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Functional Integrals in Geometric Approach to Quantum Theory

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Abstract: In quantum mechanics, one can express the evolution operator and other quantities in terms of functional integrals. The main goal of this paper is to prove corresponding results in geometric approach to quantum theory. We apply these results to the formalism of L-functionals.

Keywords: functional integral; banach space; geometric approach; MISC 81T

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1. Introduction

In geometric approach [1–3] the evolution operator of physical system obeys the equation of motion

$$\frac{d\sigma}{dt} = H(t)\sigma(t), \tag{1}$$

where $H(t)$ is a linear operator acting in Banach space (or, more, generally, in complete topological vector space) $\mathcal{L}$. We say that $H(t)$ is the "Hamiltonian" of the physical system. In what follows we assume that $H(t) = H$ does not depend on time t. This condition is imposed only to simplify notations; all results can be proved also for time- dependent "Hamiltonian".

In the standard approach to quantum mechanics the evolution operator acts in Hilbert space; it obeys the Equation (1) where $H(t)$ is a skew-adjoint operator. It can be represented by a functional integral. One of the ways to obtain such a representation is based on the notion of a symbol of an operator; this way was suggested by F. Berezin [4] (see [5] for details). We use the same ideas to obtain a representation of the evolution operator in Banach spaces (or in topological vector spaces) in terms of functional integrals. Such a representation was considered in numerous mathematical papers. (See, in particular, [6–9]. Our approach is close to the ideas of these papers.) Our main result is a construction of functional integrals in the formalism of L-functionals (Section 5).

A symbol of an operator A is a function $\underline{A}$ defined on some measure space. It should depend linearly on A. We assume that the symbol of the identity operator 1 is equal to 1 and the composition of operators corresponds to the operation on symbols denoted by $*$: if $C = AB$ then $\underline{C} = \underline{A} * \underline{B}$.

The simplest way to construct symbols of operators in quantum mechanics is to use the fact that the Fourier transform of delta-function is a constant. The matrix (the kernel in the language of mathematics) of the unit operator is $\langle \mathbf{q}_2|1|\mathbf{q}_1\rangle = \delta(\mathbf{q}_1 - \mathbf{q}_2)$ in coordinate representation and $\langle \mathbf{p}_2|1|\mathbf{p}_1\rangle = \delta(\mathbf{p}_1 - \mathbf{p}_2)$ in momentum representation. Taking Fourier transform of matrix $\langle \mathbf{q}_2|A|\mathbf{q}_1\rangle$ of the operator A with respect to the variable $\mathbf{q}_1 - \mathbf{q}_2$ we obtain $p - q$ symbol:

$$\underline{A}^{p-q}(\mathbf{p}, \mathbf{q}) = \int d\mathbf{y}\langle \mathbf{y}|A|\mathbf{q}\rangle e^{i\mathbf{p}(\mathbf{q}-\mathbf{y})}.$$

Similarly taking Fourier transform of $\langle \mathbf{p}_2|A|\mathbf{p}_1\rangle$ with respect to variable $\mathbf{p}_1 - \mathbf{p}_2$ we obtain $q-p$-symbol:

$$\underline{A}^{q-p}(\mathbf{q},\mathbf{p}) = \int d\mathbf{y}\langle \mathbf{y}|A|\mathbf{p}\rangle e^{-i\mathbf{q}(\mathbf{p}-\mathbf{y})}.$$

If A is a differential operator with polynomial coefficients we can express it as a polynomial of operators $\hat{q}^j$ (operators corresponding to the coordinates q^j) and $\hat{p}_j = \frac{1}{i}\frac{\partial}{\partial q^j}$ (momentum operators) Representing A in $q-p$ form (coordinate operators from the left of momentum operators) and "removing hats" we obtain $q-p$-symbol. Representing A in $p-q$-form (placing momentum operators from the left of coordinate operators) and "removing hats" we get $p-q$ symbol.

Notice that in our notations $\hbar = 1$. Sometimes it it is convenient to consider families of symbols $\underline{A}_\hbar^{q-p}(\mathbf{q},\mathbf{p})$ and $\underline{A}_\hbar^{p-q}(\mathbf{p},\mathbf{q})$ depending on parameter $\hbar$.

We will describe a very general construction of symbols. We will use this construction to represent physical quantities in terms of functional integrals. Our results generalize the results by F. Berezin [4,5] proved for operators in Hilbert spaces. They can be applied also to coherent states [10] and their generalizations.

We did not try to give rigorous proofs of our results, however imposing some conditions one can make our exposition rigorous (for example, using the ideas of [6–8]). We use our general results to obtain a representation of physical quantities in terms of functional integrals in the formalism of L-functionals [11–13].

2. Functional Integrals

Let us consider symbols of linear operators acting in the space $\mathcal{L}$.

The evolution operator can be represented in the form

$$\sigma(t) = e^{tH} = \lim_{N\to\infty}(1+\frac{tH}{N})^N.$$

For $N\to\infty$ the symbol of the operator $1+\frac{tH}{N}$ can be approximated by $\exp\frac{t}{N}\underline{H}$:

$$\underline{1+\frac{tH}{N}} = e^{\frac{t}{N}\underline{H}} + O(N^{-2}).$$

Using this relation we obtain an expression for the symbol of the evolution operator;

$$\underline{\sigma(t)} = \lim_{N\to\infty} I_N(t), \tag{2}$$

where

$$I_N(t) = e^{\frac{t}{N}\underline{H}} * \ldots * e^{\frac{t}{N}\underline{H}}$$

(N factors).

In many cases, $I_N(t)$ can be interpreted as an approximation of a functional integral. Notice, however, that even without this interpretation, we can apply the Laplace or stationary phase method to the calculation of $I_N(t)$. This allows us to obtain some results that often are obtained in the language of functional integrals without using this language.

Let us consider a class of symbols generalizing $q-p$ symbols and Wick symbols of quantum mechanics.

We start with complete topological vector space $\mathcal{L}$.

We assume that the symbol of an operator A acting in $\mathcal{L}$ is a (generalized) function $\underline{A}(\alpha,\beta')$ of two variables (a function on $\mathcal{M}\times\mathcal{M}'$) and that the symbol of the product $C = AB$ of operators A and B can be expressed in terms of the symbols of operators A and B by the formula

$$\underline{C}(\alpha,\beta') = \int d\gamma d\gamma' \underline{B}(\alpha,\gamma')R(\gamma,\gamma')\underline{A}(\gamma,\beta')e^{c(\alpha,\gamma')+c(\gamma,\beta')-c(\alpha,\beta')}, \tag{3}$$

where $c(\gamma,\gamma'), R(\gamma,\gamma')$ are functions on $\mathcal{M}\times\mathcal{M}'$ and $d\gamma d\gamma'$ is the measurement on this space. (We assume that $\mathcal{M}\times\mathcal{M}'$ is a measure space. This assumption can be weakened to allow infinite-dimensional spaces with integration defined for some class of functions on these spaces.) Taking $A=B=C=1$ in (3) we obtain the following restriction on these functions:

$$1=\int d\gamma d\gamma' e^{c(\alpha,\gamma')+c(\gamma,\beta')-c(\alpha,\beta')}R(\gamma,\gamma'). \tag{4}$$

If $R(\gamma,\gamma')$ is represented in the form

$$R(\gamma,\gamma')=e^{-r(\gamma,\gamma')}$$

the formulas ((3) and (4)) can be rewritten as

$$\underline{C}(\alpha,\beta')=\int d\gamma d\gamma' \underline{B}(\alpha,\gamma')\underline{A}(\gamma,\beta')e^{c(\alpha,\gamma')+c(\gamma,\beta')-c(\alpha,\beta')-r(\gamma,\gamma')}, \tag{5}$$

$$1=\int d\gamma d\gamma' e^{c(\alpha,\gamma')+c(\gamma,\beta')-c(\alpha,\beta')-r(\gamma,\gamma')}. \tag{6}$$

It follows that the symbol $\underline{C}(\alpha,\gamma')$ of the product C of N operators $A_1,\dots,A_N$ is given by the formula

$$\underline{C}(\gamma,\gamma')=\int d\gamma_1 d\gamma_1'\dots d\gamma_{N-1}d\gamma_{N-1}'\underline{A}_N(\gamma,\gamma_{N-1}')\dots\underline{A}_2(\gamma_2,\gamma_1')\underline{A}_1(\gamma_1,\gamma')e^{\rho_N}, \tag{7}$$

where

$$\rho_N=c(\gamma,\gamma_{N-1}')+c(\gamma_{N-1},\gamma_{N-2}')+\dots+c(\gamma_1,\gamma')-c(\gamma,\gamma')-r(\gamma_1,\gamma_1')-\dots-r(\gamma_{N-1},\gamma_{N-1}'). \tag{8}$$

We see that in our case

$$I_N(t)=\int d\gamma_1 d\gamma_1'\dots d\gamma_{N-1}d\gamma_{N-1}' e^{\frac{t}{N}(\underline{H}(\gamma,\gamma_{N-1}')+\underline{H}(\gamma_{N-1},\gamma_{N-2}')+\dots+\underline{H}(\gamma_1,\gamma'))}e^{\rho_N}. \tag{9}$$

Notice that assuming that operators at hand have trace we can express the trace in terms of symbols:

$$TrA=\int d\alpha d\beta' \underline{A}(\alpha,\beta')e^{\tau(\alpha,\beta')}, \tag{10}$$

where $\tau(\alpha,\beta')=c(\alpha,\beta')-r(\alpha,\beta')$. To justify this formula we verify that $TrAB=TrBA$ using (5). (A trace on an algebra is defined as a linear functional that vanishes on commutators. We are checking that the RHS of (10) is a trace in this general sense. It seems that in our situation this property specifies the trace up to a numerical factor).

Using (10) we obtain that

$$Tre^{tH}=\lim_{N\to\infty}J_N(t), \tag{11}$$

where

$$J_N(t)=\int d\gamma d\gamma' d\gamma_1 d\gamma_1'\dots d\gamma_{N-1}d\gamma_{N-1}\exp\left(\frac{t}{N}(\underline{H}(\gamma,\gamma_{N-1}')+\dots+\underline{H}(\gamma_1,\gamma'))\right)e^{\tilde{\rho}_N}, \tag{12}$$

$$\tilde{\rho}_N=c(\gamma,\gamma_{N-1}')+\dots+c(\gamma_1,\gamma')-r(\gamma,\gamma')-r(\gamma_1,\gamma_1')-\dots-r(\gamma_{N-1},\gamma_{N-1}'). \tag{13}$$

In the case when $c(\alpha,\beta')$ is a quadratic function one can prove that one of solutions of the relation (6) has the form

$$c(\alpha,\beta')=r(\alpha,\beta')+const,$$

where the constant can be absorbed in the definition of the measure $d\gamma d\gamma'$.

Let us show that one can use (5) to express the symbol of the evolution operator in terms of functional integrals. We assume that $\mathcal{M}$ and $\mathcal{M}'$ are smooth manifolds, the function c is differentiable, and $r = c$). Then the symbol $\underline{\sigma(t)}(\gamma, \gamma')$ of operator $\sigma(t) = e^{tH}$ can be represented as a functional integral:

$$I(\gamma, \gamma') = \int \prod d\gamma(\tau) d\gamma'(\tau) e^{S[\gamma(\tau), \gamma'(\tau)]}, \tag{14}$$

where

$$S[\gamma(\tau), \gamma'(\tau)] = S_0[\gamma(\tau), \gamma'(\tau)] + c(\gamma(t), \gamma') - c(\gamma, \gamma'), \tag{15}$$

$$S_0 = \int_0^t \left(\underline{H}[\gamma(\tau), \gamma'(\tau)] - \dot{\gamma}'(\tau) \frac{\partial}{\partial \gamma'(\tau)} c(\gamma(\tau), \gamma'(\tau)) \right) d\tau. \tag{16}$$

This integral depends on $\mathcal{M}$-valued function $\gamma(\tau)$ and $\mathcal{M}'$-valued function $\gamma'(\tau)$. Here $0 \le \tau \le t$ and we integrate over the set of functions obeying boundary conditions $\gamma(0) = \gamma, \gamma'(t) = \gamma'$. To prove this statement we notice that the expression

$$\frac{t}{N} (\underline{H}(\gamma, \gamma'_{N-1}) + \underline{H}(\gamma_{N-1}, \gamma'_{N-2}) + \ldots + \underline{H}(\gamma_1, \gamma')) + \rho_N$$

approximates the integral sum for the integral (16). (To define the functional integral we represent it as a limit of finite-dimensional integrals. This definition depends on the choice of approximation of functional integral by finite-dimensional integrals).

The last two terms in (15) cancel in the formula for the trace of the operator e^{tH}. Using this remark or formulas (11)–(13) we obtain

$$Tr(e^{tH}) = \int \prod d\gamma(\tau) d\gamma'(\tau) e^{S_0[\gamma(\tau), \gamma'(\tau)]}, \tag{17}$$

where S_0 is given by the formula (16).

(We integrate over the set of functions obeying boundary conditions $\gamma(0) = \gamma(t)$, $\gamma'(0) = \gamma'(t)$).

It is easy to check that the formula (3) is valid for $p - q$-symbols in n-dimensional space with:

$$\gamma = \mathbf{p}; \;\; \gamma' = \mathbf{q}; \;\; c(\gamma, \gamma') = -i(\mathbf{p}, \mathbf{q}); \;\; r(\gamma, \gamma') = -i(\mathbf{p}, \mathbf{q}); \;\; d\gamma d\gamma' = d^n \mathbf{p} d^n \mathbf{q} / (2\pi)^n$$

and for $q - p$-symbols with:

$$\gamma = \mathbf{q}; \;\; \gamma' = \mathbf{p}; \;\; c(\gamma, \gamma') = i(\mathbf{p}, \mathbf{q}); \;\; r(\gamma, \gamma') = i(\mathbf{p}, \mathbf{q}).$$

Using (9) we can get functional integrals of quantum mechanics.

One more case when it is possible to obtain functional integrals from (9) and (11) is the situation when $\mathcal{M} = \mathcal{M}'$, $R(\alpha, \beta') = \delta(\alpha, \beta')$ and $c(\gamma, \gamma) \equiv 0$. In this situation we have

$$I_N(t) = \int d\gamma_1 \ldots d\gamma_{N-1} \exp(\frac{t}{N} (\underline{H}(\gamma, \gamma_{N-1}) + \underline{H}(\gamma_{N-1}, \gamma_{N-2}) + \ldots + \underline{H}(\gamma_1, \gamma'))) e^{\rho_N}, \tag{18}$$

where

$$\rho_N = c(\gamma, \gamma_{N-1}) + c(\gamma_{N-1}, \gamma_{N-2}) + \ldots + c(\gamma_1, \gamma') - c(\gamma, \gamma')). \tag{19}$$

When $N \to \infty$

$$c(\gamma, \gamma') = c(\gamma, \gamma + \Delta\gamma) \approx \Delta\gamma \frac{\partial}{\partial \tilde{\gamma}} c(\gamma, \tilde{\gamma})|_{\tilde{\gamma} = \gamma}$$

$$I_N(t) \simeq \int d\gamma_1 \ldots d\gamma_{N-1} \exp(\frac{t}{N} \sum_i (\underline{H}(\gamma_i, \gamma_i) + \frac{t}{N} \sum_i \dot{\gamma}_i \frac{\partial}{\partial \tilde{\gamma}} c(\gamma_i, \tilde{\gamma})|_{\tilde{\gamma} = \gamma_i} - c(\gamma, \gamma')). \tag{20}$$

It follows that the symbol $\underline{\sigma(t)}(\gamma, \gamma')$ of operator $\sigma(t) = e^{tH}$ can be represented as a functional integral

$$I(\gamma, \gamma') = \int \prod d\gamma(\tau) e^{S[\gamma(\tau)]}, \tag{21}$$

where

$$S[\gamma(\tau)] = \int_0^t \left(\underline{H}[\gamma(\tau), \gamma(\tau)] + \dot{\gamma}(\tau) \frac{\partial}{\partial \tilde{\gamma}} c(\gamma(\tau), \tilde{\gamma})|_{\tilde{\gamma}=\gamma(\tau)} \right) d\tau - c(\gamma, \gamma') \tag{22}$$

with $\gamma(0) = \gamma, \gamma(t) = \gamma'$. Similarly in this case

$$Tr(e^{tH}) = \int \prod d\gamma(\tau) e^{S_0[\gamma(\tau)]}, \tag{23}$$

where

$$S_0[\gamma(\tau)] = \int_0^t \left(\underline{H}[\gamma(\tau), \gamma(\tau)] + \dot{\gamma}(\tau) \frac{\partial}{\partial \tilde{\gamma}} c(\gamma(\tau), \tilde{\gamma})|_{\tilde{\gamma}=\gamma(\tau)} \right) d\tau. \tag{24}$$

3. Covariant and Contravariant Symbols

Let us consider Banach spaces $\mathcal{L}$ and $\mathcal{L}'$ and systems of vectors $e_\alpha \in \mathcal{L}$, $e'_{\beta'} \in \mathcal{L}'$. Here $\alpha \in \mathcal{M}, \odot' \in \mathcal{M}', \mathcal{M} \times \mathcal{M}'$ is a measure space. (Again we can consider a more general case when $\mathcal{M} \times \mathcal{M}'$ is a space with integration defined for some class of functions on this space). We assume that linear combinations of vectors e_α are dense in $\mathcal{L}$ and linear combinations of vectors $e'_{\beta'}$ are dense in $\mathcal{L}'$. (In other words these systems of vectors are overcomplete).

Let us fix a non-degenerate pairing $\langle l, l' \rangle$ between $\mathcal{L}$ and $\mathcal{L}'$. (We can consider either bilinear pairing or a pairing that is linear with respect to one argument and antilinear with respect to the second argument).

We assume that

$$\langle l, l' \rangle = \int_{\mathcal{M} \times \mathcal{M}'} dm dm' \langle l, e'_{m'} \rangle R(m, m') \langle e_m, l' \rangle, \tag{25}$$

where $dmdm'$ is the measure on $\mathcal{M} \times \mathcal{M}'$. In different notations, this formula can be written as

$$\int dm dm' \, |e'_{m'}\rangle R(m, m') \langle e_m| = 1. \tag{26}$$

We define covariant symbol $\underline{A}(\alpha, \beta')$ of operator A acting in $\mathcal{L}$ by the formula

$$\underline{A}(\alpha, \beta') = \frac{\langle A e_\alpha, e'_{\beta'} \rangle}{\langle e_\alpha, e'_{\beta'} \rangle}. \tag{27}$$

In bra-ket notations

$$\underline{A}(\alpha, \beta') = \frac{\langle e'_{\beta'} | A | e_\alpha \rangle}{\langle e'_{\beta'} | e_\alpha \rangle}. \tag{28}$$

In particular, we can assume that $\mathcal{L} = \mathcal{L}'$ is a Fock space (Hilbert space of Fock representation of canonical commutation relations) and the overcomplete system of vectors in this space coinsists of eigenvectors of annihilation operators (of Poisson vectors). Then the covariant symbol coincides with Wick symbol.

Recall that the Wick symbol can be defined in the following way. Represent the operator in normal form (creation operators $\hat{a}^*(f)$ from the left, annihilation operators $\hat{a}(g)$ from the right). Remove hats. Resulting polynomial of a^*, a is a Wick symbol of the operator.

Notice that the spaces $\mathcal{L}$ and $\mathcal{L}'$ are on equal footing in our construction; hence we can define the covariant symbol of an operator B acting in $\mathcal{L}'$ in a similar way. We say that operators A and B are dual if $\langle Ax, y\rangle = \langle x, By\rangle$, it is easy to check that symbols of dual operators coincide

$$\underline{B}(\alpha,\beta') = \frac{\langle e_\alpha, Be'_{\beta'}\rangle}{\langle e_\alpha, e'_{\beta'}\rangle} = \underline{A}(\alpha,\beta'). \tag{29}$$

The covariant symbol $\underline{C} = \underline{A} * \underline{B}$ of operator $C = AB$ is given by the formula

$$\underline{C}(\alpha,\beta') = \int dm dm'\, \underline{B}(\alpha, m')\underline{A}(m,\beta')R(m,m')\frac{\langle e_\alpha, e'_{m'}\rangle\langle e_m, e'_{\beta'}\rangle}{\langle e_\alpha, e'_{\beta'}\rangle}. \tag{30}$$

This formula agrees with (3) if we take

$$\langle e_m, e'_{m'}\rangle = e^{c(m,m')}.$$

We define contravariant symbol $\mathring{A}(\alpha,\beta')$ of operator A acting in $\mathcal{L}$ by the formula

$$Ae_\alpha = \int d\gamma d\gamma' \mathring{A}(\gamma,\gamma')\langle e_\alpha, e'_{\gamma'}\rangle R(\gamma,\gamma')e_\gamma. \tag{31}$$

It is easy to express covariant symbols in terms of contravariant symbols

$$\underline{A}(\alpha,\beta') = \int d\gamma d\gamma' \mathring{A}(\gamma,\gamma')\frac{\langle e_\alpha, e'_{\gamma'}\rangle R(\gamma,\gamma')\langle e_\gamma, e'_{\beta'}\rangle}{\langle e_\alpha, e'_{\beta'}\rangle}.$$

To calculate the contravariant symbol $\mathring{C}$ of the product $C = AB$ of operators A, B we notice that

$$Ce_\alpha = \int d\gamma_1 d\gamma'_1 d\gamma_2 d\gamma'_2 \mathring{A}(\gamma_2,\gamma'_2)\langle e_\alpha, e'_{\gamma'_2}\rangle R(\gamma_2,\gamma'_2)\mathring{B}(\gamma_1,\gamma'_1)\langle e_{\gamma_2}, e'_{\gamma'_1}\rangle R(\gamma_1,\gamma'_1)e_{\gamma_1}. \tag{32}$$

Hence $\mathring{C}$ can be expressed in terms of contravariant symbols of factors by the formula

$$\mathring{C}(\gamma,\gamma') = \frac{1}{R(\gamma,\gamma')}\int d\beta d\beta' \langle e_\beta, e'_{\beta'}\rangle \mathring{A}(\beta,\gamma')R(\beta,\gamma')\mathring{B}(\gamma,\beta')R(\gamma,\beta').$$

This expression has the form (5) with

$$e^{c(\gamma,\gamma')} = R(\gamma,\gamma'), \quad e^{-r(\gamma,\gamma')} = \langle e_\gamma, e'_{\gamma'}\rangle. \tag{33}$$

Notice that a bounded operator always has a covariant symbol. Moreover, an operator A has a covariant symbol if it is unbounded but the vectors e_α belong to the domain where A is defined. However, it is non-trivial to say whether there exists an operator with a given covariant symbol. For contravariant symbols, the situation is the opposite. It is easy to give conditions for the existence of an operator with a given contravariant symbol, but if we know an operator it is non-trivial to say whether is has a contravariant symbol.

4. Coherent States

As in Section 3 we consider Banach spaces $\mathcal{L}$ and $\mathcal{L}'$, non-degenerate pairing $\langle l, l'\rangle$ between these spaces , and systems of vectors $e_\alpha \in \mathcal{L}$, $e'_{\beta'} \in \mathcal{L}'$. Here $\alpha \in \mathcal{M}$, $\odot' \in \mathcal{M}'$.

We fix a representation T of Lie group G in the space $\mathcal{L}$ and representation T' in the space $\mathcal{L}'$ in such a way that G acts transitively on the set of vectors e_α and on the set of vectors $e'_{\beta'}$. This means that $\mathcal{M}$ and $\mathcal{M}'$ can be considered as homogeneous spaces: $\mathcal{M} = G/H$ and $\mathcal{M}' = G/H'$. We assume that the representations T' and T are dual: $\langle Tl, l'\rangle = \langle l, T'l'\rangle$.

We require that linear combinations of vectors e_α are dense in $\mathcal{L}$ and linear combinations of vectors $e'_{\beta'}$ are dense in $\mathcal{L}'$; it follows that representations T and T' are irreducible.

To define covariant and contravariant symbols starting with vectors $e_\alpha, e'_{\beta'}$ we need the relation (3). If there exist a G- invariant measure $dmdm'$ on $\mathcal{M} \times \mathcal{M}'$ one can obtain such a relation taking $R = 1$. To prove this fact we notice that the expression

$$\int dmdm' \, |e'_{m'}\rangle\langle e_m| \tag{34}$$

is G-invariant. The integrand of (34) specifies an operator in $\mathcal{L}$; we assume that the integral is converging, hence (34) can be regarded as an operator in $\mathcal{L}$ commuting with all operators $T(g), g \in G$. It follows from the irreducibility of the representation T that (34) is a constant. Multiplying the measure $dmdm'$ by a constant factor we obtain (26).

If $\mathcal{M} = \mathcal{M}'$ and there exists a G-invariant measure dm on $\mathcal{M}$ then we get the relation (3) with $R = \delta(\alpha, \beta')$ (assuming convergence of the integral). Using formulas (21) and (23) we obtain functional integrals in this situation.

One says that the vectors e_α and $e'_{\beta'}$ considered in the present section are coherent states. This definition generalizes the definition of coherent state in [10] where $\mathcal{L} = \mathcal{L}'$ is a Hilbert space, T and T' are unitary operators, $e_\alpha = e'_\alpha$. Notice, that in ([10]) the group G transforms the vector e_α in a vector, proportional to the vector of the same kind (it acts transitively on corresponding elements of projectivization of $\mathcal{L}$). In our setting, we also can consider a similar situation.

5. L-Functionals

5.1. First Definition

Let us consider a unital associative algebra with generators $\gamma(f)$ obeying canonical commutation relations (CCR):

$$\gamma(f)\gamma(g) - \gamma(g)\gamma(f) = i(f, g) \tag{35}$$

(Weyl algebra).

Here, f and g are elements of real vector space E equipped with non-degenerate antisymmetric inner product $(\cdot, \cdot)$, generators $\gamma(f)$ depend linearly on f. We assume that Weyl algebra is complex algebra equipped with an antilinear involution and that generators $\gamma(f)$ are self-adjoint with respect to this involution.

Let us fix a representation of the Weyl algebra (representation of canonical commutation relations) in Hilbert space $\mathcal{F}$. We assume that generators are represented by self-adjoint operators $\hat{\gamma}(f)$; hence we can consider unitary operators $V_f = \exp(i\hat{\gamma}(f))$. It is easy to check that

$$V_f V_g = V_{f+g} \exp(\frac{i}{2}(f, g)). \tag{36}$$

These relations are formally equivalent to (35). We consider the smallest linear subspace of the space of bounded linear operators in $\mathcal{F}$ containing all operators V_f; the closure of this space in norm-topology is a C^*-algebra that can be regarded as an exponential form of Weyl algebra (see, for example, [14] and references therein for the mathematical theory of Weyl algebra). We will work with this algebra denoted by $\mathcal{W}$. The space of continuous linear functionals on $\mathcal{W}$ will be denoted by $\mathcal{L}$. Notice that a functional $L \in \mathcal{L}$ is determined by its values on operators V_f, therefore we can consider L as a non-linear functional $\mathbf{L}(f) = L(V_f)$ on E (the representation of states of Weyl algebra by means of non-linear functionals was rediscovered and studied in [14]).

In particular, positive functionals on the algebra $\mathcal{W}$ (quantum states) can be represented by non-linear functionals; we will use the term L-functional for non-linear functionals representing states. (Recall that a linear functional L on $*$-algebra $\mathcal{A}$ is positive if $L(A^*A) \geq 0$ for every $A \in \mathcal{A}$.)

If we have a normalized vector Φ or, more generally, a density matrix K in representation space of some $*$-algebra $\mathcal{A}$ we can obtain a quantum state ω by the formulas

$$\omega(A) = \langle \Phi, \hat{A}\Phi \rangle, \tag{37}$$

$$\omega(A) = Tr\hat{A}K, \tag{38}$$

where $\hat{A}$ stands for the operator representing an element $A \in \mathcal{A}$.

Every quantum state can be represented by a vector in some representation of $*$-algebra (Gelfand-Naimark-Segal construction).

If $\mathcal{A}$ is the Weyl algebra $\mathcal{W}$ we represent a density matrix K in any representation of canonical commutation relations (= in any representation of $\mathcal{W}$) by L-functional

$$\mathbf{L}_K(f) = TrV_f K.$$

One can say that L-functionals describe states in all representations of canonical commutation relations.

The evolution operators of quantum theory constitute a one-parameter group of automorphisms of the algebra $\mathcal{W}$ generated by an infinitesimal automorphism H. They induce evolution operators acting on quantum states; these operators can be extended to $\mathcal{L}$. To find evolution operators one should solve the equation of motion (1). We apply the methods of preceding sections assuming that $\mathcal{L}' = \mathcal{W}$. We define covariant symbols of operators acting in $\mathcal{L}$ using systems of vectors $e_f \in \mathcal{L}$ and vectors $e'_{f'} \in \mathcal{L}'$ that are defined in the following way. We assume that $f, f' \in E$, $e_f(V_g) = \exp \frac{i}{2}(f,g)$, $e'_{f'} = V_{f'}$. It follows that $\langle e_f, e'_{f'} \rangle = \exp \frac{i}{2}(f,f')$. To get a function R obeying (25) we can take $R(f,f') = C\exp(-\frac{i}{2}(f,f'))$ where the constant C is chosen in such a way that

$$\int dfdf' \langle e_g, e'_{f'} \rangle R(f,f') \langle e_f, e'_{g'} \rangle = \langle e_g, e'_{g'} \rangle. \tag{39}$$

Here $dfdf'$ is a measure on $E \times E$ or at least a rule that allows us to calculate integrals of some functions defined on this space (in (39) we need only integrals of quadratic exponents).

In what follows we assume that the antisymmetric inner product is represented in the form $(f,g) = f^i \sigma_{ij} g^j = f\sigma g$ and $\dim E = 2n < \infty$; then $C = |det(\sigma)|^{1/2}/(2\pi)^n$. We assume that $C = 1$ by changing the measure $dfdf'$. Then in the notations of Section 2 we have $c(f,f') = r(f,f') = \frac{i}{2}(f,f')$.

Let us suppose that the evolution is specified by an infinitesimal automorphism of Weyl algebra $\mathcal{W} = \mathcal{L}'$ represented as a commutator H of the element of $\mathcal{W}$ with $i\hat{H}$. Here $\hat{H}$ is a self-adjoint element of $\mathcal{W}$:

$$\hat{H} = \int d\beta h(\beta) V_\beta, \tag{40}$$

where $h(-\beta) = \bar{h}(\beta)$. (Notice that one can consider also a more general case when $\hat{H}$ is a formal expression such that the commutator with $\hat{H}$ makes sense.)

It is easy to check that the covariant symbol of the operator H has the form

$$\underline{H}(f,f') = i\int d\beta h(\beta)(e^{\frac{i}{2}((f,\beta)-(\beta,f'))} - e^{\frac{i}{2}((f,\beta)+(\beta,f'))}) = 2\int d\beta h(\beta) e^{\frac{i}{2}(f,\beta)} \sin\frac{(\beta,f')}{2}. \tag{41}$$

Using (9) we obtain a representation of the symbol of the evolution operator in $\mathcal{W} = \mathcal{L}'$ in terms of functional integrals

$$\begin{aligned} \underline{e^{tH}}(f,f') &= \int \prod df(\tau) df'(\tau) e^{S[f(\tau),f'(\tau)]}, \\ S[f(\tau),f'(\tau)] &= \int_0^t \left(\underline{H}(f(\tau),f'(\tau)) - \frac{i}{2}(f(\tau),\dot{f}'(\tau)) \right) d\tau + \frac{i}{2}((f(t),f') - (f,f')), \end{aligned} \tag{42}$$

where we integrate over the set of functions obeying conditions $f(0) = f; f'(t) = f'$.

The evolution operator e^{tH} in the algebra $\mathcal{W}$ is dual to the evolution operator e^{tK} in the space $\mathcal{L}$ of linear functionals on $\mathcal{W}$:

$$\langle e^{tK}x, y\rangle = \langle x, e^{tH}y\rangle,$$

hence the operator K entering the equation of motion for L-functionals is dual to the infinitesimal automorphism H. Using the formula (29) we can say the symbol of K coincides with the symbol of H and the symbols of operators of evolution e^{tK} and e^{tH} coincide. This remark allows us to say that *the symbol of the operator of evolution in the formalism of $L-$ functionals is expressed in terms of functional integrals by the formula* (42).

The same statement can be obtained from the equation of motion in the formalism of L-functionals. The time derivative of $\mathbf{L}(f) = L(V_f)$ can be written in the form

$$i\frac{dL(V_f)}{dt} = \int d\beta h(\beta)L(V_f V_\beta - V_\beta V_f) = \int d\beta h(\beta)(e^{\frac{i}{2}f\sigma\beta} - e^{\frac{i}{2}\beta\sigma f})L(V_{f+\beta}),$$

hence

$$\frac{d\mathbf{L}(f)}{dt} = 2\int d\beta h(\beta)\sin(\frac{1}{2}f\sigma\beta)\mathbf{L}(f+\beta). \tag{43}$$

We can write (43) In the form in the form

$$\frac{d\mathbf{L}(f)}{dt} = K\mathbf{L}(f),$$

where

$$(K\mathbf{L})(f) = 2\int d\beta h(\beta)\sin(\frac{1}{2}f\sigma\beta)(T_\beta\mathbf{L})(f).$$

Here $(T_\beta\mathbf{L})(f) = \mathbf{L}(f+\beta)$.

It is easy to check that the symbol of the operator K is given by the formula (41). We obtain another derivation of the functional integral (42) for the evolution operator in the formalism of L-functionals.

Sometimes it is convenient to introduce the Planck constant $\hbar$ in the formula (43) assuming that in the defining relations of Weyl algebra we have $\hbar$ in the right-hand side:

$$\gamma(f)\gamma(g) - \gamma(g)\gamma(f) = i\hbar(f,g),$$

where $(f,g) = f\sigma g = f^i\sigma_{ij}g^j$ and replacing $\hat{H}$ with $\hat{H}/\hbar$. Then

$$\frac{d\mathbf{L}(f)}{dt} = \int d\beta h(\beta)\frac{2\sin(\frac{\hbar}{2}f\sigma\beta)}{\hbar}\mathbf{L}(f+\beta). \tag{44}$$

It follows from (44) that the equation of motion for L-functionals has a limit as $\hbar \to 0$.

5.2. Second Definition

Let us consider another form of canonical commutation relations:

$$[a(f), a(f')] = [a^*(g), a^*(g')] = 0, \quad [a(f), a^*(g)] = (f,g), \tag{45}$$

where (f,g) is a non-degenerate pairing between vector space $\mathcal{E}$ and complex conjugate vector space $\bar{\mathcal{E}}$. (Here $f, f' \in \mathcal{E}, g, g' \in \bar{\mathcal{E}}$.) We will assume that this paring is defined on E; then it is linear with respect to the first argument and antilinear with respect to the second argument.(Notice that in our notations E and $\bar{E}$ consist of the same elements but have different complex structures). Then (45) can be represented takes the form

$$[a(f), a(f')] = [a^*(g), a^*(g')] = 0, \quad [a(f), (a(g))^*] = (f,g). \tag{46}$$

(The involution * transforms $a(f)$ into $a^*(f^*)$. We assume that $a(f)$ is linear with respect to f, then $a^*(f^*) = (a(f))^*$ is antilinear with respect to $f \in E$.) We do not assume that the pairing (f, g) is well-defined for all pairs f, g; in particular, (f, f) can be infinite).

If the space $\mathcal{E}$ consists of functions on measure space $\mathcal{M}$ then $a(f), a^*(g)$ should be regarded as generalized functions: $a(f) = \int f(k)a(k)dk, \quad a^*(g) = \int g(k)a^*(k)dk$. Then canonical commutation relations (45) can be written in the form

$$[a(k), a(k')] = [a^*(k), a^*(k')] = 0, \quad [a(k), a^*(k')] = \delta(k, k'). \tag{47}$$

If k is a discrete parameter (i.e., $\mathcal{M}$ is a discrete set with counting measure) the above relations can be written as follows

$$[a_k, a_{k'}] = [a_k^*, a_{k'}^*] = 0, \quad [a_k, a_{k'}^*] = \delta_{k,k'}.$$

The relations (45) are obviously equivalent to the relations (35) (to get (35) from (45) we can consider self-adjoint elements $a(f) + a^*(f), i(a(f) - a^*(f))$.

The relations (45) are especially convenient in the case of an infinite number of degrees of freedom. In this situation one should use the original definition of L-functional (see [11–13]).

Again we can write canonical commutation relations in exponential form introducing expressions

$$W_\alpha = e^{-a^*(\alpha)}e^{a(\alpha^*)}.$$

Notice that W_α is not holomorphic with respect to α therefore it would be more appropriate to use the notation $W_{\alpha^*,\alpha}$ as we are doing in similar situations below.

It is easy to check that

$$W_\alpha W_\beta = e^{-(\alpha^*,\beta)}W_{\alpha+\beta}. \tag{48}$$

Notice that in the case when (α, α) is finite W_α coincides with V_f up to a finite constant factor. However, we do not assume that $(\alpha, \alpha) < \infty$. (This is important for applications to string theory).

We define vector space $\mathcal{W}$ as a space of linear combinations of expressions of the form $P_\alpha W_\alpha$ where $\alpha \in E$ and P_α belongs to some class of polynomials with respect to a^*, a. The relations (45), (48) specify multiplication in $\mathcal{W}$, but this multiplication is not always defined. Nevertheless one can consider $\mathcal{W}$ as a version of Weyl algebra. (Better to say that our construction gives various versions of Weyl algebra because we did not specify the class of polynomials and topology in $\mathcal{W}$). We fix some topology in $\mathcal{W}$ in such a way that W_α is infinitely differentiable with respect to α, α^*. Then the elements W_α are dense in $\mathcal{W}$ (diffrentiating W_α we obtain polynomials of a^*, a). This means that a continuous linear functional L on $\mathcal{W}$ is specified by non-linear functional $\mathbf{L}(\alpha^*, \alpha) = L(W_\alpha)$ (by values of L on W_α). The space of of continuous linear functionals on $\mathcal{W}$ is denoted by $\mathcal{L}$. Notice that we can say that $\mathbf{L}(\alpha^*, \alpha) = \langle L, W_\alpha \rangle$ where $\langle \cdot, \cdot \rangle$ stands for the standard pairing between $\mathcal{L}$ and $\mathcal{W}$.

We say that non-linear functionals corresponding to quantum states (to positive functionals =elements of $\mathcal{L}$ obeying $L(A^*A) \geq 0$) are L-functionals.

We represent a density matrix K in any representation of canonical commutation relations (45), (48) by L-functional

$$\mathbf{L}_K(\alpha^*, \alpha) = TrW_\alpha K = Tre^{-a^*(\alpha)}e^{a(\alpha^*)}K.$$

Every element $B \in \mathcal{W}$ specifies two operators acting in the space of linear functionals $\mathcal{L}$. The first operator transforms the functional $\omega(A)$ into the functional $\omega(AB)$. Applying this construction to the cases $B = a(f)$ and $B = a^*(f)$ we obtain operators denoted by $b^+(f)$ and $b(f)$. The second operator transforms the functional $\omega(A)$ into the functional $\omega(B^*A)$; if we start with $B = a(f), B = a^*(f)$ we get operators denoted by $\tilde{b}(f), \tilde{b}^+(f)$.

The evolution operators of quantum theory constitute a one-parameter group of automorphisms of $\mathcal{W}$ generated by an infinitesimal automorphism H. They induce evolution operators acting on $\mathcal{L}$ and transforming quantum states to quantum states.

Let us suppose that the evolution is specified by an infinitesimal automorphism of Weyl algebra $\mathcal{W} = \mathcal{L}'$ represented as a commutator H of the element of $\mathcal{W}$ with $\frac{1}{i}\hat{H}$. Here $\hat{H}$ is a self-adjoint element of $\mathcal{W}$ or a self-adjoint formal expression

$$\hat{H} = \sum_{m,n} \int H_{m,n}(k_1,...k_m|l_1,...,l_n)a^*(k_1)...a^*(k_m)a(l_1)...a(l_n) \prod_{\substack{1\leq i\leq m\\1\leq j\leq n}} dk_i dl_j \tag{49}$$

such that the commutator H is a well-defined derivation of $\mathcal{W}$ that can be regarded as an infinitesimal automorphism (i.e. solving the equations of motion we obtain a one-parameter group of evolution operators e^{tH}). This allows us to write an equation of motion (1) in the space $\mathcal{L}$ taking $H = H_L - H_R$ where

$$H_R = \sum_{m,n} \int H_{m,n}(k_1,...k_m|l_1,...,l_n)b^+(k_1)...b^+(k_m)b(l_1)...b(l_n) \prod_{\substack{1\leq i\leq m\\1\leq j\leq n}} dk_i dl_j,$$

$$H_L = \sum_{m,n} \int H_{m,n}(k_1,...k_m|l_1,...,l_n)\tilde{b}^+(k_1)...\tilde{b}^+(k_m)\tilde{b}(l_1)...\tilde{b}(l_n) \prod_{\substack{1\leq i\leq m\\1\leq j\leq n}} dk_i dl_j.$$

We solve the equation of motion (1) applying the methods of Section 2 and assuming that $\mathcal{L}' = \mathcal{W}$. We define covariant symbols of operators acting in $\mathcal{L}$ using systems of vectors $e_f \in \mathcal{L}$ and vectors $e'_{f'} \in \mathcal{L}'$ that are defined in the following way. We assume that $e_f \in \mathcal{L}$ corresponds to a non-linear functional $e_f(W_\alpha) = \exp i((f,\alpha^*) + (\alpha, f^*))$ and that $e'_{f'} = W_{f'}$. It follows that $\langle e_f, e'_{f'}\rangle = \exp i((f, f^{*'}) + (f', f^*))$. To get a function R obeying (25) we take $R(f, f') = C\exp(-i(f, f^{*'}) - i(f', f^*))$ where the constant C is chosen in such a way that the formula (39) is satisfied.

It is easy to calculate the covariant symbol of the operator H :

$$\underline{H}(f,f') = i\sum_{m,n}\int \prod_{\substack{1\leq i\leq m\\1\leq j\leq n}} (dk_i dl_j) H_{m,n}(k_1,...k_m|l_1,...,l_n)\times$$
$$\left(f^*(k_1)...f^*(k_m)f'(l_1)...f'(l_n) - f'^*(k_1)...f'^*(k_m)f(l_1)...f(l_n) \right).$$

This allows us to get a representation of the symbol of the evolution operator in terms of functional integrals

$$\begin{aligned}\underline{e^{tH}}(f,f') &= \int \prod df(\tau)df'(\tau)e^{S[f(\tau),f'(\tau)]},\\ S[f(\tau),f'(\tau)] &= \int_0^t \left(\underline{H}(f(\tau),f'(\tau)) - i(f(\tau),\dot{f}'^*(\tau)) - i(f^*(\tau),\dot{f}'(\tau))\right)d\tau+\\ &+i((f(t),f'^*) + (f^*(t),f') - (f,f'^*) - (f^*,f')).\end{aligned} \tag{50}$$

In particular, if $\hat{H} = \int dk\epsilon(k)a^*(k)a(k)$ is a quadratic translation-invariant Hamiltonian we obtain

$$\underline{H}(f,f') = i\int dk\epsilon(k)\left(f^*(k)f'(k) - f'^*(k)f(k)\right).$$

Here $k \in \mathbb{R}^d$. We can consider also a more general case when $k = (\mathbf{k}, s)$ where s is a discrete index and $\mathbf{k} \in \mathbb{R}^d$.

Let us consider a general translation-invariant Hamiltonian (49). In other words, we assume that the integrand in (49) contains delta-function $\delta(\mathbf{k_1} + ...\mathbf{k_m} - \mathbf{l_1} - ... - \mathbf{l_n})$ (the arguments l, l are points of $\mathbb{R}^d$ plus discrete indices).

We represent $\hat{H}$ as a sum of the quadratic part $\hat{H}_0$ and perturbation $g\hat{V}$. Then we can consider time-dependent Hamiltonian $\hat{H}(t) = \hat{H}_0 + h(at)g\hat{V}$ where $h(0) = 1, h(-\infty) = 0$. For $a \to 0$ this means that we switch on the interaction $\hat{V}$ adiabatically. If $U_a(t, -\infty)$ denotes the evolution operator for the corresponding "Hamiltonian" $H(t)$ and Φ stands for a translation-invariant stationary state of quadratic "Hamiltonian" H_0. Then $\Psi = \lim_{a \to 0} U_a(0, \infty)\Phi$ is a translation-invariant stationary state of the "Hamiltonian" $H_0 + gV$.

In the derivation of the formula (50) we assumed that the "Hamiltonian" H does not depend on time but this formula can be applied also to time-dependent "Hamiltonians". This remark allows us to express Ψ in terms of functional integrals. If we start with an equilibrium state Φ the state Ψ is also an equilibrium state (in general with different temperature), however, the above considerations can be applied also in non-equilibrium situations. They can be considered as justification of Keldysh formalism in non-equilibrium statistical physics and lead to the same Feynman diagrams. (See [12] for another derivation of Keldysh diagram techniques in the formalism of L-functionals.)

The above formulas were written in the assumption that $\hbar = 1$. In general we should include the factor $\hbar$ into the right-hand side of the formula (45) and into the left-hand side of the equation of motion (1).

Itf we represent elements of $\mathcal{L}$ by non-linear functionals $\mathbf{L}$ the operators $b(f), b^+(f)$, $\tilde{b}(f), \tilde{b}^+(f)$ can be represented in the form

$$b^+(k) = -\hbar\alpha_k + \frac{\partial}{\partial\alpha_k^*} \equiv -\hbar c_2^*(k) + c_1(k), \quad b(k) = -\frac{\partial}{\partial\alpha_k} \equiv -c_2(k),$$

$$\tilde{b}^+(k) = \hbar\alpha_k^* - \frac{\partial}{\partial\alpha_k} \equiv \hbar c_1^*(k) - c_2(k), \quad \tilde{b}(k) = \frac{\partial}{\partial\alpha_k^*} \equiv c_1(k),$$

where $c_i^*(k)$ are operators of multiplication by α_k^* for $i = 1$ and by α_k for $i = 2$, and $c_i(k)$ are derivatives taken, respectively, with respect to α_k^* and α_k. (To simplify notations we assumed that E consists of functions on discrete space $\mathcal{M}$; points of $\mathcal{M}$ are labeled by index k).

It is easy to derive from these formulas that the equations of motion for functionals $\mathbf{L}(\alpha^*, \alpha)$ have a limit as $\hbar$ tends to zero.

5.3. Clifford Algebra

Clifford algebra is defined by canonical anticommutation relations

$$[a(f), a(f')]_+ = [a^*(g), a^*(g')]_+ = 0, \quad [a(f), a^*(g)]_+ = (f, g) \tag{51}$$

In other words to define Clifford algebra we take the definition of Weyl algebra and replace commutators with anticommutators.

The results above can be generalized to Clifford algebra. The main difference is that the symbols should be considered as functions of anticommuting variables.

The simplest way to understand this is to notice that Clifford algebra can be regarded as super Weyl algebra.

Recall that for $\mathbb{Z}_2$-graded space E and Grassmann algebra Λ one can define a Λ-point of E as a formal linear combination $\sum \lambda_A e_A$ where e_A is a basis of E and the coefficients λ_A are even for even e_A, odd for odd e_A. The set $E(\Lambda)$ of Λ- points can be regarded as vector space. If E is an algebra, this set can also be considered as an algebra. If for all Grassmann algebras Λ, the set $E(\Lambda)$ is a Lie algebra, one says that E is a super Lie algebra. Similarly, if for all Grassmann algebras the set $E(\Lambda)$ is a Weyl algebra, one can say that E is a super Weyl algebra. In particular, if f $\mathbb{Z}_2$-graded space E is purely odd and equipped with a structure of Clifford algebra then the set of Λ-points is a Weyl algebra. This means that Clifford algebra can be considered as super Weyl algebra.

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