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Maxwell-Dirac Isomorphism Revisited: From Foundations of Quantum Mechanics to Geometrodynamics and Cosmology

Arkady L. Kholodenko 

375 H.L. Hunter Laboratories, Clemson University, Clemson, SC 29634, USA; string@clemson.edu

Abstract: Although electrons (fermions) and photons (bosons) produce the same interference patterns in the two-slit experiments, known in optics for photons since the 17th Century, the description of these patterns for electrons and photons thus far was markedly different. Photons are spin one, relativistic and massless particles while electrons are spin half massive particles producing the same interference patterns irrespective to their speed. Experiments with other massive particles demonstrate the same kind of interference patterns. In spite of these differences, in the early 1930s of the 20th Century, the isomorphism between the source-free Maxwell and Dirac equations was established. In this work, we were permitted replace the Born probabilistic interpretation of quantum mechanics with the optical. In 1925, Rainich combined source-free Maxwell equations with Einstein's equations for gravity. His results were rediscovered in the late 1950s by Misner and Wheeler, who introduced the word "geometrodynamics" as a description of the unified field theory of gravity and electromagnetism. An absence of sources remained a problem in this unified theory until Ranada's work of the late 1980s. However, his results required the existence of null electromagnetic fields. These were absent in Rainich–Misner–Wheeler's geometrodynamics. They were added to it in the 1960s by Geroch. Ranada's solutions of source-free Maxwell's equations came out as knots and links. In this work, we establish that, due to their topology, these knots/links acquire masses and charges. They live on the Dupin cyclides—the invariants of Lie sphere geometry. Symmetries of Minkowski space-time also belong to this geometry. Using these symmetries, Varlamov recently demonstrated group-theoretically that the experimentally known mass spectrum for all mesons and baryons is obtainable with one formula, containing electron mass as an input. In this work, using some facts from polymer physics and differential geometry, a new proof of the knotty nature of the electron is established. The obtained result perfectly blends with the description of a rotating and charged black hole.

Keywords: Maxwell and Dirac equations; foundations of quantum mechanics; differential geometry; knots and links; Chladni patterns; Dupin cyclides; polymer physics; neutrinos; axions; geometrodynamics



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1. Introduction

If electrons and photons and other massive particles produce the same interference patterns in the two-slit experiments [1–5], why then can the optics formalism not be applied unchanged to electrons and heavier particles? What makes the use of Born's probabilistic interpretation of the wave function in quantum mechanics superior to the intensity interpretation of the Maxwellian wave functions in optics? In particular, what makes the use for a description of the two-slit experiments in which the results are visibly the same for light and massive particles? A study of this issue was initiated by David Bohm in his monograph on quantum mechanics ([6], pp. 97–98). From reading these pages, it follows that the differences do exist but in just a few places. The updated theoretical comparison more recently was made by Sanz and Miret-Artes, in Chapters 4 and 7 of their book [7]. From the results of these chapters, it follows that all objections made by Bohm in the remaining few places of his analysis can be removed. Fortunately, the results of [7]

along with those originating from references on which these results are based (and even more recent ones), can still be improved. It is the first purpose of this paper to perform just this. We are able to develop the unifying quantum mechanical–optical formalism for photons and massive particles with or without spin. Since, nowadays, the sophisticated quantum mechanical experiments are typically conducted optically [8,9], the results of our paper may provide additional guidelines for the interpretation of these optical experiments and vice versa. In light of the correspondence just stated, it makes sense to claim that our understanding of subtleties of quantum mechanics is contingent upon our understanding of subtleties of the developed optical formalism adapted for quantum mechanical needs. In this paper, we discuss several features of this formalism which were not yet discussed (to our knowledge) in the context of quantum mechanics or quantum field theories.

Even though the two-slits interference experiments visibly produce the same results in formalisms of both optics and quantum mechanics, it is not immediately possible to adopt word-for-word the conventional optical formalism for quantum mechanics. This is because Maxwell's equations contain vector quantities, such as $\mathbf{E}$ and $\mathbf{H}$. Moreover, the polarization in optics, whose quantum analog is spin [10], causes difficulties, since at the nonrelativistic level at which Schrödinger's equation is being used, there is no mention of spin. At the same time, the two-slit interference fringe patterns for monochromatic light depend strongly upon the light polarization. For spinless particles in nonrelativistic quantum mechanics used in the two-slit experiments, this effect of polarization is absent, while in optics, depending on polarization, there are four distinct cases of study. These were discovered by Fresnel and Arago at the beginning of the 19th Century [11,12]. For the record, these are: (1) Two rays of light polarized in the same plane. They interfere with like rays of ordinary (unpolarized) light. (2) Two rays polarized at right angles to each other. They do not interfere. (3) Two rays originally polarized at right angles and then brought into the same plane of polarization. They do not produce interference pattern. (4) Two rays originally polarized at right angles; if derived from the same linearly polarized wave and subsequently brought into the same plane, can interfere.

These and other facts, e.g., masses and spins, relativistic effects, etc., complicate the comparison between the two-slit interference experiments in optics and in quantum mechanics. Fortunately, there are ways out of these difficulties. In this work, we discuss these ways. Surprisingly, the solutions for these difficulties have resulted in solutions for many other things presented in this paper. Thus far, they were considered as unrelated. Previously, the solution of the first task required us to make several unexpected steps. The first step is described in Section 2. Its purpose is to introduce known results to be used in the remainder of this paper. Next, in Section 3, we discuss the condition (absence of sources/charges) under which the vector electromagnetic field can be replaced by the complex scalar field [13,14]. This feature was known but not used for many years. The existence of source-free Maxwellian fields is of central importance for all results of this paper. In Section 4, we use this complex scalar field to reobtain known results for nonrelativistic Schrödinger's quantum mechanics. The use of complex scalar fields (seemingly, of electromagnetic origin) in quantum mechanics allows us to extend known quantum mechanical formalism in Section 5. In it, we introduce the Chladni patterns and discuss the existence of knotted and linked configurations for the wave functions. The existence of these configurations in Schrödinger's version of quantum mechanics makes Schrödinger's and Heisenberg's interpretations of quantum mechanics not equivalent. Section 6 is devoted to the considerable extension of results of Ranada [15,16]. They are linked to the results of previous sections because, previously, in his first paper [15], Ranada used the complex scalar fields as descriptions of the electromagnetic field. He did this with the purpose of designing knotted and linked structures out of electromagnetic fields. In his second paper [16], he demonstrated that these knotty structures act like particles with masses, spins and charges. Additional helpful technical details of these particle-like properties of knotty electromagnetic fields are presented in [17]. In light of these results and in the spirit of Einstein's quantization program outlined in the same section,

the Maxwell–Dirac isomorphism is introduced and treated. Then, using the notion of electromagnetic duality, the axions are introduced. These are particles believed to be linked with the *dark matter*. The evidence is provided in favor of the statement that the axions are responsible for the Chladni patterns in the sky (that is, in the space–time of the Universe). Section 7 provides additional support (culminating in Section 9) of Einstein’s quantization program. It discusses the Maxwell–Dirac isomorphism from various angles—all based on known rigorous mathematical results which have not been collected yet in one place thus far. In particular, in this section, we discuss peculiar results coming from treating the mass in the Dirac equation. Although making the mass go to zero mathematically is perfectly permissible in order for it to be made as small as possible, physically, such a limiting process loses its meaning. This fact is not logical but experimental. After a certain threshold (theoretically not known yet), the Dirac equation must be replaced by the Majorana equation. It is commonly believed that neutrinos are described by the Majorana equations with vanishingly small but nonzero masses. This fact causes some serious difficulties in uses of the Maxwell–Dirac correspondence. To resolve these difficulties, the neutrino theory of light was proposed (e.g., read Wikipedia). This theory is not working, however, because neutrinos have masses (very small but nonzero), while the light is massless. The story with neutrino also teaches us about the notion of chirality. However, this phenomenon exists not only for elementary particles (the Standard Model of particle physics is built on chiralities). It also exists in the macroscopic world. How this property is showing itself in the macroscopic nature and how to account for chirality in the Maxwell–Dirac (actually Majorana) formalism is also discussed in Section 7. In Section 8, we develop our own mathematical proof of the Maxwell–Dirac isomorphism, keeping in mind physical applications discussed in Section 9. Surprisingly, the arguments presented in this section are based on some results from polymer physics, which were developed by the author in the past. Moreover, in this section we use some not so widely known results by Dirac, which belong to the domain of projective relativity. This field of study had emerged before Rainich–Misner–Wheeler geometrodynamics and is also aimed at designing the unified field theory of gravity and electromagnetism. However, it does have significant merit on its own, especially because it is technically connected with the Lie sphere geometry (Section 8.1). Dupin cyclides-invariants of this geometry are also mentioned in Section 8. Although Section 9 is a discussion section, it is of an independent value of major importance. We advise our readers to read it immediately after reading this section, as in a way, it could be looked upon as an extension of this section. At the same time, it should be read subsequently after Section 8. This is especially true in light of the fact that Varlamov’s mass formula, presented in Section 9, is obtained with the help of the group-theoretic methods applied to the Lorentz group. For its use, it requires at least the mass of the electron as an input. The model of the electron obtained in Section 8 fits Varlamov’s formula, since both the ground and the excited states which this formula describes have a knot-theoretic interpretation. All knots are divided into two groups with respect to the knot-theoretic operation of the pass equivalence. The Arf invariant of a knot/link (e.g., read Wikipedia) takes two values -0 and 1 , with respect to the pass-equivalence operation. Those knots which are pass-equivalent to the unknot have the Arf invariant 0 , while those which are pass-equivalent to the trefoil knot have the Arf invariant equal to 1 . As demonstrated in Section 8, the description of the electron utilizes the trefoil knot. It is the simplest torus knot, and the torus is the simplest Dupin cyclide. The mass excitation formula by Varlamov generates unstable particles, all with Arf invariant 1 , and all eventually decaying back to the trefoil. The pass-equivalence operation can be interpreted physically in terms of the equidistant spectrum of the harmonic oscillator. Indeed, Varlamov’s mass formula involves some kinds of harmonic oscillator spectrum results as inputs.

Although the Kaluza–Klein, projective relativity and the geometrodynamics serve the same purpose—to unify gravity with electromagnetism—only Rainich–Misner–Wheeler’s geometrodynamics could be considered as the theory in which such unification was finally achieved. The major difficulty of the initial Misner–Wheeler version of geometrodynamics

associated with the mathematically rigorous description of the sources of electromagnetic fields was solved by Ranada in late 1980s, when he discovered particle-like behaving knotty solutions of the source-free electromagnetic fields. Neither Ranada, nor the rest of researchers/developers of his results, myself included, were aware of a connection between Ranada's results and geometrodynamics. The connection is made only now, in this paper. Much later on, it was realized that for the existence and stability of Ranada's knots/links, it is required that these should come out as solutions involving the null fields, which are the fields for which $\mathbf{E} \cdot \mathbf{H} = 0$ and $|\mathbf{E}|^2 = |\mathbf{H}|^2$. The existence of null fields in geometrodynamics was a problem in the Rainich–Misner–Wheeler version of geometrodynamics. It was proven subsequently by Geroch (Section 9) in the mid 1960s. The problem of the stability of Ranada's knots (or their ability to brake) was seriously treated mathematically only a few years ago. The problem of the existence of sources, especially for the extended objects in gravitation and non Abelian gauge theories, to our knowledge, is not solved satisfactorily thus far. At the same time, from the very beginnings of the development of quantum field theory, Einstein objected to the existing methods of the quantization of sources and fields in the quantum field theory separately. With the results of Section 8, in which the electron is represented as the trefoil knot made of null fields and living on the Dupin cyclide, superimposed with Varlamov's mass formula, the source problem is solved in geometrodynamics in accordance with Einstein's vision of the quantization of quantum fields. Furthermore, in the same section, it is demonstrated that the developed description of the electron blends seamlessly with the description of the charged rotating black hole. With these accomplishments, the description of neutinos requires uses of Majorana equation (since neutrinos are neutral particles of very small masses). This fact apparently restricts uses of the Maxwell–Dirac isomorphism. The first steps toward solving this problem (for massless neutrinos) were made only in 2023.

2. Derivation of the Schrödinger Equation for a Single Photon

We begin with writing down Maxwell's equations without sources and currents in the vacuum. These are:

$$\begin{aligned}\nabla \cdot \mathbf{E} &= 0, \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}, \\ \nabla \cdot \mathbf{H} &= 0, \nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}.\end{aligned}\quad (1)$$

In the above equations, we keep only the speed of light c , while in the rest of the constants, we remain equal to unity. These can always be restored if needed. Incidentally, the situation described by (1) is in accord with that known in standard quantum mechanics. This was emphasized by Bohm [6] and Bohm and Hiley [18]. The De Broglie and Schrödinger's waves in quantum mechanics do not have the sources or sinks (charges) and, accordingly, the currents. In the absence of knotted/linked configurations, the Maxwellian fields also do not have sources or sinks. In Section 5, we argue that the knotted/linked Maxwellian field configurations could be reinterpreted as charges. Thus, if the charges in electrodynamics are of topological origin, their movements could then be interpreted as the currents. However, charges do have masses. Thus, such defined masses are of topological origin and could be looked upon as linked knotted configurations of electromagnetic fields. Details will be given below, in Sections 5–7. Reciprocally, these results are possible to apply to Schrodinger's quantum mechanics.

Next, we introduce the Riemann–Silberstein complex vector

$$\mathbf{F}_\lambda = \frac{1}{\sqrt{2}}(\mathbf{E}_\lambda + i\lambda\mathbf{H}_\lambda), \quad (2)$$

where $\lambda = 1(-1)$ corresponds to the positive (negative) helicity. The concept of helicity is related to the concept of polarization, which for the light is equivalent [16,17] to spin. All

this is described in detail in [10]. Following Kobe [19], and Smith and Raymer [20,21], we temporarily suppress the subscript λ and introduce the energy density ε , as follows

$$\mathbf{F}^* \cdot \mathbf{F} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{H}^2) \equiv \varepsilon. \quad (3)$$

This result then allows us to introduce the photon wave function

$$\Psi_i = \sqrt{\frac{1}{\mathcal{E}}} F_i, \quad (4)$$

Here $i = 1, 2, 3$ labels the Euclidean coordinates, while $\mathcal{E}$ is defined in (6), below. Using (4), we require

$$\sum_{i=1}^3 \int d^3x \Psi_i^* \Psi_i = 1 \quad (5)$$

with the total energy $\mathcal{E}$ defined by

$$\int d^3x \varepsilon = \mathcal{E}. \quad (6)$$

By design, the wave function Ψ_i is satisfying the Schrödinger's equation for the photon

$$i \frac{\partial}{\partial t} \mathbf{F} = c \nabla \times \mathbf{F}, \quad (7)$$

$$\text{or, } i\hbar \frac{\partial}{\partial t} \mathbf{F} = i c \mathbf{p} \times \mathbf{F}, \mathbf{p} = -i\hbar \nabla, \quad (8)$$

provided that

$$\nabla \cdot \mathbf{F} = 0. \quad (9)$$

Equations (7) and (9) are equivalent to Maxwell's Equation (1) as required. The continuity equation for the probability now reads

$$\frac{\partial \rho}{\partial t} + \text{div} \cdot \mathbf{j} = 0, \mathbf{j} = \alpha \mathbf{E} \times \mathbf{H} = \alpha \mathbf{F}^* \times \mathbf{F}, \rho = \sum_{i=1}^3 \Psi_i^* \Psi_i, \alpha = \frac{c}{\mathcal{E}}. \quad (10)$$

These results are perfectly fine as long as the quantization of the electromagnetic field is of interest only. They are not exhibiting the universal connection with quantum mechanics of particles, though. They were initially designed for photons only. This gives us an opportunity to describe such a connection in the next section.

3. Electromagnetic Field in the Absence of Currents and Sources as Complex Scalar Field

Following Green and Wolf [13,14], we notice that in a region of space free of currents and charges, the electromagnetic field is fully specified by the single vector potential $\mathbf{A}$. For such a case, we can write: $\mathbf{E} = -\frac{1}{c} \frac{\partial}{\partial t} \mathbf{A} \equiv -\frac{1}{c} \dot{\mathbf{A}}, \mathbf{H} = \nabla \times \mathbf{A}$. This observation then allows us to reobtain the vector potential $\mathbf{A}$ from a single (in general complex) scalar field potential $V(\mathbf{x}, t)$. It can be demonstrated that the total energy $\mathcal{E}$, defined in (3) and (6), can be rewritten in terms of $V(\mathbf{x}, t)$, as

$$\int d^3x \varepsilon \equiv \frac{1}{2} \int d^3x \left(\frac{1}{c^2} \dot{V} \dot{V}^* + \nabla V \cdot \nabla V^* \right) = \mathcal{E} \quad (11)$$

Accordingly, the Poynting flux $\mathbf{j}$ of the electromagnetic energy density is given by

$$\mathbf{j} = -\frac{1}{2} (\dot{V}^* \nabla V + \dot{V} \nabla V^*). \quad (12)$$

Therefore, the analog of the continuity Equation (10) now reads as

$$\frac{\partial \rho}{\partial t} + \text{div} \cdot \mathbf{j} = 0, \rho = V^* V. \quad (13)$$

It should be clear then that the obtained results are equivalent to those presented in Section 2. The advantage of having these results rewritten with the help of the scalar potential V lies in the opportunity of bringing them into correspondence with quantum mechanics and of introducing knots and links into the discussion. In fact, as we soon demonstrate, such a scalar form of Maxwell's equations allows us to accomplish much more. To begin, we represent $V(\mathbf{x}, t)$ in the form

$$V(\mathbf{x}, t) = \int d^3k [\alpha(\mathbf{k}, t) \cos(\mathbf{k} \cdot \mathbf{x}) + \beta(\mathbf{k}, t) \sin(\mathbf{k} \cdot \mathbf{x})] \equiv \int d^3k V(\mathbf{k}, t), \quad (14)$$

at the same time,

$$\mathbf{A}(\mathbf{x}, t) = \int d^3k [\mathbf{a}(\mathbf{k}, t) \cos(\mathbf{k} \cdot \mathbf{x}) + \mathbf{b}(\mathbf{k}, t) \sin(\mathbf{k} \cdot \mathbf{x})] = \int d^3k \mathbf{A}(\mathbf{k}, t). \quad (15)$$

From electrodynamics, it is known that the vector potential $\mathbf{A}$ is satisfying the wave equation for the vector field $\mathbf{A}$:

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A} - \nabla^2 \mathbf{A} = 0. \quad (16)$$

Use of (15) in (16) leads to

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A}(\mathbf{k}, t) + \mathbf{k}^2 \mathbf{A}(\mathbf{k}, t) = 0. \quad (17)$$

By comparing Equations (14) and (15), it follows that

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} V(\mathbf{k}, t) + \mathbf{k}^2 V(\mathbf{k}, t) = 0 \quad (18)$$

and, therefore,

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} V(\mathbf{x}, t) - \nabla^2 V(\mathbf{x}, t) = 0. \quad (19)$$

Next, suppose that the solution of (19) can be presented in the form

$$V(\mathbf{x}, t) = v(\mathbf{x}) \exp\{i\phi(\mathbf{x}, t)\}, \phi(\mathbf{x}, t) = k\Phi(\mathbf{x}) - \omega t. \quad (20)$$

It should be noted that such a representation is physically motivated by its direct connection with Huygens' principle. The details are best explained in monographs by Hilbert and Courant [19] and by Luneburg [22]. Some of their ideas were subsequently developed by Maslov [23]. Different/alternative interpretation of (20), adopted for the formalism of quantum mechanics, was independently developed by Bohm in his Bohmian mechanics [6,18]. Using (20) in (19), the following two equations are obtained:

$$(\nabla\Phi)^2 - \frac{1}{k^2 v} \nabla^2 v = n^2, \quad (21)$$

$$\nabla\Phi \cdot \nabla v + \frac{1}{2} (\nabla^2 \Phi) v = 0. \quad (22)$$

Here, $k = c/\omega$, $n^2 = 1$. In the case of quantum mechanics, typically, $n^2 = n^2(\mathbf{x})$. This fact will be discussed further below. For now, however, consider both of these equations

in the limit $k^2 \rightarrow \infty$. Such a limit is typical for phenomena described by the methods of geometrical optics [24]. In this limit, (21) is converted into:

$$(\nabla\Phi)^2 = n^2. \quad (23)$$

Physics behind Equations (21) and (22) is predetermined by the properties of geometric optics' limit. Specifically, in this limit the surfaces $\Phi = \text{const}$ represent the *wave fronts*, while their duals represent the *light rays*. These are orthogonal to the wavefronts. Following [23,24], we notice that: (a) (23) is known as the *eikonal equation* and (b) (22) is known as the *transport equation*. This equation can be simplified with the help of the following arguments. If $\frac{\partial}{\partial\tau}$ denotes the differentiation along a particular ray, then according to [23,24], we write,

$$\frac{\partial}{\partial\tau} \dots = \nabla\Phi \cdot \nabla \dots. \quad (24)$$

The displayed identity allows us to rewrite (22) as

$$\frac{\partial v}{\partial\tau} + \frac{1}{2}(\nabla^2\Phi) = 0. \quad (25)$$

The integration of the last equation is straightforward and is yielding the result

$$v(\tau) = v(\tau_0) \exp\left[-\frac{1}{2} \int_{\tau_0}^{\tau} d\tau' \nabla^2\Phi\right]. \quad (26)$$

Next, following [24], chr.7, we notice that the ray trajectory $\mathbf{x}(\tau)$ can be derived from the equations of motion

$$\frac{d\mathbf{x}}{d\tau} = \nabla\Phi. \quad (27)$$

By combining Equations (25) and (27), the solution, Equation (26), can be rewritten as

$$v(\mathbf{x}(\tau)) = v(\mathbf{x}(0)) \exp\left[-\frac{1}{2} \int_{\tau_0}^{\tau} d\tau' \nabla^2\Phi(\mathbf{x}(\tau'))\right]. \quad (28)$$

Notice next that $\mathbf{x}(\tau) = \mathbf{x}(\mathbf{x}_0, \tau)$, so that, by definition, $\mathbf{x}(\mathbf{x}_0, \tau_0) = \mathbf{x}_0$. This observation allows us to introduce the Jacobian J as

$$J = \det\left(\frac{\partial x^i(\mathbf{x}_0, \tau)}{\partial x_0^j}\right). \quad (29)$$

In view of (27), we then obtain, as well

$$\frac{1}{J} \frac{dJ}{d\tau} = \nabla \cdot \nabla\Phi = \nabla^2\Phi(\mathbf{x}(\tau))$$

implying

$$J(\mathbf{x}_0, \tau) = \exp\left[\int_{\tau_0}^{\tau} d\tau' \nabla^2\Phi(\mathbf{x}(\tau'))\right] \equiv J(\mathbf{x}(\tau)). \quad (30)$$

By combining (28) and (30), we finally obtain:

$$v(\mathbf{x}(\tau)) = \frac{v(\mathbf{x}(\tau_0))}{\sqrt{J(\mathbf{x}(\tau))}}. \quad (31)$$

In view of this result, it is always possible to normalize $v(x(\tau))$, that is to require

$$\int_{\Delta} d^3x v^2 = 1 \quad (32)$$

where Δ is the domain of the integration determined by a particular problem to be solved. According to [13], the energy density ε defined in (11), can be represented in view of (23), as

$$\varepsilon = \frac{1}{2} [k^2 v^2 + \frac{k^2}{n^2} v^2 ((\nabla \Phi)^2 + \frac{1}{k^2} (\nabla \ln v)^2)] \rightarrow_{k \rightarrow \infty} k^2 v^2. \quad (33)$$

In the limit $k \rightarrow \infty$, the analog of the wavefunction density ρ , (10), is given now by

$$\rho = \frac{\varepsilon}{\mathcal{E}} \rightarrow_{k \rightarrow \infty} \frac{v^2}{\int d^3x v^2} \rightarrow \Psi^* \Psi, \quad \Psi = v(x(\tau)) \exp\{i\phi(x, t)\}. \quad (34)$$

The substitution of the ansatz (20) into the expression for the energy current (12) results in

$$\frac{j}{\mathcal{E}} = \frac{1}{2} v^2(x) \nabla \Phi. \quad (35)$$

The associated Hamilton-Jacobi (H-J) equation is given by (23). In the limit $k \rightarrow \infty$, its solutions are those of the Schrödinger equation. This fact is well known from the WKB theory, where the $k \rightarrow \infty$ limit is the same as the $\hbar \rightarrow 0$ limit (the classical limit). We shall recover below Schrödinger's equation without recourse to the $k \rightarrow \infty$ (or $\hbar \rightarrow 0$) limit.

4. Schrödinger's Quantum Mechanics in Terms of Complex Scalar Fields of Electromagnetic Origin

To our knowledge, the interpretation of electromagnetic fields in terms of complex scalar fields was not in use in physics literature until the late 1950s [13,14]. Roman—one of the founders of modern quantum field theory wrote a paper [25] on this topic in 1959 and made the results of [13,14] perfectly rigorous. However, his efforts to write source-free electrodynamics were left unnoticed until the work by Ranada [15], who rediscovered this possibility without actually being guided by [13,14,25]. It is quite remarkable that Max Born—the designer of the probabilistic interpretation of quantum mechanics—later on in life wrote a book called the “Principles of Optics” [26] (along with Emil Wolf), in which only implicitly (that is, without any mention of quantum mechanics), on page 430 and in Section 8.4, the scalar field theory interpretation of the electromagnetic field can be found. In recent literature, this mention is briefly presented in [27]. While the papers [13,14,25] were left unnoticed, much more cumbersome Duffin–Kemmer (D-K) formalism has become in vogue for electromagnetic and other bosonic fields [28–30]. A consistent relativistic quantum mechanics/quantum field theory for spin 0 and 1 bosons was developed with help of the D-K formalism [28–30]. In it, the Dirac-looking first order equation is used, in which the Dirac γ matrices are replaced with β matrices obeying commutation relations different from those known for the Dirac matrices. In dealing with the (anti) self dual electromagnetic fields, the twistorial interpretation of equations for these fields is also possible [31]. The comparison between the vector (Maxwell), complex scalar, Dirac and twistorial fields is seemingly possible only for the massless versions of these fields. The inclusion of masses into results just mentioned can be conducted either by hands or topologically, e.g., as was performed by Ranada [16,17]. The role of masses will be further studied in Sections 5–7.

In standard field-theoretic notations [10,28] (making, for a moment, all constants equal to unity), the Lagrangian $\mathcal{L}$ for the complex massless scalar field is given by, e.g., ([28], p. 32).

$$\mathcal{L}[\varphi, \varphi^*] = \sum_n \frac{\partial \varphi}{\partial x^n} \frac{\partial \varphi^*}{\partial x_n}. \quad (36)$$

By varying the fields φ and φ^* in the action $\mathcal{A}$,

$$\mathcal{A} = \int d^4x \mathcal{L}[\varphi, \varphi^*], \quad (37)$$

while assuming that these fields are independent and decaying at infinity, this leads to the following equations of motion

$$\square \varphi = 0, \quad (38)$$

$$\square \varphi^* = 0. \quad (39)$$

Here, the d'Alembertian $\square$ is defined as: $\square = \frac{\partial^2}{\partial t^2} - \nabla^2$, with ∇^2 being the 3-dimensional Laplacian. In his work [25], Roman was not interested in the standard field-theoretic analysis of $\mathcal{L}[\varphi, \varphi^*]$, e.g., that which was conducted in [28] (p. 32). He was interested in proving that the gauge transformations of the Maxwellian fields, when rewritten in the formalism of complex scalar fields, will keep the action $\mathcal{A}$ form (or gauge)-invariant (up to the total divergences vanishing at the boundary of space-time). With this result proven, the task of this paper is different. Given that the gauge-invariant action $\mathcal{A}$ is gauge-invariant, we apply the standard field-theoretic treatment to (30) with purposes which will become clear upon reading. From [28], we find the time component T^{00} of the energy-momentum tensor (that is the energy density). It is given by

$$T^{00} = \frac{\partial \varphi}{\partial t} \frac{\partial \varphi^*}{\partial t} + \nabla \varphi \cdot \nabla \varphi^*. \quad (40)$$

Notice now that T^{00} coincides with ε defined in (11), where we temporarily put $c = 1$. Accordingly, the momentum density $T^{0\alpha}$ given by

$$T^{0i} = -\left(\frac{\partial \varphi^*}{\partial t} \nabla_i \varphi + \frac{\partial \varphi}{\partial t} \nabla_i \varphi^*\right), i = 1, 2, 3, \quad (41)$$

is up to a constant and coincides with the flux $\mathbf{j}$ defined in (12). Evidently, in this setting, the continuity Equation (13) is nothing but the law of conservation of the energy-momentum tensor [28]

$$\frac{\partial}{\partial x^\mu} T^\mu_\nu = 0. \quad (42)$$

For the record, we use the Minkowski-space metric tensor $g_{\mu\nu}$ of signature $(+, -, -, -)$, so that $g_{\mu\nu} = g^{\mu\nu}$. For the 4-vector a_μ , we have $a_\mu = g_{\mu\nu} a^\nu$, etc.

Because we are dealing with the complex scalar field, there is also a current vector J_μ responsible for carrying the charge (recall that the description of charges and currents associated with them is always associated with the existence of the global gauge symmetry inseparably linked with uses of complex scalar fields [28]). Mathematically, the conservation of current J_μ is expressible by analogy with (36) as

$$\frac{\partial}{\partial x^\mu} J^\mu = 0. \quad (43)$$

Explicitly, the current J_μ is given by

$$J_\mu = i\left(\varphi^* \frac{\partial \varphi}{\partial x^\mu} - \frac{\partial \varphi^*}{\partial x^\mu} \varphi\right). \quad (44)$$

Let $Q = \int d^3x J_0$ be the total charge. Then, (37) is the continuity equation associated with the charge conservation. It is well known [10,28,32] that J_0 is not always a positively defined quantity. This fact is caused by the observation that at any given time t both φ and $\frac{\partial \varphi}{\partial x^0}$ may independently have arbitrary values and signs. It is important because of the following. For the sake of generality and comparison, we include the mass m^2 term into

both Equations (32) and (33) thus converting them into the Klein–Gordon (K-G) equations ([28], p. 32),

$$(\square + m^2)\varphi = 0, \quad (45)$$

$$(\square + m^2)\varphi^* = 0. \quad (46)$$

Since the time of Schrödinger’s discovery of the equation bearing his name, the K-G equation was considered as the relativistic analog of Schrödinger’s equation. In analogy with a nonrelativistic case, by multiplying (39) by φ and (40) by φ^* and subtracting (40) from (39), we are repeating the same steps as for the nonrelativistic Schrödinger equation in order to obtain the continuity Equation (37). In the nonrelativistic case (that is for the Schrödinger equation), this procedure yields: $j_0 = \rho = \psi^*\psi$ and $\vec{j} = \frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*)$. The continuity equation in the nonrelativistic case is in its standard form: $\frac{\partial}{\partial t}\rho + \nabla \cdot \vec{j} = 0$. To compare this result with Equations (37) and (38), we must multiply (37) by $\frac{\hbar}{2m}$. Then, we obtain, respectively, $J_0 = \frac{i\hbar}{2m}(\varphi^*\frac{\partial\varphi}{\partial x^0} - \frac{\partial\varphi^*}{\partial x^0}\varphi)$ and $J_k = \frac{\hbar i}{2m}(\varphi^*\frac{\partial\varphi}{\partial x^k} - \frac{\partial\varphi^*}{\partial x^k}\varphi) = -j_k, k = 1, 2, 3$. Thus, even though (up to a sign) the currents $\vec{j}$ and $\vec{J}$ coincide, the densities j_0 and J_0 are noticeably different. This fact matters when the time-dependent problems are discussed for the K-G fields. Because of the nonpositivity of J_0 , the full time-dependent K-G equation was discarded (historically) from consideration as the relativistic analog of the Schrödinger equation. In the time-independent case, the situation is not as dramatic. Specifically, suppose that the field φ in (39) can be written as $\varphi(\mathbf{x}, t) = \psi(\mathbf{x}) \exp(-i\omega t)$, then, the K-G equation is converted into the Helmholtz equation

$$(\nabla^2 + \tilde{m}^2)\psi = 0, \quad \omega^2 - m^2 = \tilde{m}^2. \quad (47)$$

Accordingly, the field φ^* can be written now as $\varphi^*(\mathbf{x}, t) = \psi^*(\mathbf{x}) \exp(i\omega t)$. In view of these results, J_0 now acquires the form, $J_0 = \frac{\hbar\omega}{m}(\psi^*\psi)$. At the same time, $J_k = 0$. This result allows us to study all kinds of stationary K-G equations [32], and they all make physical sense. The obtained result raises the following question. In Section 3, we demonstrated that the continuity Equation (13) is exactly the same as the energy-momentum conservation in Equation (36). At the same time, the continuity in Equation (10) for the photon is the same as the continuity in Equation (13) for the complex scalar field. This means that this equation can be used instead of the continuity in Equation (37) for development of the time-dependent quantum mechanical formalism for the complex scalar K-G field. It is appropriate at this point to mention that, historically, the complex massive scalar field was used in the Yukawa theory of strong interactions, where it is known as the pi meson field. By extending the results of Harish-Chandra [29] for this field written in the language of Duffin–Kemmer formalism, Tokouoka [33] studied in detail the meson–nucleon interactions. Much more recent results are discussed, for example, in [34] and references therein. Since the meson–nucleon interactions are described nowadays with the help of quantum chromodynamics (QCD), the complex scalar field acts effectively as the abelianized version of the non Abelian Yang–Mills gauge field. More on this is discussed in the next section, where we shall also explain how to get rid of the mass term in the K-G equation. According to [32], page 99, every spinor component of the Dirac equation with a nonzero mass is satisfying the massive K-G equation. In the zero mass limit, this fact creates the equivalence class between the massless K-G and Dirac equations. Apparently the components of equations for higher spin particles, e.g., spin-2 gravitons, also belong to the same equivalence D-K class [10,28,32]. In this section, we still need to demonstrate how results of Sections 3 and 4 are connected with the nonrelativistic time-independent Schrödinger equation (TISE). In the next section, we shall present evidence that such an equation also belongs to the same equivalence class (in the sense of Hadamard) as the rest of basic equations.

To connect results of Sections 3 and 4 with TISE, just describing the treatment of the stationary K-G equation is the most helpful. After the mass term is eliminated in this equation (read the next section) and the constant c is restored, the K-G equation is converted into the wave Equations (32) or (33). This fact allows us to use the results of Schrödinger's second foundational paper on quantum mechanics [35] (pp. 13–40). In (32), we replace the speed of light c by $u = c/n$, where $n = n(x, y, z)$ is the effective refractive index. Next, we use the same ansatz $\varphi(\mathbf{x}, t) = \psi(\mathbf{x}) \exp(-i\omega t)$ for the wave function. Moreover, we replace the dispersion relation $\omega = ck$ for the electromagnetic waves in the vacuum by $\omega = uk$ and use the de Broglie-type relation $k = p/\hbar$ along with the fact of the mechanical system $E = \mathbf{p}^2/2m + V$. Thus, we obtain: $\mathbf{p}^2 = 2m(E - V)$. With help of these results, the TISE follows from the wave Equation (32) under conditions just described. Thus, we obtain,

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V - E\right]\psi = 0. \quad (48)$$

The normalized result for J_0 can now be used immediately so that the continuity Equation (37) works in this case. Nevertheless, the question remains: what to do with the continuity in Equation (36)? This equation is used for the development of quantum mechanics of photons. Can it be used in the nonrelativistic case for the stationary Schrödinger equation? Equations (32)–(36) suggest that this could be possible. Nothing meaningful is obtained if we act straightforwardly, though. Indeed, using the K-G ansatz for the wave function in (34) and (35) and restoring c in these equations yields:

$$T^{00} = \left(\frac{\omega}{c}\right)^2 \psi^* \psi + \nabla \psi \cdot \nabla \psi^*, \quad (49)$$

$$T^{0i} = 0. \quad (50)$$

The result obtained for T^{00} is different from that for J_0 . Unlike [14], we are not going to look for arguments in favor of $\nabla \psi \cdot \nabla \psi^* \simeq \left(\frac{\omega}{c}\right)^2 \psi^* \psi$ relation. This was conducted already in Section 3, where the results were obtained in the regime of geometrical optics. Instead, now we are going to use the original Schrödinger's ideas again. This time, we are going to use the methods developed in his first paper on quantum mechanics ([35], pp. 1–12), as well as from the already discussed second paper. In particular, we again replace c by u in (43) and then replace $\left(\frac{\omega}{u}\right)^2$ by $\left(\frac{p}{\hbar}\right)^2$. Next, by looking at (43), we require

$$\nabla \psi \cdot \nabla \psi^* = \left(\frac{p}{\hbar}\right)^2 \psi^* \psi$$

and use $\left(\frac{p}{\hbar}\right)^2 = \frac{2m(E-V)}{\hbar^2}$. These results can be rewritten in terms of the H-J equation if we temporarily use only the real valued functions: $\psi(\mathbf{x}) = \psi^*(\mathbf{x})$

$$(\nabla \psi)^2 = \frac{2m(E-V)}{\hbar^2} \psi^2. \quad (51)$$

Notice that this H-J equation contains explicitly Planck's constant $\hbar$ while the H-J equation used in the semiclassical WKB calculations is $\hbar$ -independent by design. Nevertheless, (45) coincides exactly with the equation (1'') of Schrödinger's first paper on quantum mechanics [27] (pp. 1–12). This difference has a profound effect on the rest of Schrödinger's calculations. It allows him and hence, us, to restore the stationary Schrödinger equation without any approximations. For this purpose, Schrödinger introduces the functional $J[\psi]$

$$J[\psi] = \frac{1}{2} \int_{\Delta} d^3x \left[(\nabla \psi)^2 - \frac{2m(E-V)}{\hbar^2} \psi^2 \right] \quad (52)$$

which he is minimizing under the subsidiary condition

$$\int_{\Delta} d^3x \psi^2 = 1 \quad (53)$$

thus producing the stationary Schrödinger Equation (42). Therefore, the continuity Equations (36) and (37) both can be used in conjunction with the nonrelativistic stationary Schrödinger's Equation (42).

5. Chladni Patterns and Knotty Wavefunction Configurations

To begin, we would like to now discuss the following issue: How the presence (absence) of masses is affecting topics discussed in previous sections. Surprisingly, the link between the discussed topics and masses can be established with help of the Huygens' principle. This principle can be discussed purely mechanically with methods of contact geometry and topology as was conducted, for example, by Arnol'd [36]. Reader's friendly basics of contact geometry and topology are provided in our book [37]. The same principle could be discussed using the theory of partial differential equations. It is beautifully discussed, for example, in books by Hilbert and Courant [38], Luneburg [22], Hadamard [39] and later, by many others. For the sake of space, we shall not go into the mathematical details of Huygens' principle in this paper. Details are given in our paper [40]. We only mention that Equations (32) and (33) do obey the Huygens' principle so that *all equations which can be reduced to (32) and (33) do obey this principle*. According to Hadamard [39], only three operations are allowed for the conversion of a given PDE to the "trivial" PDE Equations (32) and (33).

Specifically, the essence of *Huygens' equivalence principle* lies in the following. Let $L[\phi]$ be the Huygens-equivalent operator (that is, the operator which is written as in Equations (32) and (33)) and let $\tilde{L}[\phi]$ be another operator (and equation) to be investigated. Then, the operators $L[\phi]$ and $\tilde{L}[\phi]$ are Huygens-equivalent if:

- (a) $\tilde{L}[\phi]$ can be obtained from $L[\phi]$ by the nonsingular transformations of independent variables;
- (b) $\tilde{L}[\phi] = \lambda^{-1} L[\lambda\phi]$ for some positive, smooth function λ of independent variables;
- (c) $\tilde{L}[\phi] = \rho L[\phi]$ for some positive smooth function ρ of independent variables.

With these rules established, let us consider as an example an invertible sequence of transformations from the d'Alembert Equation (32) to the massive K-G equation. Let us use the transformation $\tilde{L}[\phi] = \lambda^{-1} L[\lambda\phi]$ with $\lambda = e^{\alpha t}$, with α being some constant. Then, we obtain:

$$e^{-\alpha t} \left\{ \left[\frac{\partial^2}{\partial t^2} e^{\alpha t} \phi \right] - [\nabla^2 e^{\alpha t} \phi] \right\} = \frac{\partial^2}{\partial t^2} \phi + 2\alpha \frac{\partial}{\partial t} \phi - \nabla^2 \phi + \alpha^2 \phi = 0.$$

Next, let $\lambda_1 = e^{ibx}$ and $\lambda_2 = e^{-icx}$. Upon substitution of these factors into the previous equation, while using the rule b, we obtain after some calculation

$$\frac{\partial^2}{\partial t^2} \phi + 2\alpha \frac{\partial}{\partial t} \phi - \nabla^2 \phi + 2ib \frac{\partial}{\partial x} \phi - 2ic \frac{\partial}{\partial x} \phi + (a^2 - b^2 - c^2) \phi = 0.$$

If now $b = c$ and $a^2 = 2b^2$, we obtain the telegrapher equation

$$\frac{\partial^2}{\partial t^2} \phi + 2\alpha \frac{\partial}{\partial t} \phi - \nabla^2 \phi = 0.$$

The K-G equation is obtained now upon substitution of $\phi = e^{mt} \psi$ into the telegrapher's equation with a subsequent replacement of $a = -2m$ and m by im . Thus, the K-G and d'Alembert equations are Huygens-equivalent. If this is so, it would be possible to use the twistor formalism [41] for the K-G equation.

The question arises: Is the stationary Schrödinger equation Huygens-equivalent to the d'Alembert equation? In 1935, Fock initiated a study of this problem (with different

goals in mind, however). While studying the (accidental) degeneracy of the stationary Schrödinger equation for the hydrogen atom, he converted this equation into the integral equation, which looked very similar to the Poisson integral in the theory of functions of one complex variable. Recall that harmonic functions (that is, functions satisfying the Laplace equation) inside the circle (and, thus, also in any domain which can be conformally mapped into circle) can be presented via the Poisson integral. Fock initiated the study of what is known now as dynamical symmetry groups. These groups are allowing us to solve quantum mechanical problems group-theoretically thus bypassing uses of the Schrödinger equation. This direction of research has grown into a large field nowadays [42]. Fock's work is presented in Volume 1, pages 400–410 of [42]. The group-theoretic analysis had lead Fock to the conclusion that the bound states of the hydrogen atom should be studied in the four-dimensional Euclidean space, while scattering states should be studied in the 3+1 dimensional hyperbolic (Lobachevski) space. The Euclidean four-dimensional version of the Schrödinger equation (for the bound states) was reduced by Fock to the study of solutions of the four-dimensional Laplacian. The description of the scattering states was only outlined by Fock. He suggested that this case requires the study of solutions of the d'Alembertian. From here is a connection with the Huygens principle. This suggestion was waiting for its solution for 31 years. In 1966, it was finally solved by Itzykson and Bander [43]. Although Itzykson and Bander [43] obtained the d'Alembertian for scattering states, thus making the stationary Schrödinger equation Huygens-equivalent to the d'Alembertian equation, the results of [43] happened to be very cumbersome. The transparency of results for both bound and scattering states of the hydrogen atom was achieved only in 2008 in the paper by Frenkel and Libine [44]. Using quaternions in four dimensions, Frenkel and Libine extended the Poisson formula for the harmonic functions—from two to four dimensions. This allowed them to easily extend the obtained results from the Euclidean four dimensional to Minkowski 3+1 spaces.

With these results presented, now we are in a position to explain the quantum mechanical significance of null fields. These fields were known since 1787, when Chladni, a German physicist, studied nodal lines of a vibrating metal plate by stroking this plate covered by sand with a violin bow [45]. After studying a variety of nodal patterns systematically, he wrote a book in 1802, where all these patterns were systematized. Remarkably, the 1802 edition of Chladni's book was translated into English and published in 2015 [46]. Nowadays, Chladni patterns can be observed on You Tube [47] and, more scientifically, in plasmonic charge density waves [48], etc.

With help of results of previous sections, we are now in a position to provide an explanation of the connection between Chladni's patterns, null fields, knots/links and quantum mechanics. To set up notations, following [49], we begin with the description of two-dimensional Chladni problems. It is convenient to think about a given Riemannian surface Σ with metric g_Σ as a vibrating membrane with $u(\mathbf{x}, t)$, ($\mathbf{x} \in \Sigma$) being a displacement at time t of the membrane from its original position. The function u is a solution to the wave equation

$$\frac{\partial^2}{\partial t^2} u = \nabla_\Sigma^2 u. \quad (54)$$

By representing a solution in the form $u(\mathbf{x}, t) = v(t)w(\mathbf{x})$, the above equation splits into two equations, e.g.,

$$\frac{\partial^2}{\partial t^2} v = \lambda v \quad (55)$$

and

$$\nabla_\Sigma^2 w = \lambda w. \quad (56)$$

The (null) zero set, $\Xi(w) := \{\mathbf{x} \in \Sigma : w(\mathbf{x}) = 0\}$, is called the *nodal* (Chladni) set. The definitions just described need to be supplemented with the boundary (e.g., Dirichlet or Neumann) conditions so that the above nodal problem is known as the *fixed membrane problem*. In the case when the membrane is a closed surface, the above problem is known

as a *free membrane problem*. The first detailed calculation of the Chladni pattern was made by Poisson in 1829. The chronology of subsequent developments along with detailed numerous examples can be found in the encyclopedic book [50] by Rayleigh, in chapters 9 and 10. The same results were recently reproduced in the book [51]. At a much more advanced level, Chladni patterns were studied by Cheng [52], who proved that for an arbitrary smooth Riemannian surface (Σ, g_Σ) , the nodal set is a collection of immersed closed curves. Cheng's result is remarkable, since later on, the analogous results were obtained in three dimensions in [53]. This time, since the nodal curves are closed, they can generate knots and links of any complexity.

The results of [53] can be reobtained with help of results obtained in this work superimposed with [41,54,55]. Following [56], we write

$$\mathbf{F}(\mathbf{r}, t) = \mathbf{F}_+ + \mathbf{F}_-. \quad \mathbf{F}_\pm(\mathbf{r}, t) = \int_0^\infty d\omega e^{\mp i\omega t} \mathbf{F}_\pm(\mathbf{r}). \quad (57)$$

In such notations, (7) and (9) can be rewritten as

$$\nabla \cdot \mathbf{F}_\pm = 0, \quad (58)$$

$$\nabla \times \mathbf{F}_\omega = k\mathbf{F}_\omega, \quad (59)$$

with $k = \omega/c$. In plasma physics, (53) is known as a “force-free equation” while in hydrodynamics, it is known as the “Beltrami” equation. It was discussed in detail in our book [37] in the context of methods of contact geometry/topology, while in [54,55] these methods were used for the generation of all kinds of knots and links. Applying the operator $\nabla \times$ to (53) while taking into account (52), results in the following equation

$$\nabla^2 \mathbf{F}_\omega + k^2 \mathbf{F}_\omega = 0, \quad (60)$$

to be compared with (50). This (vector) version of the Helmholtz equation is known as the Chandrasekhar–Kendall (CK) equation [37]. These authors noticed that every solution of (53) is a solution of (54) but the converse is not true. This happens to be of a fundamental importance for our tasks. In [37], p. 30, it was stated that the solution of (53) is a composition of fields of three types: (a) solenoidal (52) lamellar $\mathbf{F}_\omega \cdot (\nabla \times \mathbf{F}_{-\omega}) = 0$, (b) Beltrami $\mathbf{F}_\omega \times (\nabla \times \mathbf{F}_{-\omega}) = 0$. The null fields used in creation of “linked and closed beams of light” [41] are lamellar. In terms of notations set up in [56], they are defined as

$$\mathbf{F}_\omega \cdot \mathbf{F}_{-\omega} = 0. \quad (61)$$

As explained in [37], the same classification of fields exist in physics of liquid crystals. In [37] (pp. 32–34), it was demonstrated that the Faddeev–Skyrme (F-S) knot/link generating model [57] is of the liquid crystalline origin. It also can be looked upon as originating from the Abelian reduction of the NAYM fields, to be discussed further in Section 6. Therefore, all results of [41], as well as of [54,55], are compatible with those originating from the F-S model [57]. Presently, we can study lamellar (null) solutions of the C-K equation. Being logically guided by [53,56], our results differ in details of results of these works. This difference permits us to inject new physics absent in these references.

We begin with (55). It can be rewritten as

$$|\mathbf{E}_\omega|^2 - |\mathbf{H}_\omega|^2 + 2(\mathbf{E}_\omega \cdot \bar{\mathbf{H}}_\omega + \bar{\mathbf{E}}_\omega \cdot \mathbf{H}_\omega) = 0. \quad (62)$$

This equation is surely satisfied if $|\mathbf{E}_\omega|^2 = |\mathbf{H}_\omega|^2$ and $\mathbf{E}_\omega \cdot \bar{\mathbf{H}}_\omega + \bar{\mathbf{E}}_\omega \cdot \mathbf{H}_\omega = 0$. Next, without a loss of generality and following [56], it is permissible to assume that $\mathbf{E}_\omega = \bar{\mathbf{E}}_\omega$ and $\mathbf{H}_\omega = \bar{\mathbf{H}}_\omega$. This then leads us to $\mathbf{E}_\omega \cdot \mathbf{H}_\omega = 0$. We should avoid the use of time Fourier transforms defined in (51), and just present two (null) equations (that is $|\mathbf{E}_\omega|^2 = |\mathbf{H}_\omega|^2$

and $\mathbf{E}_\omega \cdot \mathbf{H}_\omega = 0$) that would be sufficient for the generation of all kinds of torus knots and Hopf links evolving in time [41,58]. These conditions play the central role in works by Ranada [15,16,58].

These results do not let us to make a connection with Chladni patterns yet and with physics associated with these patterns. This will be conducted in this and the next section. In this section, using [59], we consider the following remarkable identity

$$(\nabla^2 + k^2)(\mathbf{r} \cdot \mathbf{v}) = 2\nabla \cdot \mathbf{v} + \mathbf{r} \cdot (\nabla^2 + k^2)\mathbf{v}. \quad (63)$$

Here, $\mathbf{v}$ is either $\mathbf{E}_\omega$ or $\mathbf{H}_\omega$. The scalar $(\mathbf{r} \cdot \mathbf{F}_\omega)$ is convenient to rewrite in the notations of [53], i.e., $(\mathbf{r} \cdot \mathbf{F}_\omega) = u = u_1 + iu_2$. Evidently, in view of (19), the scalar u can be identified with the $V(\mathbf{x}, t)$ defined in (14) so that $\mathbf{E}_\omega$ and $\mathbf{H}_\omega$ can be recovered if needed. Fortunately, this will not be necessary. By combining (54) and (57), we obtain Chladni-type (that is (50)) equations

$$(\nabla^2 + k^2)u_{k1} = 0 \text{ and } (\nabla^2 + k^2)u_{k2} = 0. \quad (64)$$

These equations were introduced and discussed in [53] using purely mathematical arguments. Since these are the same equations we just reobtained with the help of physical arguments, this allows us to extend the results of [53].

First, we notice that both equations have *the same eigenvalue* $\lambda_1 = \lambda_2 = k^2$. Having the same eigenvalue (of multiplicity 2), the wavefunctions u_{k1} and u_{k2} are *not the same*, as demonstrated in [53] and reaffirmed below. Although the presence of the i -factor is essential, the difference goes beyond this fact. This difference should not be confused with the degeneracy concept in standard quantum mechanics. The presence of eigenvalues having multiplicities is responsible for the effects of entanglements (more accurately—the self-entanglements in the present case). This is explained in detail in our book ([37] pp. 386–395).

Presently, we are in the position to demonstrate that: (a) Quantum mechanical self-entanglement is equivalent to the entanglement in the topological knot-theoretic sense; (b) the presence of eigenvalues with a multiplicity larger than one makes Schrödinger's and Heisenberg's interpretation of quantum mechanics not equivalent. The last statement follows immediately from the detailed Heisenberg-style calculations presented in [60]. Thus, we are only left with an explanation of (a).

Following [54], let us consider the force-free Equation (53), where, temporarily, we replace $\mathbf{F}_\omega$ by $\mathbf{v}$, as in (57). Then, by applying to both sides of (53) the operator div and by assuming that $k = \text{const} = \kappa(x, y, z)$, we obtain $\text{div}(\kappa\mathbf{v}) = \mathbf{v} \cdot \nabla\kappa = 0$. Let $\mathbf{r}(t) = \{x(t), y(t), z(t)\}$ be some trajectory on the surface $\text{const} = \kappa(x, y, z)$. In such a case, $\frac{d}{dt}\kappa\{x(t), y(t), z(t)\} = v_x\kappa_x + v_y\kappa_y + v_z\kappa_z = \mathbf{v} \cdot \nabla\kappa = 0$. This means that the “velocity” $\mathbf{v}$ is always tangential to the surface $\text{const} = \kappa(x, y, z)$. Since the vector field $\mathbf{v}$ is being assumed to vanish nowhere, the surface $\text{const} = \kappa(x, y, z)$ can only be a torus T^2 . The field lines of $\mathbf{v}$ on T^2 should be closed if const is a rational number. Thus, we just demonstrated that the force-free Equation (53) supplies us with the condition of existence of all possible torus knots for all rational κ 's. Starting with works by Ranada [15,16], such torus knots were explicitly designed both in [54,55] and [41].

Presently, it remains to be demonstrated that the Chladni-type Equations (50) and (58) lead to the three-dimensional Chladni patterns associated with these equations. Since this was conducted already in [53], our task is only to supply some physics to the results of [53]. Since the surface $\text{const} = \kappa(x, y, z)$ is T^2 , while the solid torus is defined by $\hat{T}^2 = D^2 \times S^1$, with D^2 being a disc, it is convenient to introduce a cylindrical system of coordinates and to consider the Neumann-type boundary value problem for the Helmholtz Equation (58) written in cylindrical coordinates. This task is facilitated by the accumulated knowledge about circular waveguides in electrodynamics. The solutions $u_{k1} = J_1(\rho\sqrt{\tilde{\lambda}})\cos\varphi$ and $u_{k2} = J_1(\rho\sqrt{\tilde{\lambda}})\sin\varphi$ discussed in [53]. Here, $J_1(x)$ is the standard Bessel function and $\tilde{\lambda}$ are adjusted (with an account of cylindrical symmetry) eigenvalues, which are just the TE and TM-type solutions known for circular waveguides [61]. For small ρ 's, they are rep-

resented by $u_{k1} \simeq \frac{\sqrt{\lambda}}{2} \rho \cos \varphi$ and $u_{k2} \simeq \frac{\sqrt{\lambda}}{2} \rho \sin \varphi$. Both functions become zero for $\rho = 0$. Thus, the individual nodal Chladni sets are, respectively, given by $u_{k1}^{-1}(0)$ and $u_{k2}^{-1}(0)$, while the Chladni centerline S^1 for the solid torus is determined by the transversality condition: $u(0) = u_{k1}^{-1}(0) \cap u_{k2}^{-1}(0)$. It is indeed the transversality condition, since we can plot both $\rho \cos \varphi$ and $\rho \sin \varphi$ on the complex plane $\mathbb{C}$ so that they are transversal to each other [62]. The transversality condition is needed for the validity of Thom's isotopy theorem [53,62] (assuring that the obtained results are stable with respect to possible perturbations). These perturbations will occur because, by definition, every knot is an embedding of S^1 into $S^3 = \mathbb{R}^3 \cup \{\infty\}$. The validity of Thom's theorem is required to assure that the embedding of solid tori $\hat{T}^2$ into S^3 could be conducted in such a way that it shall produce knots of any complexity, as long as they are not wild. Although [53] claims that all Chladni knotted sets can be obtainable this way, physically, this requires more explanations. These are presented in the next two sections.

6. Chladni Patterns in the Sky, or How Topological Nature of Masses in the Universe Is Linked with Axions of Dark Matter

6.1. Summary of Basic Facts

At this point, our readers might ask a question: How are the obtained results related to the real currents and charges present in the electromagnetic field? Recall that Bohm [6,18] was concerned exactly with this issue when he compared the quantum mechanical and optical formalisms. Since the currents originate from moving charges, the above question should be restated accordingly. This is one of the topics which will lead us to the discussion of dark matter.

In [54,55], we developed further results by Ranada [15,16,41,58]. By skillfully using the electric-magnetic duality for the source-free electromagnetic fields, Ranada obtained stable solutions for the electromagnetic fields. In [54,55], they were reinterpreted, respectively, as Dirac monopoles, electrons and dyons (particles possessing simultaneously both electric and magnetic charges). Topologically, all these objects are represented by the interlinked (Hopf) rings made of closed electric (for the electrical monopole being interpreted as stable electron), or magnetic (the Dirac monopole) lines and, for dyons, are modelled by two linked electric and two linked magnetic closed lines. All these are pass-equivalent to trefoil knots. More complicated objects are also possible [37] but dynamically they are not stable [55,63]. That is to say, their lifetimes are finite, as it happens experimentally. All of these stable objects (electric and magnetic monopoles; dyons) are made of null fields [41,58]. Even though they were discussed in Section 5, they will be further discussed below, in this section. Ref. [41] also describes knotted structures which are not null. These were not discussed by Ranada and they are not of instanton origin [54,64]. Since the electromagnetic fields are Abelian gauge fields, their description should be made by instanton-type calculations normally used for the description of the non-Abelian gauge fields. This makes the existence of non-null structures [41] both physically and mathematically questionable. Therefore, they are not discussed further in this work.

The question remains: How do the masses enter into this, purely topological, picture? This issue was touched upon already by Ranada. In [16], he demonstrated that his electromagnetic knots do have momentum, charge and spin. *That is, they have particle-like properties.* These results by Ranada's were elaborated further in [65,66]. *Furthermore, Ranada's knotted structures are made out of two complex scalar fields [15]. These are exactly the same fields as those described in previous sections in connection with quantum mechanics.* Unlike our works [54,55], Ranada's knots were not obtained as a result of instanton calculations. This circumstance prevented him from seeing general trends—patterns/analogies between the electromagnetism, hydrodynamics of incompressible fluids, contact mechanics, the Abelian and non-Abelian gauge fields and gravity. All of these analogies are discussed in our works [37,54,55]. In these papers, we argued, based on known results from knot theory, complements of knots and links make the ambient Minkowski

space-time curved in the same way as it is curved by masses in Einsteinian gravity. That is, in compliance with general relativity, for knots/links, the familiar equation

$$\text{Spacetime Curvature} = \text{Matter} \quad (65)$$

should work. This can be easily understood by studying the dynamics of point-like masses on the thrice-punctured sphere. Three punctures make dynamics on the sphere hyperbolic. Accordingly, knots/links make the fundamental group of space nontrivial, thus causing the effects of curvature. For knots/links, the ambient space-times are in one-to-one correspondence with topologies of these knots/links. Therefore, when written mathematically, Equation (59) is representing Einstein's equations of general relativity. In knot theory, there are no characteristic scales for these knots/links (unless they are considered as made out of some physical material). This property is indicative of conformal invariance. It is also possible to develop gravity with an emphasis on conformal invariance [67]. Exactly for this reason, knotted and linked objects of microscopic sizes can be interpreted as particles but knotted/linked structures at larger scales are being interpreted as space-time (topological) defects [68]. Since all physical masses should be positive, not all types of knots/links act like masses. This physical restriction on masses excludes from consideration all types of hyperbolic knots/links. Incidentally, mathematical methods described in [54,55] for the generation of knots/links tell us that the creation of hyperbolic knots/links is possible only in the presence of boundaries. Details can be found in [69]. In [54,55], all knots/links were generated dynamically. Thus, cosmologically, it is physically plausible that massive links/knots do not require the existence of boundaries for their generation. The introduction of finite size masses into general relativity as well as of finite size charges into the theory of non-Abelian Yang-Mills (NAYM) fields is associated with the fundamental technical (theoretical) difficulties. These are described in [37] (p. 97), and references therein. In the case of NAYM theories, the problem of charges is by-passed by treating only *source-free* NAYM fields. Thus, if we are thinking about the NAYM fields as just "more complicated" AYM fields (represented by the pair of complex source-free fields), then it should not come as a surprise that the method of Abelian reduction [70] applied to the NAYM fields permits us to obtain the Dirac monopoles—exactly those known in AYM theory. These were obtained by Ranada without any reference to the method of Abelian reduction. Ranada obtained these results without the usage of the instanton methods, while Trautman [64] and Kholodenko [54] obtained these (Dirac) monopoles by instanton methods from the AYM theory using standard instanton methods [70]. Historically, these methods were used first for obtaining the non-Abelian t'Hooft-Polyakov monopoles. How the Dirac monopole is obtainable from the t'Hooft-Polyakov monopole using the method of Abelian reduction is explained in detail in [69], (p. 174). Since gravity can be rewritten in the language of NAYM gauge fields, as was already known to Utiyama [70,71] in 1956, it is only natural that Robinson, in 1961, was able to find a place of source-free Maxwell's null fields for gravity [72,73].

With the background just provided, we are now in the position to talk briefly about the connections between quantum mechanics and dark matter. Very mysteriously, the quantum mechanics is connected with dark matter from its inception. Max Planck's 1900 law of blackbody radiation happens to be directly connected with the dark matter. This happened because of the following. (1) The theory of the cosmic microwave background is based on the theory of blackbody radiation. It is fully experimentally supported by this theory [74]. (2) The experimental data providing information about the dark matter are taken from the data on the cosmic microwave background [75]. (3) The axions, to be defined below, and in the next section, are directly connected with the dark matter [76]. (4) Quantization of the source-free electromagnetic fields makes bosons from these fields. This causes us to think about the superfluidity, Bose condensation, etc. However, as is well known, the chemical potential of Planckian photons is zero. Therefore, for Planckian photons, there is no Bose condensation, etc. The situation is not at all as simple as it is looking based on information from student's textbooks. Without any references to

knots, links and foundations of quantum mechanics, such a conclusion was reached by astrophysicists [77–79].

Thus, it remains to demonstrate that the conclusions of astrophysicists are strongly backed up by the fundamental laws/principles of quantum mechanics and electrodynamics. This then will help us to strengthen the arguments by astrophysicists by accounting for the knotty nature of the Maxwellian electrodynamics [15,41,54,55,63,80]. The existing quantization for electromagnetic fields do not contain any information about knots/links. Therefore, even though the links between the dark matter and cosmic microwave background are hinting toward the connections with the Bose condensation, superfluidity, etc., they leave the role of knots/links out of these studies.

6.2. Restoring Kelvin's/Thompson's Vortex Atoms with Help of Quantization Recommendations by Einstein

Results presented in Sections 2–5 suffer from several serious drawbacks, even though they are fully sufficient for the development of Bohmian mechanics [7]. On the one hand, we demonstrated that optical and quantum mechanical formalisms can be reunited rigorously. On the other, such a reunification is possible only for the source-free Maxwellian fields. This immediately raises a question: where is the place of mass, charge, angular momentum and spin in such designed quantum mechanics? The answer thus far was as follows. When Ranada used for his electromagnetic knots/links source-free scalar fields, he demonstrated [16] that these knots/links do act as if they have mass, charge, angular momentum and spin. This is all fine, but these particle-like objects have a life of their own that is apparently totally disconnected from the designed quantum mechanical formalism.

To repair this deficiency, it is appropriate to recall some comments by Einstein in his two lectures at the Institute for Advanced Study in the late 1940s [81], (p. 383). Einstein noticed that in contemporary quantum theory, we first develop the theory of electrons via the Schrödinger equation, and work out its consequences for atomic spectra and chemical bonding with great success. Then, we develop the theory of the free quantized EM field independently, and discuss it as a separate thing. Only at the end do we, almost as an afterthought, decide to couple them together by introducing a phenomenological coupling constant “ e ” and call the result “Quantum Electrodynamics”. Einstein suggested the way out of this logic. He said: **“ I feel that it is a delusion to think of the electrons and the fields as two physically different, independent entities. Since neither cannot exist without the other, there is only one reality to be described, which happens to have two different aspects; and theory ought to recognize this from the start instead of doing things twice”**. Einstein himself worked on this problem and wrote a paper on this subject in 1919. In doing so, he had Poincaré as his predecessor and Pauli as his successor [82,83]. The Einstein results are nicely summarized in [83] (pp. 795–796). His *purely electromagnetic particle* (PEP) is made out of source-free Maxwellian fields obeying the Beltrami equation, our Equation (53), that is of source-free electromagnetic fields (1). As was demonstrated in [50,51] all electromagnetic knots/links that are obtained (generated) with the help of this equation are of a nonhyperbolic nature. Therefore, Equation (59) implies that such PEP's are all massive nonhyperbolic knots. In the presence of boundaries, the formation of hyperbolic knots is also possible, as discussed in detail in [55] and differently in [80]. However, the negativity of masses is unphysical and, therefore such (hyperbolic) knots must be discarded. Important additional connections between the knot/link topology and particle masses were developed in [63].

Thus, Einstein obtained the correct answer, Equation (53), but had not linked it with the quantum mechanics and with knot/link topology. Such linkages are already presented in this paper. Fortunately, there is a room, still, for the further development of these results, thanks, for instance, to the series of papers by Sallhofer, [84,85] and references therein. No connections with knots/links was made in his papers, though. We would now like to sketch the results of [84,85]. By doing so, we shall complete the fundamentals of the

Einstein quantization program and, as a by-product, we shall effectively restore (at a more advanced level) the seemingly outdated Kelvin's (Thomson's) theory of vortex atoms [86].

Going back to Section 4, we notice that the complete correspondence between source-free Maxwellian electrodynamics and Schrodinger's quantum mechanics had been achieved due to the replacement of the speed of light c by the speed $u = c/n$, with $n = n(x, y, z)$ being the effective refractive index. The same idea can be observed to be present in the works by Sallhofer [84,85]. Specifically, instead of our system of Maxwell's Equation (1), he begins with the analogous system

$$\begin{aligned}\nabla \cdot \epsilon \mathbf{E} &= 0, \nabla \times \mathbf{E} = -\frac{\mu}{c} \frac{\partial \mathbf{H}}{\partial t}, \\ \nabla \cdot \mu \mathbf{H} &= 0, \nabla \times \mathbf{H} = \frac{\epsilon}{c} \frac{\partial \mathbf{E}}{\partial t}, \\ \nabla \cdot \mathbf{E} &= 0, \nabla \cdot \mathbf{H} = 0.\end{aligned}\quad (66)$$

Here, the magnetic and electric permeabilities μ/c and ϵ/c could be made as functions of spatial coordinates. They are playing the same role as the previously defined refractive index n . The last two equations emphasize the source-free nature of the electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$.

Next, introduce the set of Pauli matrices $\boldsymbol{\alpha} = (\sigma_1, \sigma_2, \sigma_3)$ in a usual way:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\quad (67)$$

For any vector $\mathbf{A}$, take into account that

$$(\boldsymbol{\alpha} \cdot \nabla)(\boldsymbol{\alpha} \cdot \mathbf{A}) = \mathbf{1} \nabla \cdot \mathbf{A} + i \boldsymbol{\alpha} \cdot (\nabla \times \mathbf{A}).\quad (68)$$

Multiply (via scalar multiplication) the first and the second rows of (60) by $\boldsymbol{\alpha}$ and apply the matrix identity (62). The result comes out as

$$\begin{aligned}(\boldsymbol{\alpha} \cdot \nabla)(\boldsymbol{\alpha} \cdot \mathbf{H}) - \frac{\epsilon}{c} \frac{\partial}{\partial t}(i \boldsymbol{\alpha} \cdot \mathbf{E}) &= 0, \\ (\boldsymbol{\alpha} \cdot \nabla)(i \boldsymbol{\alpha} \cdot \mathbf{E}) - \frac{\mu}{c} \frac{\partial}{\partial t}(\boldsymbol{\alpha} \cdot \mathbf{H}) &= 0.\end{aligned}\quad (69)$$

Using the standard Dirac matrices

$$\mathbf{f} = \begin{pmatrix} 0 & \boldsymbol{\alpha} \\ \boldsymbol{\alpha} & 0 \end{pmatrix}\quad (70)$$

the above system of equations can be rewritten in the familiar form of the Dirac equation:

$$\left[\mathbf{f} \cdot \nabla - \begin{pmatrix} \epsilon \mathbf{1} & 0 \\ 0 & \mu \mathbf{1} \end{pmatrix} \frac{1}{c} \frac{\partial}{\partial t} \right] \Psi = 0,\quad (71)$$

$$\Psi = \begin{pmatrix} (i \boldsymbol{\alpha} \cdot \mathbf{E}) \\ (\boldsymbol{\alpha} \cdot \mathbf{H}) \end{pmatrix} = \begin{pmatrix} iE_3 & i(E_1 - E_2) \\ i(E_1 + iE_2) & -iE_3 \\ H_3 & H_1 - iH_2 \\ H_1 + iH_2 & -H_3 \end{pmatrix}.\quad (72)$$

The substitution of $\Psi = \psi \exp(-i\omega t)$ into (65) produces

$$\left[\mathbf{f} \cdot \nabla + i \frac{\omega}{c} \begin{pmatrix} \epsilon \mathbf{1} & 0 \\ 0 & \mu \mathbf{1} \end{pmatrix} \right] \Psi = 0.\quad (73)$$

The obtained (Dirac) equation is the relativistic analog of (TISE) Equation (42). Following the same logic as in the nonrelativistic case, the analog of Equation (42) can be written explicitly, as follows

$$\left[\mathbf{f} \cdot \nabla + i \frac{\omega}{c} \begin{pmatrix} (1 - \frac{V-mc^2}{\hbar\omega}) \mathbf{1} & \mathbf{0} \\ \mathbf{0} & (1 - \frac{V+mc^2}{\hbar\omega}) \mathbf{1} \end{pmatrix} \right] \Psi = 0. \quad (74)$$

Detailed calculations leading to the known Dirac spectrum for the relativistic electron can be found in [87]. It should be noted that the idea to relate the Dirac and Maxwell's equations can be traced back to 1931. In this year, two fundamental papers were published in a sequel. One, [88]—by Oppenheimer and, another, [89]—by Laporte and Uhlenbeck. The results, just presented, in effect are abbreviated versions of cited papers only. Other similar papers will be discussed in the next section. The references [84,85] were used instead only because they are immediately linked with the Beltrami Equation (53). Since this equation links the Maxwell and Dirac equations with knots/links, we just outlined Einstein's quantization program.

6.3. Chladni Patterns in the Sky Caused by Axions of Dark Matter

It is believed that some pseudo-scalar particles, called “axions” [75,90], are directly responsible for measurable effects attributed to dark matter. Incidentally, the same particles are also believed to play an essential role in the properties of topological insulators [75,91]. In view of results presented in Sections 5 and 6, we would like to make several comments about the axion electrodynamics following, in part, [90].

We begin with the observation that the source-free Maxwell's electrodynamics possesses a special kind of symmetry—the electric-magnetic duality. That is to say, Equation (1) remain form-invariant with respect to transformations (duality rotations)

$$\begin{bmatrix} \mathbf{E}' \\ \mathbf{H}' \end{bmatrix} = \begin{pmatrix} \cos \xi & \sin \xi \\ -\sin \xi & \cos \xi \end{pmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}. \quad (75)$$

Here, ξ is an arbitrary angle. The presence of sources and/or axions formally destroys this symmetry. Nevertheless, it will eventually be restored in a way that is consistent with results of Section 5. The axions can be introduced as follows. The four-dimensional electromagnetic Lagrangian $\mathcal{L}_0$ is given in the standard form as

$$\mathcal{L}_0 = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - A_\mu J_e^\mu, \quad (76)$$

where, as usual, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, and the electric current $J_e^\mu = (c\rho_e, \mathbf{J}_e)$. Suppose that there are magnetic monopoles, then one can introduce the magnetic charge density ρ_m and the magnetic current $\mathbf{J}_m$, so that $J_m^\mu = (c\rho_m, \mathbf{J}_m)$. Maxwell's equations accounting for the electric and magnetic charges can be written now as

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= \mu_0 J_e^\nu, \\ \partial_\mu \tilde{F}^{\mu\nu} &= \mu_0 J_m^\nu / c, \\ \tilde{F}^{\mu\nu} &= \frac{1}{2} \epsilon^{\mu\nu\sigma\rho} F_{\sigma\rho}. \end{aligned} \quad (77)$$

Clearly, in the presence of magnetic monopoles along with electric charges, Maxwell's electrodynamics will regain the duality again. The situation changes when the Lagrangian $\mathcal{L}_0$ (with electric and magnetic charges) is extended by adding to it the axion-like interaction term

$$\mathcal{L}_\theta = -\frac{\kappa}{c\mu_0} \theta(x) \mathbf{E} \cdot \mathbf{H} \equiv \frac{\theta(x)\kappa}{4\mu_0} \tilde{F}^{\mu\nu} F_{\mu\nu}. \quad (78)$$

Here, $\theta = \theta(x)$ is the pseudoscalar field representing axions. The total Lagrangian becomes

$$\mathcal{L}^T = \mathcal{L}_0 + \mathcal{L}_\theta + \mathcal{L}_a = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} + \frac{\theta(x)\kappa}{4\mu_0} F_{\mu\nu} \tilde{F}^{\mu\nu} - A_\mu J_e^\mu + \mathcal{L}_a, \quad (79)$$

where

$$\mathcal{L}_a = \frac{1}{2} [\partial^\mu \theta \partial_\mu \theta - m^2 \theta^2]. \quad (80)$$

The minimization of $\mathcal{L}^T$ leads to the following set of equations (in terms of $\mathbf{E}$ and $\mathbf{H}$ variables)

$$\begin{aligned} \nabla \cdot (\mathbf{E} - c\kappa\theta\mathbf{H}) &= \rho_e/\varepsilon_0, \\ \nabla \cdot (c\mathbf{H} + \kappa\theta\mathbf{E}) &= c\mu_0\rho_m, \\ \nabla \times (c\mathbf{H} + \kappa\theta\mathbf{E}) &= \partial_t(\mathbf{E} - c\kappa\theta\mathbf{H})/c + c\mu_0\mathbf{J}_e, \\ \nabla \times (\mathbf{E} - c\kappa\theta\mathbf{H}) &= -\partial_t(c\mathbf{H} + \kappa\theta\mathbf{E})/c - \mu_0\mathbf{J}_m, \\ \square\theta &= -\frac{\kappa}{\mu_0 c} \mathbf{E} \cdot \mathbf{H} - m^2\theta. \end{aligned} \quad (81)$$

The system of Equation (75) possesses the electric-magnetic duality symmetry. Using (69) and rotating the fields $\mathbf{E}$ and $\mathbf{H}$, it is possible to find such an angle $\tilde{\xi}$ that $\mathbf{J}_m$ is eliminated. Thus, our arguments become independent of the existence of Dirac monopoles. Working with such calibrated fields, and applying the curl operator to the third and fourth equations in (75), while electing to work with the source-free electromagnetic fields ($\mathbf{J}_e = 0$), we end up with the following system of equations

$$\begin{aligned} \nabla \cdot \hat{\mathbf{E}} &= 0, \\ \nabla \cdot \hat{\mathbf{H}} &= 0, \\ \square\hat{\mathbf{E}} &= 0, \\ \square\hat{\mathbf{H}} &= 0, \\ (\square + m^2)\theta &= 0. \end{aligned} \quad (82)$$

Here, we put $c = 1$, $\hat{\mathbf{E}} = \mathbf{E} - \kappa\theta\mathbf{H}$, $\hat{\mathbf{H}} = \mathbf{H} + \kappa\theta\mathbf{E}$, and had selected only the null electromagnetic fields for which $\mathbf{E} \cdot \mathbf{H} = 0$. It is clear then that the first four equations are exactly the same as Equations (52)–(54). In view of results of Section 5, in which we discussed the Huygens equivalence, as well as in view of Equation (57), the last of the equations of (76) is the same as in the previously derived equation $(\nabla^2 + k^2)(\mathbf{r} \cdot \mathbf{v}) = 0$, while $\nabla \cdot \mathbf{v} = 0$ in (57) is the same as the first two of the equations in (76). Thus, we just demonstrated that the system of equations in (76) coincides exactly with the system of equations presented at the end of Section 5. Thus, axions are describing knotty Chladni (dark matter) patterns in the sky.

7. Attempt at Synthesis: Inclusion of Chirality

7.1. General Comments

Recall that the chirality originates from the lack of spatial reflection symmetry of the system in question. What does this subject have to do with what was said above? In this section, we make an attempt to complete Einstein's program of quantization that is outlined in the previous section. This means that Einstein's *purely electromagnetic particle* (PEP) should be made out of source-free Maxwellian fields obeying the Beltrami equation, which is our Equation (53). As described in Section 5, this equation is equivalent to the system of source-free Maxwell's Equation (1). The Beltrami Equation (53) is one of the major equations in contact geometry [33]. Moreover, beginning with our work [54], we demonstrated in detail that the Beltrami equation is involved in a description of all kinds of torus knots. It is known [92] that all torus knots are chiral. The simplest chiral torus knot is the trefoil knot. The chirality associated with the Beltrami equation can be spotted

already at the lowest level of instanton calculations [54,93]. The relevance of the chirality of torus knots to elementary particle physics was previously recognized by Sakharov in the late 1960s [94]. Since the Beltrami equation was used in previous sections, we now need to explain why uses of this equation still require more details.

We begin by posing a question: *If in Sections 2–4 and 6, we were able to map the source-free Maxwell’s equations into the Schrödinger (or Dirac) equation, was this mapping subjective, injective or bijective?* Fortunately, this question was studied already in [95–97]. The imposition of the requirement of the local gauge invariance on the Schrödinger (or Dirac) equation and the requirement of the relativistic covariance of the emerging source-free Maxwell’s equations as result of this imposition, makes such a mapping bijective.

The obtained answer still requires explanations. First, mathematically, the results of [95–97] can be classified only as a proof of existence. They need to be contrasted with the constructive results of Sections 2–6. In these sections, we built this mapping explicitly but injectively. Some practical applications resulting from this injectivity are described in [98]. Second, it is time now to demonstrate that the mapping is indeed bijective.

7.2. The Proof of Bijectivity of Maxwell-Dirac Mapping

Very fortunately, this task was completed for the massless Dirac equation. To our knowledge, there are no results for the massive Dirac equation, with one exception [99] known to us. If m is the mass parameter in the Dirac equation, then results formally obtainable in the $m \rightarrow 0$ limit are associated with neutrino physics [100]. From the very beginning of the invention of the neutrino theory by Pauli, it was known that, like the photon, the neutrino is also neutral. Thus, in the limit $m \rightarrow 0$, the charged Dirac particle becomes neutral. There is no difficulty to understand this result if $m = 0$. Nature, however, is more intricate. It prepared surprises for us. First, the mass of neutrino is small but nonzero. This fact has no explanation within the scope of the Standard Model. Second, there are three types of neutrinos—all having small different masses. There are no explanations within the Standard Model of why masses must be different or why masses should be small but nonzero. Third, while the Dirac electron has a nonzero electric charge, the massive neutrino(s) is neutral [100]. Because the neutrino is neutral, it interacts with the electromagnetic field quantum mechanically through radiation corrections only. The stated bijectivity [99] permits us to treat the Dirac mass as an adjustable parameter. The absence of charge for the **massive** neutrino presents much harder theoretical obstacles for the implementation of the Dirac-Maxwell bijectivity. This circumstance puts the whole Einstein’s quantization program into jeopardy. The situation is repairable to a some extent by noticing that both the neutrinos (ν_e, ν_μ, ν_τ) and the leptons (e, μ, τ) associated with them are fermions. Here, we are dealing with the peculiar situation of having massless photons (bosons) and massive neutrinos (fermions), both as neutral particles. Furthermore, both leptons and neutrinos are chiral particles (e.g., read below), and both have **the same** chirality (experimental fact: there is no theory for this). Thus, by using the Dirac mass m as a parameter in the Dirac equation, we are confronted with the following *problems*:

- (a) When treated as a parameter, at the stage the limiting $m \rightarrow 0$ process loses its continuity is when the massive electron e (or μ, τ leptons) abruptly loses its charge and becomes a chargeless massive neutrino.
- (b) Being chargeless, the neutrinos obtained from the massive Dirac-like leptons will remain as Dirac-like or Majorana-like fermions.

It is well known [100] that the Majorana-like fermions are neutral by design. Thus, it appears that all neutrinos should be of the Majorana type. The experiment does not demonstrate this directly, however, due to the effect of the neutrino mixing and neutrino oscillations. This circumstance makes the implementation of Einstein’s quantization program very difficult. The situation is repairable, nevertheless, as will be explained below. After these comments, we are ready to demonstrate in some detail the Maxwell–Dirac and the Maxwell–Majorana bijectivity.

Our readers might object at this point. If this is so, what is wrong with the results presented in Section 6.2? The answer is: These results are injective only. The bijective treatment in [99] is not readily suitable for the treatment of Majorana neutrinos. The bijective correspondence discussed in [95–97] cannot be used for the case of neutral Majorana neutrinos. Thus, more work is needed.

To begin, following [32] (p. 100), we notice that every component ψ_σ , $\sigma = 0 \div 3$, of the spinor ψ of the massive Dirac equation satisfies the massive K-G equation. From Section 5, we know that, using the Hadamard transformation, the massive K-G can be reversibly transformed into the D'Alembert equation. The massless K-G (that is the D'Alembert) equation is obtainable from the massless Dirac Equation ($\hbar = 1, c = 1$) – i

$$i \sum_{i=0}^3 \gamma^i \frac{\partial}{\partial x^i} \psi = 0. \quad (83)$$

Here, the Dirac matrices are given in their standard form [28]

$$\gamma^0 = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix}, \gamma^k = \begin{pmatrix} \mathbf{0} & \sigma^k \\ -\sigma^k & \mathbf{0} \end{pmatrix}, k = 1 \div 3, \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \mu, \nu = 0 \div 3, \quad (84)$$

and the Pauli matrices are also given in their standard form as

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (85)$$

Using these standard notations and choosing

$$\psi = \begin{pmatrix} iH^3 \\ iH^1 - H^2 \\ E^3 \\ iE^2 + E^1 \end{pmatrix} \quad (86)$$

in Equation (77) converts this (Dirac) equation into the system of Maxwell's Equation (1) [101]. The wave function (80) is not the only one converting the massless Dirac equation into the system of source-free Maxwell's equations. Following [102], we write down the set of all wave functions accomplishing the same task. These are

$$\begin{aligned} \chi_1 &= \begin{pmatrix} iH^3 \\ iH^1 - H^2 \\ E^3 \\ iE^2 + E^1 \end{pmatrix}, \chi_2 = \begin{pmatrix} -iE^3 \\ -iE^1 + E^2 \\ H^3 \\ iH^2 + H^1 \end{pmatrix}, \chi_3 = \begin{pmatrix} E^2 + iE^1 \\ -iE^3 \\ -H^1 - iH^2 \\ H^3 \end{pmatrix}, \\ \chi_4 &= \begin{pmatrix} iH^1 + H^2 \\ -iH^3 \\ -iE^2 + E^1 \\ -E^3 \end{pmatrix}, \chi_5 = \begin{pmatrix} -iH^3 \\ -H^1 - iH^2 \\ iE^3 \\ -iE^2 + iE^1 \end{pmatrix}, \chi_6 = \begin{pmatrix} iE^3 \\ iE^1 + E^2 \\ iH^3 \\ -iH^2 + iH^1 \end{pmatrix}, \\ \chi_7 &= \begin{pmatrix} iE^2 - E^1 \\ E^3 \\ -iH^1 - H^2 \\ iH^3 \end{pmatrix}, \chi_8 = \begin{pmatrix} +iH^2 - H^1 \\ H^3 \\ E^2 + iE^1 \\ -iE^3 \end{pmatrix}. \end{aligned} \quad (87)$$

The just described wave functions are connected with each other group-theoretically. Details are provided in [103].

7.3. Some Facts about Chirality

The four-components function ψ in (77) is splittable into two two-component (Weyl) equations typically used in the theory of the massless neutrino [28,100]. This is being achieved via the introduction of the chirality operator. In notations of [28], this operator can be defined as follows. First, let $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$, then the projection operators are defined via $P_L = \frac{1}{2}(1 - \gamma_5)$, $P_R = \frac{1}{2}(1 + \gamma_5)$, so that we obtain

$$\gamma_5 = -\begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \quad (88)$$

and

$$P_L P_R = P_R P_L = 0, P_L + P_R = 1, P_L^2 = P_L, P_R^2 = P_R. \quad (89)$$

If ψ is the Dirac spinor, then

$$\psi_L = P_L \psi, \psi_R = P_R \psi, P_L \psi_R = P_R \psi_L = 0, \quad (90)$$

and

$$\gamma_5 \psi_{L,R} = \pm \psi_{L,R}. \quad (91)$$

Clearly, because of (85), it makes sense to call γ_5 as the *chirality operator*. The eigenvalues ± 1 are the chirality eigenvalues and $\psi_{L,R}$ are the chiral projections. Using these results, the Dirac equation can now be rewritten in a manifestly chiral form. Indeed, since

$$\psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}, \text{ with } \psi_a = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \psi_b = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} \quad (92)$$

and, using (82), we can write

$$\psi_{L,R} = P_{L,R} \psi = \frac{1}{2}(1 \pm \gamma_5) \psi = \frac{1}{2} \begin{pmatrix} I & \mp I \\ \mp I & I \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}. \quad (93)$$

Thus,

$$\psi_L = \frac{1}{2} \begin{pmatrix} \psi_a - \psi_b \\ -\psi_a + \psi_b \end{pmatrix} = \begin{pmatrix} \phi \\ -\phi \end{pmatrix}, \text{ where } \phi = \frac{1}{2}(\psi_a - \psi_b) = \frac{1}{2} \begin{pmatrix} \psi_1 - \psi_3 \\ \psi_2 - \psi_4 \end{pmatrix}, \quad (94)$$

$$\text{and, } \psi_R = \frac{1}{2} \begin{pmatrix} \psi_a + \psi_b \\ \psi_a + \psi_b \end{pmatrix} = \begin{pmatrix} \chi \\ \chi \end{pmatrix}, \text{ where } \chi = \frac{1}{2}(\psi_a + \psi_b) = \frac{1}{2} \begin{pmatrix} \psi_1 + \psi_3 \\ \psi_2 + \psi_4 \end{pmatrix}. \quad (95)$$

With help of these results, Equation (77) can now be rewritten as

$$\begin{aligned} E\phi &= -\boldsymbol{\alpha} \cdot \mathbf{p} \phi, \\ E\chi &= +\boldsymbol{\alpha} \cdot \mathbf{p} \chi. \end{aligned} \quad (96)$$

The obtained result prompts us to introduce the helicity operator $\mathcal{H}$

$$\mathcal{H} = \frac{\boldsymbol{\alpha} \cdot \mathbf{p}}{|\mathbf{p}|}. \quad (97)$$

Thus, ψ_L is the eigenspinor representing the helicity $\mathcal{H} = +1$ for particles and $\mathcal{H} = -1$ for antiparticles. Accordingly, ψ_R is the eigenspinor representing the helicity $\mathcal{H} = -1$ for particles and $\mathcal{H} = +1$ for antiparticles. When $m > 0$, the chirality eigenspinors ψ_R and ψ_L no longer describe particles with a fixed helicity. In this case, the helicity is no longer a good quantum number.

The two-component theory of the Dirac-type electron neutrino ν_e implies that the spinor ψ_ν representing neutrino ν and participating in the weak interactions is **always** in the form

$$\psi_{\nu_e} = P_L \psi. \quad (98)$$

The Dirac electron associated with ν_e is also described by the left-handed spinor.

7.4. Some Facts about the Charge Conjugation, Parity, Time Reversal and the Majorana Condition

Our readers might have noticed already that the massive Dirac equation does not explicitly carry the information about the charge $\mp q$ of the electron/positron. The simplest way to detect the presence of a charge is to turn on the external electromagnetic field A_μ . That is, to study the equation

$$(i \sum_{i=0}^3 \gamma^i \frac{\partial}{\partial x^i} \psi \mp q \gamma^\mu A_\mu - m) = 0. \quad (99)$$

For the massive Majorana neutrino, the charge $\mp q$ is zero. Moreover, the bijective Maxwell–Dirac isomorphism [95–97] breaks down instantly as soon as the external electromagnetic field is on. This is especially true for neutral neutrinos. Thus, the major problem is the following.

If Einstein’s quantization program involves knotted/linked structures representing particles, then (a) how do these structures interact with the electromagnetic field and, (b) how stable are these structures with or without the presence of an electromagnetic field?

The answer to these questions is presented in Section 8. In the meantime, we return to topics of this subsection. The adjoint $\bar{\psi}$ to Dirac spinors are defined by

$$\bar{\psi}(x) \equiv \psi^\dagger(x) \gamma^0. \quad (100)$$

The charge conjugation C is defined as

$$\begin{aligned} \psi^C(x) &= \xi_C C \bar{\psi}^T(x) = -\xi_C \gamma^0 C \psi^*(x), \\ C \gamma_\mu^T C^{-1} &= -\gamma_\mu, C^\dagger = C^{-1}, C^T = -C, C(\gamma^5)^T C^{-1} = \gamma^5, \\ \xi_C C \bar{\psi}^T(x) &\rightarrow {}^C |\xi_C|^2 \psi(x), |\xi_C|^2 = 1. \end{aligned} \quad (101)$$

Under a parity P transformation (space inversion) $x^\mu = (x^0, \vec{x}) \rightarrow^P x_P^\mu = (x^0, -\vec{x})$. The spinor field $\psi(x)$ transforms as [100] (p. 52),

$$\psi^P(x_P) = \xi_P \gamma^0 \psi(x), \xi_P = \pm 1, \pm i. \quad (102)$$

Under a time reversal T transformation $x^\mu = (x^0, \vec{x}) \rightarrow^T x_T^\mu = (-x^0, \vec{x})$. The spinor field $\psi(x)$ transforms as [100] (p. 55),

$$\psi^T(x_T) = \xi_T \gamma^0 \gamma^5 C \bar{\psi}^T(x) = \xi_T \gamma^5 C \psi^*(x), |\xi_T|^2 = 1. \quad (103)$$

With these definitions, the Majorana condition is given by

$$\psi^C(x) = \psi(x). \quad (104)$$

The physics of this condition is easily checkable in the example of the electromagnetic current $J = q \bar{\psi}(x) \gamma^\mu \psi(x)$. Indeed, we have

$$\bar{\psi}(x) \gamma^\mu \psi(x) = -\psi(x)^T C^\dagger \gamma^\mu C \bar{\psi}^T(x) = \bar{\psi}(x) C (\gamma^\mu)^T C^\dagger \psi(x) = -\bar{\psi}(x) \gamma^\mu \psi(x) = 0.$$

Using (95) and (98), we obtain

$$\psi(x) = -\xi_C \gamma^0 C \psi^*(x) \quad (105)$$

and, because [100] (p. 48),

$$C = -i \begin{pmatrix} \mathbf{0} & \sigma^2 \\ -\sigma^2 & \mathbf{0} \end{pmatrix} \text{ so that } \gamma^0 C = -i \begin{pmatrix} \mathbf{0} & \sigma^2 \\ \sigma^2 & \mathbf{0} \end{pmatrix}. \quad (106)$$

By selecting $\xi_C = i$, Equation (99) acquires the following form

$$\psi(x) \equiv \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \sigma^2 \\ \sigma^2 & \mathbf{0} \end{pmatrix} \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} = \begin{pmatrix} \sigma^2 \phi_2^* \\ \sigma^2 \phi_1^* \end{pmatrix}, \phi_1 = \begin{pmatrix} \varphi_{11} \\ \varphi_{12} \end{pmatrix}, \phi_2 = \begin{pmatrix} \varphi_{21} \\ \varphi_{22} \end{pmatrix}. \quad (107)$$

In view of the definition of the charge conjugation, Equation (95), it is clear that this operation commutes with the Dirac operator defined (77). Therefore, its action on Equation (101) again produces the set of Maxwell's source-free equations. Evidently, Equation (101) is just the duality rotation (69). The presented result "apparently" completes Einstein's quantization program. The explanation of quotation marks will be given in Section 8. To finish this section, we need to explain the physical meaning of the duality rotation in the present case. To do so, we need to return to Ref. [102] and to point out that the duality rotation is one of the symmetries of Maxwell (or massless Dirac) equations. As it is known, with each symmetry is associated a conservation law. Accordingly, the just uncovered rotation symmetries lead to conservation laws carrying useful information about the system. At the physical level of rigor, such (duality) symmetries were discussed in detail in [101]. The discovered new invariants found their macroscopic uses in studies of electromagnetically chiral media [103], (p. 7). How the macroscopic chiral Maxwell's equations originate microscopically in chiral media is nicely explained in [104]. Macroscopically in the absence of chirality, we have

$$\mathbf{D} = \varepsilon \mathbf{E}, \mathbf{B} = \mathbf{H}. \quad (108)$$

In the presence of chirality we have instead [104], (p. 66) the Drude–Born–Fedorov (DBF) equations

$$\begin{aligned} \mathbf{D} &= \varepsilon(\mathbf{E} + \beta \nabla \times \mathbf{E}), \\ \mathbf{B} &= \mu(\mathbf{H} + \beta \nabla \times \mathbf{H}). \end{aligned} \quad (109)$$

It is clear that effects of chirality in this formalism are being controlled by the parameter β . There is yet another set of equations [105], (p. 69)

$$\begin{aligned} \mathbf{D} &= \varepsilon \mathbf{E} + i\zeta \mathbf{B}, \\ \mathbf{H} &= (1/\hat{\mu}) \mathbf{B} + i\eta \mathbf{E}. \end{aligned} \quad (110)$$

These are known as chiral constitutive equations (ChC eq.s). The second ones might be related to the first ones but are considered to be more fundamental. The chirality built into the neutrino formalism just described can be used to recover the ChC equations [105].

8. Completion of the Einstein Quantization Program

8.1. Conformal Invariance, Lie Sphere Geometry, Dupin Cyclides and Chladni Patterns

Bateman and Cunningham discovered in 1910 that Maxwell's equations (even with sources) are invariant under the action of the conformal group $SO(4,2)$ of the Minkowski space. A summary of their achievements is presented in [106]. From this reference, we also find a link to the paper by Gross [107]. He proved that solutions of source-free Maxwell's equations provide unitary representations of the conformal group of the Minkowski space and then extended these results to other massless relativistic equations using the Bargmann–Wigner [108] description of particles with discrete spin. In [106], these results were considerably simplified, culminating with a proof that any zero-mass, discrete spin representation of the Poincaré group admits an unitary representation of the conformal group. The group $SO(4, 2)$ plays the central role in the Lie sphere geometry. A concise introduction to this topic is given in our work [40], Section 7. In the field of our study, the Lie sphere geometry

reveals itself in the form of *Dupin cyclides*. These are invariants of the conformal group $SO(4, 2)$, that is of the Lie sphere geometry [109,110]. Within the context of hyperbolic (wave-like) equations, Dupin cyclides were discovered by Friedlander in 1946. Within the context of quantum mechanics, they were discovered in our work [40]. Brief but informative information about Dupin cyclides is contained in [111]. From this reference, it follows that the simplest Dupin cyclides are planes, spheres, cylinders, cones and tori. In [112], the method of designing more complicated Dupin cyclides from cylinders, tori and cones by applying the Möbius inversion is described in detail.

In the next subsection, we shall discuss some results coming from the use of the AdS-CFT correspondence. To our knowledge, no mention exists in physics literature about the connections between the AdS/CFT and the Lie sphere geometry. A collection of mathematically rigorous results on this topic is provided in [113]. Here, by combining some results from our work in [40,113], we provide the absolute essentials. We begin with the *Poincaré disc* model $\mathbf{D}^2$ of the hyperbolic space $\mathbf{H}^2$. Geodesics in this model are made of horocycles. These are circular segments whose both ends lie at the circular boundary S_∞^1 of $\mathbf{D}^2$. The boundary is considered as a “spatial infinity.” By appropriately choosing constants a, b, c, d in the Möbius transformation

$$f(z) = (az + b)/(cz + d), z \in \mathbf{C} = \mathbb{R}^2 \cup \{\infty\},$$

the disc model $\mathbf{D}^2$ can be transformed into the Poincaré half plane model of the hyperbolic space $\mathbf{H}^2$. This two-dimensional model is generalizable to higher dimensions, where it is known as the *hyperbolic ball model*. Thus, the two-dimensional combination $(\mathbf{H}^2, S_\infty^1)$ is being replaced by $(\mathbf{H}^{n+1}, S_\infty^n)$ in spaces of higher dimensions. The idea of AdS/CFT can already be observed in two dimensions. In it, the deformations of S_∞^1 (of the boundary) leads to the Virasoro algebra and, hence, to two-dimensional conformal field theories. By the analogy with two dimensions, it was expected in higher dimensions that the hyperbolic-like behavior in anti-de Sitter spacetime (AdS) is affected by (linked with) the conformal field theory (CFT) residing at its boundary. The Mostow rigidity theorem makes the deformations of the conformal three-sphere S_∞^3 impossible to perform. To bypass this difficulty [113], the hyperbolic space $\mathbf{H}^n$ is replaced by the anti-de Sitter space—the Einstein space of constant negative curvature. This space is obtained from the vacuum Einstein equations with an added (negative) cosmological constant. By doing so, the hyperbolic sphere at infinity S_∞^n is replaced by the space of conformally flat solutions $\text{Ein}^{n,1}$ of Einstein’s equations. Being conformally flat, these are conformally equivalent to the Minkowski spacetime. That is the spacetime described by the conformal symmetry group $SO(4, 2)$. The objects living in such spacetimes are invariants of the Lie sphere geometry [113,114].

Being armed with such information, we would like to reobtain the results of Bateman, Cunningham and Ranada [15,16] from the point of view of Dupin cyclides. In Section 5, we had demonstrated that the presence of Beltrami equations (53) implies the existence of Chladni patterns in the form of torus knots. By definition, torus knots are living on surfaces of toruses, which are on Dupin cyclides. Since equations (53) are equivalent to the set of source-free Maxwellian equations (1) and since the Dupin cyclides are invariants of the conformal group $SO(4, 2)$, we just proved the main results of Bateman and Cunningham. Since in his works [15,16] Ranada was also using source-free Maxwell’s equations, the just obtained results are consistent with those by Ranada. A few additional comments are required, however. They will be supplied in the rest of this section. The results, just described, lead to Chladni patterns, while Randada’s knots are *real* (not empty spaces). The question arises: Is there way to correct this situation? The answer is: “Yes, there is”. The correction was made in our works [54], and in [55] (Sections 1–4 and Appendix B). The obtained results still admit a different and profoundly important interpretation.

8.2. Statistical Mechanics of the Dirac Equation, Differential Geometry of Elastic Curves and Kirkhoff's Elastic Rods

Hundreds upon hundreds of papers and many books have been written on the topic of (the classical analog of) electrons. The latest representative books are [115–118]. While the number of papers, even the most recent ones, is overwhelmingly large, the contribution into this topic that we already made differs from all of these since our purpose was to apply the Dirac results (Dirac equation) to polymer solutions, e.g., read [119–122]. Polymers typically are being modelled as random walks on the lattice. The Dirac polymer chains (semiflexible polymers) differ from Schrödinger's (fully flexible) polymer chains by the energetical requirement for each consecutive step of the random walk. There are two options for the consecutive lattice step: (a) to continue to go straight; (b) to go sideways, or even completely back. If (a) and (b) are not regulated energetically, then we are dealing with the classical random walk. If going sideways is energetically controlled, we are dealing with semiflexible chains. The connection between such a *biased* random walk and the Dirac equation was independently studied for the first time in [123,124], problem 2.6. In the case of semiflexible polymers, there is a one-to-one correspondence between the rigidity of these polymers and the mass m of the Dirac particle: for $m = 0$ the random walk degenerates into a straight line (the neutrino path), while for $m \rightarrow \infty$, we are dealing with the fully flexible polymers, that is, with the nonrelativistic (Schrödinger-like) limit of the Dirac propagator. To facilitate our readers' understanding, we would like now to provide some needed facts from polymer physics and differential geometry.

Specifically, random walks on the lattice are characterized by the end-to-end distribution (moment generating) function $G_0(\mathbf{p}, N) = \int d^d \mathbf{r} e^{i\mathbf{p} \cdot \mathbf{r}} G_0(\mathbf{r}, N)$. For the random (Gaussian chains) walks of N steps on the, say, the square/cubic lattice whose local step is l , typically we need to consider the walk starting at the origin $\mathbf{0}$ and ending at some distance $\mathbf{r}$ from the origin. In such a case, the distribution function $G_0(\mathbf{p}, N)$ is known to be: $G_0(\mathbf{p}, N) = \exp(-\mathbf{p}^2 l N / 2d)$. Here, d is the lattice dimensionality. With such a generating function, it is of interest to calculate the averages (the moments), e.g., $\langle \mathbf{R}^2 \rangle = \int d^d \mathbf{r} r^2 G_0(\mathbf{r}, N)$, etc. Using the generating function $G_0(\mathbf{p}, N)$ in three dimensions, we obtain: $\langle \mathbf{R}^2 \rangle = Nl$. More physically interesting is to evaluate the experimentally measurable scattering function $S_0(\mathbf{p}, N)$. For the fully flexible (Gaussian) polymer chains, it is given by $S_0(\mathbf{p}, N) = \frac{1}{N^2} \int_0^N d\tau \int_0^N d\tau' G_0(\mathbf{p}, |\tau - \tau'|) = \frac{2}{x^2} (x - 1 + e^x)$, $x = (\mathbf{p}^2 / 2d) Nl$. To extend this result to semiflexible chains, we notice that the Laplace transform $G_0(\mathbf{p}, s)$ of $G_0(\mathbf{p}, N)$ is known to be (in the appropriately chosen system of units) given by $G_0(\mathbf{p}, s) = (\mathbf{p}^2 + s)^{-1}$, where s is the Laplace variable conjugate to N . Our readers immediately recognize in $G_0(\mathbf{p}, s)$ the Green's function for the massive Klein–Gordon (K-G) operator (Euclidean version). We can obtain the Dirac propagator from the K-G propagator following Dirac's logic. This then will enable us to obtain the (Dirac) generating function $G_D(\mathbf{p}, N)$ for the semiflexible polymers and, therefore, the scattering function $S_D(\mathbf{p}, N)$. Such an approach is logical but is not physically illuminating. This happens because the statistical properties of both the Gaussian and the Dirac polymer chains can be computer simulated. Moreover, they can be experimentally measured by analyzing the light (or neutron) scattering data for $S_D(\mathbf{p}, N)$. This circumstance leads us to study the transition from a discrete to continuous limit for various models of polymer chains relying on results of computer simulations and light scattering experiments. Both are based on these models. Although for the Gaussian chains there is only one (random walk) model, there is a *countable infinity* of models [125] for semiflexible chains and the *Dirac chain is just one model out of this infinity*. Other than Dirac, in polymer physics, the most popular is the Kratky–Porod (K-P) model [126]. It is based on applications of Euler's elastica to polymer physics. Details are given below. It will be compared with the Dirac model because: (a) it will help us to extend the limits of the Maxwell–Dirac bijectivity; (b) it will help us to model the Dirac fermions on computers in terms of random walks with rigidity. Before making such a comparison, several results are helpful to discuss. For instance, the Dirac

propagator—the solution of the problem 2.6. from Feynman–Hibb’s book [124] (presented in detail in [119])—is read as

$$\frac{1}{2}G_D(\mathbf{p}, N) = \cosh(mEN) + \frac{1}{E} \sinh(mEN), E^2 = 1 - \frac{\mathbf{p}^2}{m^2}. \quad (111)$$

Its Laplace transform is

$$\frac{1}{2}G_D(\mathbf{p}, N) = \frac{s + m}{\mathbf{p}^2 + s^2 - m^2}. \quad (112)$$

In the limit $m \rightarrow 0$, this result can be rewritten as

$$\frac{1}{2}TrG_D(\mathbf{p}, N) = Tr \frac{i\tilde{\boldsymbol{\alpha}} \cdot \mathbf{p} + s}{(i\tilde{\boldsymbol{\alpha}} \cdot \mathbf{p} + s)(-i\tilde{\boldsymbol{\alpha}} \cdot \mathbf{p} + s)}. \quad (113)$$

Here, the Pauli matrices $\tilde{\boldsymbol{\alpha}} = (\sigma_1, \sigma_2, \sigma_3)$ are the same as in (61). The trace operation Tr reflects the averaging over directions of the ends of the polymer chain [119]. Evidently, the Dirac propagator $G_D(\mathbf{p}, s) = (-i\tilde{\boldsymbol{\alpha}} \cdot \mathbf{p} + s)^{-1}$ is now replacing the Klein–Gordon propagator $G_0(\mathbf{p}, s) = (\mathbf{p}^2 + s)^{-1}$ introduced previously for the description of Gaussian chains. Since the effects of rigidity are contained in the mass parameter m , the just presented result should be slightly modified to account for the parameter m . For this purpose, following our work [121], we have to use the Euclideanized version of the 3+1 dimensional Dirac propagator, that is, we have to replace $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$ (where $g^{\mu\nu}$ is the diagonal Minkowski matrix with $\{1, -1, -1, -1\}$ on the diagonal) with its Euclidean version $\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu}$. This leads us to

$$G_D(p, m) = \frac{-i}{\gamma^\mu p_\mu + m} = i \frac{\gamma^\mu p_\mu - m}{-p^2 + m^2}, \quad p^2 = \mathbf{p}^2 + s^2. \quad (114)$$

This happens because, instead of the Fourier transforms (used in the high energy physics), in the polymer physics, we have to use the Laplace transforms. To bring this result in correspondence with (106), it is sufficient to make yet another replacement, $m \rightarrow im$, and to take the trace. This produces:

$$TrS_D(\mathbf{p}, m) = \frac{m}{\mathbf{p}^2 + s^2 - m^2}. \quad (115)$$

The inverse Laplace transforming this result leads to $G_D(\mathbf{p}, |\boldsymbol{\sigma} - \boldsymbol{\sigma}'|, m)$. This result should be used instead of its Gaussian/Schrodinger version $G_0(\mathbf{p}, |\tau - \tau'|)$ in calculations of the scattering function. The results of such a calculation are presented in [121]. They were compared against all others coming from all known models of polymer chains. The scattering function for the Dirac chain happen to fit ideally with both the numerical and real experimental data. It was already included in the standard toolkit manual (for polymer experimentalists) distributed world-wide [127] by the Paul Scherrer Institute (Switzerland). In this manual and, therefore in the polymer literature, it is known as the “Kholodenko worm” (because semiflexible polymers are called “wormlike” in the polymer community) or “Kholodenko-Dirac”.

Armed with these results, we are now ready to talk about how the discussed results can be reproduced differential-geometrically. Since this topic was discussed in great detail in our works [37,54,55,119], this again allows us to be brief. The task is reduced to the discussion of some path integrals equivalently representing the Dirac propagator. With an analogy of the results just described, we begin with the path integral for the K-G propagator $G_0(\mathbf{p}, s) = (\mathbf{p}^2 + s)^{-1}$. Such a path integral and its calculation is described in great detail in

the book by Polyakov [128] (pp. 151–169). Equivalently, one can use [129]. The end-to-end distribution function, the K-G propagator, is formally given by

$$G(r, m^2) = \int \frac{\mathcal{D}[\mathbf{r}(\tau)]}{\mathcal{D}[f(\tau)]} \exp\{-\mu_0 i \int_0^1 d\tau \sqrt{\langle \mathbf{v} \cdot \mathbf{v} \rangle}\}, \mu_0 = m^2 \sim s, \mathbf{v} = \frac{d\mathbf{r}}{d\tau}. \quad (116)$$

The exponent in (110) is nothing but the length of the curve. By design, it is written in a reparametrization-invariant form. That is, it is invariant under the transformations of the type

$$\mathbf{r}(\tau) \rightarrow \mathbf{r}(f(\tau)), f(0) = 0, f(1) = 1, \frac{df}{d\tau} > 0. \quad (117)$$

This result has its origins in differential geometry and reflects the fact that the total length of the curve is an invariant scalar. In the case of semiflexible polymers, the following (Kratky–Porod) path integral is typically used [126].

$$G(\mathbf{v}(N), \mathbf{v}(0)) = \int_{\mathbf{v}=\mathbf{v}(0)}^{\mathbf{v}=\mathbf{v}(N)} \mathcal{D}[\mathbf{v}(\tau)] \exp\{-\frac{\eta}{2} \int_0^N d\tau \langle \dot{\mathbf{t}} \cdot \dot{\mathbf{t}} \rangle\} \prod_{\tau} \delta(\mathbf{v}^2(\tau) - 1), \dot{\mathbf{t}} = \frac{d\mathbf{v}}{d\tau}. \quad (118)$$

Here, the exponent is not a reparametrization invariant. It represents Euler’s elastica [130]. As before, here, N is the number of monomer units in the polymer chain, η is the rigidity parameter (related to mass m in the Dirac propagator). The constraint $\mathbf{v}^2(\tau) = 1$ reflects the differential-geometric requirement for the natural parametrization of the curve. If the natural parametrization is used in (110), then the exponent in (110) becomes a number and the path integral is unnecessary. In (112), the expression in the exponent under the “time” integral is the square of the local curvature of the curve as a function of the trajectory parameter τ . Mathematically, the path integral (112) describes a Brownian motion (diffusion) on the sphere (of the unit radius). Therefore, for short “times”, the Gaussian result

$$\langle [\mathbf{v}(\tau) - \mathbf{v}(0)]^2 \rangle = \frac{2}{\eta} \tau \quad (119)$$

follows. This result is consistent with the (already presented) result: $\langle \mathbf{R}^2 \rangle = Nl$, $N \rightarrow \tau$. Accordingly, the statistical average in (113) is performed with $G_0(\mathbf{r}, N)$ (up to a change of scale), as before. Since $\mathbf{v}^2(\tau) = 1$, we can rewrite (113) as

$$\langle 2 - 2\mathbf{v}(\tau) \cdot \mathbf{v}(0) \rangle = \frac{2}{\eta} \tau$$

yielding upon exponentiation

$$\langle \mathbf{v}(\tau) \cdot \mathbf{v}(0) \rangle = \exp(-\frac{\tau}{\eta}). \quad (120)$$

A closed analytical form for the Kratky–Porod propagator (112) is the same as for the rigid rotator in quantum mechanics, that is

$$G(\mathbf{v}(N), \mathbf{v}(0)) = \sum_{l=0}^{\infty} \exp\{-(l(l+1)N)\} \sum_{m=-l}^l Y_{lm}(\theta, \phi) Y_{lm}^*(\theta_0, \phi_0). \quad (121)$$

Working with a propagator is not only more difficult technically [126,131] than with the Dirac propagator but, since every three-dimensional curve is characterized by its curvature and torsion, and the torsion is absent in the exponent of the path integral (112), it is surprising that the comparison between the scattering functions $S_D(\mathbf{p}, N)$ (exact analytic result [121]) for the Dirac chains and that $S_{K-P}(\mathbf{p}, N)$ for the Kratky–Porod chains (the exact calculations for $S_{K-P}(\mathbf{p}, N)$ are not available, [126,131]) and produce closely similar results [132]. Since the discretized models of the Dirac propagator [124,125] involve the study of the biased random walks and since the random walks on the sphere, Equation

(112), still cannot be considered as biased walks, the question arises: is it possible to improve the Kratky–Porod propagator, equation (112), so that the improved propagator becomes that for the Dirac chains? This improvement, evidently, should involve the inclusion of some kind of bias terms in the exponent in (112). Without any reference to the Equation (105), such an improvement was indeed attempted in [133] with the result of Equation (3) of [133], which indeed coincides with our (105). Provided that constants in (3) are appropriately reinterpreted, as it is conducted in detail in (Section 8.4.3 of our book [37]), the complete agreement between Equation (3) and (105) is achieved. The obtained results causes us to make some important comments. These are presented in the next subsection.

8.3. Differential Geometry of Elastic Curves and Kirchhoff's Elastic Rods Emergence of the Elastic Torus Knots

From differential geometry, it follows that the bias in Equation (3) of [133] makes the polymer conformation three dimensional. This happens because we must take into account that [119]:

Theorem 1. A curve $\vec{\gamma}(s)$ in the three dimensional space is fully determined by its curvature $\mathfrak{R}(s)$ and torsion $\mathfrak{T}(s)$.

Theorem 2. The torsion $\mathfrak{T}(s)$ of planar curves is zero.

These results are correct only in a flat space. If the curve is embedded into the space of nonzero curvature, the above theorems must be amended [134]. Moreover, they must be amended if the curve is replaced by a beam of, say, the circular cross-section α . More complications will follow if the cross-section is represented by some scalar function $\alpha(s)$ of its location along the curve $\vec{\gamma}(s)$. Complications associated with such thickening of a curve are described by the theory of Kirchhoff rods [135]. Developments of rod theory depend upon the condition of rod being stretchable or nonstretchable. If we are dealing with nonstretchable rods (open or closed), it is convenient to introduce the concept of the *centerline*. It is the unit speed curve along the axis of the rod. In the context of semiflexible polymers, the presence of this centerline is accounted by the constraint $\prod_s \delta(\mathbf{v}^2(s) - 1)$ in the path integral, as was performed for the Kratky–Porod model (112).

The treatment of polymer chains of finite thicknesses is described in the book by Yamakawa [126]. In this work, we shall rely, however, on rigorous mathematic results. In particular, we begin with

Theorem 3 ([136]). Every torus knot type is realized by a smooth closed elastic rod centerline.

If $\vec{\gamma}(s)$ is the centerline of a uniform symmetric (that is α is s – independent) of the Kirchhoff elastic rod, then it is providing an extremum of the functional $\mathcal{F}[\vec{\gamma}(s)]$ defined by

$$\mathcal{F}[\vec{\gamma}(s)] = \lambda_1 \int_{\gamma} ds + \lambda_2 \int_{\gamma} ds \mathfrak{T}(s) + \lambda_3 \int_{\gamma} ds \mathfrak{R}^2(s), \text{ with } \lambda_3 \neq 0. \quad (122)$$

Here, $\lambda_1, \lambda_2, \lambda_3$ are some constants (Lagrangian multipliers) and $\mathfrak{R}$ and $\mathfrak{T}$ are the curvature and torsion in the Serret–Frenet (S-F) equations given by

$$\frac{d\vec{\gamma}}{ds} = \mathbf{e}_1, \frac{d\mathbf{e}_1}{ds} = \mathfrak{R}\mathbf{e}_2, \frac{d\mathbf{e}_2}{ds} = -\mathfrak{R}\mathbf{e}_1 + \mathfrak{T}\mathbf{e}_3, \frac{d\mathbf{e}_3}{ds} = -\mathfrak{T}\mathbf{e}_2, \quad (123)$$

while the mutually orthogonal vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ compose the S-F moving frame. The following theorem is of major physical importance (to be explained below and in the next subsection).

Theorem 4. *An elastic rod centreline of a non-constant curvature can intersect itself only at the origin of the natural system of cylindrical coordinates. If this does not occur, and the centerline is closed, then it is embedded and lies on an embedded torus of revolution.*

Building on information supplied by this theorem, the next theorem makes things perfect for applications.

Theorem 5. *Given any relatively prime integers k, n , such that $|k/n| < 1/2$, there exists a unique smooth closed elastic rod of constant torsion with the knot type of a (k, n) torus knot.*

The results presented in this subsection allow us to reach the following conclusions.

1. The source-free Maxwell equations (1) can be rewritten in the form of the Beltrami equation. This was achieved in our works [54] and in [55] (Sections 1–4 and Appendix B). These results allow us to bypass uses of the Fourier transformed form of the Beltrami Equation (53). In our book ([37], p. 3), it was explained that the Beltrami equation is just the London equation of superconductivity. This observation leads naturally to the discussion of knotted vortex filaments.
2. By repeating arguments of Section 5, that is, by applying the operator *div* to both sides of the Beltrami equation $\nabla \times \mathbf{F} = k\mathbf{F}$, and by assuming that $k = k(x, y, z)$, we obtain $\mathbf{F} \cdot \nabla \mathbf{k} = 0$. Let now $\vec{\gamma}(s) = \{x(s), y(s), z(s)\}$ be a trajectory on the surface $k(x, y, z) = k = \text{const}$. This then implies $\frac{d}{ds} \kappa \{x(s), y(s), z(s)\} = v_x \kappa_x + v_y \kappa_y + v_z \kappa_z = \mathbf{v} \cdot \nabla \kappa = 0$. Now, we have to take into consideration that $\mathbf{F} = \mathbf{v}$ is in our case. Thus, the “velocity” $\mathbf{v}$ is always tangential to the surface $\text{const} = \kappa(x, y, z)$. We also have to assume that the vector field $\mathbf{v}$ is not vanishing on the surface $\text{const} = \kappa(x, y, z)$. This is possible only if the surface is torus T^2 . The field lines of $\mathbf{v}$ on T^2 should be closed if const is a rational number. This condition is essential for the closed filament $\vec{\gamma}(s)$ to be a torus knot.
3. The facts just stated are entirely compatible with Theorems 3–5. The just presented summary of results allows us to present our next, and the final subsection of this section.

8.4. From Kirchhoff to Dirac. Solution of Electron Problem by Methods of Projective Relativity

In Section 8.2, we mentioned that the amount of papers and books about the nature of the electron is overwhelmingly large. This means only that the proposed models of the electron were always incomplete in some way or another. It is hoped that the results presented below may close the existing gaps, thus making the microscopic picture of the electron complete. More on this topic is also presented in the next section.

The following results are based on the isomorphism proven by Kirchhoff between the description of statics and dynamics of rods with thickness, as well as statics and dynamics of rigid bodies [135]. Our uses of Kirchhoff isomorphism are based on the utilization of Theorems 3–5. They serve to demonstrate the usefulness of the (Kirchhoff) functional (116) in quantum mechanics. Specifically, its uses in the path integral leads to the Dirac equation. By establishing this fact below, in this subsection, the task of establishing the Maxwell–Dirac correspondence discussed in previous sections is going to be completed. Doing so is inseparably linked with fundamentally clarifying the meaning of the electron as a quantum particle.

To begin, we shall use some results from our paper [119]. In particular, we begin with the presentation of the system of F-S Equation (117) in the form known in the dynamics of the rigid body, that is, we write

$$\frac{de_i}{ds} = \sum_{j=1}^3 \omega_{ij} e_j \quad (124)$$

where

$$\omega_{ij} = \begin{pmatrix} 0 & \Re & 0 \\ -\Re & 0 & \Im \\ 0 & -\Im & 0 \end{pmatrix}, \quad (125)$$

and $e_i \cdot e_j = \delta_{ij}$. According to the book by Arnol'd [137], only in three dimensions the operations $\times$ and $\wedge$ are equivalent. This observation makes the F-S vectors e_i act as elements of the Clifford algebra. That is, the property $e_1 \wedge e_2 = -e_2 \wedge e_1$ allows us to write the defining anticommutator of this algebra as

$$\{e_i, e_j\}_+ = 2\delta_{ij}. \quad (126)$$

Incidentally, the same conclusion was reached independently by Martin [138]. The set of Equation (118) can be equivalently rewritten as

$$\frac{de_i}{ds} = \sum_{i,j}^3 \varepsilon_{ijk} B_j e_k, \quad (127)$$

where the components of the “magnetic” field $\mathbf{B}$ are defined now as $B_1 = -\Im, B_2 = 0, B_3 = -\Re$. With Feynman’s path integral method of obtaining quantum mechanical results in mind, it is instructive to reobtain Equation (121) variationally from the action functional $\mathcal{A}(t_f, t_i)$ defined by [139].

$$\mathcal{A}(s_f, s_i) = \frac{i}{2} \int_{s_i}^{s_f} ds [\tilde{\omega}^{kl} e_k \dot{e}_l - H(\{e_i\})], \quad \tilde{\omega}^{kl} = \delta^{kl}. \quad (128)$$

In (122), the Hamiltonian is given by

$$H(\{e_i\}) = - \sum_{i,j,k=1}^3 \varepsilon_{ijk} B_i e_j e_k \quad (129)$$

The action $\mathcal{A}(t_f, t_i)$ has an imaginary i in front since the quantum mechanical path integrals come with i in front of the action functional and the S-F Equations (118) or (121) does not contain the imaginary parts, e.g., the K-P path integral (112). Suppose that the Equation (121) is solved. Then, by multiplying (121) from the left by e_i , summing over the index i and remembering the noncommutativity of $e_i, \dot{e}_i$ and also of the antisymmetry of the Kronecker symbol ε_{ijk} , we obtain

$$\mathcal{A}(s_f, s_i) = i \sum_{i=1}^3 \int_{s_i}^{s_f} ds e_i \dot{e}_i. \quad (130)$$

In view of (117), we realize that the vector $\mathbf{e}_1$ is directed along the rod’s centerline so that the mutually perpendicular vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ are rotating in the plane perpendicular to $\mathbf{e}_1$, while $\mathbf{e}_1$ is moving along the centerline. The obtained result should be made compatible with the action in the exponent in Equation (110) since the total action is the sum of both actions coming from (110) and (124). In Equation (110), the worldline is “living” in four-dimensional Minkowski space-time, while in (124), the wordline is “living” in three-dimensional Euclidean space. To make these actions compatible with each other, first of all, we notice that both actions are reparametrization-invariant. Therefore, we can put the time limits $s_i = 0, s_f = 1$ in (124) in accordance with (110). Then, we have to study the extension of S-F equations to four dimensions. It happens that: (a) the S-F equations exist in spaces of any dimensionality and any signature [139]; (b) although the equivalence of $\times$ and $\wedge$ is lost, the noncommutativity property survives, especially upon quantization [137]; (c) the problem of the dynamics of a charged particle in the presence of joint electric and magnetic fields is described by the S-F equations in the four-dimensional Minkowski space [140] (this observation makes the topic of this subsection compatible

with that in the previous subsection). In Minkowski four-space, let the vector $\mathbf{e}_0$ be directed along the four-dimensional world line (four-velocity); then, the vectors $\mathbf{e}_i, i = 1 \div 3$ must rotate in the hyperplane perpendicular to $\mathbf{e}_0$. Therefore, to account for this fact, the constraint

$$\eta_{\mu\nu} e^\mu \dot{x}^\nu = 0 \quad (131)$$

should be imposed. Following [139], we choose $\eta_{\mu\nu} = \text{diag}\{-1, +1, +1, +1\}$. At the same time, owing to [138], we know that the vectors e^μ must be treated noncommutatively, e.g., see Equation (132), below. To move forward, it is helpful to utilize some facts by studying the action $S[x(\tau)]$ in the exponent of (110)

$$S[x(\tau)] = -m \int_0^1 d\tau \sum_{\mu=0}^3 \sqrt{\dot{x}^\mu \dot{x}_\mu} \equiv \int_0^1 d\tau \mathcal{L}[x(\tau)]. \quad (132)$$

Since

$$\frac{m \dot{x}_\mu(\tau)}{\sqrt{\dot{x}^\mu \dot{x}_\mu}} = p_\mu \quad (133)$$

the Hamiltonian for this action is zero. Indeed

$$H = p_\mu \dot{x}_\mu + m \sqrt{\dot{x}^\mu \dot{x}_\mu} = 0. \quad (134)$$

At the same time,

$$p_\mu p^\mu + m^2 = 0. \quad (135)$$

This equation is yet another constraint. For the compatibility of (125), (127) and (129), it is useful to rewrite the constraint (125) as

$$\eta_{\mu\nu} e^\mu p^\nu = 0 \quad (136)$$

while the action (126) is rewritten as

$$S[x(\tau), p(\tau)] = \int_0^1 d\tau [p_\mu \dot{x}_\mu - \frac{e}{2} (p_\mu p^\mu + m^2)]. \quad (137)$$

Next, for compatibility with (131), we elevate the action (124) to four dimensions. This is permissible in view of [141]. Accordingly, the three-element Clifford algebra (120) has now acquired the fourth element. To account for Minkowski space, we have to replace (120) by

$$\{e_\mu, e_\nu\}_+ = 2\eta_{\mu\nu}. \quad (138)$$

With this anticommutator, it is clear that the noncommutative elements e_μ in fact are the Dirac and the gamma matrices [142]. However, from the information presented in Section 7 previously, we know that there is also an independent “chiral” gamma matrix γ_5 , thus the dimensionality of the space of e'_μ s (or γ'_μ s) is **five**. The question immediately emerges: Is there any additional physics behind this fifth dimension? Surprisingly, in 1919 this question was already independently raised by Kaluza, who published his findings in 1921. His results were elevated to the quantum level by Klein in 1926 [143]. Both of these authors were concerned with the usefulness of the fifth dimension for the unification of gravity and electromagnetism. Unfortunately, further refinements demonstrated that many results of the Kaluza–Klein theory, and especially, the predicted mass of the electron, are in dramatic disagreement with the experiment [144]. Quite independently, many mathematicians and physicists such as Pauli, Dirac and Schrödinger, had chosen another path to the fifth dimension (e.g., the study of dS and AdS spaces that are mentioned in Section 8.1, etc.) and the process is still ongoing. The initial stages of these alternative developments are beautifully summarized in the book by O. Veblen. This direction of research is known in the literature as the “Projective theory of relativity” [145]. The latest important developments are presented in [146] and summarized in [147]. For the purposes

of this work, we only need to use the results of reference [148] by Dirac. It is mentioned as Ref. 12 in [147]. Although [146] is the precursor of Dirac's theory of constraints [149], it has its own great merit, especially since it utilizes Eisenhart's theory of contact transformations [150], which is the major theme of our book [37].

We just have discussed (128) the situation, when the dynamical system has a Lagrangian but not a Hamiltonian. What information does this, in fact, provide? Dirac studied this phenomenon in his book [149] on the treatment of dynamical systems with constraints. The situation when both the Lagrangian and the Hamiltonian are zero is much more exotic. Exactly this case is studied in [148]. By studying the examples from this reference, not only we shall understand the origins of the fifth dimension in our problem, but we shall also learn how to deal with situations when they happen. We begin with the study of the lightcone equation

$$c^2 ds^2 = c^2 dt^2 - \sum_{i=1}^3 dx_i^2. \quad (139)$$

It is related to the action (126). We choose the signature of Minkowski space-time in accordance with the fact that, in Dirac's paper, we obtain the Lagrangian

$$\mathcal{L} = -mc(c^2 \dot{x}_0^2 - \sum_{i=1}^3 \dot{x}_i^2)^{\frac{1}{2}}. \quad (140)$$

Suppose now that the particle has a zero rest mass, $m = 0$. In such a case, the Lagrangian $\mathcal{L}$ is zero. That is

$$\mathcal{L} = c^2 \dot{x}_0^2 - \sum_{i=1}^3 \dot{x}_i^2 = 0. \quad (141)$$

This is surely the case studied in projective geometry. This is because, since the multiplication by some constant λ of the Equation (135) changes nothing, we are dealing with the equivalence class made of the infinity of Lagrangians, which differ from each other by the actual value of λ .

However, since such λ containing expressions represents the Lagrangian (the equivalence class of), the associated momenta are obtained as usual. That is

$$\begin{aligned} p_0 &= \lambda \frac{\partial \mathcal{L}}{\partial \dot{x}_0} = 2\lambda c^2 \dot{x}_0, \\ p_r &= \lambda \frac{\partial \mathcal{L}}{\partial \dot{x}_r} = -2\lambda \dot{x}_r, r = 1, 2, 3. \end{aligned} \quad (142)$$

Employing (135) combined with the Equation (136) produces, after some algebra, the Hamiltonian, which is also zero

$$H = p_0^2 - c^2 \sum_{r=1}^3 \dot{x}_r^2 = 0. \quad (143)$$

Let us now look at the Equation (133) as an equation in a projective space. That is, it is more convenient to rewrite (133) as

$$c^2 ds^2 - c^2 dt^2 + \sum_{r=1}^3 dx_r^2 = 0. \quad (144)$$

This homogenous equation is manifestly fifth-dimensional. By keeping this in mind, let us consider its generalization yielding another Lagrangian equal to zero in the five-dimensional projective space. That is, this time, we are discussing classical electrodynamics in a projective space. We have

$$\mathcal{L} = -mc(c^2 \dot{x}_0^2 - \sum_{r=1}^3 \dot{x}_r^2)^{\frac{1}{2}} + e/c \sum_{r=1}^3 A_r \dot{x}_r - eA_0 \dot{x}_0 - \dot{x}_5 = 0. \quad (145)$$

When A_0 and A_r are zero, to be in agreement with (135), we have to assume that $\dot{x}_5$ is proportional to m . Then, (139) is zero to be in accord with (138). Following the same steps as in derivations in (136), we obtain:

$$\begin{aligned} p_0 &= \lambda \frac{\partial \mathcal{L}}{\partial \dot{x}_0} = -\lambda (mc^2 \frac{dx_0}{ds} + eA_0), \\ p_r &= \lambda \frac{\partial \mathcal{L}}{\partial \dot{x}_r} = \lambda (m \frac{dx_r}{ds} + eA_r), \\ p_5 &= \lambda \frac{\partial \mathcal{L}}{\partial \dot{x}_5} = -\lambda. \end{aligned} \quad (146)$$

By keeping in mind how the Hamiltonian (137) was obtained, by repeating the same steps in the present case, we obtain yet another zero Hamiltonian:

$$(p_0 - eA_0 p_5)^2 - c^2 \sum_{r=1}^3 [p_r + e/c A_r p_5]^2 - m^2 c^4 p_5^2 = 0. \quad (147)$$

Again, let now A_0 and A_r be zero. Then, if we put $c = 1$, we reobtain the constraint in Equation (129). This time it can be reinterpreted as a homogenous equation in the projective space $\mathbf{RP}^4$. Without a loss of generality in this space, we fix, say, the fifth coordinate in $\mathbf{R}^5$ and interpret it as a mass m in accord with (139); this provides us with the mass shell condition leading to the massive Klein–Gordon equation at the quantum level. Interestingly enough, the constraint (130) is compatible with the constraint (129) under conditions just described. To demonstrate this, we have to place the constraint (130) into $\mathbf{RP}^4$. Then, to avoid mistakes, following [139], we relabel e_i as ξ_i so that the anticommutator (120) is rewritten as

$$\{\xi_\mu, \xi_\nu\} = \hbar \eta_{\mu\nu}, \quad \{\xi_5, \xi_5\} = \hbar \quad (148)$$

so that

$$\xi_\mu = (\hbar/2)^{\frac{1}{2}} \gamma_5 \gamma_\mu, \quad \xi_5 = (\hbar/2)^{\frac{1}{2}} \gamma_5, \quad \{\gamma_\mu, \gamma_\nu\} = -2\eta_{\mu\nu}. \quad (149)$$

At the quantum level, the constraints (129) and (130) are transformed into the following equations

$$(\hat{p}^2 + m^2)\psi = 0, \quad (150)$$

$$(\hat{p}^\mu \xi_\mu + m \xi_5)\psi = 0. \quad (151)$$

By multiplying (145) by ξ_5 from the left and, taking into account (142) and (143), the standard massive Dirac equation is obtained

$$(\hat{p}^\mu \gamma_\mu + m)\psi = 0. \quad (152)$$

Finally, we can assemble the total action in the path integral for the Dirac propagator. It is given by

$$S[x(\tau), \xi(\tau), e, \chi] = \int_0^1 d\tau [p_\mu \dot{x}_\mu - \frac{e}{2}(p_\mu p^\mu + m^2) + i \xi_\mu \dot{\xi}^\mu + i \xi_5 \dot{\xi}^5 + i \chi (p_\mu \xi^\mu + m \xi^5)]. \quad (153)$$

The same action (up to the signature) can be found in the book by Polyakov [129], (p. 224). By writing the action in exactly the same form as on page 224, we have to use Equation (131) to obtain

$$\frac{\delta S[x, p]}{\delta p_\mu} = \dot{x}_\mu - e p_\mu = 0 \quad (154)$$

and use it back in (131), implying

$$S[x] = \int_0^1 d\tau \frac{1}{2} [e^{-1} \dot{x}^\mu \dot{x}_\mu - e m^2]. \quad (155)$$

When this result is substituted back to (147), we obtain the action

$$S[x(\tau), \zeta(\tau), e, \chi] = \int_0^1 d\tau \left\{ \frac{1}{2} [e^{-1} \dot{x}^\mu \dot{x}_\mu - em^2] + i\zeta_\mu \dot{\zeta}^\mu + i\zeta_5 \dot{\zeta}^5 + i\chi(p_\mu \zeta^\mu + m\zeta^5) \right\} \quad (156)$$

in the form given by Polyakov. While Polyakov uses the concepts of supersymmetry in his calculations, in this work, we used the concepts of projective relativity to arrive at the same results. Only through uses of projective relativity it is possible to connect the differential-geometric results of the previous subsection with the quantum mechanical results of this subsection. Moreover, mathematically, the Equation (149) is consistent with the Equation (124). In both cases, the final total (combined) action used in the path integral contains dynamical variables defined on respective orbits determined from the respective equations of motion. These are obtained variationally from the respective actions, Equations (122) and (131).

9. Instead of a Discussion

9.1. Work by Varlamov

We have just demonstrated the place of the electron in the Maxwell–Dirac correspondence. Thanks to a series of works [151–153] by Varlamov, it possible to find the place of the obtained results inside the Standard Model of particle physics. The Standard Model consists of 17 fundamental particles. Only two of these—the electron and the photon—were known 100 years ago. Already, at this level of knowledge, they represented two major groups of particles: the fermions and the bosons. The fermions are the building blocks of matter. The 12 fermions known today are split into six quarks and six leptons. The electron is a lepton. Protons and neutrons are not a part of the Standard Model because they are made out of quarks. All bigger particles and all matter are made out of just quarks and leptons. Only five bosons (including the photon) are responsible for all of the interactions between matter (leptons and quarks). They carry three of the four fundamental forces in nature: the strong force, the weak force and the electromagnetism. The fourth force is gravity. Varlamov’s achievement lies in demonstrating that the mass spectrum of all mesons (made of two quarks) and baryons (made of three quarks) is possible to obtain with the help of just one formula containing the mass of electron m_e as an input. The formula

$$m^s = m_e \left(l + \frac{1}{2} \right) \left(\tilde{l} + \frac{1}{2} \right), s = |l - \tilde{l}| \quad (157)$$

fits the experimental data with a margin of errors less than or at most 1%. The formula is obtained as some Lorentz group representation characterized by the pair $(l, \tilde{l})$, where each l and $\tilde{l}$ are nonnegative pairs of numbers (m, n) , $(\frac{m}{2}, n)$, $(m, \frac{n}{2})$, or $(\frac{m}{2}, \frac{n}{2})$. In this case, if there is a division by two, the respective numbers are odd.

In Section 8.1, we introduced the Dupin cyclides. These are invariants of the Lie sphere geometry. The Möbius group is a subgroup of the Lie sphere geometry isomorphic to the Lorentz group [40]. Therefore, all results of Varlamov are transferable to results under the umbrella of the Lie sphere geometry. Thus, the Dupin cyclides are also invariants of the Lorentz group, as was previously established by Friedlander in 1946 [40]. Since topologically the electron is a torus knot—the trefoil, living on the simplest Dupin cyclide—the rest of the short lived particles are also living on the Dupin cyclides and all of them have the trefoil knot (that is the electron) as their precursor in accord with (129). As is known in knot theory [55,63], all knots are obtainable either from trefoils or from figure 8 knots. However, the figure 8 knot is a hyperbolic knot. Therefore, it cannot serve as a precursor and must be excluded from consideration for reasons explained in Section 6.1.

9.2. From Projective Relativity to Knotty Geometrodynamics

It is worth mentioning that the results of the Lie sphere geometry also have their origins in projective geometry. They, as well as the results of projective relativity (touched upon in

Section 8.4) serve well for the Minkowski and curved spaces of the Minkowski signature. The purpose of both the Kaluza–Klein and the projective relativity is to provide a logically consistent way to unify gravity and electromagnetism. The alternative logically consistent way to unify gravity and electromagnetism was made by Rainich in 1925 [154]. His results were rediscovered by Misner and Wheeler [155] and the ongoing development had acquired the name “geometrodynamics” [156]. We would like to provide some details explaining why we believe that this approach to the unification of gravity and electromagnetism is the most promising. First, as noticed already in our book [37], (p. 97), the existing non-Abelian gauge theories, and also gravity, viewed as one of these non-Abelian gauge theories [70,71], are having profound difficulties in including the extended geometrical objects into the formalism of these theories. It is hard for us to tell whether Rainich was aware of these difficulties but in his theory [154], he discussed only *source-free Maxwellian fields*, self-consistently creating some gravitational background. That is to say, in Einstein’s field Equation (59), the source-free electromagnetism is placed on the right-hand side in these equations as the source of gravity. In their paper [155], the authors claimed that the dynamics of the source-free electric and magnetic fields is described in terms of the “rate of change of curvature of pure Riemannian geometry, and nothing more”, page 530. On the same page, we find “... the electromagnetic field leaves an imprint upon the metric that is so characteristic, that from that imprint one can read back to find out all that one needs to know about the electromagnetic field.” The authors were concerned about the meaning of charges in Rainich theory. As we know now, the answer was provided by Ranada [15,16], and was elaborated by many authors afterwards, myself included. Furthermore, as it is demonstrated in this work, *only source-free electromagnetic fields make the probabilistic and optical interpretations of quantum mechanics equivalent*. Because the Maxwell–Dirac isomorphism is the major topic of this work, the topological and Diracian nature of Ranada charges provides the answer to the question about the charges in geometrodynamics: all charges, without exception, are of topological origin. These are either individual nonhyperbolic knots or nonhyperbolic links. However, Rainich–Misner–Wheeler’s “*Already Unified Field Theory*” (AUFT) had one significant drawback. It had not found a place for the null electromagnetic fields. However, as it was extensively discussed in this work, it is because they are essential for the existence of Ranada’s knotty charges. Furthermore, the stability of these knotty structures (or their ability to disintegrate) is also linked with the existence of null fields [80,157]. Moreover, stabilities of knots require applications of contact geometry and topology [37,158]. Recall (Section 8.4) that this discipline was used not only in our works on Ranada’s knots [54,55] but also in Dirac’s work [148] on Lagrangian/Hamiltonian dynamics in projective spaces. In his studies, Dirac employed the paper by Eisenhart [150] on contact transformations. This usage of contact geometry and topology is not some artefact, because the source-free Maxwellian field equations can be rewritten as Beltrami field equations, as we demonstrated in Appendix B of [55]. Furthermore, because of this, based on the results of this work, we can state that the Beltrami field equations can be translated one-to-one into the Dirac equation, thus providing yet another outlet for connecting contact geometry and topology with quantum mechanics. In quantum mechanics, the Beltrami equation is known as the London equation of conventional superconductivity [37], (p. 3), which is inseparably linked with the Meissner’s effect. Going back to AUFT, the place for null fields in AUFT was found in a beautiful paper by Geroch [159]. Subsequently, many authors simplified and improved his results, e.g., in [160].

9.3. Cosmological Significance/Meaning of the Obtained Results

9.3.1. Dirac–Kerr–Newman Electron

Recently, Burinskii, in a series of papers, e.g., [161], demonstrated that “the Dirac equation can be naturally incorporated into the Kerr–Schild formalism as a master equation controlling the Kerr–Newman geometry” (describing the Kerr–Newman solution of Einstein equations for the charged rotating black hole). As a result, the Dirac electron

acquires an extended space–time structure of the Kerr–Newman geometry: a singular ring of Compton size and a twistorial polarization of the gravitational and electromagnetic fields. The behavior of this Dirac–Kerr–Newman system in weak and slowly changing electromagnetic fields **is determined by the wave function of the Dirac equation and is indistinguishable from the behavior of the Dirac electron**. The wave function of the Dirac equation plays the role of an “order parameter” in this model, which controls the dynamics, spin polarization and the twistorial nature of space–time. This information is useful to supplement with results by Dalhuisen [73,162] that connect twistors with electromagnetic knots. In addition, in [163,164], the connection between twistors, geometric algebra and cyclides is discussed.

9.3.2. Geometrodynamics of Neutrino, Majorana Fermions and Knot Theory

Rainich–Misner–Wheeler’s AUFT inspired others to think about the development of the analogous unified field theory of gravitation and neutrino, e.g., in [165,166]. This idea seemed meaningful until the discovery of neutrino oscillations [100], when everybody believed that the neutrino was massless. By an analogy with a photon, the massless neutrino does not carry charge but it is a fermion, while the photon is a boson. Nevertheless, by analogy with superconductivity in which Cooper pairs are bosons, the neutrino theory of light was created [167]. After discoveries of neutrino masses, this theory was abandoned. However, it is commonly believed that neutrinos are Majorana fermions, that is, these are neutral fermions with very small masses [100]. Since they are neutral, uses of the concepts of gauge invariance [95] for establishing the analog of the Dirac–Maxwell correspondence cannot be used. Nevertheless, for the nonexisting massless neutrinos described by the massless Majorana equation, very recently, the correspondence analogous to the Dirac–Maxwell was found [168]. In view of the results of Section 5, the presence of mass in the Majorana equation can be eliminated so that the correspondence found in [169] can be extended to the massive case. Because of this, one can think about the relationship between the Majorana fermions and knots. This thought encounters many difficulties. First, there is no analog of the Varlamov’s formula (129) for the Majorana fermion, second, the path integrals for Dirac and Majorana electrons are not the same, e.g., see [168]. Therefore, the differential-geometric results of Sections 8.3 and 8.4 cannot be used. The results of knot theory can still be used, however, on Maxwell’s side of the Majorana–Maxwell correspondence, because the source-free set of Maxwell’s equations is exactly equivalent to the Beltrami equation, as explained above. On the Majorana side, connections with knot theory are already known in two dimensions [170]. This happens because the knot-theoretic Arf invariant [63,171] is connected with the 2D-using model. Moreover, the solution of this model in two dimensions involves the uses of massive Dirac fermions [131]. Using the logic behind the Equations (86)–(89) (see Section 7.3), Shankar demonstrated in his book [172] (p. 153), that the action for the 2D Dirac fermion can be represented by the sum of actions for two decoupled Majorana fermions:

$$\int dx d\tau \bar{\Psi}(\partial + m)\Psi = \frac{1}{2} \int dx d\tau [\bar{\eta}(\partial + m)\eta + \bar{\chi}(\partial + m)\chi]. \quad (158)$$

Here, $\Psi = \frac{1}{\sqrt{2}}(\chi + i\eta)$, $\bar{\Psi} = \frac{1}{\sqrt{2}}(\bar{\chi} + i\bar{\eta})$, $\bar{\chi} = \chi^T \sigma_2$, $\bar{\eta} = \eta^T \sigma_2$, and, $\partial = \sigma_2 \partial_\tau + \sigma_1 \partial_x$. The same result is presented in the book by Mussardo in [173], (p. 337), in a somewhat more focused form.

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