have the same (resp. opposite) orientation.

If we talk about the paths corresponding to the boundary cycles on the fatgraph, $q([\gamma]) = (-1)^k$, where k is a number of edges with an orientation opposite to the canonical orientation of γ. One can identify them with R and NS punctures in the uniformization picture [14], so that k is even for R punctures and odd for NS punctures.

In fact, there is another way of thinking about the spin structures, using graph connections [10,26,27].

Definition 2 ([10]). *Let G be a group. A G-*graph connection *on τ is the assignment $g_e \in G$ to each oriented edge e of τ so that $g_{\bar{e}} = g_e^{-1}$ if $\bar{e}$ is the opposite orientation to e. Two assignments $\{g_e\}, \{g'_e\}$ are equivalent iff there are $t_v \in G$ for each vertex v of τ such that $g'_e = t_v g_e t_w^{-1}$ for each oriented edge $e \in \tau_1$ with initial point v and terminal point w.*

Therefore, we obtain the following description of spin structures.

Corollary 1 ([13]). *The space of spin structures on F is identified with $\mathbb{Z}_2$-graph connections on a given fatgraph τ of F.*

We will refer to the fatgraph with the associated spin structure/$\mathbb{Z}_2$-graph connection as a *spin fatgraph*.

4. Complex Structures and Strebel Differentials

4.1. Gluing of Riemann Surfaces

Consider the fatgraph τ corresponding to an s-punctured surface F of genus g. Let us assign a positive real parameter L_j associated with every edge j. We will refer to the resulting object as a *metric fatgraph*.

It is known from the works of Penner (the so-called convex hull construction) that the fatgraphs with a valence of every vertex greater or equal to 3 or dual ideal cell decompositions of Riemann surfaces describe the mapping class group-invariant cell decomposition of the decorated Teichmüller space (see, e.g., [10,28]), a universal cover of $\mathbb{R}^s_+ \otimes \mathcal{M}_{g,s}$, where $\mathcal{M}_{g,s}$ is a moduli space of Riemann surfaces of genus g with s marked points. Then, the trivalent fatgraphs correspond to the higher-dimensional cells of dimension $6g - 6 + 3s$.

An important problem is how to reproduce inequivalent complex structures based on the data of metric fatgraphs. An important work by Mulase and Penkava [17], based on earlier ideas of Kontsevich [16], allows us to construct the appropriate covering of a Riemann surface and the transition functions associated with a given fatgraph, thus exhausting all possible complex structures. Let us have a look at those in detail.

Fixing an orientation on τ, we consider a neighborhood U_v with coordinate w corresponding to the fixed m-valent vertex v, so that the vertex is placed at the point $w = 0$. One can describe that neighborhood by considering stripes

$$\{z_j, \in \mathbb{C}, 0 < \mathrm{Re}(z_i) < L_j\}, \quad j = 1, \dots, m \tag{17}$$

glued together via the formula

$$w = e^{\frac{2\pi i(j-1)}{m}} z_j^{\frac{2}{m}}, \quad j = 1, \dots, m \tag{18}$$

if all m edges point out from vertex v. In the case that one or more of them is pointing towards vertex v, we substitute the above formulas by

$$w = e^{\frac{2\pi i(j-1)}{m}} (L_j - z_j)^{\frac{2}{m}}, \quad j = 1, \dots, m. \tag{19}$$

One can construct such coordinate patches around every such vertex. The overlaps $U_v \cap U_{v'}$ are described by the corresponding stripes associated with the edge j of the fatgraph running between v and v'. Note that there are no triple intersections on such a punctured surface and that the vertices of the fatgraph belong to the boundary of the intersections.

Let us look at the transition functions on the overlaps between two such coordinate neighborhoods U_v, $U_{v'}$ around neighboring vertices v and v', assuming the edge is pointing from v to v'. We note that both coordinates w and w' are expressed in terms of z_j in the following way:

$$w = c_j z_j^{2/m}, \quad w' = c'_j (L_j - z_j)^{2/m}, \tag{20}$$

where c_j, c'_j are the mth and m'th roots of unity. The resulting overlap coordinate transformation $f_{v'v}$ between patches is given by the following formula:

$$w' = f_{v',v}(w) = c'_j(L_j - c_j^{-m/2} w^{m/2})^{2/m}, \tag{21}$$

where $-i\pi/2 < \arg(w^{m/2}) < i\pi/2$. This completely describes the transition functions between charts for the punctured Riemann surface F.

Note that if the consecutive edges $L_1, L_2, \ldots, L_n$ correspond to the boundary piece of the fatgraph associated with puncture p, one can glue the following coordinate neighborhood V_p with coordinate y covering the puncture:

$$y = e^x = \exp\Big(\frac{2\pi i}{a_B}(L_1 + \ldots L_{k-1} + z_k)\Big), \text{ where } a_B = L_1 + \cdots + L_n, \tag{22}$$

so that one glues together the top or bottom part of the stripes based on orientation and the x-variable is on a cylinder $x \sim x + 2\pi i$. Suppose the k-th strip is glued to the vertex v_k with coordinate w_k as above, then the transition function f_{pv} is given by

$$y = f_{p,v}(w) = \exp\Big(\frac{2\pi i}{a_B}(L_1 + \ldots L_{k-1} + w_k^{m/2})\Big). \tag{23}$$

Below, we will adopt the following notation for L-parameters: if the edge connects two vertices v, v', we will denote the corresponding parameter as $L_{v,v'}$.

4.2. Strebel Differentials

An important object in the constructions of [16,17] is the *Strebel differentials*, the quadratic meromorphic differentials with special properties. A nonzero quadratic differential is a holomorphic section μ of $K^{\otimes 2}$, where K stands for a canonical bundle on F. It defines a flat metric on the complement of the discrete set of its zeroes, written in local coordinates as $|\mu(z)|dzd\bar{z}$, where $\mu = \mu(z)dz^2$.

A *horizontal trajectory* of a quadratic differential is a curve along which $\mu(z)dz^2$ is real and positive. The *Strebel* differential is the one for which the union of nonclosed trajectories has a measure of zero. Non-closed trajectories of a given Strebel differential decompose the surface into the maximal ring domains swept out by closed trajectories. These ring domains can be annuli or punctured disks. All trajectories from any fixed maximal ring domain have the same length, the circumference of the domain.

The following theorem is from Strebel:

Theorem 2 ([29])**.** *For any connected closed Riemann surface F^c with s distinct points $p_1, \ldots, p_s$, $s > 0$ and genus g, $s > \chi(F^c) = 2 - 2g$ and n positive real numbers $a_1, \ldots, a_s$, there exists a unique Strebel differential on $F = F^c \backslash \{p_1, \ldots, p_s\}$, whose maximal ring domains are s punctured disks surrounding p_i's with circumference a_i's.*

The unions of non-closed trajectories of Strebel differentials together with their zeroes define a graph, embedded into a Riemann surface, thus giving it a fatgraph structure. Every vertex of a fatgraph corresponds to the zero of the Strebel differential of degree $m - 2$, where $m \geq 3$ is the valence of the vertex. The length of each edge gives the graph a metric structure.

For every such Strebel differential, one can construct the covering associated with the corresponding fatgraph, described in the previous section and vice versa, so that in the charts U_v, Strebel differential μ has the following explicit form:

$$\mu|_{U_v} = w^{m-2}dw^2. \tag{24}$$

It also has a pole of order 2 at punctures so that in y-coordinates for each neighborhood V_p, the differential looks as follows:

$$\mu|_{V_p} = -\frac{a_p^2}{4\pi^2} y^{-2} dy^2. \tag{25}$$

One can then formulate the following Theorem.

Theorem 3 ([16]). *Let $\mathcal{M}_{g,s}^{\text{comb}}$ denote the set of equivalence classes of connected ribbon graphs with metrics and with a valency of each vertex greater than or equal to 3, such that the corresponding noncompact surface has a genus g and s punctures numbered $1, \ldots, s$. The map $\mathcal{M}_{g,s} \times \mathbb{R}^s \longrightarrow \mathcal{M}_{g,s}^{\text{comb}}$ is associated with the surface F^c, and the numbers $a_1, \ldots a_s$ of the critical graph of the canonical Strebel differential from Theorem 2 are one-to-one.*

In this paper, we do not need more properties of Strebel differentials; however, we refer the reader to [17], as well as original source [29], for more information.

5. Complex Structures on $(1|1)$ Supermanifolds

5.1. Split Case

Let us consider the punctured Riemann surface glued as in the previous subsection using a metric fatgraph and overlapping neighborhoods U_v corresponding to vertices. To construct the coordinate transformations for a split $(1|1)$ supermanifold SF with such a base complex manifold, one has to consider a line bundle $\mathcal{L}$ over F^c. Then, the coordinate transformations for the coordinates (w', ξ'), (w, ξ), (y, η) corresponding to neighborhoods $U_v, U_{v'}, V_p$ of vertices v, v' and puncture p are given by the following formulas:

$$\begin{aligned} &\xi' = g_{v',v}(w)\xi, \quad w' = f_{v',v}(w), \\ &\eta = g_{p,v}(w)\xi, \quad y = f_{p,v}(w), \end{aligned} \tag{26}$$

where $g_{v',v}$, $g_{p,w}$ is the holomorphic function, serving as a transition function of bundle $\mathcal{L}$. The collection $\{g_{v',v}, g_{v,p}\}$ generates Čech cocycles

$$g_{v,v'}|_{U_v \cap U_{v'}} \in H^0(U_v \cap U_{v'}, \mathcal{O}^*), \quad g_{v,p}|_{U_v \cap V_p} \in H^0(U_v \cap V_p, \mathcal{O}^*), \tag{27}$$

representing the Picard group of F^c, i.e., $\check{H}^1(F, \mathcal{O}^*)$, if the following constraint on $g_{v',v}$ and $\{g_{v,p}\}$ is imposed around the given puncture p:

$$g_{v,v'}|_{U_{v'} \cap U_v \cap V_p} = g_{v,p} g_{p,v'}. \tag{28}$$

Then, the following Proposition holds.

Proposition 2. *When $\mathcal{L}$ is degree 0 over F^c, the fatgraph data describing it aare a $U(1)$ graph connection with a trivial monodromy around every boundary piece.*

Proof. Notice that one can choose $g_{v'v}$ to be constant functions with values on a unit circle, which on the level of a fatgraph is described by $U(1)$-graph connection, so $g_{v,v'} = e^{ih_{v,v'}}$, where $h_{v,v'} \in \mathbb{R}$. Indeed, the corresponding holomorphic equivalences for the corresponding Čech cocyle reduce to constant $U(1)$ gauge transformations at the vertices. However, according to the condition (28) that we imposed, we have to have $g_{v_1,v_2} g_{v_2,v_3} \ldots g_{v_{n-1},v_n} g_{v_n,v_1} = 1$, which is exactly the trivial monodromy condition. □

In order to describe any line bundle of degree d, one has to carry out the following. First, choose a fixed divisor of degree d, say, a linear combination of puncture points. Then, multiplying it by the appropriate bundle of degree 0, one can reproduce the original bundle. Since we described the moduli spaces of degree 0 bundles in Proposition 2 above, we can now characterize split punctured supermanifolds.

Theorem 4. *Consider the following data on a fatgraph τ:*

- *Metric structure.*
- *Flat $U(1)$-connection with zero monodromies around the punctures.*
- *Fixed divisor M of degree d, which is a linear combination of puncture points.*

The data above determine the complex split $(1|1)$-supermanifold corresponding to the line bundle of degree d on F. For a fixed divisor M, metric fatgraphs with $U(1)$ connections describe the moduli space of split $(1|1)$ supermanifolds.

5.2. Infinitesimal Deformations and Various Types of Punctures

As usual, the infinitesimal deformations of the above formulas leading to the generic non-split structure are described by $H^1(SF^c, ST)$, where ST is a tangent bundle of SF^c, where SF^c is a split $(1|1)$ supermanifold, which we discussed in the previous section. Since we are deforming the split case, one can describe infinitesimal deformations $\rho \in H^1(SF^c, ST)$ using Čech cocycles, i.e., in coordinates (ξ, w) of U_v on $U_v \cap U_{v'}$ and $U_v \cap V_p$:

$$\begin{aligned} \rho_{v,v'} &= \mathrm{v}_{v,v'}(w)\partial_w + \xi\alpha_{v,v'}(w)\partial_w + \beta_{v,v'}(w)\partial_\xi + \mathrm{u}_{v,v'}(w)\xi\partial_\xi, \\ \rho_{p,v} &= \mathrm{v}_{p,v}(w)\partial_w + \xi\alpha_{p,v}(w)\partial_w + \beta_{p,v}(w)\partial_\xi + \mathrm{u}_{p,v}(w)\xi\partial_\xi. \end{aligned} \tag{29}$$

where the indices v, v' and p, v here mean that the corresponding elements are the corresponding Čech cocycles considered on the intersections $U_v \cap U_{v'}$ and $V_p \cap U_v$.

Now, we need to specify the behavior at the punctures to describe the cocycles ρ leading to deformations of SF in terms of cocyles on F^c.

There are two types of punctures we want to consider:

- A puncture as a $(0|1)$-dimensional divisor on SF^c. We denote the number of such punctures as r.
- A puncture as a $(0|0)$-dimensional divisor, or in other words, just a point on SF^c. We denote the number of such punctures as n.

Let T be the tangent bundle of F^c, D_{n+r} be the divisor corresponding to the sum of all points on F^c corresponding to punctures, and D_n is the sum of the ones corresponding to point-punctures on SF. Let us look in detail at the components of (29):

$$\begin{aligned} &\mathrm{v} \in \check{Z}^1(F^c, T \otimes \mathcal{O}(-D_{n+r})), \quad \mathrm{u} \in \check{Z}^1(F^c, \mathcal{O}), \\ &\beta \in \Pi\check{Z}^1(F^c, \mathcal{L} \otimes \mathcal{O}(-D_n)), \quad \alpha \in \Pi\check{Z}^1(F^c, T \otimes \mathcal{L}^{-1} \otimes \mathcal{O}(-D_{n+r})), \end{aligned} \tag{30}$$

where $\check{Z}^1$ is the notation for Čech cocycles of degree 1. Note that we need to impose the constraints on cocycles on $V_p \cap U_v \cap U_{v'}$:

$$s_{v,v'}|_{U_{v'} \cap U_v \cap V_p} = s_{v,p} + s_{p,v'}, \tag{31}$$

where $s = \mathrm{v}, \mathrm{u}, \alpha, \beta$. Here, the u and v terms correspond to the deformations of the original manifold F, and Notice, that we already incorporated the moduli for the base manifold F and the line bundle $\mathcal{L}$ in the Formula (26). The odd deformations, provided by the cycles α, β give the following deformations for the upper line of (26)

$$\xi' = g_{v'v}(w)(\xi + \beta_{v',v}(w)), \quad w' = f_{v'v}(w + \xi\alpha_{v',v}(w)), \tag{32}$$

which describes (in the first order in complex parameters) all possible complex structures on the punctured supermanifold.

If we remove the infinitesimality condition, the formulas above will be deformed. Let us formulate it in a precise form.

Theorem 5. *1. Consider the following data:*

- *A metric fatgraph with a $U(1)$-connection with trivial monodromy around boundary pieces, a fixed divisor, which is a linear combination of puncture points of degree d, which* *defines a split punctured* $(1|1)$ *supermanifold determined by base Riemann surface* F *and line bundle* $\mathcal{L}$*.*

- *Čech cocycles*

$$\tilde{\beta} = \sum_k \sigma_k^{\beta} b_k,\ \tilde{\alpha} = \sum_k \sigma_k^{\alpha} a_k,$$

so that $\{\sigma_k^{\beta}\}$, $\{\sigma_k^{\alpha}\}$ *are two sets of odd parameters,*

$$b_k \in \check{Z}^1(F^c, \mathcal{L} \otimes \mathcal{O}(-D_n)),$$
$$a_k \in \check{Z}^1(F^c, T \otimes \mathcal{L}^{-1} \otimes \mathcal{O}(-D_{n+r})),$$

where D_{n+r} *is the divisor corresponding to the sum of all* $s = n + r$ *punctures on the closed surface* F^c, D_n *is the sum of the certain subset of the set of punctures, and the cohomology classes of* $\{b_k\}$ $\{a_k\}$ *form a basis in the corresponding cohomology spaces.*

These data give rise to a family of complex structures on SF, the $(1|1)$*-supermanifold with n point punctures and r* $(0|1)$*-divisor punctures, so that the transition functions on SF are given with the following formulas on the overlaps* $\{U_v \cap U_{v'}\}$*:*

$$\xi' = g_{v'v}^{(\alpha,\beta)}(w)(\xi + \beta_{v',v}(w)), \quad w' = f_{v'v}^{(\alpha,\beta)}(w + \xi\alpha_{v',v}(w)), \tag{33}$$

where $g_{v',v}^{(\alpha,\beta)}, f_{v',v}^{(\alpha,\beta)}$*, are holomorphic functions on the overlaps, depending on the parameters* σ_k^{α} *and* σ_k^{β} *such that:*

$$g_{v',v}^{(0,0)} = g_{v',v},\ f_{v',v}^{(0,0)} = f_{v',v},$$

where $\{f_{v'v}\}$, $\{g_{v',v}\}$ *define the split supermanifold with the line bundle* $\mathcal{L}$ *and s punctures so that, in the first order in* $\{\sigma_k^{\alpha}\}$ *and* $\{\sigma_k^{\beta}\}$*, we have*

$$\tilde{\beta}_{v',v}(w) = \beta_{v',v}(w),\ \tilde{\alpha}_{v',v}(w) = \alpha_{v',v}(w).$$

2. *Let us fix the choice of transition functions in (33), for every metric fatgraph* τ *with the U(1)-connection, divisor of degree d, and the odd data given by the cocycles* $\tilde{\beta}, \tilde{\alpha}$ *on* F^c*.*
The complex structures constructed in such a way are inequivalent to each other, and the set of such complex structures constructed by varying τ *and the data on it, form a dense subset of maximal dimension in the moduli space of punctured* $(1|1)$ *supermanifolds with underlying line bundles of degree d.*

Proof. Let us look at the Formula (33) as a generic one for arbitrary holomorphic functions $\{\alpha_{v'v}\}$, $\{\beta_{v'v}\}$ on overlaps. There are a finite number of odd parameters that parametrize all $\{\alpha_{v',v}\}$, $\{\beta_{v',v}\}$ corresponding to inequivalent complex structures. Expanding the Formula (33) in terms of these parameters, we obtain that, in the linear order, $\beta \in \Pi\check{Z}^1(F^c, \mathcal{L} \otimes \mathcal{O}(-D_n))$ and $\alpha \in \Pi\check{Z}^1(F^c, T \otimes \mathcal{L}^{-1} \otimes \mathcal{O}(-D_{n+r}))$, as in the infintesimal case. Conversely, since α, β represent the tangent space to the moduli space of complex structures, parameters σ^{α}, σ^{β} serve as coordinates there. Considering the corresponding 1-parametric subgroups generated by $\tilde{\alpha}$, $\tilde{\beta}$, we obtain formulas from (33). The fact that the cohomologically equivalent cocycles lead to the equivalent complex structures is justified by dimensional reasons. □

It is, however, nontrivial to explicitly parametrize those cocycles α, β. In the next subsection, we will analyze the special case of supermanifolds with the line bundle $\mathcal{L}$ of negative degree.

5.3. $(1|1)$ *Supermanifolds with* $\deg(\mathcal{L}) = 1 - g - r/2$

It is not easy to explicitly parametrize cocycles α, β from fatgraph data if one does not fix a degree. From now on, we will be interested in the case when $\deg(\mathcal{L}) = 1 - g - k$, where $s \geq k \geq 0$ on F^c. Assuming that the number of divisor punctures is even and setting $k = r/2$, both bundles $\mathcal{L} \otimes \mathcal{O}(-D_n)$ and $\mathcal{L}^{-1} \otimes T \otimes \mathcal{O}(-D_{n+r})$ have equal degree $1 - g - r/2 - n$ on F^c.

Let us be generic enough first and characterize the cycles in $\Pi\check{Z}^1(F^c, \mathcal{L} \otimes \mathcal{O}(-D_n))$, where $s \geq k = g - 1 - \deg \mathcal{L} \geq 0$, using the data from the fatgraph. To do this, we define a cocycle ρ, a representative of $\Pi\check{H}^1(F^c, \mathcal{L} \otimes \mathcal{O}(-D_n))$, as follows:

$$\begin{aligned} &\rho_{v,v'}|_{U_v \cap U_{v'}} = \rho_v - \rho_{v'}, \text{ so that } \rho_v|_{U_v} = \frac{\sigma_v(w)}{w^{m_v - 2}}, \ \rho_{v'}|_{U'_v} = \frac{\sigma_{v'}(w')}{w'^{\,m_{v'}-2}}, \\ &\rho_{v,p}|_{U_v \cap V_p} = \rho_v, \end{aligned} \tag{34}$$

where ρ_v, $\rho_{v'}$ are meromorphic sections of $\mathcal{L} \otimes \mathcal{O}(-D_n)$ on U_v, $U_{v'}$, correspondingly, so that m_v is the valence of the given vertex v,

$$\sigma_v(w) = \sum_{i=0}^{m_v - 3} \sigma_v^i w^i$$

are the polynomials with odd coefficients, assigned to each fatgraph vertex v of degree at most $m_v - 3$, Let us denote, for simplicity, $\tilde{\mathcal{L}} = \mathcal{L} \otimes \mathcal{O}(-D_n)$.

Then, the following proposition holds.

Theorem 6. *1. The cycles (34) are uniquely defined by the numbers σ_v at the fatgraph vertices, thus forming a complex vector space of dimension $4g - 4 + 2s$.*

2. Cycle ρ is cohomologous to cycle $\tilde{\rho}$ in $\Pi\check{H}^1(F^c, \tilde{\mathcal{L}})$ if and only if

$$\sigma_v(w) - \tilde{\sigma}_v(w) = \gamma^{(m-3)}(w), \tag{35}$$

for every vertex v, where $\gamma \in \Pi H^0(F^c, \tilde{\mathcal{L}} \otimes K^2 \otimes \mathcal{O}(2D_{n+r}))$, so that $\gamma|_{U_v} = \gamma(w)$, $\gamma^{(m-3)}(w)$ is the Taylor expansion of $\gamma(w)$ up to order $m - 3$.

3. The cohomology classes of cycles ρ span $\Pi\check{H}^1(F^c, \tilde{\mathcal{L}})$.

Proof. To prove part (1), one just has to count the number of vertices and parameters at vertices. An elementary Euler characteristic computation shows that

$$2g - 2 + s = \sum_{j \geq 3} \Big(\frac{j}{2} - 1\Big) \mathcal{V}_j(\tau), \tag{36}$$

where $\mathcal{V}_j(\tau)$ is the number of j-valent vertices in τ. Notice that for a j-valent vertex v, we have exactly $j - 2$ odd parameters from the expansion of $\sigma_v(w)$, which immediately leads to the necessary parameter count, giving $4g - 4 + 2s$.

To prove (2), on each coordinate neighborhood U_v, Strebel differential μ has the form $\mu|_{U_v} = w^{m_v - 2} dw^2$, and $\mu|_{V_p} = -\frac{a_p^2}{4\pi^2} \frac{dy^2}{y^2}$, which means that one can rewrite the formula for the cocycle

$$\rho_{v,v'} = (\gamma_v - \gamma_{v'})/\mu, \quad \rho_{v,p} = (\gamma_v - \gamma_p)/\mu, \tag{37}$$

where $\gamma_v|_{U_v} = \sigma_v(w)$, $\gamma_p|_{V_p} = 0$, so that $\gamma_v \in \Pi\check{H}^0(U_v, \tilde{\mathcal{L}} \otimes K^2)$ and $\gamma_p = 0 \in \Pi\check{H}^0$ $(V_p, \tilde{\mathcal{L}} \otimes K^2)$.

Suppose that such a cocycle is exact, namely,

$$\gamma_v/\mu - \gamma_{v'}/\mu = (a_v - a_{v'})|_{U_v \cap U'_v}, \quad \gamma_v/\mu - \gamma_p/\mu = (a_v - a_p)|_{U_v \cap U_p} \tag{38}$$

for all v and v', so that $a_v \in \Pi\check{H}^0(U_v, \tilde{\mathcal{L}})$, $a_p \in \Pi\check{H}^0(V_p, \tilde{\mathcal{L}})$. It is equivalent to $(\gamma_v - a_v\mu) = (\gamma_{v'} - a_{v'}\mu)|_{U_v \cap U'_v}$, $(\gamma_v - a_v\mu) = (\gamma_p - a_p\mu)|_{U_v \cap V_p}$; i.e., formulas

$$\gamma_v - a_v\mu = \gamma|_{U_v}, \quad \gamma_p - a_p\mu = \gamma|_{V_p}$$

define γ as a holomorphic section on F, i.e., $\gamma \in \Pi\check{H}^0(F, \tilde{\mathcal{L}} \otimes K^2)$. Assuming $\gamma|_{U_v} = \gamma(w)$ and $a_v|_{U_v} = a(w)$, the identity $\gamma_v = a_v\mu + \gamma|_{U_v}$ is only possible if

$$a_v(w) = \frac{\gamma(w) - \gamma^{(m-3)}(w)}{w^{m-2}} \text{ and } \sigma_v(w) = \gamma^{(m-3)}(w),$$

where $\gamma^{(m-3)}(w)$ is the Taylor expansion of $\gamma(w)$ up to order $m-3$. Also, the identity $\gamma_p = a_p\mu + \gamma|_{V_p}$, i.e.,

$$a_p\mu + \gamma|_{V_p} = 0$$

is possible only if $\gamma|_{V_p}$ has poles not greater than 2 at $y = 0$, or, more precisely, $\gamma \in H^0(V_p, \tilde{\mathcal{L}} \otimes K^2 \otimes \mathcal{O}(2D_{n+r}))$. Therefore, cycles ρ and $\tilde{\rho}$ are cohomologous to each other iff the relation between the parameters on the fatgraph $\{\sigma_v\}$ and $\{\tilde{\sigma}_v\}$ correspondingly parametrizing them is as follows:

$$\tilde{\sigma}_v(w) - \sigma_v(w) = \gamma^{(m-3)}(w) \tag{39}$$

where $\gamma \in H^0(F^c, \tilde{\mathcal{L}} \otimes K^2 \otimes \mathcal{O}(2D_{n+r}))$ on F^c, such that $\gamma|_{U_v} = \gamma(w)$ with poles at the punctures of F of order less than or equal to 2 so that $\gamma^{(m-3)}(w)$ is the Taylor expansion of $\beta(w)$ up to order $m-3$.

Now, to prove part (3), we need to show that such classes of cocycles form a $2g - 2 + k + n$-dimensional complex space as elements of $\Pi\check{H}^1(F^c, \tilde{\mathcal{L}})$. For a given section γ of $\tilde{\mathcal{L}} \otimes K^2 \otimes \mathcal{O}(2D_{n+r})$, the collection of the coefficients in $\gamma^{(m-3)}$, for each vertex v, form a vector in our $4g - 4 + 2s$-dimensional space of σ-parameters. The space, spanning all such vectors, is a complex $2g - 2 + 2s - k - n$-dimensional space. Indeed, it cannot be of smaller dimension, since we know that $dim_{\mathbb{C}}\check{H}^1(F^c, \mathcal{L} \otimes \mathcal{O}(-D_n)) = 2g - 2 + k + n$; at the same time, it cannot be of greater dimension, since we know that the dimension of space of such meromorphic global sections of $\tilde{\mathcal{L}} \otimes K^2 \otimes \mathcal{O}(2D_{n+r})$ is $2g - 2 + 2s - k - n$ via the Riemann-Roch theorem. $\square$

Now, we are ready to formulate a Theorem regarding the parametrization of complex structures via fatgraph data.

Theorem 7. *1. Consider the following data associated with the fatgraph τ:*

- *Metric structure and a $U(1)$-connection on τ with zero monodromy around punctures and a fixed divisor of degree $d = 1 - g - r/2$ at the punctures.*
- *Two complex odd parameter sets $\{\sigma^{\alpha}_{v,k}\}$, $\{\sigma^{\beta}_{v,k}\}$ at each vertex v, so that $k = 0, \ldots, m_v - 3$.*

We will call two sets of data from (1) *associated with fatgraph τ equivalent if the odd data are related as in Theorem 6.*

Constructing transition functions $f_{v',v}$ and $g_{v',v}$ from the even fatgraph data and cocycles $\tilde{\alpha}$, $\tilde{\beta}$, corresponding to r $(0|1)$-divisor punctures and n point punctures from the odd data, one obtains a family of complex structures on the $(1|1)$ supermanifold in the framework of Theorem 5.

2. *Fixing the transition functions in (33) and considering one such complex structure per equivalence class of data for every fatgraph τ, we obtain a set of inequivalent complex structures, which is a dense subspace of odd complex dimension $4g - 4 + 2n + r$ in the space of all complex structures on $(1|1)$-supermanifolds with a baseline bundle of degree $d = 1 - g - r/2$ and $s = n + r$ punctures, where n is the number of point punctures and r is the number of $(0|1)$-divisor punctures.*

Proof. The first part of the data allows us to construct a split $(1|1)$ supermanifold as we know from previous sections, and the odd data from the second part allow us to construct the corresponding cycles $\tilde{\alpha} \in \Pi\check{Z}^1(F^c, \mathcal{L} \otimes \mathcal{O}(-D_n))$ and $\tilde{\beta} \in \Pi\check{Z}^1(F^c, \mathcal{L}^{-1} \otimes T \otimes \mathcal{O}(-D_{n+r}))$. If we choose an orientation on the fatgraph, the formulas (see Theorem 5)

$$\xi' = g^{(\alpha,\beta)}_{v',v}(w)(\xi + \beta_{v',v}(w)), \quad w' = f^{(\alpha,\beta)}_{v',v}(w + \xi\alpha_{v',v}(w)) \tag{40}$$

produce the transition functions on $U_v \cap U_{v'}$ for the vertex oriented from v to v'. □

Remark. Note that the gauge equivalence for the $U(1)$ connection produce the following identification. If real numbers $h_{v,v'}$ parametrize the $U(1)$ connection, then the transformations

$$\begin{aligned} h_{v,v'} &\longrightarrow h_{v,v'} + t_v - t'_v, \\ \sigma^\alpha_v, \sigma^\beta_v &\longrightarrow e^{it_v}\sigma^\alpha_v, e^{-it_v}\sigma^\beta_v \end{aligned} \tag{41}$$

produce equivalent configuration for infinitesimal parameters σ. In the paper [13], the uniformization version of $N = 2$ Teichmüller space was constructed (see also [3,15]), which corresponds exactly to $(1|1)$-supermanifolds, which serves as a universal cover for the one we use here in the case of $\deg(\mathcal{L}) = 1 - g - r/2$. The above identifications played an instrumental role in the construction.

In the next two sections, we will use the obtained results to describe transition functions for the corresponding dual supermanifold and the $N = 2$ super Riemann surface following [7].

5.4. *Dual* $(1|1)$ *Supermanifold*

Finally, we give a description of the concept dual $(1|1)$ supermanifold, which is a supermanifold of $(0|1)$ divisors of SF. To describe the explicit coordinates and coordinate transformations on such an object, one can use a very simple equation (see, e.g., [8]),

$$w = a + \zeta\xi \tag{42}$$

where a, ζ are the coordinates parametrizing such a $(0|1)$ divisor. Let us derive the formulas for the transformations of a, ζ variables, for the transformation between the charts with coordinates (a, ζ) and (a', ζ'), so that $w' = a + \zeta'\xi'$.

$$\begin{aligned} \xi' &= g^{(\alpha,\beta)}_{v'v}(a + \zeta\xi)(\xi + \beta_{v',v}(a + \zeta\xi)), \\ a' + \zeta'\xi' &= f^{(\alpha,\beta)}_{v',v}(a + \zeta\xi + \xi\alpha_{v',v}(a + \zeta\xi)). \end{aligned}$$

We will now substitute the first equation into the second and obtain

$$\begin{aligned} &a' + \zeta' g^{(\alpha,\beta)}_{v'v}(a + \zeta\xi)(\xi + \beta_{v',v}(a + \zeta\xi)) = \\ &f_{v'v}(a + \zeta\xi + \xi\alpha_{v',v}(a)). \end{aligned}$$

which leads to two equations:

$$\begin{aligned} &a' + \zeta' g^{(\alpha,\beta)}_{v'v}(a)\beta_{v',v}(a) = f^{(\alpha,\beta)}_{v'v}(a) \\ &\zeta' g^{(\alpha,\beta)}_{v'v}(a) + \zeta'\zeta\partial_a(g^{(\alpha,\beta)}_{v'v}(a)\beta_{v',v}(a)) = \zeta\partial_a f^{(\alpha,\beta)}_{v'v}(a) - \partial_a f^{(\alpha,\beta)}_{v'v}(a)\alpha_{v',v}(a) \end{aligned} \tag{43}$$

The latter equation immediately gives the transformation for ζ:

$$\zeta' = g^{(\alpha,\beta)}_{vv'}(a)(1 + \zeta g^{(\alpha,\beta)}_{vv'}(a)\partial_a(g^{(\alpha,\beta)}_{v'v}(a)\beta_{v',v}(a)))(\partial_a f_{v'v}(a)\zeta - \partial_a f_{v'v}(a)\alpha_{v',v}(a))$$

which could be simplified as follows:

$$\begin{aligned} \zeta' &= g_{v,v'}^{(\alpha,\beta)}(a)\partial_a f_{v'v}^{(\alpha,\beta)}(a)\Big(s_{v,v'}(a)\zeta - \alpha_{v',v}(a)\Big) \\ s_{v,v'} &= \Big(1 - g_{v,v'}^{(\alpha,\beta)}(a)\partial_a(g_{v',v}^{(\alpha,\beta)}(a)\beta_{v',v}(a))\alpha_{v',v}(a)\Big). \end{aligned} \tag{44}$$

Now, substituting that into the Equation (43) for a', we obtain:

$$a' + \partial_a f_{v',v}^{(\alpha,\beta)}(a)\Big(s_{v,v'}(a)\zeta - \alpha_{v',v}(a)\Big)\beta_{v',v}(a) = f_{v'v}^{(\alpha,\beta)}(a),$$

which is equivalent to

$$\begin{aligned} a' &= f_{v'v}^{(\alpha,\beta)}(a) - \\ &\Big(\partial_a f_{v'v}(a)\Big(1 - \partial_a\beta_{v',v}(a)\alpha_{v',v}(a))\Big)\zeta - \partial_a f_{v',v}^{(\alpha,\beta)}(a)\alpha_{v',v}(a)\Big)\beta_{v',v}(a), \end{aligned}$$

and simpler as

$$a' = f_{v'v}^{(\alpha,\beta)}\Big(a - (1 - \partial_a\beta_{v',v}(a)\alpha_{v',v}(a))\zeta\beta_{v',v}(a) + \beta_{v',v}(a)\alpha_{v',v}(a)\Big).$$

One can see from the transformations we obtained that the self-dual $(1|1)$ supermanifolds are indeed $N = 1$ SRS. Let us combine all that in the following theorem.

Theorem 8. *Given the coordinate transformations (40) for SF, the coordinate transformations for the dual manifold $\widetilde{SF}$ of $(0|1)$ divisors are given by the formulas*

$$\begin{aligned} \zeta' &= g_{v,v'}^{(\alpha,\beta)}(a)\partial_a f_{v'v}^{(\alpha,\beta)}(a)\Big(s_{v,v'}(a)\zeta - \alpha_{v',v}(a)\Big), \text{ where} \\ s_{v,v'} &= \Big(1 - g_{v,v'}^{(\alpha,\beta)}(a)\partial_a(g_{v',v}^{(\alpha,\beta)}(a)\beta_{v',v}(a))\alpha_{v',v}(a)\Big), \\ a' &= f_{v'v}^{(\alpha,\beta)}\Big(a - (1 - \partial_a\beta_{v',v}(a)\alpha_{v',v}(a))\zeta\beta_{v',v}(a) + \beta_{v',v}(a)\alpha_{v',v}(a)\Big). \end{aligned} \tag{45}$$

Remark. Note that, in the case of a dual manifold, $\mathcal{L}$ is replaced by $\mathcal{L}^{-1} \otimes T$.

6. N = 2 Super Riemann Surfaces

In this section, we write the coordinate transformations for punctured $N = 2$ supermanifold $SF_{N=2}$, corresponding to SF, based on the equivalence between complex structures on $(1|1)$ supermanifolds and superconformal structures on $N = 2$ supermanifolds discovered in [7].

Let us write the transition functions between the chart with coordinates (z, θ) and the chart with coordinates (u, η) on $(1|1)$ supermanifold in the following way:

$$u = S(z) + \theta V(z)\varphi(z), \quad \eta = \psi(z) + \theta V(z), \tag{46}$$

where $S(z)$, $V(z)$, and $\varphi(z)$, $\psi(z)$ are correspondingly even and odd analytic functions. On the other hand, the superconformal coordinate transformations for $N = 2$ SRSs between the charts with coordinates (z, θ_+, θ_-) and $(z', \theta'_+, \theta'_-)$ are

$$\begin{aligned} z' &= q(z) + \frac{1}{2}\theta_-\epsilon_+(z)q_-(z) + \frac{1}{2}\theta_+\epsilon_-(z)q_+(z) + \frac{1}{4}\theta_+\theta_-\partial_z(\epsilon_+(z)\epsilon_-(z)) \\ \theta'_+ &= \epsilon_+(z) + \theta_+q_+(z) + \frac{1}{2}\theta_+\theta_-\partial_z\epsilon_+(z) \\ \theta'_- &= \epsilon_-(z) + \theta_-q_-(z) + \frac{1}{2}\theta_-\theta_+\partial_z\epsilon_-(z) \\ q_+(z)q_-(z) &= \partial_z q(z) + \frac{1}{2}(\epsilon_+(z)\partial_z\epsilon_-(z) + \epsilon_-(z)\partial_z\epsilon_+(z)). \end{aligned} \tag{47}$$

The following Theorem matches these transformations.

Theorem 9 ([7])**.** *There is a one-to-one correspondence between $N = 2$ SRSs from $(1|1)$ supermanifolds. The explicit correspondence between transition functions is given by the following formulas:*

$$\begin{aligned} &\epsilon_+(z) = \psi(z), \quad q_+(z) = V(z) \\ &\epsilon_-(z) = \varphi(z), \quad q_-(z) = (\partial_z S(z) - \partial_z \psi(z)\varphi(z))V^{-1}(z), \\ &q(z) = S(z) + \frac{1}{2}\varphi(z)\psi(z). \end{aligned} \tag{48}$$

Let us now describe how it works for the transition functions we introduced in the previous section. In our case,

$$\begin{aligned} &V(w) = g^{(\alpha,\beta)}_{v',v}(w), \quad \psi(w) = g^{(\alpha,\beta)}_{v',v}(w)\beta_{v',v}(w), \\ &S(w) = f^{(\alpha,\beta)}_{v',v}(w), \quad \varphi(w) = \partial_w f^{(\alpha,\beta)}_{v,'v}(w)\alpha_{v',v}(w)g^{(\alpha,\beta)}_{v,v'}(w). \end{aligned} \tag{49}$$

Therefore, we can write, for the transition functions of $SF_{N=2}$,

$$\begin{aligned} &\epsilon_+(w) = g^{(\alpha,\beta)}_{v',v}(w)\beta_{v',v}(w), \\ &\epsilon_-(w) = \partial_w f^{(\alpha,\beta)}_{v',v}(w)\alpha_{v',v}(w)g^{(\alpha,\beta)}_{v,v'}(w), \\ &q_+(w) = g^{(\alpha,\beta)}_{v'v}(w), \\ &q_-(w) = \Big(\partial_w f^{(\alpha,\beta)}_{v'v}(w) - \\ &\partial_w(g^{(\alpha,\beta)}_{v',v}(w)\beta_{v',v}(w))\partial_w f^{(\alpha,\beta)}_{v',v}(w)\alpha_{v',v}(w)g^{(\alpha,\beta)}_{v,v'}(w)\Big)g^{(\alpha,\beta)}_{v,v'}(w), \\ &q(w) = f^{(\alpha,\beta)}_{v',v}(w) + \frac{1}{2}\partial_w f^{(\alpha,\beta)}_{v,'v}(w)\alpha_{v',v}(w)\beta_{v',v}(w). \end{aligned}$$

This can be rewritten in a simpler way:

$$\begin{aligned} &\epsilon_+(w) = g^{(\alpha,\beta)}_{v',v}(w)\beta_{v',v}(w), \\ &\epsilon_-(w) = \partial_w f^{(\alpha,\beta)}_{v',v}(w)\alpha_{v',v}(w)g^{(\alpha,\beta)}_{v,v'}(w), \\ &q_+(w) = g^{(\alpha,\beta)}_{v',v}(w) \\ &q_-(w) = \partial_w f^{(\alpha,\beta)}_{v',v}(w)g^{(\alpha,\beta)}_{v,v'}(w)(1 + \alpha_{v',v}(w)\partial_w\beta_{v',v}(w)) + \\ &\partial_w g^{(\alpha,\beta)}_{v,v'}(w)\partial_w f^{(\alpha,\beta)}_{v',v}(w)\beta_{v',v}(w)\alpha_{v',v}(w), \\ &q(w) = f^{(\alpha,\beta)}_{v',v}(w) + \frac{1}{2}\partial_w f^{(\alpha,\beta)}_{v,'v}(w)\alpha_{v',v}(w)\beta_{v',v}(w). \end{aligned} \tag{50}$$

Hence, we obtain the following theorem.

Theorem 10. *Formula (50) produces the transition functions describing the superconformal structure on $N = 2$ SRSs with punctures, corresponding to $(1|1)$ supermanifolds with transition functions (40). That is, the transition function corresponding to oriented edge v, v' of the fatgraph, i.e., the overlap $U_v \cap U_{v'}$, is described using the functions $\epsilon_\pm(w)$, $q_\pm(w)$ from (50).*

7. Involution and $N = 1$ Super Riemann Surfaces with NS and R Punctures

7.1. Involution: R vs. NS Punctures

There is an involution I on the moduli space of super Riemann surfaces such that

$$I : D_\pm \longrightarrow D'_\mp, \tag{51}$$

where $D'_{\mp}$ is the corresponding operator after the $N = 2$ superconformal transformation.

Such an involution takes $N = 2$ super Riemann surface to the *dual*, which on the level of $(1|1)$ supermanifolds produces a manifold of $(0|1)$ divisors, which we discussed earlier. The self-dual supermanifolds are known to be $N = 1$ super Riemann surfaces.

Let us describe how this works on a $N = 2$ supertube (or $N = 2$ punctured disk) with coordinates (x, η_+, η_-), where $x \sim x + 2\pi i$. Let us consider an obvious choice of how involution could act in these coordinates:

$$D_+ \longrightarrow D_-,\ D_- \longrightarrow D_+, \tag{52}$$

For self-duality, one has to identify η_+ and η_-, i.e., $(x, \eta_+, \eta_-) \sim (x + 2\pi, \eta_-, \eta_+)$. The operator $D = D_+ + D_-$ gives a standard superconformal structure on a supertube. We see that, in this case, the puncture is a *Ramond puncture*. Let us perform an elementary $N = 2$ superconformal transformation, amounting to reflection, so that involution is

$$D_+ \longrightarrow -D_-,\ D_- \longrightarrow -D_+, \tag{53}$$

i.e., $\eta_\pm \longrightarrow -\eta_\mp$. The invariance under this involution gives the identification $(x, \eta_+, \eta_-) \sim (x + 2\pi, -\eta_-, -\eta_+)$, so that the operator $D = D_+ + D_-$ gives a superconformal structure around the NS puncture.

Note that the two examples of the action of involution that we considered in this section are the only ones that preserve the base manifold.

7.2. Split $N = 1$ SRS

Let us now discuss the split $N = 2$ SRS, which implies that we let cocycles $\alpha, \beta = 0$. The involution

$$I : D_\pm \longrightarrow D_\mp, \tag{54}$$

acts on the level of transition functions as follows:

$$q_\pm(z) \longrightarrow q_\mp(z)$$

Therefore, for fixed points of the involution, we have

$$g^2_{v',v}(w) = \partial_w f_{v',v}(w). \tag{55}$$

This means that $g^{(\alpha,\beta)}_{v',v}(w) = \mathrm{sign}(v', v)\sqrt{\partial_w f_{v',v}(w)}$, where $\mathrm{sign}(v', v)$ is the notation for the sign of the square root, so that on a resulting $N = 1$ SRS, we have

$$\xi' = \mathrm{sign}(v', v)\sqrt{\partial_w f_{v',v}(w)}\,\xi. \tag{56}$$

The choice of signs for such square roots is the same as the choice of the spin structure on the punctured surface. However, we have already discussed that problem on the level of fatgraphs (Section 3), which leads to the following Theorem.

Theorem 11. *Consider a metric fatgraph τ with a spin structure ω provided by the orientation, as discussed in Section 3. These data define the superconformal structure on the split $N = 1$ SRS. For every boundary cycle on the fatgraph corresponding to puncture p, let m_p be the number of oriented edges, which are opposite to the orientation induced by the one on the surface. The corresponding puncture is Ramond or Neveu–Schwarz, depending on whether m_p is even or odd.*

Proof. So let us consider the metric graph τ with orientations on edges. Our problem is to use orientations for definitions. To do that, for each overlap, we will look at the z

coordinates on stripes discussed in Section 4. For given vertices v and v', the transformation between z and z' coordinates is given by

$$z' = \tilde{f}_{v',v}(z) = L_{v,v'} - z$$

We will define the value of the $\sqrt{\partial_z \tilde{f}_{v',v}(z)} = \pm i$ in the following way. If the orientation is from vertex v to v', we choose the positive sign ($\mathrm{sign}(v,v') = 1$) and otherwise choose the negative sign ($\mathrm{sign}(v,v') = -1$). One can prove that such a choice does not depend on the choice of orientation for a given spin structure; that is, a different choice, corresponding to a fatgraph reflection, will just result in a reflection of an odd coordinate for a given vertex.

Regarding R and NS punctures, one can deduce immediately that the statement is correct by using a simple condition that there is a natural combinatorial constraint on the punctures with m_p being odd on a fatgraph (see Section 4), matching the one for Ramond punctures on a surface. Nevertheless, let us prove this directly.

For a given choice of spin structure, let us superconformally continue $g^{(\alpha,\beta)}_{v'v}$ cocycles by constructing $\tilde{g}_{p,v}(z) = \mathrm{sign}(p,v)\sqrt{\partial_z \tilde{f}_{p,v}(z)}$ on $U_v \cap U_{v'} \cap V_p$, where we recall that

$$x = \tilde{f}_{p,v}(z) = \frac{2\pi i}{a_B}(L_{v_1,v_2} + ... + L_{v_k,v_{k-1}} + z),$$

where z is the coordinate on the consequtive stripe v_k, v_{k+1} and $v_1, \ldots, v_m$ are consecutive vertices around the puncture. Now, we obtain R and NS punctures by gluing the supertube with a proper twist of the odd variable. That will of course depend on the number m_p of the edges $\{v_i, v_{i+1}\}$, which have opposite orientations with respect to the orientation induced on the cycle by the one on the surface. Note that, in terms of z-variables, $\sqrt{\partial_z \tilde{f}_{p,v}(z)}$ is a constant, so one can again make a choice of signs explicitly. We have the following identity

$$\mathrm{sign}(v_1,v_2)\mathrm{sign}(v_2,v_3)\ldots\mathrm{sign}(v_{n-1},v_n)\mathrm{sign}(v_n,v_1) = \pm 1, \tag{57}$$

where the sign is positive for even N and negative otherwise. In the case of even m_p, we can choose $\{\mathrm{sign}(p,v)\}$ so that for $\mathrm{sign}(v,v')$, so that v, v' are neighboring vertices, so that $\mathrm{sign}(v,v') = \mathrm{sign}(p,v)\mathrm{sign}(p,v')$, thus gluing the stripes into a supertube. However, this is not possible in the case of odd m_p. In this case, we have to assume that $\mathrm{sign}(v_n,v_1) = -\mathrm{sign}(p,v_1)\mathrm{sign}(p,v_n)$, thus gluing the stripes into a twisted supertube corresponding to an NS puncture. $\square$

Remark. One can of course superconformally transform the twisted supertube in the NS puncture case into the disk, the same way we did in the introduction, thus making the corresponding cocycle $\{g_{v,p}(y)\} = \{\pm\sqrt{\partial_y f_{v,p}(y)}\}$. In the Ramond case, this is of course impossible. We see that if p is an R puncture,

$$g^2_{v,p}(x) = y\partial_y f_{v,p}(y).$$

Therefore, for the bundle $\mathcal{L}$, we have a condition:

$$\mathcal{L}^2 = T \otimes \mathcal{O}\Big(-\sum_{i=1}^{n_R} p_i\Big), \tag{58}$$

which is possible only when the n_R is divisible by 2.

7.3. $N = 1$ SRS: Non-Split Case

In order to construct a nonsplit $N = 1$ SRS, we first will carry it out at an infinitesimal level near the split $N = 1$ SRS. So let us look at the Formula (50) when $\alpha_{v',v}$ $\beta_{v',v}$ are infinitesimal:

$$\begin{aligned}
&\epsilon_+(w) = g_{v',v}(w)\beta_{v',v}(w),\\
&\epsilon_-(w) = \partial_w f_{v',v}(w)\alpha_{v',v}(w)g^{(\alpha,\beta)}_{v,v'}(w),\\
&q_+(w) = g_{v',v}(w),\\
&q_-(w) = \partial_w f_{v',v}(w)g_{v,v'}(w),\\
&q(w) = f_{v',v}(w).
\end{aligned} \tag{59}$$

An invariance under a simple involution $D_\pm \longrightarrow D_\mp$ allows us to identify $\alpha_{v',v}$ and $\beta_{v',v}$, and, as before, $g^2_{v',v}(w) = \partial_w f_{v',v}(w)$; thus, infinitesimally, the transformations for the resulting $N = 1$ SRS on the overlap $U_v \cap U_{v'}$ are given by

$$\begin{aligned}
w' &= f_{v'v}(w + \xi\rho_{v',v}(w))\\
\xi' &= \pm\sqrt{\partial_w f_{v',v}(w)}(\xi + \rho_{v',v}(w)),
\end{aligned} \tag{60}$$

so that the signs are prescribed as in the Theorem 11, where

$$\rho \in \Pi\check{Z}^1(F^c, \mathcal{L} \otimes O(-D_{NS})) \text{ and } \mathcal{L}^2 = T \otimes O(-D_R),$$

so that D_R and D_{NS} are the divisors corresponding to the sum of all NS and R punctures correspondingly. We have described such cocycles using odd-number decorations at the vertices of the fatgraph in Theorem 6. Formula (60) is not hard to continue to full superconformal transformations for transition functions (one can obtain them by applying involution $D_\pm \longrightarrow D_\mp$ invariance to Formula (50) as well):

$$\begin{aligned}
w' &= f^{(\rho)}_{v'v}(w + \xi\lambda^{(\rho)}_{v',v}(w))\\
\xi' &= \pm\sqrt{\partial_w f^{(\rho)}_{v',v}(w)}(1 + \frac{1}{2}\lambda^{(\rho)}_{v',v}(w)\partial_w\lambda^{(\rho)}_{v',v}(w))(\xi + \lambda_{v',v}(w)).
\end{aligned}$$

Combining the parametrization data for cocyles ρ from Theorem 34 with the results of this section, we obtain the following omnibus theorem, describing the dense set of superconformal structures inside a moduli space of $N = 1$ SRS.

Theorem 12. *Consider the following data on a fatgraph τ:*

1. *Metric structure.*
2. *Spin structure, as an equivalence class of orientations on the fatgraph. The cycles on the fatgraph encircling the punctures are divided into two subsets, NS and R, depending on whether there is an odd or even number of edges oriented opposite to the surface-induced orientation of the correspondingly appropriate boundary piece of a fatgraph. We denote the number of the corresponding boundary pieces as n_R and n_{NS}.*
3. *Ordered set $\{\sigma^k_v\}_{k=0,\ldots,m_v-3}$ of odd complex parameters for each vertex v, where m_v is the valence of the vertex v.*

   Then, the following points are true:

   (a) *Data from* (1) *and* (2) *uniquely determine the split Riemann surface with n_R Ramond and n_{NS} Neveu–Schwarz punctures with the transition functions given by*

$$w' = f_{v',v}(w) \quad \xi' = \pm\sqrt{\partial_w f_{v',v}(w)}\xi, \tag{61}$$

one for each overlap $U_v \cap U_{v'}$. The sign of the square root is given by the spin structure on the fatgraph, making an odd coordinate a section of a line bundle $\mathcal{L}$ on the corresponding closed Riemann surface F^c, such that $\mathcal{L}^2 = T \otimes \mathcal{O}(-D_R)$, where D_R is a divisor, which is a sum of points corresponding to the Ramond punctures.

(b) *Part* (3) *of the above data allows to construct Čech cocycles on a Riemann surface F, which are the representatives of $\Pi\check{H}^1(F^c, \mathcal{L} \otimes \mathcal{O}(-D_{NS}))$, where D_{NS} is a divisor corresponding to the sum of the points corresponding to NS punctures:*

$$\rho_{v,v'}|_{U_v \cap U_{v'}} = \rho_v - \rho_{v'}, \text{ so that } \rho_v|_{U_v} = \frac{\sigma_v(w)}{w^{m_v - 2}}, \ \rho_{v'}|_{U'_v} = \frac{\sigma_{v'}(w')}{w'^{\ m_{v'} - 2}},$$
$$\sigma_v(w) = \sum_{i=0}^{m_v - 3} \sigma_v^i w^i, \quad \sigma_{v'}(w') = \sum_{i=0}^{m_{v'} - 3} \sigma_{v'}^i {w'}^i, \tag{62}$$

where ρ_v, $\rho_{v'}$ are the meromorphic sections of $\mathcal{L} \otimes \mathcal{O}(-D_{NS})$ on U_v, $U_{v'}$, respectively, so that m_v is valence of the given vertex v.
The cocycles defined by configurations described by $\{\sigma_v\}$ and $\{\tilde{\sigma}_v\}$ are equivalent to each other if and only if

$$\sigma_v(w) - \tilde{\sigma}_v(w) = \gamma^{(m-3)}(w), \tag{63}$$

for every vertex v, $\gamma \in \Pi H^0(F^c, \mathcal{L} \otimes K^2 \otimes \mathcal{O}(D_{NS} + 2D_R))$, $\gamma|_{U_v} = \gamma(w)$. so that $\gamma^{(m-3)}(w)$ is the Taylor expansion of $\gamma(w)$ up to order $m - 3$.
We call two sets of data associated to the fatgraph τ equivalent if they are related as in (63).

(c) *There exists a superconformal structure for $N = 1$ super Riemann surface SF with n_R Ramond punctures and n_{NS} Neveu–Schwarz punctures so that the superconformal transition functions for each overlap $U_v \cap U_{v'}$ are*

$$w' = f^{(\sigma)}_{v'v}(w + \xi\lambda^{(\sigma)}_{v',v}(w))$$
$$\xi' = \pm\sqrt{\partial_w f^{(\sigma)}_{v',v}(w)}\left(1 + \frac{1}{2}\lambda^{(\sigma)}_{v',v}(w)\partial_w\lambda^{(\sigma)}_{v',v}(w)\right)(\xi + \lambda^{(\sigma)}_{v',v}(w)), \tag{64}$$

where the deformed functions $f^{(\sigma)}_{v',v'}$, $\lambda^{(\sigma)}_{v',v}$ depend on odd parameters $\{\sigma^k_v\}$, characterizing the Čech cocylce $\{\rho_{v',v}\}$, with $f^{(0)}_{v',v} = f_{v',v}$ and in the first order in $\{\sigma^k_v\}$ variables $\lambda^{(\sigma)}_{v',v} = \rho_{v',v}$.

(d) *To describe the non-split SRS, we fix the choice of transition functions in* (c) *for every metric spin fatgraph τ with the odd data from* (3)*. We consider the set of superconformal structures constructed by picking one superconformal structure per equivalence class of data for every fatgraph τ. The points in this set represent inequivalent superconformal structures, and together, they form a dense subspace of odd complex dimension $2g - 2 + n_{NS} + n_R/2$ in the space of all superconformal structures with n_{NS} Neveu–Schwarz and n_R Ramond punctures associated with F.*

Author Contributions: Conceptualization: A.S.S.; Methodology: A.S.S. and A.M.Z.; Investigation: A.M.Z. All authors have read and agreed to the published version of the manuscript.

Funding: A.M.Z. is partially supported by Simons Collaboration Grant 578501 and NSF grant DMS-2203823.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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