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Neutron Stars in the Context of $f(\mathbb{T},\mathcal{T})$ Gravity

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Abstract: In this work, we investigate the existence of neutron stars (NS) in the framework of $f(\mathbb{T},\mathcal{T})$ gravity, where $\mathbb{T}$ is the torsion tensor and $\mathcal{T}$ is the trace of the energy–momentum tensor. The hydrostatic equilibrium equations are obtained, however, with p and ρ quantities passed on by effective quantities $\bar{p}$ and $\bar{\rho}$, whose mass–radius diagrams are obtained using modern equations of state (EoS) of nuclear matter derived from relativistic mean field models and compared with the ones computed by the Tolman–Oppenheimer–Volkoff (TOV) equations. Substantial changes in the mass–radius profiles of NS are obtained even for small changes in the free parameter of this modified theory. The results indicate that the use of $f(\mathbb{T},\mathcal{T})$ gravity in the study of NS provides good results for the masses and radii of some important astrophysical objects, as, for example, the NS of low-mass X-ray binary in NGC 6397, the millisecond pulsar PSR J0740+6620 and the GW170817 event. In addition, radii results inferred from the Lead Radius Experiment (PREX-2) can also be described for certain parameter values.

Keywords: general relativity; modified gravity; neutron stars

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1. Introduction

In recent years, there have been a growing number of ideas exploring modifications and alternative formulations of General Relativity (GR) emerging from different contexts. In fact, GR is a theory well tested, providing an interesting description of the space–time nature as a dynamical stage where physical phenomena takes place. In parallel to the advances in GR, the quantization of the gravitational field remains an open problem. With respect to this issue, it was pointed out that the action for gravity should be constructed with higher-order curvature terms in the context of renormalization at one loop level [1]. In the literature, there are some formulations of gravity where the usual Einstein–Hilbert action is supplemented by higher-order curvature terms, as for example in the context of the $f(R)$ theory, in which case the Ricci scalar R in the action is replaced by a general function $f(R)$ [2].

On the other hand, there are questions concerning the content of energy and matter in the universe that, at the moment, are not satisfactorily explained in the scope of standard theories. The observed rotation curves of galaxies [3] and the "missing mass" of galaxy clusters [4] suggest the dark matter hypothesis, while the accelerated expansion of the universe observed today can be interpreted as an effect of the so-called dark energy [5,6]. Unexpectedly these observations reveal that the ordinary baryonic matter corresponds to only 4% of content of energy of the universe while the dark matter and dark energy correspond to 20% and 76%, respectively. In this sense, there are studies considering the possibility of modified theories of gravity which may help to alleviate the need for dark components of energy of the universe beyond the scope of GR.

The late-time acceleration of the universe can be interpreted under two points of view. In the first one, it is introduced a dark energy sector in the energy content of the universe through a type of field. In the second one, the gravitational field itself is modified. In addition, there may be combinations of both approaches depending on the couplings between gravitational and non-gravitational sectors of theory [7–10]. Thus, it is expected that different formulations of gravity imply that standard results in astrophysics suffer modifications. Compact objects as neutron stars (NS), have been studied considering effects of such modifications [11–20]. NS in the context of $f(R)$ gravity were studied in [21–23] and in $f(R,T)$ gravity in the papers [24–28]. In common, all of these works have considered effects on NS due to the modification of the gravitational field that include extra terms in the action. In the scheme of non-conservative gravity, the modification of the gravitational field can be performed through a reinterpretation of the conservation law, as was considered in the papers [29,30] (for a review on non-conservative theories of gravity, see [31]). Usually, the non-conservation of the stress-energy tensor is proportional to the matter density and pressure themselves. For this reason, an environment such as a compact object like a NS turns out to be an appealing laboratory for testing such theories.

In the context of modified theories of gravity, the so-called $f(\mathbb{T},\mathcal{T})$ gravity is a class of such theories, free of ghosts and instabilities which, when applied to cosmological problems, leads to interesting results [32]. In this formulation, the action depends on the torsion scalar $\mathbb{T}$ and on the trace of the energy–momentum tensor $\mathcal{T}$. As in the case of $f(\mathbb{T})$, gravity where the action is an arbitrary function of the torsion, in $f(\mathbb{T},\mathcal{T})$ gravity, the action is a arbitrary function of both the trace of the energy–momentum tensor and the torsion scalar.

In this paper, we study an important context, not yet explored in the literature, that are the implications of the $f(\mathbb{T},\mathcal{T})$ gravity on NS. In particular, we obtain the mass–radius relation of NS in the context of this modified gravity and compare our results with recent astrophysical observations and experiments.

This work is organized as follows. In Section 2 we expose a summary of the $f(\mathbb{T},\mathcal{T})$ gravity. In Section 3, we derive the equations describing static, spherically symmetric stars in this modified theory of gravity. In Section 4 we present our results and in Section 5 we close with our final remarks.

2. Gravitational Field Equations of $f(\mathbb{T},\mathcal{T})$ Gravity

Given a line element describing a space-time we want to study

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{AB}e^A{}_\mu e^B{}_\nu dx^\mu dx^\nu \tag{1}$$

where $g_{\mu\nu}$ and $\{e^A_\mu\}$ are, respectively, the metric tensor and the components of the tetrad associated to space-time geometry, and $\eta_{AB} = diag(1,-1,-1,-1)$ is the Minkowski metric. The signature $(+\ -\ -\ -)$ and geometrized units, that is, $G = c = 1$, will be taken into account. In GR, we assume that gravity is associated with the curvature of the space-time and, thus, we use the Levi–Civita's connection

$$\overset{\circ}{\Gamma}{}^\rho{}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}) \tag{2}$$

to compute quantities associated with the curvature such as the Ricci scalar, R, that is present in the GR's action.

On the other hand, in teleparallel theory one assumes that gravity is associated to the torsion of the space-time and thus the Weizenbock's connection

$$\Gamma^{\lambda}_{\mu\nu} = e_A{}^{\lambda}\partial_\mu e^A{}_\nu = -e^A{}_\mu \partial_\nu e_A{}^{\lambda} \tag{3}$$

is used to construct quantities associated with the torsion, as the torsion scalar $\mathbb{T}$ that appears in the teleparallel gravity action. In the modified teleparallel theories, it is assumed that the action depends on a arbitrary function of $\mathbb{T}$. In our case, we are going to consider a modified action given by [32]

$$\mathbb{S} = \int d^4x \ e\left[\frac{\mathbb{T} + f(\mathbb{T}, \mathcal{T})}{16\pi} + \mathcal{L}_m\right], \tag{4}$$

where e is the determinant of the tetrads $e = \det(e^A{}_\mu) = \sqrt{-g}$ and $\mathcal{T} = g^{\mu\nu}T_{\mu\nu}$ is the trace of the energy–momentum tensor $T_{\mu\nu}$, which can be obtained from the Lagrangian for the matter distribution $\mathcal{L}_m$ in the following way

$$T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_m - 2\frac{\partial \mathcal{L}_m}{\partial g^{\mu\nu}} \, . \tag{5}$$

Let us assume that the function $f(\mathbb{T}, \mathcal{T})$ is given by

$$f(\mathbb{T}, \mathcal{T}) = \overline{\omega}\, \mathbb{T}^n \, \mathcal{T} - 2\Lambda \, , \tag{6}$$

where $\overline{\omega}$, n, and Λ are arbitrary constants, specifically $\overline{\omega}$ can be interpreted as a coupling constant of geometry with matter fields, n is a pure number (assumed to be unity here) and Λ can be recognized as the cosmological constant as discussed in [32,33]. Notice that since this approach reduces to general relativity, this model is compatible with classical tests of GR.

We are interested in matter that can be described by a perfect fluid, so that $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = -pg_{\mu\nu} + (p + \rho)u_\mu u_\nu, \tag{7}$$

where p is the pressure and ρ is the energy density of the fluid. By varying the action from Equation (4) with respect to the tetrad we find the following field equation

$$G_{\mu\nu} = 8\pi T^{eff}_{\mu\nu} \, , \tag{8}$$

where the effective energy–momentum tensor $T^{eff}_{\mu\nu}$ is

$$T^{eff}_{\mu\nu} = g_{\mu\nu}\left[\frac{\Big(-\overline{\omega}(\rho - 3p) + 2\Lambda\Big)}{16\pi} + \frac{\overline{\omega}p}{8\pi}\right] + T_{\mu\nu}\left(1 + \frac{\overline{\omega}}{8\pi}\right). \tag{9}$$

Calculating the covariant derivative of the energy–momentum tensor given by Equation (7), we obtain the following result

$$\nabla_\mu T_\nu{}^\mu = \frac{1}{\Big(4\pi + (1/2)\overline{\omega}\Big)}\left\{\frac{\overline{\omega}}{4}(\partial_\nu \mathcal{T}) - \frac{\overline{\omega}}{2}\partial_\nu p\right\}, \tag{10}$$

where the covariant derivative is defined with the Levi–Civita connection. In a cosmological context, Equation (10) can be associated to creation or destruction of matter throughout the universe evolution. As discussed in [26], the interpretation of creation or destruction of matter particles in the NS level encounters difficulties in a static framework, as occurs in the study of the hydrostatic equilibrium expression, i.e, the Tolman–Oppenheimer–Volkof equation. These difficulties arise because, in the cosmological context, we can assume that the Universe as a whole is an open thermodynamic system where the particle number N is time dependent, which allows us to construct a balance equation associated to creation of particles [34]. However, in systems with no time evolution we still do not know how to interpret the non-conservation of $T_{\mu\nu}$. Additionally, it usually implies in the presence of a fifth force and non-geodesic trajectory for free particles. Naturally, results that depend on such input would also be modified correspondingly. However, this is not the case analyzed in the present paper. In the next section, we use Equations (8) to (10) to obtain and analyze the mass–radius relation of NS in the context of modified teleparallel gravity.

3. Stellar Structure Equations

In this section, we discuss some of the main procedures that leads to the deduction of the hydrostatic equilibrium equation in the context of $f(\mathbb{T},\mathcal{T})$ gravity.

To study compact stars, such as NS, magnetars, and other astrophysical structures, we assume these objects as being homogeneous, static (no rotation), isotropic, and spherically symmetric [35]. Therefore, we must use the appropriate metric in a convenient coordinate system that describes the object being studied. The most general metric describing the space-time under consideration is given by the line element

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin\theta^2 d\phi^2), \tag{11}$$

where ν and λ are radial functions that we want to determine based on the field Equation (8). Thus, using Equation (11) and substituting appropriately into Equation (8),we obtain the following results

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi\left\{\left[\frac{\left(-\overline{\omega}(\rho-3p)+2\Lambda\right)}{16\pi} + \frac{\overline{\omega}p}{8\pi}\right] + \rho\left(1+\frac{\overline{\omega}}{8\pi}\right)\right\} = 8\pi\bar{\rho}, \tag{12}$$

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = -8\pi\left\{\left[\frac{\left(-\overline{\omega}(\rho-3p)+2\Lambda\right)}{16\pi} + \frac{\overline{\omega}p}{8\pi}\right] - p\left(1+\frac{\overline{\omega}}{8\pi}\right)\right\} = 8\pi\bar{p}, \tag{13}$$

$$\begin{aligned} &\frac{e^{-\lambda}}{4r}\left[2\left(\lambda'-\nu'\right) - \left(2\nu'' + \nu'^2 - \nu'\lambda'\right)r\right] \\ &= -8\pi\left\{\left[\frac{\left(-\overline{\omega}(\rho-3p)+2\Lambda\right)}{16\pi} + \frac{\overline{\omega}p}{8\pi}\right] - p\left(1+\frac{\overline{\omega}}{8\pi}\right)\right\} = 8\pi\bar{p}, \end{aligned} \tag{14}$$

where the prime denotes a derivative with respect to the radial coordinate r. The quantities $\bar{\rho}$ and $\bar{p}$ are, respectively, the effective energy density and effective pressure, defined as

$$\bar{\rho} = \rho + \frac{\overline{\omega}\rho}{16\pi} + \frac{5\overline{\omega}\,p}{16\pi} + \frac{\Lambda}{8\pi}, \tag{15}$$

$$\bar{p} = p + \frac{\overline{\omega}\rho}{16\pi} - \frac{3\overline{\omega}\,p}{16\pi} - \frac{\Lambda}{8\pi}, \tag{16}$$

and they are such that $T^{eff}_{\mu\nu} = -\bar{p}g_{\mu\nu} + (\bar{p}+\bar{\rho})u_\mu u_\nu$. In addition to the field equations, we can also consider the conservation Equation (10) in $f(\mathbb{T},\mathcal{T})$ gravity so that we have a complete set of equations to be solved. In the case we are studying, Equation (10) has the form as follows

$$-p' - \frac{\nu'}{2}(\rho + p) = \frac{1}{\left(4\pi + (1/2)\overline{\omega}\right)} \left\{ \frac{\overline{\omega}\rho'}{4} - \frac{5\overline{\omega}\, p'}{4} \right\}. \quad (17)$$

Redefining the function $\lambda(r)$ as

$$e^{-\lambda(r)} = 1 - \frac{2M(r)}{r}, \quad (18)$$

and rearranging Equations (12) and (17), we obtain the equations required to describe static spherically symmetric stellar structures in $f(\mathbb{T}, \mathcal{T})$ gravity theory, which are given by

$$\frac{dM(r)}{dr} = 4\pi r^2 \bar{\rho}, \quad (19)$$

and

$$\frac{d\bar{p}}{dr} = -\frac{M\bar{\rho}}{r^2}\left[1 + \frac{\bar{p}}{\bar{\rho}}\right]\left[1 + \frac{4\pi r^3 \bar{p}}{M}\right]\left[1 - \frac{2M}{r}\right]^{-1}. \quad (20)$$

In the next section, we show some results obtained by solving Equations (19) and (20) for realistic EoS of NS.

4. Results

In this section, we present the results obtained from the solution of the field equations in the context of $f(\mathbb{T}, \mathcal{T})$ modified theory of gravity applied to NS.

As an input to the stellar hydrostatic equilibrium equations, we use two realistic EoS obtained from a relativistic mean field approach. Firstly, we consider the IU-FSU [36] parametrization because it is able to explain reasonably well both nuclear [37] and stellar matter properties [38]. We then compare the IU-FSU results with the ones obtained with a stiffer EoS calculated with a model of coupling of mesons and quarks, the quark–meson coupling (QMC) model [39]. (For the EoS with the QMC model, we refer the reader to refs. [39–43].) It is well known that a stiffer EoS leads to a bigger NS maximum mass in contrast to a softer one. In fact, using the EoS QMC as an input to the stellar equilibrium equations yields a maximum mass greater than 2.0 $M_\odot$, and, therefore, we want to verify that we obtain the same qualitative behavior for macroscopic properties (such as mass and radius) with parameterizations that are substantially different. For the NS crust, we use the full BPS [44] EoS.

After defining the EoS, some boundary conditions are required to solve the Equations (19) and (20) along the radial coordinate r, from the center towards the surface of the star. At the star's center $r = 0$ we take

$$M(0) = 0\,;\ \ \bar{\rho}(0) = \bar{\rho}_c\,;\ \ \bar{p}(0) = \bar{p}_c, \quad (21)$$

where we have implicitly $\bar{p}_c$ as a function of $\bar{\rho}_c$. The radius of the star $(r = R)$ is determined as the point where the pressure vanishes, i.e., $p(R) = 0$. At this point, the interior solution connects softly with the Schwarzschild vacuum solution, indicating that the potential metrics of the interior and the exterior metric are related as $e^{\nu(R)} = \frac{1}{e^{\lambda(R)}} = 1 - 2M/R$, being M the total mass of the star. As we shall discuss later, we are going to assume $\Lambda = 0$, since it has very little effects on the mass–radius profiles.

Let us discuss and compare our results with recent astrophysical observations and nuclear physics experiments. At first, the NS in low-mass X-ray binary (LMXB) NGC 6397, depicted as a green shaded area in all figures, provides a constraint at 68% confidence level over the possible values of the masses and corresponding radii of the NS [45,46]. Similarly, the millisecond pulsars are among the most useful astrophysical objects in the Universe for testing fundamental physics, because they impose some of the most stringent constraints on high-density nuclear physics in the stellar interior [47]. Recent measurements coming from the *Neutron Star Interior Composition Explorer* (NICER) mission reported pulsar observations for canonical (1.4 $M_\odot$) and massive (2.0 $M_\odot$) NS. The mass measurement and radius estimates provided for these objects, are 11.80 km $\leq R_{1.4} \leq$ 13.1 km for the 1.4 $M_\odot$ NS PSR J0030+0451 (horizontal line segment in red color shown in all Figures) and 11.60 km $\leq R \leq$ 13.1 km for a NS with mass between 2.01 $M_\odot \leq M \leq$ 2.15 $M_\odot$ PSR J0740+6620 (the rectangular region in orange color shown in all figures). However, the authors of Ref. [48] used the recent measurement of neutron skin on ^{208}Pb by PREX-2 to constrain the radius of NS, which leads to a prediction of the radius of the canonical 1.4 $M_\odot$ of 13.25 km $\lesssim R_{1.4} \lesssim$ 14.26 km (horizontal line segment in green color shown in all figures). Likewise, we also compare our results with two massive stars that had been discovered in 2010 and 2013, namely, PSR J1614+2230 [49] with mass 1.97 ± 0.04 $M_\odot$ (horizontal line in blue color shown in all figures) and PSR J0348+0432 [50] with mass 2.01 ± 0.04 $M_\odot$ (horizontal line in pink color shown in all figures). Our results are discussed in the next paragraphs.

We modelled the function $f(\mathbb{T}, \mathcal{T})$ according to Equation (6). This function model has already been used in recent works as, for example, in [32,33]. We explore the values of the parameter $\overline{\omega}$ which range from -0.2 to 0.2. On the other hand, we check that the Λ parameter has no significant effect on the mass–radius profiles of NS, since it appears as a constant in the $f(\mathbb{T}, \mathcal{T})$ function that we have chosen. In fact, as can be seen in Equations (15) and (16), while Λ only contributes additively, $\bar{\omega}$ couples to ρ and p, which is responsible for amplifying its effects in an very compact environments, such as neutron stars. Therefore, we use $\Lambda = 0$. Note that we recover the GR solution from $f(\mathbb{T}, \mathcal{T})$ theory by assuming that $\overline{\omega} = \Lambda = 0$. These plots are represented by the continuous purple lines in the Figures.

In Figure 1, we show the effects of $f(\mathbb{T}, \mathcal{T})$ theory on NS properties obtained with the IU-FSU EoS. We can see that the value of $\overline{\omega}$ has a very small influence on the maximum mass of the stars. The radius of the canonical NS (M = 1.4 $M_\odot$) is considerably affected. Note a bigger (smaller) radius for the most positive (negative) values of $\overline{\omega}$. We can observe that the results of PREX-2 cannot be described with IU-FSU EoS in the GR, but in $f(\mathbb{T}, \mathcal{T})$ theory the solutions with $\overline{\omega} = 0.08$ and $\overline{\omega} = 0.1$ produce mass and radius that agree with this constraint. However, the solutions obtained with IU-FSU EoS cannot describe the mass and radius of PSR J0740+6620, PSR J1614+2230, and NS PSR J0348+0432 neither on GR nor on $f(\mathbb{T}, \mathcal{T})$ theory.

In Figure 2, we show the mass-radius relation obtained for QMC EoS in $f(\mathbb{T}, \mathcal{T})$ gravity. Again, the effect of the parameter $\overline{\omega}$ is to increase the radius when its values increase positively and to decrease the radius when its values increase negatively. At the same time, the maximum mass changes very little with the variation of $\overline{\omega}$. We can also see that the solutions obtained with the QMC EoS in $f(\mathbb{T}, \mathcal{T})$ can accommodate almost all the constraints we are taking into consideration, and with a smaller radius than in GR, if we take $\overline{\omega} = -0.01$ or $\overline{\omega} = -0.02$. The exception is NS PSR J0030+0451 which only can be described with QMC EoS in $f(\mathbb{T}, \mathcal{T})$ gravity if we take $\overline{\omega} = -0.2$. We can note that for both EoS analyzed we could not find a configuration that satisfies all the constraints at the same time.

We can see that for both EoS's the value of $\overline{\omega}$ has a very small influence on the maximum mass of the stars, on the other hand, the value of the radius of the star with maximum mass increases when we increase the value of $\overline{\omega}$ and decreases when $\overline{\omega}$ decreases. Additionally, for both EoS's, the case $\overline{\omega} = -0.2$ produces mass–radius curves that are typical of quark stars. In fact, this effect of mimicking different matter fields is a consequence of the non-minimal coupling between matter and geometry of this model. Additionally, notice that due to such non-minimal coupling between matter and geometry if we consider different values of $\overline{\omega}$, we can have stars with different masses, but with the same radius.

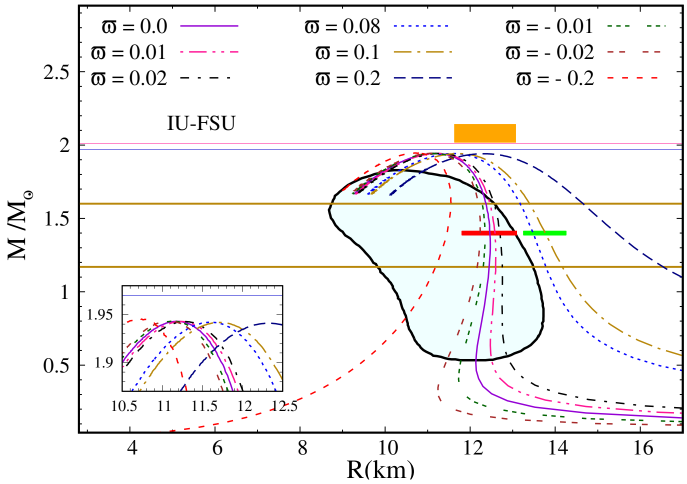

Figure 1. Mass-radius relation for families of NS described by the IU-FSU EoS. We analyze the effect of varying the parameter ϖ of the $f(\mathbb{T},\mathcal{T})$ theory. The red and green line segment represent the radius range of the 1.4 $M_\odot$ NS for PSR J0030+0451 and PREX-2, respectively. The orange rectangular region corresponds to the range of radius estimates for 2.08 $\pm$ 0.07 $M_\odot$ NS PSR J0740+6620. Similarly, the blue, pink, and golden horizontal lines stand, respectively, for the mass measurements of NS PSR J1614+2230, NS PSR J0348+0432, and GW170817 event [51]. The purple solid line curve is solution for the usual TOV equation from GR.

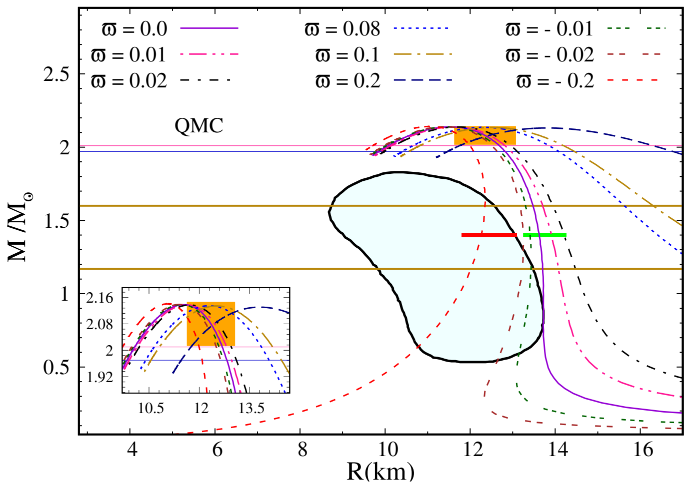

Figure 2. Mass-radius relation for families of NS described by the QMC EoS. We analyze the effect of varying the parameter ϖ of the $f(\mathbb{T},\mathcal{T})$ theory. The red and green line segment represent the radius range of the 1.4 $M_\odot$ NS for PSR J0030+0451 and PREX-2, respectively. The orange rectangular region corresponds to the range of radius estimates for 2.08 $\pm$ 0.07 $M_\odot$ NS PSR J0740+6620. Similarly, the blue, pink, and golden horizontal lines stand, respectively, for the mass measurements of NS PSR J1614+2230, NS PSR J0348+0432 and GW170817 event [51]. The purple solid line curve is the solution for the usual TOV equation from GR.

It is also important to mention that we are now in the new era of gravitational wave astronomy, and, in fact, the observation of binary system composed by two neutron stars can give new information of the maximum mass and also of the radius of neutron stars. In this sense, the gravitational event GW170817 observed by the LIGO-VIRGO collaboration, consisting of three gravitational wave interferometric detectors, have imposed more constraints on the mass and radius of neutron stars. It can be seen from that recent observation that the masses of neutron stars composing this binary system range from 0.86 to 2.26 $M_\odot$. Additionally, if theoretical and observational constraints in spins of neutron stars are considered, then the neutron star mass is inside the range 1.17 to 1.60 $M_\odot$ [51].

5. Final Remarks

We have investigated the effects of $f(\mathbb{T},\mathcal{T})$ gravity on NS assuming these compact objects as being homogeneous, static, and isotropic. In this way, we have considered a spherically symmetric space-time and solved the field equations and the hydrostatic equilibrium equation in the context of this modified theory of gravity. This type of system can be transformed into a system with effective pressure and energy density which permitted that the hydrostatic equilibrium equation was obtained through known techniques. For the choice of the $f(\mathbb{T},\mathcal{T})$ function used here, we obtained that this theory can predict NS with almost the same mass and smaller radius than in GR, for a given EoS, that is an interesting result in view of the recent observations. Considering the the NS of LMXB in NGC 6397 and the millisecond pulsar PSR J0740+6620, the results obtained using the modified hydrostatic equilibrium equations present good agreement with the observed masses and radii.

We particularize $f(\mathbb{T},\mathcal{T})$ gravity according to Equation (6). The good results obtained in comparison to GR suggest future extensions of this work, as for example, by taking into consideration different choices of the $f(\mathbb{T},\mathcal{T})$ function, which should be performed in a near future. It can be interesting to test, for example, high powers in $\mathbb{T}$ besides and new couplings between $\mathbb{T}$ and $\mathcal{T}$. In addition, we can use different EoS as input to the stellar hydrostatic equilibrium equations along the aforementioned choices of $f(\mathbb{T},\mathcal{T})$ function.

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