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Fab-Four Cosmography to Tackle the Hubble Tension

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Abstract: In the context of the Fab-Four theory of gravity in a Friedmann-Lemaître-Robertson-Walker background, in this work we use the cosmography approach to study a particular self-tuning filter solution focused on a zero-curvature fixed point to study the H_0 tension. In this scheme, the equations restrict the universe's evolution to certain scenarios, including radiation-like expansion, matter-like expansion, and late-time acceleration. Furthermore, we build the cosmographic series of the Fab-Four theory to obtain the kinematic parameters as the Hubble constant H_0 and the deceleration parameter q_0 for all the scenarios mentioned. Finally, we compare our results to find that it is possible to alleviate the current discrepancy on H_0 by considering specific requirements on the free parameters of the Fab-Four theory through a self-tuning filter.

Keywords: fab-four; horndeski gravity

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1. Introduction

The concordance model of cosmology, Λ Cold Dark Matter (ΛCDM), is thought to work well at explaining several observations of the Universe, including the observed late cosmic acceleration [1,2], the structure of the cosmic microwave background (CMB), the baryon acoustic oscillations (BAOs), among others [3]. In recent years, there have been several issues at hand [4], e.g., CMB anisotropy anomalies [5], cosmic dipoles [4], and the most notable issue: the Hubble constant tension, reaching a statistical discrepancy of $\sim 5\sigma$ [6–9] between early and late measurements.

Direct measurements of the current Hubble constant H_0, through late-universe observations, have yielded a value of $H_0 = 73.3 \pm 1.04$ km s^{-1} Mpc $^{-1}$ [8], while early-universe observations of the CMB give a lower result of $H_0 = 67.4 \pm 0.5$ km s^{-1} Mpc $^{-1}$ [6], differing by around 8–10%. This discrepancy has positioned itself as an important crisis for the standard ΛCDM model. Consequently, several theories have been proposed to solve, alleviate or give insights into the Hubble tension issue. In the same line of thought, some proposals focus on the nature of dark energy, since if the cosmological constant Λ played a bigger role than previously thought in the early stages of the expansion of the Universe, it could alter the calculation for the early-universe Hubble constant [10], as a consequence, this would bring it closer to the locally measured H_0 value. On one hand, changing the equation-of-state of dark energy by making it time-dependent, has also been proposed as a possible explanation [11]. Other theories include adding new interactions, such as interacting dark energy [12,13], interacting dark matter [14] and interacting holographic dark energy [15]; including sterile neutrinos [16], axions [17], or other theorized particles to increase the relativistic degrees of freedom; unifying dark energy and dark matter as a single fluid [18]; introducing cosmological piecewise functions [19]; using a statistical lookback time approach [20], considering the black hole shadows as standard rulers calibrators [21],

among many others [22–24]. See, for example, [25], for a summary and comparison of several proposals that involve interesting solutions to the H_0 tension.

On the other hand, alternative gravity theories also attempt to solve the Hubble tension and other problems within standard cosmology. These theories change General Relativity's (GR) Einstein-Hilbert action to introduce different dynamics in the resulting equations of motion, which can explain different phases of expansion of the universe and could explain the Hubble tension [26,27]. Allowing higher order equations of motion, there are theories such as $f(\mathcal{R})$ theories, which introduce a function of the Ricci scalar, $\mathcal{R}$, to act as the action. In the context of Teleparallel Gravity [28], it can be considered $f(\mathcal{T})$ theories, which instead use functions of the torsion scalar. Another way to extend GR is by allowing the Lagrangian density to depend not only on the metric but also on another degree of freedom such as a scalar field ϕ, these are the so-called scalar-tensor theories. If we maintain the requirement of working in four dimensions, and having second-order equations of motion—to avoid Ostrogradsky instabilities [26]—the most general of these theories is Horndeski theory [29,30].

In this work, we focus on a special case of Horndeski's theory first proposed in [29] called the *Fab Four* theory. As a first approach, in [29] a *self-tuning* filter was applied on the original Horndeski action that partially de-couples the curvature from the cosmological constant and reduces the action to only four base Lagrangians, each one a function dependent on combinations of the Riemann tensor and an arbitrary function of the scalar field ϕ. The self-tuning filter consisted in requiring the following [29]:

1. The theory admits a patch of Minkowski vacuum for any value of the cosmological constant Λ, and for instantaneous changes in its value;
2. The theory allows non-trivial cosmologies by introducing additional dynamical terms to the standard one.

According to these two requirements, in [31], an initial outlook was provided on the types of cosmology and its dynamics derived from the Fab-Four theory. Although it is a fairly recent proposal, the Fab-Four theory has been explored extensively and in different contexts. Specifically, there have been several studies involving the $\mathcal{L}_{john}$ term in isolation, since it is the one that mainly drives the self-tuning mechanism [32] and it contains a derivative coupling to the scalar field, e.g., in [33] the $\mathcal{L}_{john}$ term (see Section 2) is studied in detail, specifically its generalization as the $\mathcal{L}_5$ term in Horndeski's theory which adds a dependency on the field kinetic term $X = \nabla_\mu\phi\nabla^\mu\phi$, finding that the resulting *Fab 5* theory can be ghost-free, or stable, or self-tuning, but not all these characteristics at the same time. *Fab 5* cosmology is studied in more detail in [34–36]. After the extension of the original Horndeski Lagrangian [37], adding two more terms in Horndeski's action, it was shown in [38] that these new terms can be included in a generalized Fab-Four theory where the arbitrary functions of the field $V_j(\phi)$ and $V_p(\phi)$ become arbitrary functions of the field and its kinetic term $V_j(\phi, X)$ and $V_p(\phi, X)$. Additionally, it was shown that these new terms also present self-tuning for screening the cosmological constant. In [39], it was argued that the measurement of the speed of gravitational waves (GWs) strongly constrained *John*-like terms in Horndeski's action, along with several other scalar-tensor theories and dark energy (DE) models. However, in [32], it is shown that working in an Effective Field Theory (EFT) framework, the resulting effective action is not constrained as such, and so it still admits *John*-like terms, therefore allowing a self-tuning mechanism in the EFT regime and the possibility to be consistent with the observational data [39,40]. In order to explore GW constraints, see [41]. Furthermore, as any alternative theory of gravity could, the Fab Four proposal could be excluded if we demand that the speed of gravitational waves c_s is exactly the same as the speed of light c along the evolution of the Universe. Therefore, if we consider small deviations from c, then several models can become consistent with the observable data (and their constrictions). New self-tuning models known as *well-tempered* have been developed and deal with the GW constraints by construction [42], and also prevent the scalar field from screening not only the cosmological constant but also all other matter sources in the system. A generalized class of *well-tempered* models and their

self-tuning mechanisms has been discussed in [43]. In a quantum gravity context, the study of the quantization of the Fab-Four *John* term results in a Hamiltonian with fractional powers of the momenta, leading to the use of a conformal fractional derivative approach to get a wide range of solutions for the scale factor and the scalar field [44].

In this work, we follow closely the derivations performed for the Fab-Four cosmology [31] to obtain the corresponding equations of motion for a Friedmann-Lemaître-Robertson-Walker (FLRW) background and to apply a dynamical systems approach to solving the dynamical equations. Under this scheme and considering all four components of Fab-Four theory, we also extend this analysis by using a cosmographic approach, which will be useful to obtain kinematic quantities, such as H_0 and q_0. Finally, with these quantities at hand, we will be capable of setting general requirements on the free parameters to relax the Hubble tension and give a likelihood landscape for future works and analyses of this theory. The core idea behind this process is to facilitate the search for a stable cosmological solution through dynamical systems. The advantage of this kind of solution is the possibility to explore those that fulfill the cosmological constraint equation of interest for the analysis scheme. Once this constraint is satisfied, the analytical solution at hand can be tested with observations to establish a cutoff for the cosmological parameters of interest, namely H_0 and q_0. Although there may exist more analytical solutions for the set of equations, our cosmographic approach up to second order allows us to narrow down the options and select the ones that address the late-time cosmic acceleration.

This paper is divided as follows: In Section 2, we present a brief introduction of the results discussed in [31], where the full equations of motion of the system are given for the Fab-Four theory in a FLRW background. Also, we describe the dynamical system approach for this theory to find the dynamical parameters and the corresponding fixed points. We rewrite the equations in the zero-curvature fixed point of interest and solve them to find viable cosmological scenarios. In Section 3, we construct the cosmographic series for the Fab-Four theory and find the expressions for the Hubble constant H_0, and the deceleration parameter q_0 for each cosmological scenario described. Also, we will restrict our analysis to a couple of cases due to the solutions found in the dynamical system approach. In Section 4, we discuss how our solutions from the Fab-Four theory are able to relax the H_0 tension by following a set of requirements on the free parameters of the theory. Finally, we present our conclusions and considerations for future work in Section 5.

2. Fab-Four Cosmology as a Dynamical System

The Fab-Four model comes from a restriction set upon Horndeski's theory, which simplifies it greatly so that the Lagrangian density is now just the sum of four base Lagrangians [29]:

$$\mathcal{L}_{\text{Fab-Four}} = \mathcal{L}_{\text{john}} + \mathcal{L}_{\text{paul}} + \mathcal{L}_{\text{george}} + \mathcal{L}_{\text{ringo}}, \tag{1}$$

where each Lagrangian is defined as a combination of derivatives of the scalar field ϕ, combinations of the metric tensor, and an arbitrary function of the field $V_i(\phi)$, with $i = j, p, g, r$, namely:

$$\mathcal{L}_{john} = \sqrt{-g} V_{john} G^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi, \tag{2}$$

$$\mathcal{L}_{paul} = \sqrt{-g} V_{paul} P^{\mu\nu\alpha\beta} \nabla_\mu \phi \nabla_\alpha \phi \nabla_\nu \phi \nabla_\beta \phi, \tag{3}$$

$$\mathcal{L}_{george} = \sqrt{-g} V_{george} R, \tag{4}$$

$$\mathcal{L}_{ringo} = \sqrt{-g} V_{ringo} \hat{\mathcal{G}}, \tag{5}$$

where $G_{\mu\nu}$ is the Einstein tensor, $P_{\mu\nu\alpha\beta}$ is the double dual of the Riemann tensor, R is the Ricci scalar and $\hat{\mathcal{G}}$ is the gauss-Bonnet combination [31]. Given this notation and considering an FLRW metric whose line element is given by

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \tag{6}$$

where $a(t)$ is the scale factor, which denotes the relative size of the co-moving, constant-time hypersurfaces of space-time, $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the element of solid angle and k is the spatial curvature constant. Here, $k = 0$ gives a flat space, while $k > 0$ ($k < 0$) corresponds to positive (negative) curvature. We can express the Fab-Four Lagrangians as follows [31]:

$$\mathcal{L}_j = 3a^3 V_j(\phi)\dot{\phi}^2\left(H^2 + \frac{k}{a^2}\right), \tag{7}$$

$$\mathcal{L}_p = -3a^3 V_p(\phi)\dot{\phi}^3 H\left(H^2 + \frac{k}{a^2}\right), \tag{8}$$

$$\mathcal{L}_g = -6a^3 V_g(\phi)\left(H^2 - \frac{k}{a^2}\right) - 6a^3 V_{g,\phi}(\phi)H\dot{\phi}, \tag{9}$$

$$\mathcal{L}_r = -8a^3 V_{r,\phi}(\phi)\dot{\phi}H\left(H^2 + 3\frac{k}{a^2}\right). \tag{10}$$

Here, $V_{i,\phi}$ denotes the derivative of V_i with respect to the field, $dV_i/d\phi$, and $\dot{\phi} = U^a\nabla_a\phi$, with $U^a = \delta^a{}_t$ the 4-velocity. Following [31], we find that in the presence of standard matter and a cosmological constant Λ, the Hamiltonian density is given by the following:

$$\mathcal{H} = \mathcal{H}_j + \mathcal{H}_p + \mathcal{H}_g + \mathcal{H}_r + \rho_\Lambda = 0, \tag{11}$$

where, for convenience, we can express each Hamiltonian in conformal time by considering $N = \ln a$, where A' denotes differentiation with respect to N, such that $A' = dA/dN = \dot{A}(1/H)$:

$$\mathcal{H}_j = 3V_j\phi'^2H^2\left(3H^2 + \frac{k}{a^2}\right), \tag{12}$$

$$\mathcal{H}_p = -3V_p\phi'^3H^4\left(5H^2 + 3\frac{k}{a^2}\right), \tag{13}$$

$$\mathcal{H}_g = -6V_g\left[\left(H^2 + \frac{k}{a^2}\right) + H^2\frac{V_{g,\phi}\phi'}{V_g}\right], \tag{14}$$

$$\mathcal{H}_r = -24V_{r,\phi}\phi'H^2\left(H^2 + \frac{k}{a^2}\right). \tag{15}$$

The equations for the scalar field ϕ can be obtained through varying

$$E_i^{(\phi)} = -\frac{d}{dt}\left(\frac{\partial\mathcal{L}_i}{\partial\dot{\phi}}\right) + \frac{\partial\mathcal{L}_i}{\partial\phi}, \tag{16}$$

which gives the following expressions [31]:

$$E_j^{(\phi)}\dot{\phi}H^{-1} = \frac{1}{a^3\Delta_2}\left(\frac{\mathcal{H}_j a^6\Delta_2^2}{3\Delta_2 - 2\frac{k}{a^2}}\right)', \tag{17}$$

$$E_p^{(\phi)}\dot{\phi}H^{-1} = \frac{2}{a^{3/2}(H\Delta_2)^{1/2}}\left(\frac{\mathcal{H}_p a^{9/2}(H\Delta_2)^{1/2}\Delta_2}{5\Delta_2 - 2\frac{k}{a^2}}\right)', \tag{18}$$

$$E_g^{(\phi)}\dot{\phi}H^{-1} = -3\frac{V_g'}{a}(\Delta_2 a^4)', \tag{19}$$

$$E_r^{(\phi)}\dot{\phi}H^{-1} = \frac{\mathcal{H}_r}{H\Delta_2}\left[a^3\left(\frac{k}{a^2}\Delta_1 + \frac{1}{3}\Delta_3\right)\right]', \tag{20}$$

with $\Delta_n = H^n - (\sqrt{-k}/a)^n$. Notice that this function goes to zero as the latter equations approach a self-tuning solution corresponding to a Ricci flat scenario and the full system of equations that arises is coupled with second-order differential equations, which are extremely difficult to solve analytically. Therefore, a viable path to reach a set of possi-

ble solutions is through a dynamical systems approach[31]. In order to implement this procedure, we consider the following dynamical variables:

$$h = \frac{H'}{H}, \qquad \sigma = \frac{\sqrt{-k}}{Ha}, \tag{21}$$

notice that if h =const. we obtain that $H = H_0 a^h \Rightarrow a \propto t^{-\frac{1}{h}}$, where a matter-like expansion would arise from $h = -3/2$, radiation-like expansion from $h = -2$, and an inflationary expansion coming from $-1 < h < 0$. To build the dynamical system according to the latter variables, we define the following expressions [31]:

$$x = H^{\alpha}\phi', \tag{22}$$

$$y_j = H^{4-2\alpha}V_j, \quad y_p = H^{6-3\alpha}V_p, \quad y_g = H^2 V_g, \quad y_r = H^{4-\alpha}V_{r,\phi}, \tag{23}$$

$$\lambda_j = H^{-\alpha}\frac{V_{j,\phi}}{V_j}, \quad \lambda_p = H^{-\alpha}\frac{V_{p,\phi}}{V_p}, \quad \lambda_g = H^{-\alpha}\frac{V_{g,\phi}}{V_g}, \quad \lambda_r = H^{-\alpha}\frac{V_{r,\phi\phi}}{V_{r,\phi}}, \tag{24}$$

$$\mu_j = \frac{V_j V_{j,\phi\phi}}{(V_{j,\phi})^2}, \quad \mu_p = \frac{V_p V_{p,\phi\phi}}{(V_{p,\phi})^2}, \quad \mu_g = \frac{V_g V_{g,\phi\phi}}{(V_{g,\phi})^2}, \quad \mu_r = \frac{V_{r,\phi} V_{r,\phi\phi\phi}}{(V_{r,\phi\phi})^2}, \tag{25}$$

where α is an arbitrary constant that provides freedom in re-defining the scalar field while maintaining equations of motion that do not explicitly depend on H. This freedom will later be fixed to a particular value in Section 3.

The evolution equations for these variables are directly obtained by differentiating Equations (21)–(25) with respect to N. Finally, we obtain:

$$\sigma' = \sqrt{-k}\frac{1}{H^2 a^2}\left(H'a + Ha\right) = -\sigma(h+1), \tag{26}$$

$$x' = \alpha H^{\alpha-1}H'\phi' + H^{\alpha}\phi'' = x\left(\alpha h + \frac{\phi''}{\phi'}\right), \tag{27}$$

$$y_i' = \beta_i H^{\beta_i} h V_i + H^{\beta_i} V_{i,\phi}\phi' = y_i(\beta_i h + \lambda_i x), \tag{28}$$

$$\lambda_i' = -\alpha h H^{-\alpha}\frac{V_{i.\phi}}{V_i} + H^{-\alpha}\phi'\frac{V_{i,\phi\phi}}{V_i} - H^{-\alpha}\frac{V_{i,\phi}^2}{V_i^2}\phi' = \lambda_i(-\alpha h + \lambda_i x(\mu_i - 1)), \tag{29}$$

where $i = j, p, g, r$ and β_i denotes the exponents of H in Equation (23). Knowing how these variables evolve, we can easily find their fixed points. Considering constant μ_i allows for potentials in a power law form. For the y_i and the λ_i, these fixed points are those where

$$\beta_i h + \lambda_i x = 0, \tag{30}$$

$$\lambda_i x(\mu_i - 1) = \alpha h. \tag{31}$$

In this fixed point, combining Equations (30) and (31) yields:

$$\lambda_i\left(-\frac{\beta_i h}{\lambda_i}\right)(\mu_i - 1) = \alpha h \Rightarrow (1 - \mu_i) = \frac{\alpha}{\beta_i}, \tag{32}$$

which allows us to directly find the exponents for all four potentials, and they become

$$V_j(\phi) = c_j \phi^{\frac{4}{\alpha}-2}, \tag{33}$$

$$V_p(\phi) = c_p \phi^{\frac{6}{\alpha}-3}, \tag{34}$$

$$V_g(\phi) = c_g \phi^{\frac{2}{\alpha}}, \tag{35}$$

$$V_{r,\phi}(\phi) = c_r \phi^{\frac{4}{\alpha}-1}. \tag{36}$$

Furthermore, the fixed point for σ corresponds to $\sigma(h+1) = 0$, i.e., fixed points solutions exist for $\sigma = 0$ and $h = -1$. On this line of thought, we are interested in the case where $\sigma = 0$, which refers to vanishing spatial curvature, as it is consistent with observations [6]. In this selected fixed point, by also using Equations (30) and (31), the full scalar field equation Equation (20) can be written as follows:

$$2xy_j(h+3) - 3x^2y_p(h+3) - 2\lambda_g y_g(h+2) - 8y_r(h+1) = 0. \tag{37}$$

Similarly, the Hamiltonian constraint Equation (11) can be written as

$$9x^2y_j - 15x^3y_p - 6y_g(1-2h) - 24xy_r = -\rho_\Lambda. \tag{38}$$

Following the arguments in [31], because the self-tuning filter demands that the solutions hold for any value of the cosmological constant, the parameters in the potentials (α, c_i) should be all independent of ρ_Λ. This assumption constrains the possible values of the terms present in Equations (37) and (38) while satisfying the following:

$$(3+h)c_p(-\alpha h)^{2-\frac{2}{\alpha}} = 0, \tag{39}$$

$$-2c_j\alpha h(3+h) - 8c_r(h+1) = 0, \tag{40}$$

$$h(2+h)c_g(-\alpha h)^{-1+\frac{2}{\alpha}} = 0. \tag{41}$$

We can distinguish several values for h that fulfill these requirements, which are associated with different behaviors of a, due to $H = H_0a^h$. Two cases of interest in this article are arbitrary for $h \neq 0$ and radiation for $h = -2$.

Under the arbitrary case, we focus on three cases: Firstly, the case of $h = -1$ corresponds to an evolution where $H^2 \propto a^{-2}$, indicating an accelerated expansion. Secondly, the case of $h = -3/2$ exhibits an evolution with $H^2 \propto a^{-3}$, resembling matter-like expansion. Lastly, we consider the radiation case, which is applicable for studying the standard cosmological model. This case encompasses an inflationary period followed by a radiation-dominated epoch, then a matter-like expansion facilitating structure formation, and finally, a late-time inflationary expansion that aligns with our current observations [3]. The applicability of such cases is due to the fact that we are looking for a viable cosmological scheme that can reproduce the current cosmic dynamics. Of course, in this case, we are considering solely a particular scalar field dynamic minimally coupled to gravity; so, the possibility that this scalar field can imitate dark matter and the effects of dark energy could set up a *toy* contemporary universe model.

3. Fab Four Cosmography

In what follows, we proceed with the standard cosmographic methodology [45–47] to derive the kinematic parameters, which will be evaluated at current cosmic times. The advantage of considering this approach is the geometry construction behind it. By assuming the structure of the Cosmological Principle, we can factorize the scale factor around the present time. However, we should take into account that we must relax the issues of such an approach: (i) the convergence of the series, which can be attended by using observables in order to restrain the order of it. (ii) the cosmic systematics that could be propagated on the cosmographic parameters. In such a case, we require a control of the numerical fit. Notice that for this analysis, we must select a fixed stable point and two possible cosmological scenarios.

Staying in the fixed point $\sigma = 0$, we can start by constructing cosmographic series based on the Taylor series expansion context of the scale factor a, defined as the following:

$$a = 1 + H_0\Delta t - \frac{1}{2}q_0H_0^2\Delta t^2 + \mathcal{O}(\Delta t^3), \tag{42}$$

where q is defined as

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\dot{H}}{H^2} - 1 = -\frac{H'}{H} - 1, \tag{43}$$

and q_0 is the current deceleration parameter. To obtain H_0, we first choose an appropriate scenario as described in Section 2, substitute it into the Hamiltonian constraint Equation (11), and solve the system for the Hubble parameter H. Furthermore, with H we can find q using Equation (43). Finally, we can then evaluate both expressions at the current time to obtain H_0 and q_0, respectively.

Specifically, we chose to build the cosmographical series because it is a purely kinematic expression where other parameters such as matter density or curvature do not play any role. Therefore, we can focus solely on the kinematic terms without specifying other parameters of the model.

At this point, we consider the radiation-dominant behaviour case (for the early-universe calculation of H_0), and the arbitrary case (for the late-universe calculation of H_0). On one hand, for the arbitrary case, we have only two potential functions different from zero (as can be seen in Equations (39)–(41) by letting h be arbitrary):

$$V_j(\phi) = c_{\rm arb}(h+1)\phi^{\frac{4}{\alpha}-2}, \tag{44}$$

$$V_{r,\phi}(\phi) = -\frac{\alpha}{4}h(3+h)c_{\rm arb}\phi^{\frac{4}{\alpha}-1}, \tag{45}$$

where the subindex *arb* denotes arbitrary. Therefore, the only Hamiltonian parts considered would be $\mathcal{H}_j$ and $\mathcal{H}_r$; thus, we can write Equation (11) as follows:

$$9c_{\rm arb}(1+h)H^4\phi'^2\phi^{\frac{4}{\alpha}-2} + 6c_{\rm arb}\alpha h(3+h)H^4\phi'\phi^{\frac{4}{\alpha}-1} + \rho_\Lambda = 0, \tag{46}$$

$$-3c_{\rm arb}H^4\phi'\phi^{\frac{4}{\alpha}-1}\left[3(h+1)\phi'\phi^{-1} + 2\alpha h(3+h)\right] = \rho_\Lambda. \tag{47}$$

From the latter, we can solve for H:

$$H^{\rm arb} = \frac{\sqrt[4]{\rho_\Lambda}\phi^{-\frac{1}{\alpha}+\frac{1}{4}}}{\sqrt[4]{3c_{\rm arb}\phi'}\sqrt[4]{-3(h+1)\phi'\phi^{-1} - 2\alpha h(3+h)}}, \tag{48}$$

which, evaluating at current times, gives us the following:

$$H_0^{\rm arb} = \frac{\sqrt[4]{\rho_{\Lambda 0}}\phi_0^{-\frac{1}{\alpha}+\frac{1}{4}}}{\sqrt[4]{3c_{\rm arb}\phi_0'}\sqrt[4]{-3(h+1)\phi_0'\phi_0^{-1} - 2\alpha h(3+h)}}, \tag{49}$$

where we have discarded the negative signs of the fourth roots to ensure a real, positive value of H_0. Substituting Equation (48) in Equation (43), and evaluating at the current time gives us the following:

$$q_0^{\rm arb} = -1 - \frac{\rho'_{\Lambda 0}}{4\rho_{\Lambda 0}} - \frac{h\alpha\phi_0\phi_0'^2(3+h)(\alpha-4) + 3\phi_0'^3(h+1)(\alpha-2) - h\alpha^2\phi_0^2\phi_0''(3+h)}{2\alpha\phi_0\phi_0'\left[2h\alpha\phi_0(3+h) + 3\phi_0'(h+1)\right]} - \frac{3\alpha\phi_0\phi_0'\phi_0''(h+1)}{2\alpha\phi_0\phi_0'\left[2h\alpha\phi_0(3+h) + 3\phi_0'(h+1)\right]}. \tag{50}$$

On the other hand, for the radiation scenario, the potentials to be considered are as follows:

$$V_j(\phi) = c_1\phi^{\frac{4}{\alpha}-2}, \tag{51}$$

$$V_g(\phi) = c_2\phi^{\frac{2}{\alpha}}, \tag{52}$$

$$V_{r,\phi}(\phi) = -\frac{\alpha}{2}c_1\phi^{\frac{4}{\alpha}-1}, \tag{53}$$

which come from considering $h = -2$ in Equations (39)–(41); with the corresponding Hamiltonian parts $\mathcal{H}_j$, $\mathcal{H}_g$ and $\mathcal{H}_r$. The full Hamiltonian constraint for this case is:

$$9c_1H^4\phi'^2\phi^{\frac{4}{\alpha}-2} - 6c_2H^2\phi^{\frac{2}{\alpha}}\left(1+\frac{2}{\alpha}\phi^{-1}\phi'\right) + 12c_1\alpha H^4\phi'\phi^{\frac{4}{\alpha}-1} + \rho_\Lambda = 0, \tag{54}$$

where solving for H we obtain:

$$H^{\rm rad}_{\pm} = \sqrt{\frac{\alpha\rho_\Lambda\phi^{1-\frac{2}{\alpha}}}{3c_2(\alpha\phi+2\phi')\pm\alpha\gamma(\alpha)}}. \tag{55}$$

Here, we have defined the quantity:

$$\gamma(\alpha) = \frac{1}{\alpha}\sqrt{9c_2^2\alpha^2\phi^2 - 12\alpha\phi\phi'(-3c_2^2+c_1\alpha^2\rho_\Lambda) + 9\phi'^2(4c_2^2-c_1\alpha^2\rho_\Lambda)}, \tag{56}$$

which will appear frequently in our results. From Equation (55) and using Equation (56) we can obtain values for H_0 and q_0 for the radiation scenario:

$$H^{\rm rad}_{0\pm} = \sqrt{\frac{\alpha\rho_{\Lambda0}\phi_0^{1-\frac{2}{\alpha}}}{3c_2(\alpha\phi_0+2\phi_0')\pm\alpha\gamma_0(\alpha)}}, \tag{57}$$

$$\begin{aligned} q_0^{\rm rad} = -1+\frac{1}{2}\Bigg[&-\frac{\rho'_{\Lambda0}}{\rho_{\Lambda0}} - \frac{(\alpha-2)\phi_0'}{\alpha\phi_0} + \frac{3\alpha\phi_0'[2c_2(3\alpha c_2\phi_0+6c_2\phi_0'\pm\alpha\gamma)]}{2\alpha\gamma(3\alpha c_2\phi_0+6c_2\phi_0'\pm\alpha\gamma)} \\ &+\frac{6\phi_0''[2c_2(3\alpha c_2\phi_0+6c_2\phi_0'\pm\alpha\gamma)-\alpha^2c_1\rho_{\Lambda0}(2\alpha\phi_0+3\phi_0')]}{2\alpha\gamma(3\alpha c_2\phi_0+6c_2\phi_0'\pm\alpha\gamma)} \\ &-\frac{3\alpha^2c_1\phi_0'(\rho'_{\Lambda0}(4\alpha\phi_0+3\phi_0')+4\alpha\rho_{\Lambda0}\phi_0')}{2\alpha\gamma(3\alpha c_2\phi_0+6c_2\phi_0'\pm\alpha\gamma)}\Bigg]. \end{aligned} \tag{58}$$

Now, in the calculated values for H_0 and q_0 in Equations (49), (50) and (57)–(59) there are several free parameters, namely: $c_{\rm arb}$, c_1, c_2, α, h and the scalar field and its derivative at current time ϕ_0 and ϕ_0'. We can choose an arbitrary value of α to fix the freedom in re-defining the scalar field. Following [31], we can select α such that $\alpha h = -1$. Alternatively, we can choose values of α to ensure we can obtain power law potentials of the form $V \propto \phi^2$, which gives an inflationary behaviour. Only the cases where $\alpha = 1$ and $\alpha = 2$, give potentials of this kind, while $\alpha = 1$ is preferred, as it also fulfills $\alpha h = -1$ when $h = -1$ (inflation).

3.1. Case for $\alpha = 1$

For this case, we consider such $\alpha = 1$ value in Equations (49) and (50), which yields:

$$H_0^{\rm arb} = \frac{\sqrt[4]{\rho_{\Lambda0}}}{\phi_0^{\frac{3}{4}}\sqrt[4]{3c_{\rm arb}\phi_0'}\sqrt[4]{-3(h+1)\phi_0'\phi_0^{-1}-2h(3+h)}}, \tag{59}$$

and

$$q_0^{\rm arb} = -1 - \frac{\rho'_{\Lambda0}}{4\rho_{\Lambda0}} + \frac{3h\phi_0\phi_0'^2(3+h)+3\phi_0'^3(h+1)+h\phi_0^2\phi_0''(3+h)+3\phi_0\phi_0'\phi_0''(h+1)}{2\phi_0\phi_0'\left[2h\phi_0(3+h)+3\phi_0'(h+1)\right]}. \tag{60}$$

Now, choosing a value for h determines how the Hubble parameter depends on a according to Equation (21), and thus specifies the type of expansion. We can study the cases

for $h = -3/2$ and $h = -1$, which correspond to matter-dominated expansion and inflation, respectively. Substituting these values of h in Equations (59) and (60) gives:

$$H_0^{\text{matter}} = \frac{\sqrt[4]{2\rho_{\Lambda 0}}}{\sqrt{3}\phi_0^{\frac{3}{4}}\sqrt[4]{c_{\text{arb}}\phi_0'\left(3+\phi_0'\phi_0^{-1}\right)}}, \tag{61}$$

$$q_0^{\text{matter}} = -1 - \frac{\rho_{\Lambda 0}'}{4\rho_{\Lambda 0}} + \frac{\phi_0'^2(9\phi_0+2\phi_0') + \phi_0\phi_0''(3\phi_0+2\phi_0')}{4\phi_0\phi_0'(3\phi_0+\phi_0')}, \tag{62}$$

which correspond to $h = -3/2$, and

$$H_0^{\text{inf}} = \frac{\sqrt[4]{\rho_{\Lambda 0}}}{\sqrt{2}\phi_0^{\frac{3}{4}}\sqrt[4]{3c_{arb}\phi_0'}}, \tag{63}$$

$$q_0^{\text{inf}} = -1 - \frac{\rho_{\Lambda 0}'}{4\rho_{\Lambda 0}} + \frac{3\phi_0'}{4\phi_0} + \frac{\phi_0''}{4\phi_0'}, \tag{64}$$

which corresponds to the inflationary case $h = -1$.

Now, the case for radiation-like expansion is given by substituting $\alpha = 1$ in $\gamma(\alpha)$ as in Equations (57)–(59):

$$H_{0\pm}^{\text{rad}} = \sqrt{\frac{\rho_{\Lambda 0}\phi_0^{-1}}{3c_2(\phi_0+2\phi_0') \pm \gamma_0(1)}}, \tag{65}$$

$$\begin{aligned} q_{0\pm}^{\text{rad}} = {} & -1 - \frac{\rho_{\Lambda 0}'}{2\rho_{\Lambda 0}} + \frac{\phi_0'}{2\phi_0} + \frac{3\phi_0'\left(2c_2(3c_2\phi_0+6c_2\phi_0' \pm \gamma(1)) - c_1(\rho_{\Lambda 0}'(4\phi_0+3\phi_0') + 4\rho_{\Lambda 0}\phi_0')\right)}{4\gamma(1)\left(3c_2\phi_0+6c_2\phi_0' \pm \gamma(1)\right)} \\ & + \frac{6\phi_0''\left(2c_2(3c_2\phi_0+6c_2\phi_0' \pm \gamma(1)) - c_1\rho_{\Lambda 0}(2\phi_0+3\phi_0')\right)}{4\gamma(1)\left(3c_2\phi_0+6c_2\phi_0' \pm \gamma(1)\right)}. \end{aligned} \tag{66}$$

3.2. Case for $\alpha = 2$

Another way to obtain potentials of the form ϕ^2 is with $\alpha = 2$. Substituting this value on Equations (49), (50), and (57)–(59), and following a similar procedure as in the latter case, we obtain the following:

$$H_0^{\text{arb}} = \frac{\sqrt[4]{\rho_{\Lambda 0}}}{\sqrt[4]{3c_{\text{arb}}\phi_0\phi_0'}\sqrt[4]{-3(h+1)\phi_0'\phi_0^{-1} - 4h(3+h)}}, \tag{67}$$

$$q_0^{\text{arb}} = -1 - \frac{\rho_{\Lambda 0}'}{4\rho_{\Lambda 0}} + \frac{2h(h+3)(\phi_0'^2 + \phi_0\phi_0'') + 3(h+1)\phi_0'\phi_0''}{2\phi_0'(4h\phi_0(3+h) + 3\phi_0'(h+1))}, \tag{68}$$

and for the matter case,

$$H_0^{\text{matter}} = \frac{\sqrt[4]{\rho_{\Lambda 0}}}{\sqrt[4]{3c_{\text{arb}}\phi_0\phi_0'}\sqrt[4]{9+\frac{3\phi_0'}{2\phi_0}}}, \tag{69}$$

$$q_0^{\text{matter}} = -1 - \frac{\rho_{\Lambda 0}'}{4\rho_{\Lambda 0}} + \frac{3\phi_0'^2 + \phi_0''(\phi_0+\phi_0')}{2\phi_0'(6\phi_0+\phi_0')}. \tag{70}$$

The inflation case gives us the following:

$$H_0^{\rm inf} = \frac{\sqrt[4]{\rho_{\Lambda 0}}}{2^{\frac{3}{4}}\sqrt[4]{3c_{\rm arb}\phi_0\phi_0'}}, \tag{71}$$

$$q_0^{\rm inf} = -1 - \frac{\rho_{\Lambda 0}'}{4\rho_{\Lambda 0}} + \frac{\phi_0'}{4\phi_0} + \frac{\phi_0''}{4\phi_0'}, \tag{72}$$

$$H_{0\pm}^{\rm rad} = \sqrt{\frac{\rho_{\Lambda 0}}{3c_2(\phi_0+\phi_0') \pm \gamma(2)}}, \tag{73}$$

and for the radiation case,

$$q_{0\pm}^{\rm rad} = -1 - \frac{\rho_{\Lambda 0}'}{2\rho_{\Lambda 0}} + \frac{\pm 6c_2\phi_0'\gamma(2) \pm 6c_2\phi_0''\gamma(2) + 18c_2^2(\phi_0+\phi_0')(\phi_0'+\phi_0'')}{4\gamma(2)(3c_2(\phi_0+\phi_0')+\gamma(2))} - \frac{3c_1(8\rho_{\Lambda 0}\phi_0'^2 + \rho_{\Lambda 0}'\phi_0'(8\phi_0+3\phi_0') + 2\rho_{\Lambda 0}\phi_0''(4\phi_0+3\phi_0'))}{4\gamma(2)(3c_2(\phi_0+\phi_0')+\gamma(2))}. \tag{74}$$

For both values of α, we notice similarities in all forms of H_0, which are all proportional to $\propto \sqrt[4]{\rho_\Lambda}$. Additionally, we notice similarities in the expressions for q_0, where all of them can be written in the same functional form:

$$q_0 = -1 - A\left(\frac{\rho_\Lambda'}{\rho_\Lambda}\right) + B\left(\frac{\phi'}{\phi}\right) + C\left(\frac{\phi''}{\phi'}\right), \tag{75}$$

where A, B and C are, in general, functions of the parameters ρ_Λ, ϕ, ϕ', ϕ'' and the constants $c_{\rm arb}$, c_1 and c_2. Considering standard GR with a cosmological constant and a scalar field as the source in an FLRW background, the deceleration parameter has the same functional form.

4. Fab-Four on the H_0 Tension

According to the latter results, we can now discuss the possible requirements to relax the value of H_0 in the cosmological scenarios discussed.

With the expressions obtained for $H_0^{\rm inf}$ and $H_0^{\rm rad}$ we can now calculate the predicted discrepancy between both values. The most recent measurements [6,8] suggest a discrepancy of around $\sim$10% between the values for the Hubble constant in the early universe and those in the late universe. As such, we expect that

$$\delta_{H_0} \equiv 1 - \frac{H_0^{\rm rad}}{H_0^{\rm inf}} \approx 0.1 \tag{76}$$

For $\alpha = 1$, we can use Equations (63)–(65) to get:

$$\delta_{H_0,\alpha=1} = 1 - \frac{\sqrt{2}\sqrt[4]{3c_{\rm arb}\rho_{\Lambda 0}\phi_0\phi_0'}}{\sqrt{3c_2(\phi_0+2\phi_0') \pm \gamma(1)}}. \tag{77}$$

On the other hand, for $\alpha = 2$, using Equations (71)–(73), we obtain the following

$$\delta_{H_0,\alpha=2} = 1 - \frac{\sqrt[4]{24c_{\rm arb}\rho_{\Lambda 0}\phi_0\phi_0'}}{\sqrt{3c_2(\phi_0+\phi_0') \pm \gamma(2)}}. \tag{78}$$

Now, before proceeding, several requirements arise in the process of obtaining expressions for H_0, q_0, and δ_{H_0} to ensure that the solutions are real. These are summarized in Table 1.

Table 1. Requirements on the Fab-Four derived constants in the results for H_0, q_0, and δ_{H_0}.

Case		Requirements
General		$\rho_{\Lambda 0} > 0, \quad c_{arb}\phi_0' > 0, \quad -3(h+1)\phi_0'\phi_0^{-1} - 2\alpha h(3+h) > 0, \quad \phi_0 > 0$
$\alpha = 1$	General	$9c_2^2\phi_0^2 - 12\phi_0\phi_0'(-3c_2^2 + c_1\rho_{\Lambda 0}) + 9{\phi_0'}^2(4c_2^2 - c_1\rho_{\Lambda 0}) > 0,$ $3c_2(\phi_0 + 2\phi_0') \pm \gamma_0(1) > 0$
	$h = -3/2$ (Matter)	$3 + \phi_0'\phi_0^{-1} > 0$
	$h = -1$ (Inflation)	No extra requirements
$\alpha = 2$	General	$9c_2^2\phi_0^2 + 6\phi_0\phi_0'(3c_2^2 - 4c_1\rho_{\Lambda 0}) + 9{\phi_0'}^2(c_2^2 - c_1\rho_{\Lambda 0}) > 0$ $3c_2(\phi_0 + \phi_0') \pm \gamma_0(2) > 0$
	$h = -3/2$ (Matter)	$6 + \phi_0'\phi_0^{-1} > 0$
	$h = -1$ (Inflation)	No extra requirements

Taking into account the restrictions given by Table 1, it is possible to get values for $\delta_{H_0} = 0.1$, by considering the requirements on the five free parameters that appear in the expressions: ϕ_0, ϕ_0', c_{arb}, c_1 and c_2. We also have another two parameters, ρ_Λ' and ϕ'', which only appear in the expressions for q_0.

However, the behaviour of ρ_Λ comes into play. The Fab-Four model was introduced to handle the cosmological constant fine-tuning problem [29,31]. As such, instead of the cosmological constant being constant, it is assumed that it is a piecewise constant function, with several phase transitions that change its value. Therefore, ρ_Λ is discontinuous at these phase transitions. This means that the *r.h.s* of Equation (11) is discontinuous, but the *l.h.s* of this equation is composed of the Hamiltonian parts, which depend only on the functions a, ϕ, and ϕ'. Since a is continuous, and ϕ as well - by construction - it follows that ϕ' must be discontinuous to maintain equality at the phase transitions. Consequently, both ρ_Λ' and ϕ'' contain delta-functions localized at the transition times. These same arguments are used to initially restrict Horndeski's full theory of gravity [29].

At these phase-transitions, the self-tuning nature of the Fab-Four theory is active to still allow a standard cosmology to develop [29]. However, by construction, the self-tuning filter can only work when the curvature is non-zero. Since the latest observations assert a negligible curvature [6], we assume that we are not presently in a phase transition. For this work, we are only taking the present-day values of these parameters ($\rho_{\Lambda 0}'$, ϕ_0''), which we will assume to be zero since we are not currently in a phase-transition, where there would be a delta-function.

Therefore, substituting $\rho_{\Lambda 0}' = 0$ and $\phi_0'' = 0$, from Equations (64)–(66) and (72)–(74), we obtain:

$$q_{0\,\alpha=1}^{\text{inf}} = -1 + \frac{3\phi_0'}{4\phi_0}, \tag{79}$$

$$q_{0\,\alpha=1}^{\text{rad}} = -1 + \frac{\phi_0'}{2\phi_0} + \frac{3\phi_0'(2c_2(3c_2\phi_0 + 6c_2\phi_0' \pm \gamma(1)) - 4c_1\rho_{\Lambda 0}\phi_0')}{4\gamma(1)\left(3c_2\phi_0 + 6c_2\phi_0' \pm \gamma(1)\right)}, \tag{80}$$

$$q_{0\,\alpha=2}^{\text{inf}} = -1 + \frac{\phi_0'}{4\phi_0}, \tag{81}$$

$$q_{0\,\alpha=2}^{\text{rad}} = -1 + \frac{\pm 3c_2\phi_0'\gamma(2) + 9c_2^2\phi_0'(\phi_0 + \phi_0') - 12c_1\rho_{\Lambda 0}{\phi_0'}^2}{2\gamma(2)(3c_2(\phi_0 + \phi_0') + \gamma(2))}, \tag{82}$$

for each $\alpha = 1, 2$ case, respectively. Now, given a value of q_0 through the latest observations [8] ($q_0 \approx -0.5$), we can either force the deceleration parameters (evaluated at the current time) of both the early and late universe to be the same, or we can leave them to be different.

In this work, we assume $q_0^{\text{inf}} = q_0^{\text{rad}} \approx -0.5$ for simplicity, and we leave the other case for future work. Given this restriction, with the expressions for q_0^{inf} Equations (79) and (81),

we will get a fixed value for ϕ_0'/ϕ_0 (depending on the choice of α). Then, using the expressions for $q_0^{\rm rad}$ Equations (80) and (82), we can constrain the value of one more parameter, say, for example, c_1.

Finally, we are only left with two free parameters: $c_{\rm arb}$ and either c_1 or c_2. These are enough to get $\delta_{H_0} \approx 0.1$, and reducing the number of free parameters is important to enhance the likelihood of this model in Bayesian statistic comparisons [48].

5. Conclusions

In this work, we consider the possibility of tackling the Hubble tension using the Fab-Four theory. This scalar-tensor theory of gravity applies a self-tuning filter on the more general Horndeski theory, to solve the cosmological constant fine-tuning problem. This leaves a much more manageable theory, consisting only of four base Lagrangians, given by Equations (7)–(10). We follow the procedure discussed in [31] to obtain a simpler form of the theory, using a dynamical system approach to find the fixed points of the theory, and we concentrate on the $\sigma = 0$ fixed-point, which denotes zero curvature (as supported by observations [6]). In this fixed point, the equation of motion of the scalar field Equation (37) and the Hamiltonian restriction Equation (38) of the theory are such that we have certain allowed types of cosmological scenarios, among which are the inflationary expansion, radiation-like expansion, and a matter-like phase. This shows the utility of the Fab-Four theory, as *turning on-and-off* certain potentials lead naturally to different types of cosmological behaviors, and the full dynamics of the theory allow for a smooth, continuous shift between these types of cosmological evolution.

Furthermore, we construct the Fab-Four cosmography in the zero-curvature fixed point. In particular, we derive analytical expressions for the current Hubble constant H_0, and the deceleration parameter q_0, for the matter, inflation, and radiation scenarios, as well as for different choices of the arbitrary constant α. To do this, we take into account the forms of the potentials, substitute them into the Hamiltonian constraint, which results in quartic equations of the Hubble parameter, and then solve for H. These equations yield four solutions: *(i)* in the arbitrary case, we discard two complex roots and a negative one; *(ii)* in the radiation case, we can only discard two negative roots, leaving both signs of the expression for $H_\pm^{\rm rad}$. Examining Table 1, we also notice that the requirements do not restrict the choice of signs in the expressions for $H_\pm^{\rm rad}$ and $q_\pm^{\rm rad}$, and both signs can give $\delta_{H_0} = 0.1$, which is why we have left them written explicitly in all expressions. Afterward, we differentiate the results for H according to Equation (43) to obtain expressions for the deceleration parameter, which all ultimately take the functional form of Equation (75). Using these expressions evaluated at current time, we can compare the Hubble constant in the late universe with the Hubble constant in the early universe (radiation-like expansion).

Finally, we show that it is possible to get the desired discrepancy between the early-universe and the late-universe Hubble constants, of around $\sim$10%, showing that the Hubble constant could, in reality, be dynamical. Therefore, we have succeeded in our objective to show that it is indeed possible to get a disagreement between the observed Hubble constant if measured through local means, and the one obtained from observations of the CMB, due to the different dynamics put forward by the Fab-Four theory.

The final results contain five arbitrary parameters which are constrained by the requirements in Table 1. However, using the expressions for the deceleration parameter, we can reduce the number of free parameters in these results to only two constants. In future work, we will propose performing an analysis with observational data to obtain confidence limits on the free variables of the theory. For this, extracting other observables from the Fab Four theory, e.g., expressions for the luminosity distance, and even perturbation theory results such as the angular power spectrum from the CMB would be convenient. Afterward, a possible course of action would be to make a Bayesian statistics comparison of this model's prediction against the standard ΛCDM to test its plausibility, as well as to further constrain the possible values of the free parameters to match observations [48]. Furthermore, the

possibility to explore beyond Horndeski theories [26] could be a path in order to tackle the cosmological tensions at early cosmic times.

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Abbreviations

The following abbreviations are used in this manuscript:

FLRW	Friedmann-Lemaître-Robertson-Walker
CMB	Cosmic Microwave Background
BAO	Baryon Acoustic Oscillations
GR	General Relativity
GW	Gravitational Wave

References

1. Riess, A.G.; Filippenko, A.V.; Challis, P.; Clocchiatti, A.; Diercks, A.; Garnavich, P.M.; Gilliland, R.L.; Hogan, C.J.; Jha, S.; Kirshner, R.P. Observational evidence from supernovae for an accelerating universe and a cosmological constant. *Astron. J.* **1998**, *116*, 1009. [CrossRef]
2. Perlmutter, S.; Aldering, G.; Goldhaber, G.; Knop, R.A.; Nugent, P.; Castro, P.G.; Deustua, S.; Fabbro, S.; Goobar, A.; Groom, D.E.; et al. Measurements of Ω and Λ from 42 High-Redshift Supernovae. *Astrophys. J.* **1999**, *517*, 565–586.
3. Dodelson, S.; Schmidt, F. *Modern Cosmology*, 2nd ed.; Academic Press: Cambridge, MA, USA, 2021.
4. Perivolaropoulos, L.; Skara, F. Challenges for ΛCDM: An update. *New Astron. Rev.* **2022**, *95*, 101659. [CrossRef]
5. Schwarz, D.J.; Copi, C.J.; Huterer, D.; Starkman, G.D. CMB anomalies after Planck. *Class. Quantum Gravity* **2016**, *33*, 184001. [CrossRef]
6. Aghanim, N. et al. [Plank Collaboration] *Planck* 2018 results. *Astron. Astrophys.* **2020**, *641*, A6. [CrossRef]
7. Kamionkowski, M.; Riess, A.G. The Hubble Tension and Early Dark Energy. *arXiv* **2022**, arXiv:2211.04492.
8. Riess, A.G.; Yuan, W.; Macri, L.M.; Scolnic, D.; Brout, D.; Casertano, S.; Jones, D.O.; Murakami, Y.; Anand, G.S.; Breuval, L.; et al. A Comprehensive Measurement of the Local Value of the Hubble Constant with 1 km s^{-1} Mpc^{-1} Uncertainty from the Hubble Space Telescope and the SH0ES Team. *Astrophys. J. Lett.* **2022**, *934*, L7. [CrossRef]
9. Di Valentino, E.; Anchordoqui, L.A.; Akarsu, Ó; Ali-Haimoud, Y.; Amendola, L.; Arendse, N.; Asgari, M.; Ballardini, M.; Basilakos, S.; Battistelli, E.; et al. Snowmass2021—Letter of interest cosmology intertwined II: The hubble constant tension. *Astropart. Phys.* **2021**, *131*, 102605.
10. Karwal, T.; Kamionkowski, M. Dark energy at early times, the Hubble parameter, and the string axiverse. *Phys. Rev. D* **2016**, *94*, 103523. [CrossRef]
11. Zhao, G.B.; Raveri, M.; Pogosian, L.; Wang, Y.; Crittenden, R.G.; Handley, W.J.; Percival, W.J.; Beutler, F.; Brinkmann, J.; Chuang, C.H.; et al. Dynamical dark energy in light of the latest observations. *Nat. Astron.* **2017**, *1*, 627–632. [CrossRef]
12. Aich, A. Interacting Dark Energy: New parametrization and observational constraints. *arXiv* **2022**, arXiv:2207.09079.
13. Kim, R.G.; Jong, S.; Hyok O.G. Cosmic coincidence problem in interacting holographic dark energy models. *New Astron.* **2023**, *101*, 102015. [CrossRef]
14. Cao, S.; Zhu, Z.H.; Liang, N. Observational constraints on interacting dark matter model without dark energy. *Astron. Astrophys.* **2011**, *529*, A61. [CrossRef]
15. Escamilla-Rivera, C.; Gamboa, A. A new parameterized interacting holographic dark energy. *Eur. Phys. J. Plus* **2022**, *137*, 254.
16. Salvio, A.; Scollo, S. Axion Sterile Neutrino Dark Matter. *Universe* **2021**, *7*, 354.
17. Berlin, A.; Hooper, D. Axion-Assisted Production of Sterile Neutrino Dark Matter. *Phys. Rev. D* **2017**, *95*, 075017.
18. Farnes, J.S. A unifying theory of dark energy and dark matter: Negative masses and matter creation within a modified ΛCDM framework. *Astron. Astrophys.* **2018**, *620*, A92.

19. Sandoval-Orozco, R.; Escamilla-Rivera, C. Cosmological piecewise functions to treat the local Hubble tension. *Eur. Phys. J. Plus* **2022**, *137*, 819.
20. Escamilla-Rivera, C.; Nájera, J.A. Comment on: "Questioning the H_0 tension via the look-back time". *arXiv* **2023**, arXiv:2303.02140.
21. Escamilla-Rivera, C.; Torres Castillejos, R. H0 Tension on the Light of Supermassive Black Hole Shadows Data. *Universe* **2023**, *9*, 14.
22. Di Valentino, E.; Mena, O.; Pan, S.; Visinelli, L.; Yang, W.; Melchiorri, A.; Mota, D.F.; Riess, A.G.; Silk, J. In the realm of the Hubble tension—A review of solutions. *Class. Quantum Gravity* **2021**, *38*, 153001. [CrossRef]
23. Adhikari, R.X.; Anchordoqui, L.A.; Fang, K.; Sathyaprakash, B.S.; Tollefson, K.; Lewis, T.R.; Engel, K.; Aboubrahim, A.; Akarsu, O.; Akrami, Y.; et al. Report of the Topical Group on Cosmic Probes of Fundamental Physics for for Snowmass 2021. *arXiv* **2022**, arXiv:2209.11726.
24. Abdalla, E.; Abellán, G.F.; Aboubrahim, A.; Agnello, A.; Akarsu, Ó; Akrami, Y.; Alestas, G.; Aloni, D.; Amendola, L.; Anchordoqui, L.A.; et al. Cosmology intertwined: A review of the particle physics, astrophysics, and cosmology associated with the cosmological tensions and anomalies. *J. High Energy Astrophys.* **2022**, *34*, 49–211.
25. Schöneberg, N.; Franco Abellán, G.; Pérez Sánchez, A.; Witte, S.J.; Poulin, V.; Lesgourgues, J. The H0 Olympics: A fair ranking of proposed models. *Phys. Rept.* **2022**, *984*, 1–55.
26. Kobayashi, T. Horndeski theory and beyond: a review. *Rep. Prog. Phys.* **2019**, *82*, 086901. [CrossRef]
27. Clifton, T.; Ferreira, P.G.; Padilla, A.; Skordis, C. Modified Gravity and Cosmology. *Phys. Rept.* **2012**, *513*, 1–189.
28. Bahamonde, S.; Dialektopoulos, K.F.; Escamilla-Rivera, C.; Farrugia, G.; Gakis, V.; Hendry, M.; Hohmann, M.; Levi Said, J.; Mifsud, J.; Di Valentino, E. Teleparallel gravity: from theory to cosmology. *Rept. Prog. Phys.* **2023**, *86*, 026901.
29. Charmousis, C.; Copeland, E.J.; Padilla, A.; Saffin, P.M. Self-tuning and the derivation of a class of scalar-tensor theories. *Phys. Rev. D* **2012**, *85*, 104040. [CrossRef]
30. Horndeski, G.W. Second-order scalar-tensor field equations in a four-dimensional space. *Int. J. Theor. Phys.* **1974**, *10*, 363–384. [CrossRef]
31. Copeland, E.J.; Padilla, A.; Saffin, P.M. The cosmology of the Fab-Four. *J. Cosmol. Astropart. Phys.* **2012**, *12*, 026.
32. Latosh, B. Fab Four Effective Field Theory Treatment. *Eur. Phys. J. C* **2018**, *78*, 991.
33. Linder, E.V. How Fabulous Is Fab 5 Cosmology? *J. Cosmol. Astropart. Phys.* **2013**, *12*, 032.
34. Starobinsky, A.A.; Sushkov, S.V.; Volkov, M.S. The screening Horndeski cosmologies. *J. Cosmol. Astropart. Phys.* **2016**, *2016*, 007. [CrossRef]
35. Starobinsky, A.A.; Sushkov, S.V.; Volkov, M.S. Anisotropy screening in Horndeski cosmologies. *Phys. Rev. D* **2020**, *101*, 064039. [CrossRef]
36. Galeev, R.; Muharlyamov, R.; Starobinsky, A.A.; Sushkov, S.V.; Volkov, M.S. Anisotropic cosmological models in Horndeski gravity. *Phys. Rev. D* **2021**, *103*, 104015. [CrossRef]
37. Gleyzes, J.; Langlois, D.; Piazza, F.; Vernizzi, F. New Class of Consistent Scalar-Tensor Theories. *Phys. Rev. Lett.* **2015**, *114*, 211101. [CrossRef]
38. Babichev, E.; Charmousis, C.; Langlois, D.; Saito, R. Beyond Fab Four. *Class. Quant. Grav.* **2015**, *32*, 242001.
39. Ezquiaga, J.M.; Zumalacárregui, M. Dark Energy After GW170817: Dead Ends and the Road Ahead. *Phys. Rev. Lett.* **2017**, *119*, 251304. [CrossRef]
40. Copeland, E.J.; Kopp, M.; Padilla, A.; Saffin, P.M.; Skordis, C. Dark energy after GW170817 revisited. *Phys. Rev. Lett.* **2019**, *122*, 061301.
41. Bordin, L.; Copeland, E.J.; Padilla, A. Dark energy loopholes some time after GW170817. *J. Cosmol. Astropart. Phys.* **2020**, *2020*, 063. [CrossRef]
42. Appleby, S.; Linder, E.V. The well-tempered cosmological constant. *J. Cosmol. Astropart. Phys.* **2018**, *2018*, 034. [CrossRef]
43. Emond, W.T.; Li, C.; Saffin, P.M.; Zhou, S.Y. Well-Tempered Cosmology. *J. Cosmol. Astropart. Phys.* **2019**, *05*, 038.
44. Torres, I.; Fabris, J.C.; Piattella, O.F.; Batista, A.B. Quantum Cosmology of Fab Four John Theory with Conformable Fractional Derivative. *Universe* **2020**, *6*, 50.
45. Bamba, K.; Capozziello, S.; Nojiri, S.; Odintsov, S.D. Dark energy cosmology: the equivalent description via different theoretical models and cosmography tests. *Astrophys. Space Sci.* **2012**, *342*, 155–228.
46. Capozziello, S.; D'Agostino, R.; Luongo, O. Extended Gravity Cosmography. *Int. J. Mod. Phys. D* **2019**, *28*, 1930016.
47. Escamilla-Rivera, C.; Capozziello, S. Unveiling cosmography from the dark energy equation of state. *Int. J. Mod. Phys. D* **2019**, *28*, 1950154.
48. Escamilla-Rivera, C.; de Albornoz-Caratozzolo, J.; Najera, S. In preparation. 2023.

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