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Quantum Time and Quantum Evolution

Andrzej Góźdź [1,*,†], Marek Góźdź [2,†] and Aleksandra Pędrak [3,†]

[1] Institute of Physics, Maria Curie-Skłodowska University, pl. Marii Curie-Skłodowskiej 1, 20-031 Lublin, Poland
[2] Institute of Computer Science, Maria Curie-Skłodowska University, ul. Akademicka 9, 20-033 Lublin, Poland; mgozdz@kft.umcs.lublin.pl
[3] Department of Fundamental Research, National Centre for Nuclear Research, Pasteura 7, 02-093 Warsaw, Poland; aleksandra.pedrak@ncbj.gov.pl
* Correspondence: andrzej.gozdz@mail.umcs.pl
† These authors contributed equally to this work.

Abstract: The problem of quantum time and evolution of quantum systems, where time is not a parameter, is considered. In our model, following some earlier works, time is represented by a quantum operator. In this paper, similarly to the position operators in the Schrödinger representation of quantum mechanics, this operator is a multiplication-type operator. It can be also represented by an appropriate positive operator-valued measure (POVM) which together with the 3D position operators/measures provide a quantum observable giving a position in the quantum spacetime. The quantum evolution itself is a stochastic process based on Lüder's projection postulate. In fact, it is a generalization of the unitary evolution. This allows to treat time and generally the spacetime position as a quantum observable, in a consistent and observer-independent way.

Keywords: foundation of quantum mechanics; quantum spacetime; quantum evolution

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1. Introduction

For centuries time was treated as a universal parameter, ordering events and splitting in this way the reality into past, present, and future. This understanding of time ultimately leads to the notion of classical causality. The development of relativity theory changed this picture in a substantial way—time and space became one entity, spacetime [1].

On the other hand, one may ask if time and space positions behave in the same way in the macro- and microscopic scales?

The classical and quantum mechanics are based on different ideas: the classical description is deterministic but the quantum world is undeterministic. The classical world should emerge as averaged behaviour with small quantum fluctuations of properties of physical objects [2,3].

In the standard approach to quantum mechanics, the so-called Pauli theorem [4,5], states that the self adjoint time operator does not exist, i.e., time is not a quantum observable but, similar in classical physics it is introduced as a universal numerical parameter. This approach is inconsistent with the relativity theory. However, E.A. Galapon showed that this problem can be overcome using weaker assumptions about the observables [6].

There was an extensive discussion of how to introduce time as an observable in the theory. The earliest work was mainly devoted to the construction of various kinds of Hermitian operators and connecting them to time [7–24]. This work laid a solid foundation for attempts to construct the arrow of time and clocks [25–55]. Related topics, including the problem of time in entangled systems [56–58], the time of decoherence, the role of the energy–time uncertainty relation, the problem of time measurements [59], and time in the

context of quantum cosmology [60], have also been considered. A further discussion can be found in [61].

Experimental techniques for the time analysis of quantum phenomena are improving. They enable a better understanding of time as a quantum object. For example, using bremsstrahlung photons emitted during the studied quantum processes allows to estimate nuclear time durations [62] but using the Bose-condensed 87 Rb atoms tunnelling through a 1.3-micrometre-thick optical barrier allows to measure the tunnelling time [63].

The important role of time in quantum theories is also demonstrated by some other experiments. In [64,65], J.A. Wheeler proposed a thought experiment based on the Mach–Zehnder interferometer, consisting of two beam splitters and two mirrors. A single photon was travelling through the interferometer. During the particle's flight some changes to the setup were introduced, including the removal or insertion of the first or second beam splitter, even after the photon had classically passed that part of the machine. Wheeler argued that the final detection of the photon should be sensitive to these changes, mainly due to the spatial width of the photon wave function. This idea has been experimentally tested. An analogue of the Mach–Zehnder interferometer was used by the group of A. Aspect with the primary intention to test Bell's inequalities [66] showing that Wheeler's predictions were correct. Other groups [67–72] arrived at similar results. In order to investigate the problem further, a quantum eraser was used. Its purpose was to remove the information about an additional measurement, which was performed on the particle during the experiment. It turned out that erasing the information recreates the quantum behaviour of the system even in the case when the eraser worked after the final detection had been performed [73]. This setup has been called the delayed choice quantum eraser. In another experiment [74] the interferometer was built between an Earth station and a satellite. The photons behaved like particles or waves depending on the choices made by the investigators on Earth. The effect was visible even when the changes introduced to the setup were causally disconnected from the particles.

Another experiment was conducted using entangled pairs of photons [75,76]. The pair was created in one laboratory and one of the particles stayed there, while the other was sent to a second laboratory. The transmission took place between two islands, La Palma and Tenerife, with the distance between them around 144 km. Even though the particles were causally disconnected, the changes made in the first laboratory affected the second particle suggesting an either faster-than-light communication or the notion that the spatial and temporal localization of a quantum object should be reformulated.

If time in the quantum regime is treated as a coordinate, and in fact a quantum observable, all physical objects have to have some "width" in the time direction, which is related to the energy–time (more precisely, the temporal component of the four momentum operator versus time) uncertainty relation. This means that it should be possible to observe the interference of quantum objects through their overlap in time. One of the first experiments in which such behaviour was observed, was reported in [77], followed later by [78]. In [77], a single photon was emitted and a spinning chopper in the form of a wheel with slits was placed between the source and the detector. The energy spectrum of the detected photons was recorded as a function of the disc's rotation frequency. The spectrum clearly showed minima and maxima in a way very similar to the interference pattern. The authors were unable to fully explain this behaviour, even though they presented a simple analysis based on the Fourier transform of the energy spectrum. It was pointed out in [79,80] that after the Fourier transform of the energy-dependent function the authors worked with the time-dependent one and all could be interpreted as an interference pattern between the temporal parts of the photon wave function. Therefore, the most obvious explanation suggests the observation of the interference of the photon with itself in different time instances. This was recently observed [81] in the form of the temporal double-slit interference, which we describe in Section 7.

All these experiments suggest that time cannot be treated as a parameter, e.g., to explain interference in time in a natural way one needs to add probability amplitudes existing

in different moments which contradicts with the standard probabilistic interpretation of quantum mechanics, where the scalar product and normalization of states are determined for a fixed time parameter.

It is very difficult to answer the fundamental question: *What is time?* An interesting hypothesis is presented in [58] in which the authors proposed, that time is a consequence of the entanglement between particles in the universe.

In this paper, we present a consistent formulation of the quantum theory called the projection evolution (PEv) approach. In this formalism the spacetime emerges from a set of observables supported by the quantum state space. This can be generated from a set of self-conjugated operators or operator-valued measures possessing expected properties. As a by-product, such a construction should allow to obtain the time operator and the canonically conjugated observable which represents the temporal momentum. In addition, we show that the traditional time evolution, such as the Schrödinger, Klein–Gordon, Dirac and other equations of motion, can be obtained as special cases within our model. The problem of symmetries and conservation laws during the evolution is also briefly discussed.

The PEv does not need any external observer and similar to relational dynamics no background is required. This implies that PEv formalism can also be applied in a natural way to quantum gravity.

2. Projection Evolution of Quantum Systems

In quantum mechanics each physical system is described by a set of all possible observables associated with it, see, e.g., the algebraic approach to quantum mechanics [82–84]. The observables themselves are represented either by self-adjoint operators or, more generally, appropriate operator-valued measures (sharp or POVM). In traditional approaches to quantum mechanics time is not represented in the set of these observables, it is considered a parameter. This inconsistency leads to various quantum paradoxes and the time problem in gravity and cosmology; an extensive set of references for the latter problem can be found in [53–55].

In this section, we describe the formalism of the PEv which treats all quantum observables the same—time is considered a component of the spacetime position quantum observable.

This approach to quantum mechanics only generalizes the evolution law of quantum systems. The standard unitary evolution can be obtained as an approximation of the PEv. It is suitable for both: non-relativistic and relativistic physical systems.

The assumption that quantum time, and generally the spacetime, is the quantum observable requires a modification to some parts of the physical paradigm related to the causality and ordering of quantum events.

2.1. The Change Principle

The fundamental principle of the PEv approach, called *the change principle*, can be expressed as follows:

The evolution of any physical system is a random process caused by spontaneous changes in the Universe.

This means that we treat change, understood as any change of the quantum state, as the natural, primary process in our Universe. In this approach, time, space and spacetime have to be considered equally as other quantum observables. In other words, one expects, the full set of observables of any physical system under consideration contains a subset of the spacetime position operators and their canonically conjugated momenta. Among them the time observable and its canonical conjugate is also expected. This implies that different time characteristics of a given physical system have to be calculated, similar to other features of quantum systems.

This is in contradiction with usual thinking in which the existence of time allows the changes to happen. In our approach the changes happen spontaneously, according to the probability distribution, which is dictated by many factors describing the Universe and in the case of the subsystems of this Universe also their environments. This does not mean

that the changes of a quantum state are totally stochastic, without any constraints. They are obviously not deterministic, but because of interactions, symmetries which must be conserved, EPR correlations etc., they are related to each other.

As a consequence one may expect the existence of a kind of pseudo-causality based on the ordering of the quantum events, which leads to the causality principle in the case of macroscopic physical systems. In order to describe this property we introduce a parameter τ which orders quantum events. This numerical parameter should be common for the whole Universe. It should take values from an ordered set but it does not need to have any metric structure. The parameter τ is not an additional dimension of our space and it is not a replacement of time. It serves only to enumerate the subsequent steps of the evolution of the Universe and any of its physical subsystems. This implies, that at a given τ, the position of a physical system on the temporal axis should be calculated by making use of the same quantum rules as for other quantum observables, i.e., if the system is in the state described by the density operator ρ, the expectation value $\mathrm{Tr}(\hat{t}\rho)$ of the time operator $\hat{t}$ gives the position of this system on the time axis. In what follows, we assume that the domain of the evolution parameter τ is isomorphic to integers $\mathbb{Z}$ or their subset. In this case we can always use the notion of "the next step of the evolution," which may be problematic for real numbers. In the situation of a continuous or dense subset of real numbers as the domain for τ, there are some conceptual difficulties which should be, if needed, solved in the future.

An additional, very important feature of this approach is that this idea does not need the spacetime as the background, it is background-independent. Most of the current physical theories constructed use the spacetime as the primary object, with the dynamics built on top of it. In other words, the PEv approach is a background-free theory. In addition, it does not need an external observer as time is an internal observable. This point is extremely important in quantum gravity and cosmology.

2.2. Projection Evolution Operators

In the standard formulation of quantum physics, there are two kinds of time evolution: (i) the unitary evolution, which is a deterministic evolution of the actual quantum state, and (ii) the stochastic evolution, which takes place during a measurement. The latter process involves the projection of the quantum state onto the measured state and can be described by one of the projection postulates.

There is a common belief that every measurement process can be described by the unitary evolution of a larger system. However, this approach leads to known quantum measurement problems [85].

The change principle is incompatible with the unitary evolution, where time is considered a parameter. The idea of the change principle suggests the opposite scenario, the primary evolution is the stochastic evolution offered by a projection postulate. In the PEv formalism we propose the generalized Lüders [86]-type form of the projection postulate.

One may notice that this mechanism defines events as subsequent steps of the evolution.

In the following, we introduce the evolution operators formally responsible for the quantum evolution and dynamics of a physical object. In general these operators are different for different systems, similar to the Hamiltonian, which is a characteristic object for a given quantum system. On the other hand, in principle, one should be able to construct the PEv operators for the whole Universe containing the operators for any smaller subsystem.

The PEv operator from the evolution step τ_{n-1} to τ_n, where $n \in \mathbb{Z}$ is a family of mappings from the quantum state space at the evolution step τ_{n-1} to the quantum state space at the evolution step τ_n.

The appropriate state space, at the evolution step τ_k, denoted by $\mathcal{T}_1^+(\mathcal{K}(\tau_k))$, is assumed to be the space of trace one, positive and self-adjoint operators acting in the Hilbert space $\mathcal{K}(\tau_k)$, i.e., the space of quantum density operators. Every corresponding Hilbert space $\mathcal{K}(\tau_k)$ at the evolution step τ_k is a subspace of a single global Hilbert space $\mathcal{K}_U$.

Note, in this case the simplest Hilbert space of a single, spinless particle is not the space $L^2(\mathbb{R}^3, d^3x)$ but the space $L^2(\mathbb{R}^4, d^4x)$, where the fourth dimension is time, treated here equally as the positions in the 3D space. The fundamental difference is that the scalar product in $L^2(\mathbb{R}^4, d^4x)$, representing the probability amplitudes, contains integration over time. A more detailed discussion of this case can be found in Section 5.2.

These mappings can always be written in terms of the so-called quantum operations or their generalizations. The formalism of the quantum operations was invented around 1983 by Krauss [87], who relied on the earlier mathematical works of Choi [88].

The PEv $\mathbb{F}$-operators at the evolution step τ_n are formally defined as a family of transformations from the quantum state space (quantum density operator space) $\mathcal{T}_1^+(\mathcal{K}(\tau_{n-1}))$ to the space $\mathcal{T}^+(\mathcal{K}(\tau_n))$,

$$\mathbb{F}(\tau_n; \nu, \cdot) : \mathcal{T}_1^+(\mathcal{K}(\tau_{n-1})) \to \mathcal{T}^+(\mathcal{K}(\tau_n)), \tag{1}$$

where $\mathcal{T}^+(\mathcal{K}(\tau))$ is the space of finite trace, positive and self-adjoint operators acting in the Hilbert space $\mathcal{K}(\tau)$, $\nu \in \mathcal{Q}_n$, with $\mathcal{Q}_n \equiv \mathcal{Q}_{\tau_n}$ being a family of quantum number sets that define potentially available final states for the evolution from τ_{n-1} to τ_n.

We will denote by $\mathbb{F}(\tau_n; \nu, \rho)$ the action of the operator $\mathbb{F}(\tau_n; \nu, \cdot)$ on the density operator ρ, i.e., $\mathbb{F}(\tau_n; \nu, \cdot)\rho \equiv \mathbb{F}(\tau_n; \nu, \rho)$.

To use the generalized Lüders projection postulate as the principle for the evolution, the operators $\mathbb{F}(\tau_n; \nu, \rho)$ have to be self-adjoint, non-negative, and with finite trace:

$$\mathbb{F}(\tau; \nu, \rho)^\dagger = \mathbb{F}(\tau; \nu, \rho), \tag{2}$$

$$\mathbb{F}(\tau; \nu, \rho) \geq 0, \tag{3}$$

$$\sum_{\nu \in \mathcal{Q}_\tau} \mathrm{Tr}(\mathbb{F}(\tau; \nu, \rho)) < \infty, \tag{4}$$

for every state ρ. These three conditions allow $\mathbb{F}$ to transform the density operator ρ into another density operator, as shown in Equation (5).

Assuming that at evolution step τ_{n-1} the actual quantum state of a physical system is given by the density operator $\rho(\tau_{n-1}; \nu_{n-1})$, with $\nu_{n-1} \in \mathcal{Q}_{n-1}$. The change principle implies that every step of the evolution is similar to the measurement process whereby there exists a mechanism in the Universe, the chooser, which randomly chooses the next state of the system for $\tau = \tau_n$ from the set of states determined by the projection postulates. With these assumptions, following [86], we postulate $\rho(\tau_n; \nu_n)$, $\nu_n \in \mathcal{Q}_n$, in the form

$$\rho(\tau_n; \nu_n) = \frac{\mathbb{F}(\tau_n; \nu_n, \rho(\tau_{n-1}; \nu_{n-1}))}{\mathrm{Tr}(\mathbb{F}(\tau_n; \nu_n, \rho(\tau_{n-1}; \nu_{n-1})))}. \tag{5}$$

The expression in (5) determines the set of allowed states to which a physical system can randomly evolve from the state $\rho(\tau_{n-1}; \nu_{n-1})$. This is dependent on the dynamics and symmetries related to the structure of $\mathbb{F}(\tau; \nu, \cdot)$.

Because the chooser represents a stochastic process, to fully describe it one needs to determine the probability distribution for obtaining a given state in the next step of the evolution. An example of an evolution path is presented in Figure 1 by the solid line. The dotted lines show other potential paths.

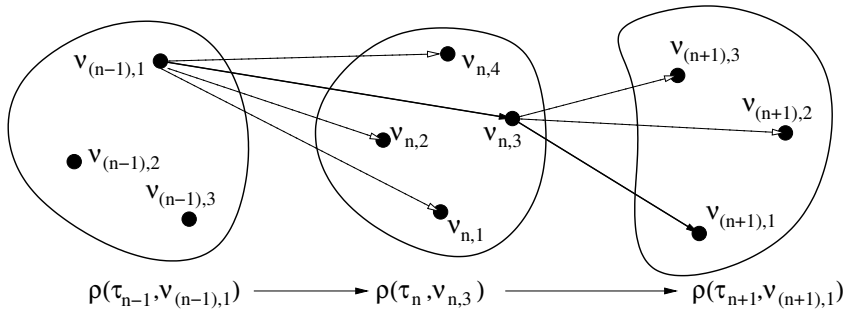

Figure 1. The density matrix ρ is randomly chosen at each evolution step τ from the possible states labelled by $\mathcal{Q}_m = \{\nu_{m,1}, \nu_{m,2}, \dots\}$, where $m = n-1, n, n+1$. Every $\nu_{m,k}$ represents a set of the quantum numbers that characterize the available quantum states.

In general, the probability distribution for the chooser is given by the quantum mechanical transition probability from the previous to the next state. This probability for pure quantum states is determined by the appropriate probability amplitudes in the form of scalar products. The transition probability among mixed states, in general, remains an open problem, see, e.g., [89,90].

In the following, we denote the transition probability (or the transition probability density) from the state labelled by the set of quantum numbers ν_{n-1} at τ_{n-1} to the state labelled by the set of quantum numbers ν_n at τ_n for a given evolution process by $\mathrm{pev}(\nu_{n-1} \to \nu_n)$. The arguments of $\mathrm{pev}()$ indicate the initial and final states of the transition.

2.3. An Example of the Evolution Operator

The most important and most interesting realization of the $\mathbb{F}$-operators can be constructed from operators $\mathbb{E}$ in the Krauss form describing the so-called quantum operations [87,88]. For every ρ and family of quantum numbers ν_n that characterize the available quantum states, i.e., $\nu_n \in \mathcal{Q}_n = \{\nu_{n,1}, \nu_{n,2}, \nu_{n,3}, \dots\}$ we have

$$\mathbb{F}(\tau_n; \nu_n, \rho) = \sum_\alpha \mathbb{E}(\tau_n; \nu_n, \alpha)\, \rho\, \mathbb{E}(\tau_n; \nu_n, \alpha)^\dagger, \tag{6}$$

where the summation over α is dependent upon the quantum numbers ν_n and chooses several evolution channels required to achieve a state described by the quantum numbers set ν_n. The operators $\mathbb{E}(\tau_n; \nu_n, \alpha)$ are transformations from the state space $\mathcal{T}_1^+(\mathcal{K}(\tau_{n-1}))$ to the non-normalized state space $\mathcal{T}^+(\mathcal{K}(\tau_n))$, i.e., $\rho \in \mathcal{T}_1^+(\mathcal{K}(\tau_{n-1}))$ and $\mathbb{F}(\tau_n; \nu_n, \rho) \in \mathcal{T}^+(\mathcal{K}(\tau_n))$.

For simplicity, the operators $\mathbb{E}$ will also be called the evolution operators.

It is easy to check that the conditions (2) and (3) are automatically fulfilled, namely:

$$\mathbb{F}(\tau_n; \nu_n, \rho)^\dagger = \sum_\alpha \mathbb{E}(\tau_n; \nu_n, \alpha)\, \rho\, \mathbb{E}(\tau_n; \nu_n, \alpha)^\dagger = \mathbb{F}(\tau_n; \nu_n, \rho) \tag{7}$$

and, since $\rho \geq 0$, we have for all $\phi \in \mathcal{K}$

$$\begin{aligned} &\langle \phi | \sum_\alpha \mathbb{E}(\tau_n; \nu_n, \alpha)\, \rho\, \mathbb{E}(\tau_n; \nu_n, \alpha)^\dagger | \phi \rangle \\ &= \sum_\alpha \langle \phi | \mathbb{E}(\tau_n; \nu_n, \alpha)\, \rho\, \mathbb{E}(\tau_n; \nu_n, \alpha)^\dagger | \phi \rangle \geq 0. \end{aligned} \tag{8}$$

Using Equation (6) and the fact that the trace is cyclical, the condition (4) takes the form

$$\begin{aligned} &\sum_{\nu_n \in \mathcal{Q}_n} \sum_\alpha \mathrm{Tr}(\mathbb{E}(\tau_n; \nu_n, \alpha)\, \rho\, \mathbb{E}(\tau_n; \nu_n, \alpha)^\dagger) \\ &= \mathrm{Tr}\left(\left[\sum_{\nu_n \in \mathcal{Q}_n} \sum_\alpha \mathbb{E}(\tau_n; \nu_n, \alpha)^\dagger \mathbb{E}(\tau_n; \nu_n, \alpha) \right] \rho \right) < \infty\,. \end{aligned} \tag{9}$$

Typical and useful examples of $\mathbb{E}$ operators are connected with the unitary evolution and orthogonal resolution of unity.

(I) In the first case the operator is

$$\mathbb{E}(\tau_n;\nu_n,\alpha) = \sqrt{p(\tau_n;\nu_n,\alpha)}U(\tau_n;\nu_n,\alpha), \tag{10}$$

where the set of ν_n labels the reaction channels and α represents the different unitary sub-channels. The numbers $p(\tau_n;\nu_n,\alpha) \geq 0$ are weights in the resulting mixed state

$$\rho(\tau_n;\nu_n) = \frac{1}{\sum_\alpha p(\tau_n;\nu_n,\alpha)} \sum_\alpha p(\tau_n;\nu_n,\alpha)U(\tau_n;\nu_n,\alpha)\, \rho(\tau_{n-1};\nu_{n-1})\, U(\tau_n;\nu_n,\alpha)^\dagger\,. \tag{11}$$

One must note that the unitary operator (10) is not parametrized by time but by the evolution parameter τ, even though, it generally acts on the spacetime-dependent states.

The PEv is not a simple generalization of the traditional unitary evolution. One must remember, that the evolution parameter τ cannot be interpreted as time, it is only a parameter which enumerates the quantum events. The traditional form of the evolution, i.e., unitary evolution driven by time interpreted as a parameter, is only an approximation valid if the following conditions are satisfied:

(i) the average values of the time operator $\hat{t}$ denoted as $t_{(observed)} = \langle\hat{t}\rangle_n = \mathrm{Tr}(\hat{t}\rho(\tau_n))$ are an increasing function of the evolution parameter, i.e., $\tau_{n-1} < \tau_n$ implies $\langle\hat{t}\rangle_{n-1} < \langle\hat{t}\rangle_n$;
(ii) the temporal spread of subsequent states is very small, i.e., the variance $\langle(\hat{t} - \langle\hat{t}\rangle_n)^2\rangle_n \sim 0$;
(iii) the probability of choosing the next state during the evolution is close to 1, i.e., $\mathrm{pev}(\nu_{n-1} \to \nu_n) \sim 1$, for all n-enumerating projection evolution steps of the system under consideration.

In this case one can find the function $f(\tau) = t_{(observed)}$ which relates τ and the observed expectation value $t_{(observed)}$ of the quantum time.

(II) In the case of the orthogonal resolution of unity with respect to the quantum numbers ν_n, the projection operators reproduce the required physical properties of some measurements. These operators are defined by the following conditions (for simplicity we have fixed the parameter α and neglect it in the notation, but the more general case can be written similarly):

$$\begin{aligned}
&\mathbb{E}(\tau_n;\nu_n)^\dagger = \mathbb{E}(\tau_n;\nu_n),\\
&\mathbb{E}(\tau_n;\nu_n)\mathbb{E}(\tau_n;\nu_n') = \delta_{\nu_n\nu_n'}\mathbb{E}(\tau_n;\nu_n),\\
&\sum_{\nu_n\in\mathcal{Q}_n} \mathbb{E}(\tau_n;\nu_n) = \mathbf{1}\,,
\end{aligned} \tag{12}$$

where $\mathbf{1}$ denotes the unit operator. Alternative choices of the quantum states are described by different sets of quantum numbers ν_n.

The probability distribution of choosing the next evolution state generated by (12) is given by the known quantum mechanical formula:

$$\mathrm{pev}(\nu_{n-1} \to \nu_n) = \mathrm{Tr}\Big(\mathbb{E}(\tau_n;\nu_n)\, \rho(\tau_{n-1},\nu_{n-1})\, \mathbb{E}(\tau_n;\nu_n)^\dagger\Big). \tag{13}$$

The above examples, even though generic for many quantum mechanical systems, are special cases of the potentially more general evolution operators.

3. The Quantum Spacetime

The quantum spacetime, similar to other properties of the quantum system can change from one step to another during its evolution. Without sacrificing generality, in this section we consider a single evolution step only, i.e., we keep the evolution parameter τ fixed.

The PEv is compatible with any reasonable model of the quantum spacetime. Here, we consider four dimensional spacetimes, but the generalization to a different number of

dimensions is simple. In the following we do not consider relations between quantum dynamics and geometrical descriptions of the spacetime. This is a very important problem which requires further consideration in future work. In this paper, we apply a general PEv idea to flat spacetime. Simplified applications of this idea in non-flat spacetime, using the expected values of appropriate observables instead of PEv evolution operators, can be found in [91,92].

To formulate a full description of the Universe one needs additional variables to describe the intrinsic properties of matter which, for simplicity, we do not take into account. However, they can be directly added to the formalism.

Let $\mathcal{K} \equiv \mathcal{K}(\tau; \mathbf{X})$ denote the Hilbert space, usually represented by square integrable functions on $\mathbf{X}$ with respect to a given measure μ. Let $\mathcal{A}$ be a σ-algebra of μ-measurable subsets of $\mathbf{X}$ such that $(\mathbf{X}, \mathcal{A}, \mu)$ represents a measurable space. The set $\mathbf{X}$ can be interpreted as a support for the classical spacetime.

3.1. Generalized Observables

The most general approach to quantum observables is given by the formalism of the positive operator-valued measures (POVMs) [85]. This is a generalization of the orthogonal operator-valued measures equivalent to using self-adjoint operators as quantum observables. In our case, POVMs allow construction of a common measure of multi-dimensional observables, e.g., four vector operators.

In the following, $\mathcal{L}(\mathcal{K})$ denotes a set of bounded operators on the state space $\mathcal{K}$.

A POVM $\hat{M} : \mathcal{A} \to \mathcal{L}(\mathcal{K})$, normalized to $\mathbf{1}$, on $(\mathbf{X}, \mathcal{A})$ is defined as [85]:

1. $\hat{M}(\Omega) \geq 0$ for all $\Omega \in \mathcal{A}$ (positivity);
2. if $\Omega_i \in \mathcal{A}$ is a countable set of disjoint sets then $\hat{M}(\bigcup_i \Omega_i) = \sum_i \hat{M}(\Omega_i)$, with weak operator topology convergence (σ–additivity);
3. $\hat{M}(\mathbf{X}) = \mathbf{1}$ and $\hat{M}(\emptyset) = 0$ (normalization).

Such measures represent a contemporary notion of quantum observables. They are related to physics by the so-called "minimal interpretation" of quantum mechanics [85], which states that the expression

$$\text{Prob}\,(\hat{M}(\Omega); \hat{\rho}) = \text{Tr}(\hat{M}(\Omega)\hat{\rho}) \tag{14}$$

gives the probability that the observable $\hat{M}$ has a value in the set Ω if the quantum system is in the state $\hat{\rho}$, where $\hat{\rho}$ is a quantum density operator.

Let us now assume that in a given model we are able to define a POVM $\hat{M}_{ST}$ that describes positions in spacetime, i.e., the operator $\hat{M}_{ST}(\Omega)$, where $\mathcal{A} \ni \Omega \subset \mathbf{X}$ measures if a system is in the spacetime region Ω.

For a given observer $\mathcal{O}$ the spacetime $\mathbf{X}$ can be decomposed into one-dimensional time space T and three-dimensional position space S, i.e., $\mathbf{X} = (T \times S)_{\mathcal{O}}$. This decomposition allows the observable to be written as $\hat{M}_T([x_A^{(0)}, x_B^{(0)}]) := \hat{M}_{ST}([x_A^{(0)}, x_B^{(0)}] \times S)$ measuring if the quantum physical system is in the time interval $[x_A^{(0)}, x_B^{(0)}]$ independent of its position in 3D space. Such an operator represents the time operator with respect to the observer $\mathcal{O}$. In this context time is a component of a compound observable that represents a place in the spacetime. The complementary operator $\hat{M}_S(Y) := \hat{M}_{ST}(T \times Y)$, where $Y \subset S$ measures if the quantum system is in region Y of the 3D space S, independent of its position in time.

In this way, for every region Ω of the classical spacetime one can ascribe the operator $\hat{M}(\Omega)$ to measure if the system is in Ω.

Intuitively, a good observable can also be an operator $\hat{M}_{ST}(x)$ to check whether the system is at a given point of the spacetime $x \in \mathbf{X}$. These operators define a more natural correspondence between the classical spacetime $\mathbf{X}$ and quantum points $\hat{M}_{ST}(x)$.

Such an operator can be imagined as a sequence of approximations $\hat{M}_{ST}(\omega_k(x))$, where $\omega_k(x) \in \mathcal{A}$ is a sequence of descending neighbourhoods of a point x, i.e., $\omega_1(x) \supset \omega_2(x) \supset \omega_3(x) \supset \ldots \omega_n(x) \supset \cdots \supset \{x\}$. $\hat{M}_{ST}(\omega_k(x))$. This is a very useful notion;

however, although one can construct well-behaving operators, the above limit can lead to operator-valued distributions [93].

A natural connection between the POVMs $\hat{M}_{ST}(\Omega)$ and $\hat{M}_{ST}(x)$ is given by

$$\hat{M}_{ST}(\Omega) = \int_{\mathbf{X}} d\mu(x)\, \chi_\Omega(x)\, \hat{M}_{ST}(x)\,, \tag{15}$$

where $\chi_\Omega(x)$ is the characteristic function of the set Ω, i.e., $\chi_\Omega(x) = 1$ if $x \in \Omega$, otherwise $\chi_\Omega(x) = 0$.

The last expression suggests that in many practical cases the components of the position operator with respect to a given observer $\mathcal{O}$ for which $x = (x^0, x^1, x^2, x^3)$ can be expressed as:

$$\hat{x}^\mu_{\mathcal{O}} = \int_{\mathbf{X}_{\mathcal{O}}} d\mu(x)\, x^\nu\, \hat{M}_{ST}(x)\,, \tag{16}$$

where $\mathbf{X}_{\mathcal{O}} = T \times S$ is the decomposition of the spacetime into its temporal and spatial parts, with respect to the observer $\mathcal{O}$. The formula (16) is compatible with the integral quantization method, see [93] and references therein.

In general, the spacetime position operator components (16) fail to commute and have no common eigenstates [91,92]. This requires the construction of preferred states representing the quantum spacetime.

3.2. Quantum Spacetime Points

The construction of preferred quantum spacetime states requires mapping κ_{ST} between the support of the classical spacetime and quantum spacetime states

$\mathbf{X}$ and $\mathcal{K}$:

$$\kappa_{ST} : \mathbf{X} \ni x \to |\eta_x\rangle \in \mathcal{K}\,. \tag{17}$$

One must remember that the state space consists of functions on the spacetime support $\mathbf{X}$. However, the same set $\mathbf{X}$ serves as the set of labels indexing the states in the mapping (17).

Using the coordinate frame corresponding to the observer $\mathcal{O}$, Equation (17) can be rewritten as

$$\kappa_{ST_{\mathcal{O}}} : \mathbf{X}_{\mathcal{O}} \ni (x^0, x^1, x^2, x^3) \to |\eta_{x^0,x^1,x^2,x^3}\rangle_{\mathcal{O}} \in \mathcal{K}\,, \tag{18}$$

where the coordinates (x^0, x^1, x^2, x^3) of a point x in the spacetime are quantum numbers enumerating the state that represents this point with respect to the observer $\mathcal{O}$. We call the vectors $|\eta_x\rangle = |\eta_{x^0,x^1,x^2,x^3}\rangle_{\mathcal{O}}$, either the position states or the quantum spacetime points.

This map has to fulfil two important conditions.

The main requirement, the self-consistency of the position state, is to obtain the appropriate expectation values of the position operators constructed with respect to a given observer $\mathcal{O}$, i.e., we require the reproduction of the classical position values as the mean values of the position operators $\hat{x}^\mu_{\mathcal{O}}$:

$$\langle \hat{x}^\mu_{\mathcal{O}}; \eta_x\rangle := \langle \eta_x|\hat{x}^\mu_{\mathcal{O}}|\eta_x\rangle = x^\mu\,, \tag{19}$$

where $x = (x^0, x^1, x^2, x^3)$ with respect to $\mathcal{O}$, and the expectation value of the operator $\hat{A}$ in the pure state ψ is defined as

$$\langle \hat{A}; \psi\rangle := \langle \psi|\hat{A}|\psi\rangle\,. \tag{20}$$

The second condition comes from the observation that every physical object has to be located somewhere in spacetime. This implies that the spacetime position states have to provide a resolution of unity:

$$\int_{\mathbf{X}} d\mu(x)|\eta_x\rangle\langle\eta_x| = \mathbf{1}\,. \tag{21}$$

Important characteristics of the spacetime position states are variances of the position observables $\mathrm{Var}(\hat{x}^{\mu}_{\mathcal{O}};\eta_x)$, where the variance of the operator $\hat{A}$ in state ψ is defined as

$$\mathrm{Var}(\hat{A};\psi) := \langle(\hat{A} - \langle\hat{A};\psi\rangle)^2;\psi\rangle\,. \tag{22}$$

The variances determine the "sizes" of the quantum points in spacetime.

In the following, to simplify notation we fix a given observer $\mathcal{O}$ and an index $\mathcal{O}$ is omitted.

If the components of the spacetime position observable commute, a possible mapping can be defined by common eigenstates of the position operators $\hat{x} = (\hat{x}^0, \hat{x}^1, \hat{x}^2, \hat{x}^3)$, where $\hat{x}^{\mu}|\eta_{x^0,x^1,x^2,x^3}\rangle = x^{\mu}|\eta_{x^0,x^1,x^2,x^3}\rangle$. The difficulty is that in some cases the eigenstates of the spacetime position operators do not belong to the Hilbert state space. In this case, not all required expressions are well defined, e.g., the expected values of the position operators within the Dirac delta-type states are indeterminable because the square of the Dirac delta-type distribution does not exist. Obviously, such problems are already well-recognized and can be solved by certain regularization procedures. From the physical perspective, such space usually consists of orthogonal operators, i.e., independent, eigenstates with extremely sharp localization. Such quantum states represent a structureless spacetime as the points with zero "size".

In models where the set of states representing quantum points of a spacetime is different from the eigenstates of position operators, the variance products (16) are bounded from below by the Heisenberg uncertainty principle [94]:

$$\mathrm{Var}(\hat{x}^{\mu};\eta_x)\mathrm{Var}(\hat{x}^{\nu};\eta_x) \geq \frac{1}{4}\langle i[\hat{x}^{\mu},\hat{x}^{\nu}];\eta_x\rangle^2\,. \tag{23}$$

In such models smeared quantum points are used which are not point-like objects. This is an important property, especially in the context of possible singularities of dynamics in the spacetime.

4. Quantum Spacetime Generated by a Set of Commuting Multiplication-Type Position Operators—The Minkowski Space

In this section, we consider the simplest and basic example of the quantum spacetime. The structureless Minkowski quantum spacetime is generated by a set of spacetime position operators in the state space $\mathcal{K} = L^2(\mathbb{R}^4, d^4x)$, where $\mathbb{R}^4$ is the support of this spacetime. This space can be considered as a homogenous space $\mathrm{P}_4/\mathrm{SO}(1,3)$, where P_4 and $\mathrm{SO}(1,3)$ denote the Poincarè and Lorentz groups, respectively. In this example we assume that these operators are commuting multiplication-type operators which, with respect to a fixed but arbitrary observer $\mathcal{O}$, can be written as

$$\hat{x}^{\mu} = \int_{\mathbb{R}^4} d^4x\, x^{\mu}\, M_X(x)\,, \tag{24}$$

where $|\eta\rangle_x := |x\rangle$ are generalized eigenstates of the traditional position operators $\hat{x}^{\mu} f(x^0,x^1,x^2,x^3) = x^{\mu} f(x^0,x^1,x^2,x^3)$ and the operators $M_X(x) = |x\rangle\langle x|$ provide the resolution of unity of the four-vector position operator $\hat{x} = (\hat{x}^0, \hat{x}^1, \hat{x}^2, \hat{x}^3)$. Note, the generalized eigenstates $|x\rangle$ are Dirac delta-type.

The observable $\hat{x}^0$ represents a quantum time and the remaining operators $\hat{x}^k$, $k = 1,2,3$ represent the 3D space position.

The operators

$$\hat{M}_{ST}(\Omega) = \int_{\mathbb{R}^4} d^4x\, \chi_{\Omega}(x) M_X(x), \tag{25}$$

where $\Omega \subset \mathbf{X}$, give the orthogonal operator-valued measure which describes the localization of points in the Minkowski spacetime.

The generated quantum states $|\eta\rangle_x = |x\rangle$, representing points of the Minkowski space, are orthogonal. There are no transitions among them so the dynamic structure has to be introduced from the outside.

Let us consider a test particle in the Minkowki quantum spacetime. By definition of the test particle we assume no back-reaction of the particle onto the spacetime.

The scalar product in the state space $\mathcal{K}$ is given by

$$\langle\Phi_2|\Phi_1\rangle = \int_{\mathcal{K}} d^4x\, \Phi_2(x)^* \Phi_1(x), \tag{26}$$

and is invariant with respect to the Lorentz transformations.

The scalar product (26) has the following probabilistic interpretation: the spacetime realization $\Psi(x) = \langle x|\Psi\rangle$ of any pure state $|\Psi\rangle \in \mathcal{K}$ represents the probability amplitude of finding the particle at the spacetime point x, i.e., $|\Psi(x)|^2$ is the probability density of finding this particle at x.

In general, the PEv breaks classical causality. The functions $\Psi(x) := \langle x^0, x^1, x^2, x^3|\Psi\rangle \in \mathcal{K}$, in their general form, connect events with space-like intervals $(x^0)^2 - \vec{x}^2 < 0$. Obviously, this can be easily removed by assuming that $\mathcal{K}$ consists of functions with only time- and zero-like support, meaning that outside the set $(x^0)^2 - \vec{x}^2 \geq 0$ the function $\Psi(x)$ is zero. However, other experimental works [95] suggest that it is a natural phenomenon for the classical causality to break in the quantum world. In general, we allow for states which break the classical causality to some acceptable extend. Within the PEv the quantum causality is realized by keeping the correct sequence of subsequent evolution steps, ordered by τ.

In general, the notion of simultaneity is observer-dependent. However, for every fixed choice of coordinates, in which one can distinguish between space and time, one can construct a spectral measure $M_T(x^0)$, which for any fixed time $t = x^0$ projects onto the space of simultaneous events:

$$M_T(x^0) = \int_{\mathbb{R}^3} d^3x\, M_X(x). \tag{27}$$

This allows one to interpret the operator $\hat{t} \equiv \hat{x}^0$ as a time operator (for a non-relativistic case a preliminary attempt is demonstrated in [96]) in the form

$$\hat{t} \equiv \hat{x}^0 = \int_{\mathbb{R}} dx^0\, x^0 M_T(x^0), \tag{28}$$

implying

$$\hat{t}\Psi(x) \equiv \langle x|\hat{t}|\Psi\rangle = \int_{\mathbb{R}^4} d^4x' t' \langle x|x'\rangle\langle x'|\Psi\rangle = x^0\Psi(x), \tag{29}$$

where the normalization of the position states $|x\rangle$ is given by $\langle x|x'\rangle = \delta^4(x - x')$.

The spectral decompositions (27), (28) allow an ideal clock to be determined. However, more realistic clocks should be described by POVMs. A good introduction to clocks is given in [53–55] and the references therein. We leave this discussion to future work.

In relativistic physics, the time operator is only determined for a given observer and cannot be considered a standalone observable, as is possible in non-relativistic cases. It must always be treated as a part of the four-vector position operator $\hat{x}$.

As a by-product of the above considerations one can construct the spectral measure which can be used as a causality measure of a given state $|\Psi\rangle$ at time x^0,

$$M_T^{(C)}(x^0) = \int_{C(x^0)} d^3x\, M_X(x)\,, \tag{30}$$

where $C(x^0) = \{\vec{x} : (x^0) - \vec{x}^2 \geq 0\}$. The expectation value of this operator,

$$\mathrm{Prob}_C[\Psi] = \langle\Psi|M_T^{(C)}(x^0)|\Psi\rangle, \tag{31}$$

gives the probability that the particle described in state $|\Psi\rangle$ is in the light cone, both in the past and future directions, with vertex x^0.

An important operator related to the time operator is the temporal component $\hat{p}_0$ of the four-momentum operator $\hat{p} = (\hat{p}_0, \hat{p}_1, \hat{p}_2, \hat{p}_3)$. In the spacetime representation, the operator, canonically conjugate to the position operator $\hat{x}^\mu$, is the translation generator in the spacetime of a single particle in the μ direction,

$$\hat{p}_\mu = i\frac{\partial}{\partial x^\mu}. \tag{32}$$

To maintain consistent interpretation, the temporal component of the momentum operator should measure, similar to the spatial components, the value of the "temporal inertia" $\times$ "speed in time" product for a particle moving along the time direction.

In addition, because the temporal linear momentum is a component of the four-momentum operator it determines the arrow of time: one direction corresponds to $p_0 > 0$, while the opposite to $p_0 < 0$.

For the traditional interpretation of p_0, the energy only holds when the equations of motion relate p_0 directly to the energy of the system, such as the Schrödinger equation $\hat{p}_0 = \hat{\mathcal{H}}$, $\hat{\mathcal{H}}$ being the Hamiltonian. A similar relationship is present in the relativistic Klein–Gordon equation, $p_0^2 = m_0^2 + \vec{p}^2$. This type of relationship exists for other physical systems. In general, one can expect that in the spacetime representation, the equation of motion of a free particle relates its four-position to its four-momenta, with the possibility that other degrees of freedom, if present, can also be involved.

Both the Schrödinger- and Klein–Gordon-type equations of motion with quantum time allow to indirectly measure the temporal component p_0 of the four-vector momentum operator $\hat{p}$. It is commonly expected that in our world the temporal momentum $p_0 \geq 0$, even though this feature does not follow the mathematical structure of the model, as the $\hat{p}_0$ operator has a full spectrum $\mathbb{R}$.

The condition $p_0 \geq 0$ can be imposed either by assuming that the equation of motion only allows for real motion if $p_0 \geq 0$, or that this condition is a more fundamental property of our Universe. A simple argument, or rather a hypothesis, supporting the latter possibility is related to the initial state of our Universe. Assuming that the four-momentum is a conserved quantity, the initial chaotic motion of matter should lead to a situation in which matter is moved in the $p_0 > 0$ and $p_0 < 0$ directions with the same probability. The spatial components lead to the expansion of matter in the $\mathbb{R}^3$ space; however, the temporal component of the four-momentum leads to separation of the Universe into two parts: one moving in the positive time direction; the other in the negative time direction. Both state subspaces are orthogonal and cannot communicate unless an interaction connecting both time directions occurs. This implies that our Universe corresponds to one of the directions of the time flow, say, $p_0 > 0$. This does not mean, obviously, that in our Universe we do not have the possibility to create particles with $p_0 < 0$. According to common interpretations, such objects are antiparticles. This strongly simplified picture requires further analysis but could provide a possible explanation of the $p_0 > 0$ phenomenon.

An interesting feature of the pair of operators $\hat{x}$ and $\hat{p}$ is that, since they fulfil the canonical commutation relations

$$[\hat{p}_\mu, \hat{x}^\nu] = i\delta_\mu^\nu, \tag{33}$$

they obey the Heisenberg uncertainty principle in the Robertson form [94],

$$\mathrm{Var}(p_\mu; \psi)\ \mathrm{Var}(x^\nu; \psi) \geq \frac{1}{4}\langle i[\hat{p}_\mu, \hat{x}^\nu]; \psi\rangle^2 = \frac{1}{4}\delta_\mu^\nu\,. \tag{34}$$

It may be interesting for future works to revisit different forms of the uncertainty principles for time, temporal component of the linear momentum, and other observables.

An interesting example is the mass operator. Assuming that the mass operator for a free particle is given by

$$\hat{m}^2 = \hat{p}_\mu \hat{p}^\mu. \tag{35}$$

Then, the uncertainty relation between the invariant mass and the position in spacetime is given by

$$\mathrm{Var}(m^2;\psi)\ \mathrm{Var}(x^\nu;\psi) \geq \langle p^\nu;\psi\rangle^2. \tag{36}$$

The width of such a mass is bounded by the ratio of the expected value of $\langle p^\nu;\psi\rangle$ and the variance $\mathrm{Var}(x^\nu;\psi)$.

When p_0 is related to the energy by the equations of motion for a given system, one can obtain the uncertainty relationship between energy and time. For example, in the Schrödinger-type equations of motion, described by $\hat{p}_0|\psi\rangle = \hat{\mathcal{H}}|\psi\rangle$, the Heisenberg relation (34) can be rewritten as

$$\mathrm{Var}(\hat{\mathcal{H}};\psi)\ \mathrm{Var}(\hat{x}^0;\psi) \geq \frac{1}{4}. \tag{37}$$

This relation is completed in the space of solutions $|\psi\rangle$ of the Schrödinger equation. Similar relationships between time and energy can be obtained from appropriate equations of motion of the considered system.

5. Generators of Projection Evolution

Within the traditional approach, the evolution of a quantum state is driven by a single operator called the Hamiltonian $e^{-i\hat{\mathcal{H}}t}$, where time t is the evolution parameter. In the PEv mechanism the changes in the system are spontaneous and time is an intrinsic variable of the physical system. However, in the orthogonal resolution of unity (12) it is possible to introduce a tool, called the PEv generator, based on already known operators used in the construction of quantum equations of motion.

The evolution operators can be obtained from a set of commuting generators $\hat{\mathbb{W}}_k(\tau)$.

For a given evolution step τ, the PEv generators $\hat{\mathbb{W}}_k(\tau)$, where $k = 1, 2, \ldots, N_W$, are defined as a set of self-adjoint operators in $\mathcal{K}(\tau)$, with a common spectral measure, which represents the set of evolution $\mathbb{E}$-operators.

When all PEv generators $\hat{\mathbb{W}}_k(\tau)$ have discrete spectra, the following relationship between $\hat{\mathbb{W}}_k(\tau)$ and the evolution operators $\mathbb{E}(\tau;\nu)$ is present:

$$\hat{\mathbb{W}}_k(\tau) = \sum_\nu w_\nu^{(k)} \mathbb{E}(\tau;\nu)\,, \tag{38}$$

where $w_\nu^{(k)}$ are the eigenvalues of the operator $\hat{\mathbb{W}}_k(\tau)$ and ν stands for a set of appropriate quantum numbers. For a continuous spectrum one must use the integral form of the spectral theorem.

Note that the generators can be subject to different constraints coming from the physics of the considered system.

5.1. Free Particles in the Flat, Structureless Spacetime

For example, let us consider a free single particle with zero spin and no intrinsic degrees of freedom. In this case the generator $\hat{\mathbb{W}}$ is dependent on the spacetime position $\hat{x}$ and the four-momentum $\hat{p}$ operators.

Accounting for the translational symmetry in our Minkowsky spacetime, the dependence of $\hat{\mathbb{W}}$ on the position operators disappears. Imposing the additional requirement for rotational symmetry of the evolution generator results in the construction of the operator $\hat{\mathbb{W}}$ as a function of the rotational invariants of the form $a^\mu \hat{p}_\mu$, $a^{\mu\nu}\hat{p}_\mu\hat{p}_\nu, \ldots$, where a^μ, $a^{\mu\nu}, \ldots$ are appropriate tensors concerning the rotational group SO(3). Basing on classical and

quantum physics one can expect that the expansion to second-order momenta should be a good approximation, leaving

$$\hat{\mathbb{W}} \overset{C}{=} a^{\mu}\hat{p}_{\mu} + a^{\mu\nu}\hat{p}_{\mu}\hat{p}_{\nu}\,, \tag{39}$$

where $\overset{C}{=}$ denotes that $\hat{\mathbb{W}}$ is equal to the right-hand side of Equation (39) only if the set of additional conditions C is fulfilled. These conditions depend on the physical properties of the studied case. This will be later used in Section 6, where the symmetries are discussed.

The additional symmetries expected for a free particle are the space inversion and anti-unitary time reversal operator. Assuming that a^{μ}, $a^{\mu\nu}, \ldots$ are invariant with respect to both of these symmetries, the linear term in the momenta reduces to $a^0\hat{p}_0$. The quadratic term splits into two parts $a^{00}(\hat{p}_0)^2 + a^{mn}\hat{p}_m\hat{p}_n$, where $m, n = 1, 2, 3$. Because of the rotational symmetry the spatial quadratic term can be written in the form $a^{mn} = B\delta^{mn}$, which casts $\hat{\mathbb{W}}$ in the form

$$\hat{\mathbb{W}} \overset{C}{=} a^0\hat{p}_0 + a^{00}(\hat{p}_0)^2 + B(\hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2)\,. \tag{40}$$

To compare Equation (40) with the standard non-relativistic quantum mechanics, one can rescale it setting $a^0 = 1$. Then, the first and third terms represent the Schrödinger equation for a free particle with mass $m = \frac{1}{2B}$. The second term is proportional to the second time derivative $(p_0)^2 \sim -\frac{\partial^2}{\partial t^2}$ and is not part of the Schrödinger equation in the standard formulation. It is probably highly suppressed by the a^{00} coefficient. By setting this coefficient to zero this term can be removed from the equation, recreating the standard Schrödinger operator. Note, that this equation is now constructed in the space $L^2(\mathbb{R}^4, d^4x)$, not in $L^2(\mathbb{R}^3, d^3x)$.

Similarly, imposing the Poincaré group invariance of $\hat{\mathbb{W}}$, one must reject the first-order term completely. Setting $a^{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$, we are left with

$$\hat{\mathbb{W}}_{KG} \overset{C}{=} \hat{p}_{\mu}\hat{p}^{\mu}, \tag{41}$$

which leads to the Klein–Gordon equation $\hat{p}_{\mu}\hat{p}^{\mu} = m^2$ with potentially additional conditions C. Assuming that C stands for the positive mass $m > 0$ and positive temporal component of the momentum operator $p_0 > 0$, the generator (41) describes the PEv of a free scalar particle, coinciding with the traditional solution of the Klein–Gordon equation considered in the Hilbert space $L^2(\mathbb{R}^4, d^x)$.

If a^{μ}, $a^{\mu\nu}, \ldots$ are tensor operators, one can reproduce the other equations of motion. For example, in the case of spin-$\frac{1}{2}$ particles, assuming $a^{\mu} = \gamma^{\mu}$, where γ^{μ} are Dirac matrices, one obtains the Dirac equation

$$\hat{\mathbb{W}}_D \overset{C}{=} \gamma^{\mu}\hat{p}_{\mu}. \tag{42}$$

We conclude that the known quantum equations, which describe specific quantum systems, imply evolution generators $\hat{\mathbb{W}}_k$, which can describe much more complicated cases.

5.2. The Schrödinger Evolution as a Special Case of PEv

To analyse the Schrödinger non-relativistic motion with quantum time one needs to assume that the corresponding quantum spacetime is generated by the Galilean group of transformations, a slight contraction of the Poincarè group in the limit $c \to \infty$, and at the same time the Minkowski relativistic quantum spacetime reduces to the non-relativistic (1 + 3)D quantum spacetime.

This ensures the appropriate transformation properties of the spacetime quantum observables used in the Section 5.1.

The generator of the Schrödinger evolution can be written as

$$\hat{\mathbb{W}}_S = i\frac{\partial}{\partial t} - \hat{\mathcal{H}} = \hat{p}_0 - \hat{\mathcal{H}}. \tag{43}$$

Let us assume that the Hamiltonian $\hat{\mathcal{H}}$ is independent of time. The eigenvalues and the corresponding orthonormal eigenvectors of $\hat{\mathcal{H}}$ are denoted by ϵ_n and $\phi_{n\mu}(\vec{x})$, respectively, such that

$$\hat{\mathcal{H}}\phi_{n\mu}(\vec{x}) = \epsilon_n \phi_{n\mu}(\vec{x}). \tag{44}$$

The action of $\hat{\mathbb{W}}_S$ on the full wavefunction results in

$$\hat{\mathbb{W}}_S\, \eta_{k_0}(x^0)\phi_{n\mu}(\vec{x}) = w(k_0, n)\, \eta_{k_0}(x^0)\phi_{n\mu}(\vec{x}), \tag{45}$$

where

$$w(k_0, n) = k_0 - \epsilon_n, \tag{46}$$

$$\eta_{k_0}(x^0) = \frac{1}{\sqrt{2\pi}} e^{-ik_0 x^0}. \tag{47}$$

The spectral decomposition of the generator $\hat{\mathbb{W}}_S$ in the form of a Riemann–Stieltjes integral can be written as

$$\hat{\mathbb{W}}_S = \int_{\mathbb{R}} w\, dE_{\hat{\mathbb{W}}}(w), \tag{48}$$

where $dE_{\hat{\mathbb{W}}}(w)$ projects onto the eigenspace of $\hat{\mathbb{W}}_S$ belonging to the eigenvalue w. This subspace is spanned by the generalized eigenfunctions of the form

$$\Phi_w(x^0, \vec{x}) = \frac{1}{\sqrt{2\pi}} \sum_n \sum_\mu c_{n\mu} e^{-i(\epsilon_n + w)x^0} \phi_{n\mu}(\vec{x})\,, \tag{49}$$

with $c_{n\mu}$ being c-number coefficients. The scalar product in the appropriate state space is given by (26).

Using the scalar product (26) we see that the eigenfunctions (49) are normalized to the Dirac delta functions,

$$\langle \Phi_{w'} | \Phi_w \rangle = \int_{\mathbb{R}^4} dx^0\, dx^1 dx^2 dx^3 \Phi_{w'}(x^0, \vec{x})^* \Phi_w(x^0, \vec{x}) = \delta(w' - w)\,. \tag{50}$$

There are a few methods of obtaining vectors belonging to the state space $\mathcal{K}$. For example, one can consider an extension of the Schrödinger equation which contains the temporal part describing the temporal dependencies of the kinetic and potential terms,

$$\hat{\mathbb{W}}_{GS}(\tau) = \hat{p}_0 - \hat{\mathcal{H}}(\tau) + \left[\frac{1}{2} B_T^{-1}(\tau)\hat{p}_0^2 + V_T(\tau, x^0)\right]. \tag{51}$$

These additional terms represent the kinematics and the possible localization of a physical object on the time axis. The parameter $B_T^{-1}(\tau)$ represents a kind of temporal inertia of the physical object.

When treating time as a parameter t, the functions (49) have to have a well-defined maximum in x^0 at t, i.e., in the limiting case they should be of the Dirac delta-type in the time variable, $\delta(x^0 - t)$. In this approximation the functions (49) stand for the general solutions to the Schrödinger equation, where the eigenvalue w determines the zero value of the energy represented by the Hamiltonian $\hat{\mathcal{H}}$. This follows the fact that the eigen equation for $\hat{\mathbb{W}}_S$, from Equation (43), can be written in the form

$$i\frac{\partial}{\partial t}\phi_w = (\hat{\mathcal{H}} + w)\phi_w, \tag{52}$$

which means that the arbitrary eigenvalue w shifts the energy spectrum. Of course, in this approximation the physics in $\mathcal{K}_3$ are independent of the chosen w because it only gives a

phase factor in the wavefunction. Note that the integration over time in the scalar product is absent,

$$\langle \Phi_2 | \Phi_1 \rangle_3 = \int_{\mathbb{R}^3} d^3x \, \Phi_2(x^0, \vec{x})^* \Phi_1(x^0, \vec{x}) \,, \tag{53}$$

because the state space $\mathcal{K}_3 = L^2(\mathbb{R}^3)$ does not contain time.

We conclude that an important difference between the PEv approach and the traditional formulation of quantum mechanics lies in the interpretation of the wavefunctions $\Psi(x^0, \vec{x})$. In the PEv formalism the function $|\Psi(x^0, \vec{x})|^2$, where $\Psi(x^0, \vec{x}) \in \mathcal{K}$, represents the joined probability density of finding the particle in the 4D spacetime point $(x^0, \vec{x})$. Such a wavefunction is normalized to the whole spacetime, which in the relativistic case can have quite a complicated structure, with different regions potentially unavailable for a given system. In these regions states should be equal to zero. The quantum states determined on the whole spacetime allow to model the spacetime behaviour of quantum systems in a much more flexible way, e.g., they allow for not only spatial but also temporal interference. The PEv states are compatible with the required spacetime symmetries in a natural way. The 4D form of the wavefunction supports the relativistic covariance of the formalism.

In traditional quantum mechanics, with time as a parameter, the function $|\Psi(x^0, \vec{x})|^2$, where $\Psi(x^0, \vec{x}) \in \mathcal{K}_3$, represents the conditional probability density of finding the particle in the 3D space point $\vec{x}$, assuming that the particle is localized at time x^0. Any traditional wavefunction can be obtained from the PEv wavefunction by its projection onto the space of simultaneous events. Such projection is obviously observer-dependent.

5.3. Relativistic Motion

To verify that the PEv approach allows the relativistic evolution equations to be described in a more natural way than the (1+3)-formalism, the Klein–Gordon equation of motion must be considered for a free scalar particle.

We use the Minkowski space with the metric tensor

$$\gamma^{00} = 1, \quad \gamma^{11} = \gamma^{22} = \gamma^{33} = -1, \quad \text{otherwise } \gamma^{\mu\nu} = 0. \tag{54}$$

We assume that all four-vectors are presented by their contravariant components as $a = (a^0, a^1, a^2, a^3) = (a^0, \vec{a})$.

The generator of the appropriate evolution is given by (41). Without any constraint the mass operator $\hat{m}^2 = \hat{p}_\mu \hat{p}^\mu$ has the following continuous spectrum and generalized eigenvectors:

$$\hat{p}_\mu \hat{p}^\mu \eta_k(x) = w \eta_k(x) \,, \tag{55}$$

where $\eta_k(x) = \exp(-ik_\mu x^\mu)/(4\pi^2)$, $k = (k^0, k^1, k^2, k^3)$, $p_\mu = k_\mu$, and $w \in \mathbb{R}$. Comparing both sides of Equation (55) one obtains the relation $k_\mu k^\mu = w$. For each w this relation determines the subspace $\mathcal{K}_w$ invariant under the Poincaré group, corresponding to states with definite w, i.e., they belong to the mass shell. This subspace consists of all generalized eigenvectors of the mass operator (55) belonging to this mass shell. They are of the form

$$\Phi_w(x) = \int_{\mathbb{R}^4} d^4k \, \delta^4(k_\mu k^\mu - w) c(k) \eta_k(x) = \int_{\mathcal{K}_w} d^4k \, c(k) \eta_k(x) \,, \tag{56}$$

where $c(k)$ is a function representing the profile of the wave package (56).

This implies that the evolution operators generated by (41) are the generalized projection operators

$$\text{Ŧ}(\tau; w) = \int_{\mathcal{K}_w} d^4k |\eta_k\rangle \langle \eta_k| \,. \tag{57}$$

Using the usual conditions that the space states is restricted to for which $\hat{m}^2 > 0$ and $\hat{p}_0 > 0$, the eigenvalues w are traditionally interpreted as the invariant mass squared, m^2. In this case, the evolution operators (57) can be rewritten as

$$\mathbb{E}_{\mathrm{C}}(\tau; w) = \int_{\mathcal{K}_{w+}} d^4k |\eta_k\rangle\langle\eta_k|, \tag{58}$$

where $\mathcal{K}_{w+}$ comprises functions $\phi(k) \in \mathcal{K}_w$ with the constraints: $k^2 > 0$ and $k_0 > 0$.

In this case the evolution operators (according to the evolution principle (5) and the definition (6)) projecting on the subspaces $\mathcal{K}_{w+}$ reproduce the known solutions for the standard scalar particle of non-zero mass,

$$\Phi_w(x) = \int_{\mathbb{R}^3} \frac{d^3k}{k_0} c(\vec{k}) \eta_{(k_0,\vec{k})}(x), \tag{59}$$

where $k = (k_0, \vec{k}) = (\sqrt{m^2 + \vec{k}^2}, \vec{k})$. Note that both vectors (56) and (59) are normalized Dirac delta-type distributions.

One can extend the Klein–Gordon equation evolution generator to a particle in the electromagnetic field by including the appropriate four-vector field A_μ. Using the minimal coupling scheme one obtains

$$\hat{\mathbb{W}} = (\hat{p}_\mu - A_\mu)(\hat{p}^\mu - A^\mu). \tag{60}$$

This vector field can play a similar role to the temporal part of the potential in the extended Schrödinger PEv generator (51), allowing for solutions in the form of square integrable states.

All other single particle relativistic equations of motion can be reproduced in a similar way, but now they are constructed in the Hilbert state space $L^2(\mathbb{R}^4, d^4x)$, where time is one of the physical degrees of freedom—time becomes a quantum observable and is treated the same as the remaining position coordinates.

6. Symmetries

As is well known, different kinds of symmetries play a fundamental role in physics. They are the most important constraints for the structure, interaction and motion of physical objects.

In the case of the PEv formalism one thinks about two distinct types of symmetries:

(A) the symmetries for a fixed step of the evolution, i.e., for a constant evolution parameter τ;

(B) the symmetries related to the transition of the system from one step of the evolution to another, i.e., when the evolution parameter changes, $\tau_{n-1} \to \tau_n$.

The first type of symmetry (A) describes the structural, spacetime and intrinsic properties of a quantum system. An important difference is that time is now a quantum observable. Taking this into account, the symmetry analysis seems to be similar to those performed in relativistic quantum mechanics. Many results remain valid, but most of them require reinterpretation.

The second type of symmetry (B) is different because the evolution operators are involved in the symmetry analysis. The operators $\mathbb{F}(\tau; \nu, \rho)$ can have different structures, they can be unitary operators, projection operators or other type of operators which allow the transformation of quantum states into new quantum states. This opens many mathematical and interpretational problems.

In this section, we analyse two elementary properties related to type (B) symmetries for fixed state space $\mathcal{K} = \mathcal{K}(\tau_n)$, i.e., the state space is the same for all τ_n. A more extensive analysis is beyond the scope of this paper.

We consider the evolution operators $\mathbb{F}(\tau_n;\nu_n)$ for which the operators $\mathbb{E}(\tau_n;\nu_n)$ form either an orthogonal resolution of unity (12) or generalized unitary PEv operators (11). Other cases will be considered in subsequent work.

The problem here is to find the physical properties which remain invariant at subsequent steps of the evolution. In other words, we are looking for the transformations which do not change the physical system.

We start by writing the definition of the transformations of the operator $\mathbb{F}(\tau_n;\nu_n, \rho(\tau_{n-1};\nu_{n-1}))$ under the action of the group G. Let us denote by $S : \mathrm{G} \to \mathcal{K}$ a unitary operator representation of the group G in the state space $\mathcal{K}$. The transformation of the evolution operator $\mathbb{F}$ is defined as:

$$\mathbb{F}'(\tau_n;\nu_n,\rho'(\tau_{n-1};\nu_{n-1})) = S(g)\ \mathbb{F}(\tau_n;\nu_n,\rho(\tau_{n-1};\nu_{n-1}))\ S(g^{-1})\,, \tag{61}$$

where

$$\rho'(\tau_{n-1};\nu_{n-1}) = S(g)\rho(\tau_{n-1};\nu_{n-1})S(g^{-1})\,. \tag{62}$$

This definition follows the idea of transformations of functions of more complex objects, e.g., the rotation $\hat{\mathcal{R}}$ of a vector function $f : \mathbb{R}^3 \to \mathbb{R}^3$. The values of the rotated function f' with the rotated argument x' should be equal to the rotation of the value of the original function with the original argument, $f'(x') = \hat{\mathcal{R}}f(x)$.

The definition (61) can be expressed in a more convenient form:

$$\mathbb{F}'(\tau_n;\nu_n,\rho(\tau_{n-1};\nu_{n-1})) = S(g)\ \mathbb{F}(\tau_n;\nu_n,S(g^{-1})\rho(\tau_{n-1};\nu_{n-1})S(g))\ S(g^{-1})\,. \tag{63}$$

A fundamental property of the definition (61) is that it allows the probability structure of transitions under the action group G to be conserved. To demonstrate this feature, we assume that the transition probability from the evolution step τ_{n-1} to τ_n is given by

$$\mathrm{pev}\left(\rho(\tau_{n-1};\nu_{n-1}) \to \rho(\tau_n;\nu_n) = \frac{\mathbb{F}(\tau_n;\nu_n,\rho(\tau_{n-1};\nu_{n-1}))}{\mathrm{Tr}[\mathbb{F}(\tau_n;\nu_n,\rho(\tau_{n-1};\nu_{n-1}))]}\right) = \mathrm{Tr}[\mathbb{F}(\tau_n;\nu_n,\rho(\tau_{n-1};\nu_{n-1}))]\,, \tag{64}$$

as it is for the PEv operators represented by an orthogonal resolution of the unit operator. Here, ν_n denotes the set of allowed quantum numbers. Similarly, the transition probability from the state $\rho'(\tau_{n-1};\nu_{n-1}) = S(g)\rho(\tau_{n-1};\nu_{n-1})S(g)^{-1}$ to the state $\tilde{\rho}(\tau_n,\nu_n)$ obtained by the transformed evolution operator $\mathbb{F}'$ is given by

$$\mathrm{pev}\left(\rho'(\tau_{n-1};\nu_{n-1}) \to \tilde{\rho}(\tau_n,\nu_n) = \frac{\mathbb{F}'(\tau_n;\nu_n\,,\rho'(\tau_{n-1};\nu_{n-1}))}{\mathrm{Tr}[\mathbb{F}'(\tau_n;\nu_n\,,\rho'(\tau_{n-1};\nu_{n-1}))]}\right) = \mathrm{Tr}[\mathbb{F}'(\tau_n;\nu_n\,,\rho'(\tau_{n-1};\nu_{n-1}))]\,. \tag{65}$$

Because $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, it follows from Equation (61) that both probabilities are equal:

$$\mathrm{pev}(\rho(\tau_{n-1};\nu_{n-1}) \to \rho(\tau_n;\nu_n)) = \mathrm{pev}\big(\rho'(\tau_{n-1};\nu_{n-1}) \to \tilde{\rho}(\tau_n,\nu_n)\big)\,. \tag{66}$$

A consequence of this symmetry for every evolution step is the fact that the action of the group G does not change the probability structure of the possible evolution paths.

The second important problem is the relation between the symmetry and conservation laws. One can intuitively say that we are looking for conditions under which the expected value of a given observable A is conserved during the PEv process:

$$\langle A;\rho(\tau_1;\nu_1)\rangle = \langle A;\rho(\tau_2;\nu_2)\rangle = \cdots = \langle A;\rho(\tau_n;\nu_n)\rangle = \ldots\,. \tag{67}$$

The required conditions may involve special relations between the evolution operators, density operators and quantum observables.

When the evolution is described by the operators $\mathbb{E}(\tau_n;\nu_n,\alpha)$ the conservation of the expected value $\langle A;\rho(\tau_k;\nu_k)\rangle$ has the following form

$$\mathrm{Tr}[A\,\rho(\tau_{n-1};\nu_{n-1})] = \mathrm{Tr}[A\,\rho(\tau_n;\nu_n)] = \frac{\sum_\alpha \mathrm{Tr}[A\,\mathbb{E}(\tau_n;\nu_n,\alpha)\,\rho(\tau_{n-1};\nu_{n-1})\,\mathbb{E}(\tau_n;\nu_n,\alpha)^\dagger]}{\sum_\alpha \mathrm{Tr}[\mathbb{E}(\tau_n;\nu_n,\alpha)\,\rho(\tau_{n-1};\nu_{n-1})\,\mathbb{E}(\tau_n;\nu_n,\alpha)^\dagger]}\,. \tag{68}$$

When we have the unitary-type operators $\mathbb{E}(\tau_n;\nu_n,\alpha) = \sqrt{p(\tau_n;\nu_n,\alpha)}\,U(\tau_n;\nu_n,\alpha)$, where $p(\tau_n;\nu_n,\alpha) \geq 0$, the condition (68) can be rewritten as

$$\mathrm{Tr}[A\rho(\tau_{n-1};\nu_{n-1})] = \frac{1}{\sum_\alpha p(\tau_n;\nu_n,\alpha)} \sum_\alpha p(\tau_n;\nu_n,\alpha)\mathrm{Tr}[U(\tau_n;\nu_n,\alpha)^\dagger\,A\,U(\tau_n;\nu_n,\alpha)\,\rho(\tau_{n-1};\nu_{n-1})]. \tag{69}$$

The expected value of the observable A is conserved if operator A commutes with the evolution operators, i.e., $[A, U(\tau_n;\nu_n,\alpha)] = 0$. This fact has its counterpart in the standard quantum mechanics—if the Hamiltonian commutes with operator A, the expected value of $\langle A\rangle$ is conserved during the unitary evolution generated by this Hamiltonian. Generally, when $\mathbb{E}(\tau_n;\nu_n,\alpha)$ is an orthogonal decomposition of unity the condition (68) to be fulfilled requires more complicated relations between the evolution operators, states and the observable A. This will be considered in further studies. However, generally the evolution operator and quantum observable are invariant under a given symmetry and can be solved. For simplicity, we fix the index α in the following.

Let a compact Lie group G be a symmetry group of the evolution generator $\hat{\mathbb{W}}(\tau_n)$, i.e., $S(g)\hat{\mathbb{W}}(\tau_n)S(g)^{-1} = \hat{\mathbb{W}}(\tau_n)$ for every $g \in \mathrm{G}$ and every τ_n, where the operators $S(g)$ play the role of a unitary operator representation of this symmetry group in the state space $\mathcal{K}$. Because G is the symmetry group of the generator $\hat{\mathbb{W}}(\tau_n)$, its eigenstates $|\tau_n;\kappa_n\Gamma_n a\rangle$ form the irreducible subspaces of the irreducible representations of this group,

$$\hat{\mathbb{W}}(\tau_n)|\tau_n;\kappa_n\Gamma_n a_n\rangle = w(\tau_n;\kappa_n\Gamma_n)|\tau_n;\kappa_n\Gamma_n a_n\rangle, \tag{70}$$

$$S(g)|\tau_n;\kappa_n\Gamma_n a_n\rangle = \sum_{a'_n} \Delta^{\Gamma_n}_{a'_n a_n}(g)|\tau_n;\kappa_n\Gamma_n a'_n\rangle\,, \tag{71}$$

where Δ^{Γ_n} denotes the irreducible representation of the symmetry group G labelled by Γ_n, the quantum number a_n labels vectors within the given irreducible representation Δ^{Γ_n}, the set of quantum numbers κ_n describes these properties of our quantum system which are independent of the symmetry, and distinguishes it among equivalent irreducible representations for fixed Γ_n. For every τ_n the vectors $|\tau_n;\kappa_n\Gamma_n a_n\rangle$ form the orthonormal bases in the state space $\mathcal{K}$ and the quantum numbers Γ_n belong to an established set of labels $\{\Gamma^{(1)}, \Gamma^{(2)}, \ldots \Gamma^{(k)}, \ldots\}$ enumerating irreducible representations of the group G. This independence of the evolution step set of labels can be determined from decomposing the state space $\mathcal{K}$ into irreducible subspaces.

In this case the spectral decomposition of the evolution generator can be written as

$$\hat{\mathbb{W}}(\tau_n) = \sum_{\kappa_n,\Gamma_n} w(\tau_n;\kappa_n,\Gamma_n)P(\tau_n;\kappa_n\Gamma_n)\,, \tag{72}$$

where $w(\tau_n;\kappa_n,\Gamma_n)$ are eigenvalues and the projectors on the eigenspaces read

$$P(\tau_n;\kappa_n\Gamma_n) = \sum_{a_n} |\tau_n;\kappa_n\Gamma_n a_n\rangle\langle\tau_n;\kappa_n\Gamma_n a_n|\,. \tag{73}$$

The decomposition (72) determines the following evolution operators

$$\mathbb{E}(\tau_n;\kappa_n\Gamma_n) = P(\tau_n;\kappa_n\Gamma_n) \tag{74}$$

for which the two orthogonal relations hold

$$\mathbb{E}(\tau_n;\kappa_n\Gamma_n)\,\mathbb{E}(\tau_n;\kappa'_n\Gamma'_n) = \delta_{\Gamma_n,\Gamma'_n}\delta_{\kappa_n,\kappa'_n}\,\mathbb{E}(\tau_n;\kappa_n\Gamma_n)\,. \tag{75}$$

and

$$\text{Ŧ}(\tau_n;\kappa_n\Gamma_n)\text{Ŧ}(\tau_{n'};\kappa_{n'}\Gamma_{n'}) = 0 \text{ if } \Gamma_n \neq \Gamma_{n'} \tag{76}$$

Using the above conditions, the Casimir operator $\mathcal{C}^2$ of the symmetry group G, an observable invariant with respect to this symmetry group, satisfies

$$\mathcal{C}^2\text{Ŧ}(\tau_n;\kappa_n\Gamma_n) = c_{\Gamma_n}\text{Ŧ}(\tau_n;\kappa_n\Gamma_n)\,, \tag{77}$$

where c_{Γ_n} are eigenvalues of the Casimir operator obtained from

$$\mathcal{C}^2|\tau_n;\kappa_n\Gamma_n a_n\rangle = c_{\Gamma_n}|\tau_n;\kappa_n\Gamma_n a_n\rangle\,. \tag{78}$$

Let ρ_0 denote the initial state. After the first step of the PEv one obtains a new state

$$\rho(\tau_1;\kappa_1,\Gamma_1) = \frac{\text{Ŧ}(\tau_1;\kappa_1\Gamma_1)\rho_0\text{Ŧ}(\tau_1;\kappa_1\Gamma_1)}{\mathrm{Tr}[\text{Ŧ}(\tau_1;\kappa_1\Gamma_1)\rho_0\text{Ŧ}(\tau_1;\kappa_1\Gamma_1)]}\,. \tag{79}$$

The expected value of the Casimir operator is

$$\mathrm{Tr}[\mathcal{C}^2\rho(\tau_1;\Gamma_1,\kappa_1)] = \frac{\mathrm{Tr}\left[\mathcal{C}^2\text{Ŧ}(\tau_1;\kappa_1\Gamma_1)\rho_0\text{Ŧ}(\tau_1;\kappa_1\Gamma_1)\right]}{\mathrm{Tr}[\text{Ŧ}(\tau_1;\kappa_1\Gamma_1)\rho_0\text{Ŧ}(\tau_1;\kappa_1\Gamma_1)]} = c_{\Gamma_1}\,. \tag{80}$$

Because of the orthogonal relations (75) and (76), one obtains

$$\mathrm{Tr}[\mathcal{C}^2\rho(\tau_2;\Gamma_2,\kappa_2)] = \frac{\mathrm{Tr}\left[\mathcal{C}^2\text{Ŧ}(\tau_2;\kappa_2\Gamma_2)\rho(\tau_1;\kappa_1,\Gamma_1)\text{Ŧ}(\tau_2;\Gamma_2\kappa_2)\right]}{\mathrm{Tr}[\text{Ŧ}(\tau_2;\kappa_2\Gamma_2)\rho_0\text{Ŧ}(\tau_2;\kappa_2\Gamma_2)]} = \delta_{\Gamma_1,\Gamma_2}\,c_{\Gamma_1}\,. \tag{81}$$

This implies that the value of this Casimir operator is fixed for all subsequent steps

$$\mathrm{Tr}[\mathcal{C}^2\rho(\tau_n;\Gamma_n,\kappa_n)] = \begin{cases} c_{\Gamma_1}, & \text{for } \Gamma_n = \Gamma_1\,, \\ 0, & \text{for } \Gamma_n \neq \Gamma_1\,. \end{cases} \tag{82}$$

We conclude that if the evolution operators are invariant with respect to G and fulfil the above conditions, the value of the Casimir operator $\mathcal{C}^2$ of G is conserved during the evolution.

This special case has its analogy in the standard quantum mechanics. Let us assume that the Hamiltonian $\hat{\mathcal{H}}$ is invariant with respect to G. The eigenvectors of $\hat{\mathcal{H}}$ belong to the invariant subspaces spanned by the bases of the irreducible representations of G. In this case the expectation value of the Casimir operator is conserved during the unitary evolution generated by this Hamiltonian.

We have presented a short outline of some problems related to the symmetry analysis within the PEv approach. PEv opens new areas for the application of symmetries and group theoretical methods in physics.

7. Interference in Time

One of the most important phenomenon in quantum physics is quantum interference. The fundamental observations of interference are obtained due to Young's double slit-type experiments. They are very well known in both the classical and quantum versions.

In the traditional approach, having two wavefunctions in the position representation, $\Psi_1(t,\vec{x})$ and $\Psi_2(t,\vec{x})$, and two spatially separated slits, one expects that under proper conditions one obtains an interference of these wavefunctions $\Psi(t,\vec{x}) = \Psi_1(t,\vec{x}) + \Psi_2(t,\vec{x})$ in the spatial domain.

This similar effect is also observed in the time domain, see, e.g., [77–80], where two types of particles, namely photons and electrons, were used. The most recent research on temporal interference was performed on photons, clearly showing temporal interference [81]. These observations generalize the idea of Young's double slit-type experiments to the spacetime domain. Having two PEv states in the spacetime representation, $\Psi_1(x)$

and $\Psi_2(x)$, and two slits in the spacetime, one obtains an interference of these states under the same proper conditions.

Let us assume that $\Psi_1(x)$ is an amplitude of a particle passing the first slit and $\Psi_2(x)$ is the amplitude of the same particle passing the second slit. The resulting probability of passing any of the slits is given by

$$|\Psi_1(x) + \Psi_2(x)|^2 = |\Psi_1(x)|^2 + |\Psi_2(x)|^2 + 2\mathrm{Re}[\Psi_1(x)^* \Psi_2(x)]. \tag{83}$$

Treating all the components of the position operator $\hat{x}^\mu$ equally, one can describe devices which open two slits not in the 3D space but in time. The temporal interference, according to Equation (83), is also expected in this case.

Temporal interference is a natural component of the PEv formalism. In the following we describe a relativistic version of the temporal double slit experiment.

7.1. Time Interference of Scalar Relativistic Particles

Assume that a relativistic spinless Klein–Gordon particle is emitted from a source at the evolution step τ_0 and propagates towards the detector. In the derivation below we use the standard condition for particles where the temporal component of the momentum four-vector is positive, i.e., $p_0 \geq 0$.

In the following we use the metric tensor (54).

Travelling towards the detector the particle meets a wall in which a slit may open for a limited time. Let the slit open two times in a fixed spatial location. When the slit is closed, the path to the detector is blocked. After successfully passing the slit, the particle, if not observed, will be in a superposition of states corresponding to the two time intervals of the opened slit. As a result, the energy spectrum measured by the detector will have the form of an interference pattern, as was reported in [77].

Let us now construct the initial state of the particle and the evolution operators $\mathbb{E}$ for the subsequent steps of this evolution.

The emission process does not happen in zero time, so the mass m_0 of the particle will be distributed around some mean value $\bar{m}_0$,

$$m_0 \in \Delta_{\bar{m}_0} = \left[\bar{m}_0 - \frac{\Gamma}{2}, \bar{m}_0 + \frac{\Gamma}{2}\right], \tag{84}$$

where $\Gamma/2 < \bar{m}_0$. It follows from the Klein–Gordon equation $k_\mu k^\mu = m_0^2$ that to fulfil (84) the particle's four-momentum must belong to the set $B_{\bar{m}_0} := \left\{k : \left(\bar{m}_0 - \frac{\Gamma}{2}\right)^2 \leq k^2 \leq \left(\bar{m}_0 + \frac{\Gamma}{2}\right)^2\right\}$. We write the initial state of the particle in the form

$$|\psi_0\rangle = \int_{B_{\bar{m}_0}} d^4k \, a(k)|k\rangle \,, \tag{85}$$

where $\langle x|k\rangle = \exp(-ik_\mu x^\mu)/(4\pi^2)$ and $a(k)$ denotes the momentum distribution function. For simplicity, we omit the normalization factor and introduce the overall normalization in the final formula (102).

The slit is open at certain spatial location during the two time periods. We denote the spacetime regions of the opened slit by Δ_1 and Δ_2. The evolution operator for the evolution step τ_1 is parametrized by the quantum number ν_1 having two values: (i) $\nu_1 = 1$ representing the passing of the particle through the slits and (ii) $\nu_1 = 0$ for the complementary case in which the particle does not pass through the slits. For the first case the evolution operator takes the form of a projection of the state onto the region $\Delta_T = \Delta_1 \cup \Delta_2$, and for the second the corresponding operator is the projection operator onto the complementary space:

$$\mathbb{E}_S(\tau_1; \nu_1 = 1) = \int_{\Delta_T} d^4x \, |x\rangle\langle x|, \tag{86}$$

$$\mathbb{E}_S(\tau_1; \nu_1 = 0) = \mathbf{1} - \mathbb{E}_S(\tau_1; \nu_1 = 1). \tag{87}$$

The operator $\mathbb{E}_S(\tau_1;\nu_1=0)$ can be decomposed into more detailed evolution operators which describe a particle that does not manage to pass the slit; however, we are specifically interested in the form (86) only. The action of (86) on the initial state $|\psi_0\rangle$ is given by

$$\mathbb{E}_S(\tau_1)|\psi_0\rangle = \int_{\Delta_T} d^4x\, |x\rangle \int_{B_{\bar{m}_0}} d^4k\, a(k)\langle x|k\rangle\,. \tag{88}$$

In the next step the particle propagates freely from the slits to the detector. During this step the Klein–Gordon equation is fulfilled, so the required evolution operator projects onto the momentum space $B_{\bar{m}_0}$,

$$\mathbb{E}_F(\tau_2;\nu_2=1) = \int_{B_{\bar{m}_0}} d^4k'\, |k'\rangle\langle k'|\,. \tag{89}$$

In what follows we neglect the off-shell case, described by the projection operator $\mathbb{E}_F(\tau_2;\nu_2=0) = \mathbf{1} - \mathbb{E}_F(\tau_2;\nu_2=1)$, as it is only a small correction to the main effect. The non-normalized state of the particle at this step reads

$$\begin{aligned}&\mathbb{E}_F(\tau_2;\nu_2=1)\mathbb{E}_S(\tau_1;\nu_1=1)|\psi_0\rangle \\ &= \int_{B_{\bar{m}_0}} d^4k'\, |k'\rangle \int_{B_{\bar{m}_0}} d^4k\, a(k) \int_{\Delta_T} d^4x\, \langle k'|x\rangle\langle x|k\rangle\,.\end{aligned} \tag{90}$$

Detection of the particle is the next step of the evolution. The detector measures the four-momentum κ of the particle, so the last evolution operator will be the following set of projections:

$$\mathbb{E}_D(\tau_3;\nu_3=\kappa) = |\kappa\rangle\langle\kappa|\,, \tag{91}$$

where $|\kappa\rangle$ is a generalized eigenstate of the four-momentum operator. The action of the operators (91) results in the final state:

$$\begin{aligned}&\mathbb{E}_D(\tau_3;\nu_3=\kappa)\mathbb{E}_F(\tau_2;\nu_2=1)\mathbb{E}_S(\tau_1;\nu_1=1)|\psi_0\rangle \\ &= |\kappa\rangle \int_{B_{\bar{m}_0}} d^4k'\, \langle\kappa|k'\rangle \int_{B_{\bar{m}_0}} d^4k\, a(k) \int_{\Delta_T} d^4x\, \langle k'|x\rangle\langle x|k\rangle \\ &= |\kappa\rangle \int_{B_{\bar{m}_0}} d^4k'\, \delta^4(\kappa-k') \int_{B_{\bar{m}_0}} d^4k\, a(k) \int_{\Delta_T} d^4x\, \langle\kappa|x\rangle\langle x|k\rangle\,,\end{aligned} \tag{92}$$

where the Dirac delta appears due to the orthogonality of the momenta.

It is worth mentioning that Equation (92) represents only one, chosen by us, possible path of evolution, the other possibilities do not lead to the time interference phenomenon.

To evaluate the expression (92) we notice that the integration over k' is equal to zero if $\kappa \notin B_{\bar{m}_0}$ and is equal to one if $\kappa \in B_{\bar{m}_0}$. We account for this fact by introducing the function $\mathrm{id}_{B_{\bar{m}_0}}(\kappa)$, defined as

$$\mathrm{id}_{B_{\bar{m}_0}}(\kappa) = \int_{B_{\bar{m}_0}} d^4k'\, \delta^4(\kappa-k') = \begin{cases} 1 & \text{if } \kappa \in B_{\bar{m}_0} \\ 0 & \text{if } \kappa \notin B_{\bar{m}_0} \end{cases}. \tag{93}$$

Let the spacetime coordinates of the opened slit be in the form

$$\begin{aligned}\Delta_i &= \left(t_i - \frac{\delta_T}{2}, t_i + \frac{\delta_T}{2}\right) \\ &\times \left(x_s^1 - \frac{\delta_1}{2}, x_s^1 + \frac{\delta_1}{2}\right) \times \left(x_s^2 - \frac{\delta_2}{2}, x_s^2 + \frac{\delta_2}{2}\right) \times \left(x_s^3 - \frac{\delta_3}{2}, x_s^3 + \frac{\delta_3}{2}\right).\end{aligned} \tag{94}$$

Since the scalar product $\langle x|k\rangle = \exp(-ik_\mu x^\mu)/(4\pi^2)$, the integration over x in (92) takes the form

$$\int_{\Delta_T} d^4x\, \langle\kappa|x\rangle\langle x|k\rangle = \left(\frac{1}{\sqrt{2\pi}}\right)^8 \int_{\Delta_T} d^4x\, e^{-i(k_\mu-\kappa_\mu)x^\mu}$$
$$= \left(\frac{1}{2\pi}\right)^4 \left(\int_{\Delta_1} d^4x\, e^{-i(k_\mu-\kappa_\mu)x^\mu} + \int_{\Delta_2} d^4x\, e^{-i(k_\mu-\kappa_\mu)x^\mu}\right). \tag{95}$$

The integrals in (95) can be evaluated using (94),

$$\int_{\Delta_i} d^4x\, e^{-i(k_\mu-\kappa_\mu)x^\mu} = \delta_T\delta_1\delta_2\delta_3 e^{-i(k_0-\kappa_0)t_i} e^{-i(k_1-\kappa_1)x_s^1} e^{-i(k_2-\kappa_2)x_s^2} e^{-i(k_3-\kappa_3)x_s^3}$$
$$\times j_0\left(\frac{k_0-\kappa_0}{2}\delta_T\right) j_0\left(\frac{k_1-\kappa_1}{2}\delta_1\right) j_0\left(\frac{k_2-\kappa_2}{2}\delta_2\right) j_0\left(\frac{k_3-\kappa_3}{2}\delta_3\right), \tag{96}$$

where $j_0(z) = \sin(z)/z$ is the spherical Bessel function of the first kind.

The measurable quantity, i.e., the probability density $\mathrm{Prob}(\kappa)$ of detecting a particle with the four-momentum κ, is given by the modulus squared of the expression (92). Using Equations (93), (95) and (96), we obtain

$$\mathrm{Prob}(\kappa) = (\delta_T\delta_1\delta_2\delta_3)^2 \left(\frac{1}{2\pi}\right)^8 \mathrm{id}_{B_{\bar{m}_0}}(\kappa)$$
$$\times \left| \int_{B_{\bar{m}_0}} d^4k\, a(k) \left(e^{-i(k_0-\kappa_0)t_1} + e^{i(k_0-\kappa_0)t_2}\right) e^{i(\vec{k}-\vec{\kappa})\vec{x}_s} \right. \tag{97}$$
$$\left. \times j_0\left(\frac{k_0-\kappa_0}{2}\delta_T\right) j_0\left(\frac{k_1-\kappa_1}{2}\delta_1\right) j_0\left(\frac{k_2-\kappa_2}{2}\delta_2\right) j_0\left(\frac{k_3-\kappa_3}{2}\delta_3\right) \right|^2.$$

The interference term $e^{-i(k_0-\kappa_0)t_1} + e^{i(k_0-\kappa_0)t_2}$ can be rewritten after the variable change,

$$t_1 = T_s - \frac{\epsilon_T}{2}, \qquad t_2 = T_s + \frac{\epsilon_T}{2}, \tag{98}$$

leading to

$$\mathrm{Prob}(\kappa) = 4(\delta_T\delta_1\delta_2\delta_3)^2 \left(\frac{1}{2\pi}\right)^8 \mathrm{id}_{B_{\bar{m}_0}}(\kappa)$$
$$\times \left| \int_{B_{\bar{m}_0}} d^4k\, a(k) e^{-i(k_0-\kappa_0)T_s} e^{i(\vec{k}-\vec{\kappa})\vec{x}_s} \cos\left(\frac{k_0-\kappa_0}{2}\epsilon_T\right) j_0\left(\frac{k_0-\kappa_0}{2}\delta_T\right) \right.$$
$$\left. \times j_0\left(\frac{k_1-\kappa_1}{2}\delta_1\right) j_0\left(\frac{k_2-\kappa_2}{2}\delta_2\right) j_0\left(\frac{k_3-\kappa_3}{2}\delta_3\right) \right|^2. \tag{99}$$

We evaluate the exact expression (99) assuming that the initial spatial momentum of the particle is directed along the z axis, $\vec{p} = (0, 0, k_z)$, i.e., the covariant components are $p_1 = 0, p_2 = 0, p_3 = -k_z$. In this case the profile $a(k)$ takes the form

$$a(k) = \tilde{a}(m^2)\delta(k_1)\delta(k_2)\delta(k_3 - k_z). \tag{100}$$

By changing the integration variables, the modulus squared in (99) reduces to

$$\left[j_0\left(\frac{\kappa_1}{2}\delta_1\right) j_0\left(\frac{\kappa_2}{2}\delta_2\right) j_0\left(\frac{k_z+\kappa_3}{2}\delta_3\right)\right]^2 \left| \int_{(\bar{m}_0-\frac{\Gamma}{2})^2}^{(\bar{m}_0+\frac{\Gamma}{2})^2} d(m^2) \frac{\tilde{a}(m^2)}{2\sqrt{m^2+k_z^2}} \right.$$
$$\left. \times e^{-i\sqrt{m^2+k_z^2}T_s} \cos\left(\frac{\sqrt{m^2+k_z^2}-\kappa_0}{2}\epsilon_T\right) j_0\left(\frac{\sqrt{m^2+k_z^2}-\kappa_0}{2}\delta_T\right) \right|^2 \tag{101}$$

This expression can be simplified further if we assume that the mass spread (84) is small, i.e., $\Gamma \approx 0$. In this case the integration can be approximated using the integral mean value theorem. As a result we obtain

$$\begin{aligned}\mathrm{Prob}(\kappa) \approx \mathcal{N}\left[j_0\left(\frac{\kappa_1}{2}\delta_1\right)j_0\left(\frac{\kappa_2}{2}\delta_2\right)j_0\left(\frac{k_z+\kappa_3}{2}\delta_3\right)\right]^2 \\ \times\, \mathrm{id}_{B_{\bar{m}_0}}(\kappa)\left[\cos\left(\frac{\sqrt{\bar{m}_0^2+k_z^2}-\kappa_0}{2}\epsilon_T\right)j_0\left(\frac{\sqrt{\bar{m}_0^2+k_z^2}-\kappa_0}{2}\delta_T\right)\right]^2 \end{aligned} \tag{102}$$

where we have introduced $\mathcal{N}$ as the overall normalization factor. In our construction this factor does not normalize (102) to 1 because not all particles are assumed to pass the slit, see Equations (86) and (87). We may however normalize (102) for all particles measured by the detector, obtaining

$$\int_{\mathbb{R}^4} d^4\kappa\ \mathrm{Prob}(\kappa) = 1\,. \tag{103}$$

7.2. A Numerical Example

As an example, let us discuss the π^+ meson in a setup with no sources of the electromagnetic interactions. During the numerical calculations the natural units $c = \hbar = 1$ will be used.

The particle is produced with the initial three-momentum $\vec{k} = (0, 0, -k_z)$. Travelling towards the detector, the particle has to pass a slit which opens twice in the same spatial location. The slit has spatial widths δ_1 and δ_2 in the plane perpendicular to the direction of motion. The width δ_3 is less important, because the highest probability for the detector to register the pion is for the incoming momentum $\kappa_3 = -k_z$, for which the term $j_0((k_z + \kappa_3)\delta_3/2) = 1$ drops out. We denote the time width of the opened slit by δ_T, and the time between the two openings by ϵ_T.

The mass of π^+ is $m_\pi \approx 139$ MeV. Its half-life is $t_\pi = 3.95 \cdot 10^7\ \mathrm{eV}^{-1}$, implying the mass spread of the order of $\Gamma \sim 1/t_\pi \approx 2.5 \cdot 10^{-8}$ eV. Because m_π and Γ differ by sixteen orders of magnitude, the pion is almost exactly on its mass shell; consequently, $B_{\bar{m}_0}$ is just the Klein–Gordon condition, and the function $\mathrm{id}_{B_{\bar{m}_0}}(\kappa) = 1$ becomes trivial. If the initial Klein–Gordon state of the particle is given by $k_0^2 = m_\pi^2 + k_z^2$, the state seen by the detector will be $\kappa_0^2 = m_\pi^2 + \kappa_1^2 + \kappa_2^2 + k_z^2$. Taking this into account, the non-normalized probability, as a function of κ_1 and κ_2, is given by the expression:

$$\begin{aligned}\mathrm{Prob}(\kappa_1,\kappa_2) \approx j_0^2\left(\frac{\kappa_1}{2}\delta_1\right)j_0^2\left(\frac{\kappa_2}{2}\delta_2\right) \\ \times \cos^2\left(\frac{\sqrt{m_\pi^2-k_z^2}-\sqrt{m_\pi^2+\kappa_1^2+\kappa_2^2-k_z^2}}{2}\epsilon_T\right) \\ \times j_0^2\left(\frac{\sqrt{m_\pi^2-k_z^2}-\sqrt{m_\pi^2+\kappa_1^2+\kappa_2^2-k_z^2}}{2}\delta_T\right). \end{aligned} \tag{104}$$

The detection probability (104) is plotted in Figure 2. The slits are 0.01 mm wide in the x and y directions. The time parameter ϵ_T ranges from 10^{-7} to 10^{-12} s, whereas δ_T is set to $\delta_T = \epsilon_T/3$. The momentum k_z is a constant number and does not play any significant role. For long opening times, the detection is only possible for very small perpendicular momenta κ_1 and κ_2. For smaller ϵ_T the inner region widens and starting from $\epsilon_T = 10^{-10}$ s higher-order maxima start to appear. They are clearly visible for $\epsilon_T = 10^{-10}$ s and shorter times.

The interference effect presented in Figure 2 comes from two sources, the spatial diffraction on the slits and the temporal interference. In Figure 3 we have drawn density

plots of the temporal part of Equation (104). For shorter opening times the axes have been re-scaled to show that the maxima appear for higher values of the momenta. Even though we have not used the Heisenberg-like condition for energy and time, high momenta, and thus high energy, are needed for short opening times. It follows from Equation (104) that the temporal part is dependent on $\kappa_1^2 + \kappa_2^2$; therefore, circles appear on the plots in Figure 3. The comparison of these diagrams with those in Figure 2 reveals that the spatial part forces the maxima to appear along the $\kappa_1 = 0$ and $\kappa_2 = 0$ directions, dominating the temporal effects.

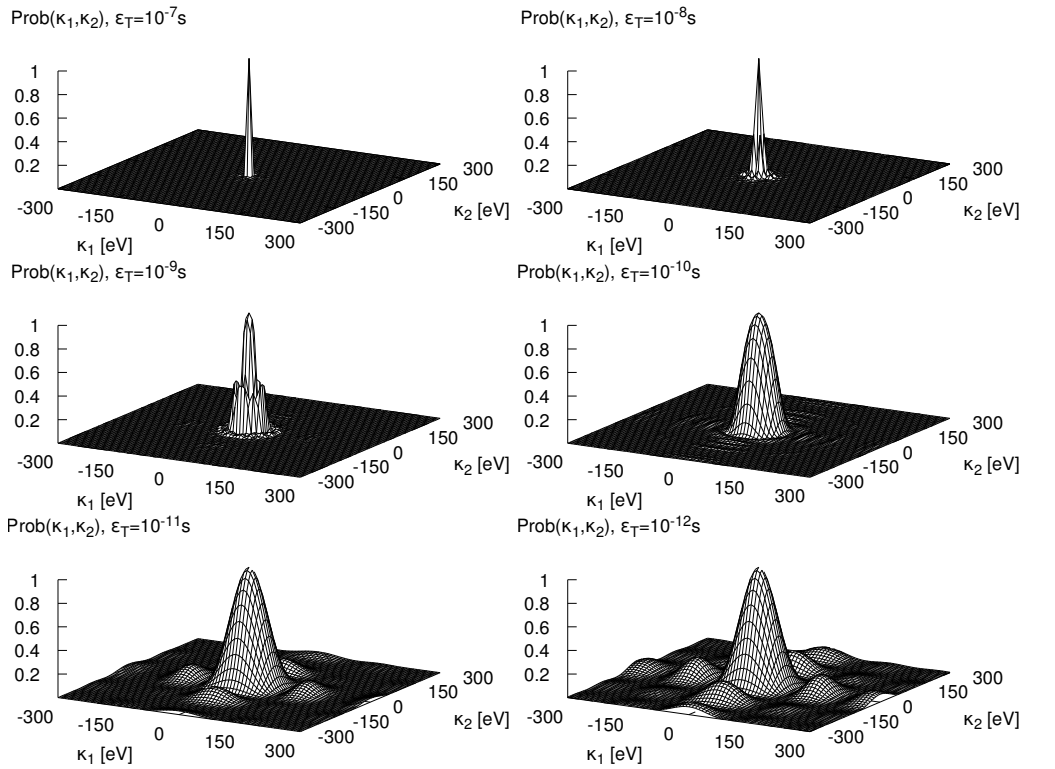

Figure 2. The detection probability as a function of κ_1 and κ_2 for different values of the opening times.

One may make the temporal effect visible by manipulating the spatial widths of the slit. In Equation (104), for small δ_1 and δ_2 the Bessel functions are close to one and do not suppress the temporal part. We show this on Figure 4, where the temporal parameters are set to $\epsilon_T = 10^{-14}$ s and $\delta_T = \epsilon_T/3$. The spatial size of the slit is $\delta_1 = \delta_2 = d$. For $d = 10^{-8}$ mm the circular maxima are clearly visible. For larger slits, $d = 10^{-7}$ mm, this changes into the four-fold shape dictated by the Bessel functions.

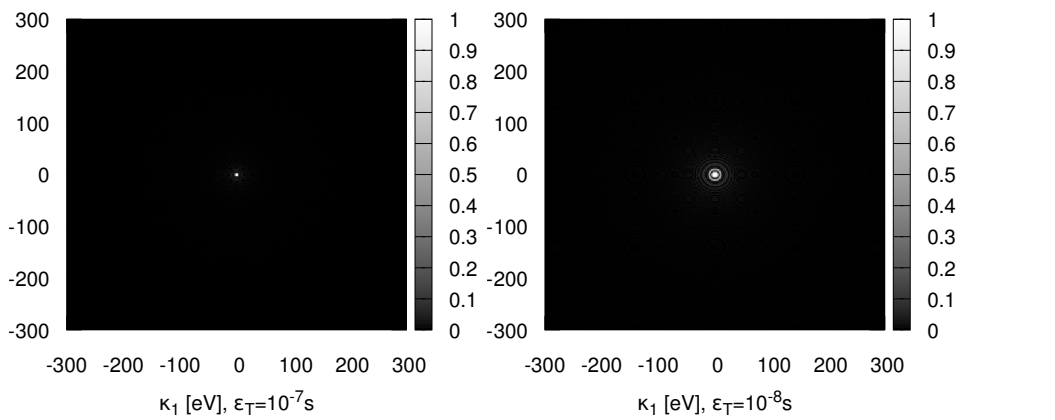

Figure 3. *Cont.*

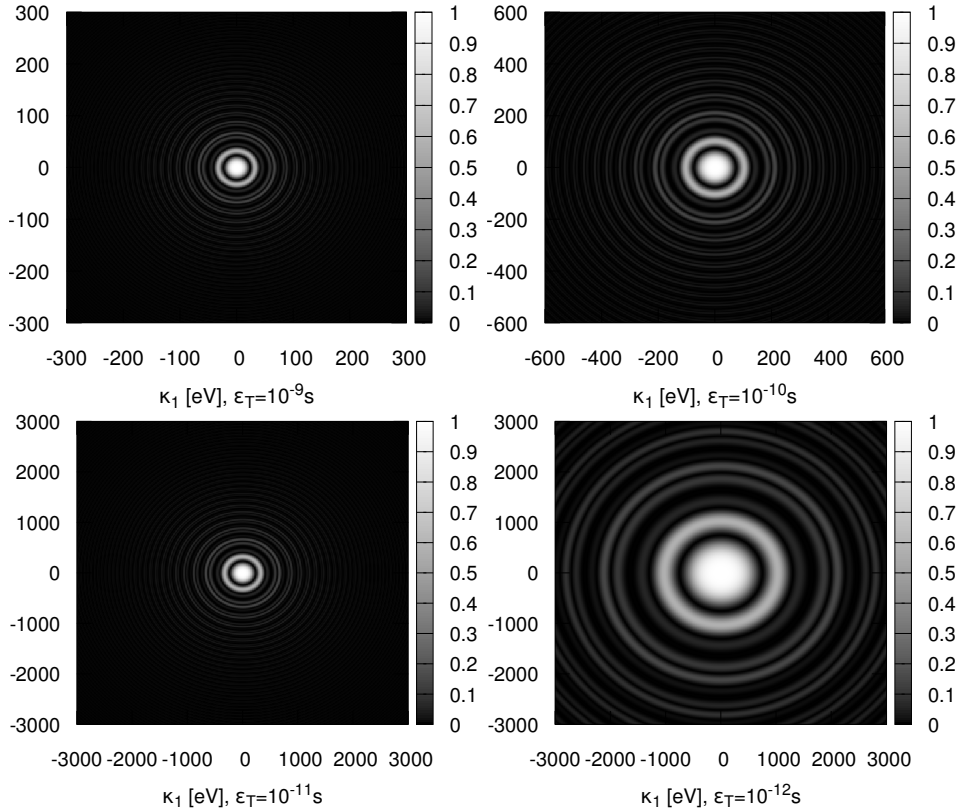

Figure 3. The temporal part of $\mathrm{Prob}(\kappa_1, \kappa_2)$ for different values of the opening times ϵ_T.

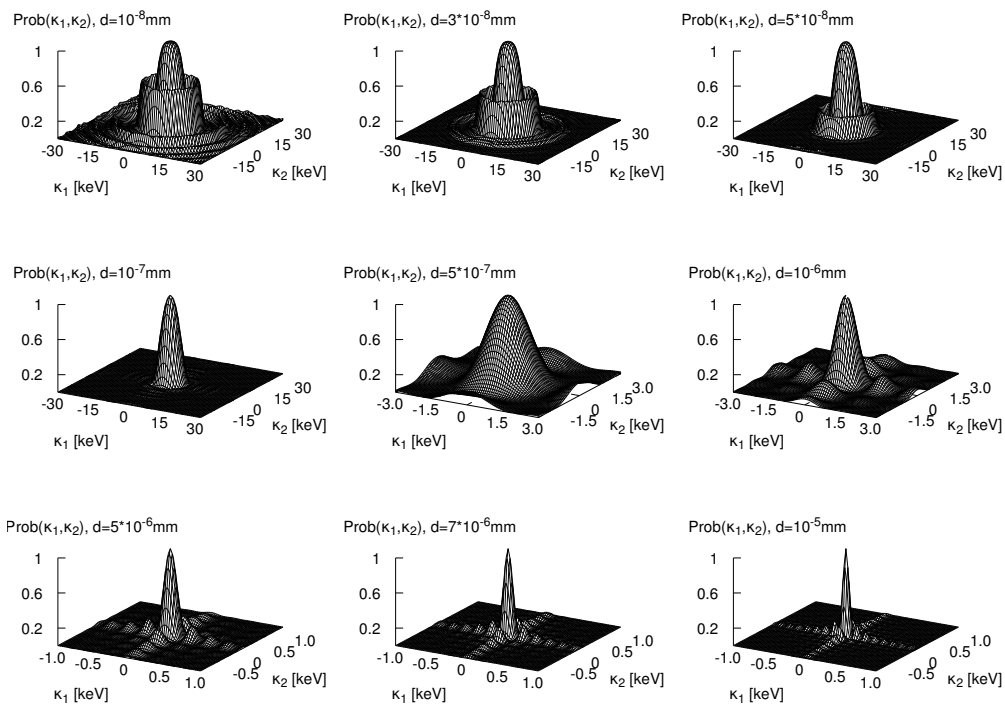

Figure 4. The detection probability as a function of κ_1 and κ_2 for different spatial widths of the slit. Here, $d = \delta_1 = \delta_2$. The time between the openings is set to $\epsilon_T = 10^{-14}$ s and the slit is open for $\delta_T = \epsilon_T/3$.

8. Concluding Remarks

The discussion of the structure and role of time is as long as the history of physics. A collection of papers devoted to different aspects of the physical time from the modern perspective can be found in [80,97]. The authors of [80] mention three types of time.

The most popular one is time considered as a parameter which is measured by an external laboratory clock, uncoupled from the measured system. This time is called the external time. In addition, time can be defined through the dynamics of the observed quantum systems, termed dynamic (or intrinsic) time. Lastly, time can be considered on the same footing as other quantum observables, especially as positions in space. This was called by P. Busch the observable (or event) time. It represents the approach considered in this paper in which we discuss the most natural model of quantum spacetime where time is a quantum observable treated on the same footing as 3D space position observables. Time considered here is an essential component of the position in spacetime. It is also important to note that it allows the temporal characteristics of a quantum system to be calculated on the same basis as for other observables.

In experiments the external time is usually used. This is introduced by constructing different kinds of semi-macroscopic clocks (because of the required interface with the macroscopic world). They are constructed to be uncoupled from the analysed physical phenomenon. Because the state of the clock in the PEv approach at the evolution step τ_n is, in principle, described by its density operator $\rho_c(\tau_n;\nu)$, this type of clock seems to keep the ordering relation in the set of all values of the evolution parameter τ, i.e., it has to fulfil the relation $\mathrm{Tr}(\hat{t}\rho_c(\tau_{n+1};\nu')) > \mathrm{Tr}(\hat{t}\rho_c(\tau_n;\nu))$, where $\hat{t}$ is the time operator. The trace $\mathrm{Tr}(\hat{t}\rho_c(\tau_n;\nu))$ denotes the expectation value of the temporal position in comparison to the average position of an object in 3D space, i.e., time measured by the clock in the state $\rho_c(\tau_n;\nu)$ at the evolution step τ_n. Having one clock, one can treat it as the standard clock. All other clocks can be constructed and synchronized to this standard clock. In this context the external time, even though very useful, is a conventional rather than a physical entity.

The intrinsic time, or times to be more precise, is determined by a set of appropriate dynamic variables. This is compatible with our "change principle", i.e., the change in states or observables are more fundamental than the time itself. However, because the physical time in our approach is a quantum observable, the required characteristic times (intrinsic times) for a given physical process can be directly calculated. In this context, the intrinsic times are not fundamental but derivable temporal observables.

In the PEv approach the spacetime is "created" in the same way as other quantum properties of our Universe. The positions in the spacetime are represented by the preferred states of the spacetime position measure. This implies that the PEv approach leads to the background-free theory. Such an approach is important not only for particle physics but also for classical and quantum relativity, and for unification of quantum mechanics with gravity.

The proposed structureless Minkowski space is the simplest background spacetime, and is used in many physical theories. In this case the self-adjoint spacetime position operator transforms to any four-vector with respect to the Poincaré group and generates an appropriate orthogonal operator-valued measure. Obviously, real physical devices corresponding to the resulting measure cannot be ideal, and in practice this measure has to be replaced by POVM-type operators. An interesting discussion, especially about clocks can be found in [53–55] and the references therein. A part of this discussion should be revisited in the context of PEv; however, this is a subject for further studies. A related problem which requires further analysis is the appearance of spacetime frames which are natural constructions in the PEv model, which support the idea of general covariance up to the transformations allowed among the quantum observables [98–103].

The PEv model demonstrates some new quantum effects. One of the most important phenomenon is the time interference described in Section 7. The experiment [78] was performed at very short, attosecond laser pulses. These time scales can be found in nuclear physics, where many nuclear processes have characteristic times a few orders of magnitude shorter. The main criterium for the existence of temporal interference is that the time width of a given physical process has to be larger than the temporal distance between the openings of the slits.

The time operator and the corresponding conjugate temporal momentum operator are natural complements to the covariant relativistic four-position and four-momentum operators. The temporal component $\hat{p}_0$ of the momentum operator is also responsible for the basic time arrow represented by the operator $\hat{p}_0/|\hat{p}_0|$, i.e, the sign of the temporal momentum determines the direction along the time axis.

The corresponding components of the spacetime position operator and the four-momentum operator fulfil the canonical commutation relations and as a consequence they obey the standard uncertainty Heisenberg relations. In addition, through the equations of motion (which usually involve the temporal momentum) this introduces the time–energy uncertainty relation.

One needs to notice that observable time also allows, in a very natural way, for the dependence of interactions on temporal distance among particles or more generally among quantum events. A schematic example of this type of potential is shown in [104], but this problem still remains to be solved.

The evolution generators used in the paper are effective tools linking different kinds of conventional equations of motions and the PEv approach. They allow the construction og evolution operators corresponding to the Schrödinger, Klein–Gordon, Dirac and other equations of motions. However, one needs to remember another interpretations of the quantum states in PEv with respect to time.

The idea of the "change principle" and the concept of quantum evolution as a stochastic process driven by the evolution ordering parameter τ, not time, is much more general than the specific model presented in this paper. However, the proposed method seems to be a minimal interpretation satisfying the main requirements concerning the quantum spacetime structure.

A series of new problems mentioned in this paper require further investigation.

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