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Scattering in Algebraic Approach to Quantum Theory—Jordan Algebras

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Abstract: Using the geometric approach, we formulate a quantum theory in terms of Jordan algebras. We analyze the notion of a (quasi)particle (=elementary excitation of translation-invariant stationary state) and the scattering of (quasi)particles in this framework.

Keywords: geometric approach; Jordan algebra; convex cone

1. Introduction

In an algebraic approach, physical observables correspond to self-adjoint elements of $*$-algebra $\mathcal{A}$. The vector space $\mathcal{B}$ of self-adjoint elements of $\mathcal{A}$ is not closed with respect to the operation of multiplication, but it is closed with respect to the operation $a \circ b = \frac{1}{2}(ab + ba)$. It was suggested long ago [1–3] that a more natural algebraic approach should be based on the axiomatization of the operation $a \circ b$. This idea led to the notion of Jordan algebra defined as a commutative algebra over $\mathbb{R}$ with multiplication $a \circ b$ obeying

$$(x \circ y)(x \circ x) = x \circ (y \circ (x \circ x)). \tag{1}$$

(Defining by R_a the operator of multiplication by a, we can express this identity by saying that operators R_a and $R_{a\circ a}$ commute). The space $\mathcal{B}$ is also closed with respect to the linear operator Q_a transforming $x \in \mathcal{B}$ into axa (here $a \in \mathcal{B}$). Another approach to Jordan algebras is based on the axiomatization of this operator. The operator Q_a is quadratic with respect to a. Imposing some conditions on Q_a we obtain the notion of quadratic Jordan algebra. Starting with the original definition of Jordan algebra, we obtain a quadratic Jordan algebra taking

$$Q_a = 2R_a^2 - R_{a\circ a}.$$

Conversely, starting with a quadratic Jordan algebra $\mathcal{B}$, we obtain a family of products obeying (1) by the formula

$$a \circ_x b = Q_{a,b}(x)$$

where $Q_{a,b} = R_aR_b + R_bR_a - R_{a\circ b}$ is an extension of quadratic operator Q_a to an operator that is symmetric and bilinear with respect to a, b (here a and b are arbitrary elements of $\mathcal{B}$).

In what follows, we work with unital topological Jordan algebra $\mathcal{B}$ specified by a product $a \circ b$. It seems, however, that quadratic Jordan algebras are more convenient to relate Jordan algebras to the geometric approach [4–7]. This relation is based on the remark that one can define a cone $\mathcal{B}_+$ in $\mathcal{B}$ as the smallest convex closed cone, invariant with respect to operators Q_a and containing the unit element. One can also consider the dual cone $\mathcal{B}_+^\vee$ consisting of linear functionals on $\mathcal{B}$ that are non-negative on $\mathcal{B}_+$. (See Section 2 for more details).

In the case when $\mathcal{B}$ consists of self-adjoint elements of C^*-algebra $\mathcal{A}$, elements of $\mathcal{B}_+^\vee$ can be identified with positive linear functionals on $\mathcal{A}$ (states). This means that when applying the geometric approach to Jordan algebras, we generalize the algebraic approach based on $*$-algebras. Notice that Jordan algebras can be regarded as the natural framework of the algebraic

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approach. This statement is prompted by the following theorem: Cones of states of two C^*-algebras are isomorphic if the corresponding Jordan algebras are isomorphic (Alfsen-Shultz).

One says that $\mathcal{B}$ is a JB-algebra if it is equipped with a Banach norm obeying

$$||x \circ y|| \leq ||x|| \cdot ||y||, ||x^2|| = ||x||^2, ||x^2|| \leq ||x^2 + y^2||.$$

For such an algebra, the cone $\mathcal{B}_+$ consists of squares. (For any Jordan algebra, an element $a \circ a = Q_a(1)$ belongs to the cone; for JB-algebras, all elements of the cone have this form). It follows that for JB-algebras, the cones are homogeneous: automorphism groups of cones act transitively on the interior of the cone.

It is natural to use homogeneous cones in the geometric approach, hence it seems that JB-algebras can lead to interesting models.

The appearance of cones in the theory of Jordan algebras allows us to apply general constructions of the geometric approach to these algebras. In Section 3, we consider (quasi)particles in the framework of Jordan algebras. In Sections 4 and 5, we consider the scattering of (quasi)particles. In Section 6, we define generalized Green functions and show that the (inclusive) scattering matrix can be expressed in terms of these functions. Section 7 is devoted to some generalizations of Jordan algebras and the above results. Some results of the theory of Jordan algebras used in Section 5 are proven in Appendix A.

We do not discuss here the numerous papers where the theory of Jordan algebras is related to physics (see, in particular, [8–22]). A short review of the theory of Jordan algebras and Jordan pairs as well as a review of various relations between Jordan algebras and physics is given in a companion paper [23].

2. Jordan Algebras

Let us consider a topological Jordan algebra over $\mathbb{R}$ denoted $\mathcal{B}$. (Recall that Jordan algebra is defined as a commutative algebra with multiplication obeying the identity $(x \circ y)(x \circ x) = x \circ (y \circ (x \circ x))$. In what follows, we consider unital Jordan algebras. If $\mathcal{B}$ is a complete topological vector space and the multiplication is continuous, we say that $\mathcal{B}$ is a topological Jordan algebra). The most important class of topological Jordan algebras consists of Jordan Banach algebras (JB-algebras). The Banach norm in JB-algebra should obey

$$||x \circ y|| \leq ||x|| \cdot ||y||, ||x^2|| = ||x||^2, ||x^2|| \leq ||x^2 + y^2||.$$

(See [3] for a review of the theory of operator Jordan algebras).

In the finite-dimensional case, the class of JB-algebras coincides with the class of Euclidean Jordan algebras classified in the famous paper by Jordan, von Neumann and Wigner [2]. The most natural simple finite-dimensional JB-algebras $\mathfrak{h}_n(\mathbb{R}), \mathfrak{h}_n(\mathbb{C}), \mathfrak{h}_n(\mathbb{H})$ consist of Hermitian matrices with real, complex or quaternion entries. One more series of simple finite-dimensional JB-algebras is the series of spin factors (or Jordan algebras of Clifford type). These algebras are generated by elements $1, e_1, .., e_n$ with relations $e_i \circ e_i = 1, e_i \circ e_j = 0$ for $i \neq j$. The last simple finite-dimensional JB-algebra is Albert algebra $\mathfrak{h}_3(\mathbb{O})$, which can be realized as an algebra of 3×3 Hermitian matrices with octonion entries. This 27-dimensional algebra is exceptional (it cannot be embedded into matrix Jordan algebra with the operation $a \circ b = \frac{1}{2}(ab + ba)$).

The structure semigroup $Str(\mathcal{B})$ is defined as a semigroup generated by automorphisms of $\mathcal{B}$ and operators Q_a that can be expressed in terms of the Jordan triple product $\{a, x, b\}$ by the formula $Q_a(x) = \{a, x, a\}$. (Jordan triple product is defined by the formula $\{a, x, b\} = (a \circ x) \circ b + (x \circ b) \circ a - (a \circ b) \circ x$.)

The operator Q_a is quadratic with respect to a; we use the notation $Q_{\tilde{a},a}$ for the corresponding bilinear operator: $Q_{\tilde{a},a} = \{\tilde{a}, x, a\}$.

An element $B \in Str(\mathcal{B})$ (a structural transformation) obeys

$$Q_{Ba} = BQ_aB^t \tag{2}$$

where $B \to B^t$ denotes an involution in the structure semigroup transforming every operator Q_a into itself and every automorphism into inverse automorphism. The structure group $Strg(\mathcal{B})$ is generated by automorphisms of $\mathcal{B}$ and invertible operators Q_a.

The inner structure semigroup $iStr(\mathcal{B})$ is generated by operators Q_a. The inner structure group $iStrg(\mathcal{B})$ consists of invertible elements of the inner structure semigroup.

We define the positive cone in Jordan algebra as the smallest closed convex subset of $\mathcal{B}$ containing the unit element and invariant with respect to operators Q_a.[1] One can also say that the positive cone is the smallest closed convex subset of $\mathcal{B}$ that contains the unit element and is invariant with respect to the inner structure semigroup. The positive cone is invariant with respect to automorphisms, hence it is also invariant with respect to the structure semigroup. It is obvious that the cone contains all squares; this follows from $Q_a(1) = a \circ a$. For JB-algebras, all elements of the cone can be represented as squares and the structure group acts transitively on the interior of the cone.

The positive cone is denoted by $\mathcal{B}_+$ and the dual cone is denoted by $\mathcal{B}_+^\vee$.

Let us suppose that the Jordan algebra $\mathcal{B}$ is obtained from associative algebra $\mathcal{A}$ as a set of self-adjoint elements with respect to involution *; this set is equipped with the operation $a \circ b = \frac{1}{2}(ab + ba)$. Then, $Q_a(x) = axa, Q_{\tilde{a},a} = \frac{1}{2}(\tilde{a}xa + ax\tilde{a})$. It follows that the structure semigroup of $\mathcal{B}$ contains all maps $x \to A^*xA$, where A is a self-adjoint or unitary element of $\mathcal{A}$ (for a self-adjoint element we obtain a map Q_A, for a unitary element we obtain an automorphism).

If $\mathcal{A}$ is a C^*-algebra, then every element of the form A^*A can be represented as a square of a self-adjoint element, hence the cone in $\mathcal{B}$ is the smallest closed convex set containing all elements of the form A^*A where $A \in \mathcal{A}$. The dual cone consists of all linear functionals on $\mathcal{B}$ that are non-negative on the elements of the form A^*A (in agreement with the standard definition of a positive linear functional on associative algebra with involution).

We also consider complexifications of Jordan algebras considered as complex Jordan algebras with involution. (If we start with JB-algebra, then the complexification is called JB^*-algebra.) The cones associated with these algebras can be defined as the cones of their real parts. The Jordan triple product $\{a, x^*, b\}$ is defined as an operation antilinear with respect to the middle arguments and linear with respect to other arguments. We introduce the notation $Q_a(x) = \{a, x, a^*\}$ where x is real. It is easy to check that Q_a belongs to the structure semigroup (it is equal to $Q_\alpha + Q_\beta$ where α and β are real and imaginary parts of a). It follows that, in the case when x is real and $b = a^*$, the triple product belongs to the positive cone. Notice that the map Q_a is Hermitian with respect to a.

In the geometric approach to scattering theory [7], we are starting with a cone of states $\mathcal{C} \subset \mathcal{L}$ and a group $\mathcal{U}$ consisting of automorphisms of the cone. (Here, $\mathcal{L}$ denotes a complete topological vector space.)

If we take as a starting point a Jordan algebra $\mathcal{B}$, we can take as $\mathcal{C}$ either the cone $\mathcal{B}_+$ or the dual cone. The group $\mathcal{U}$ can be identified with the structure group $Strg(\mathcal{B})$. Sometimes, it is convenient to fix a semiring $\mathcal{W}$ consisting of endomorphisms of the cone; if we are starting with Jordan algebra $\mathcal{B}$, the semiring $\mathcal{W}$ can be defined as the smallest semiring containing the structure semigroup $Str(\mathcal{B})$.

3. (Quasi)particles

To define (quasi)particles and their scattering, we should specify time translations T_τ and spatial translations $T_{\mathbf{x}}$ as elements of the structure group $Strg(\mathcal{B})$. In other words, we should fix a homomorphism of the commutative translation group $\mathcal{T}$ to $Strg(\mathcal{B})$. The translation group also acts on the cones. We are using the same notations $T_\tau, T_{\mathbf{x}}$ for time and spatial translations of the cones. As usual, we denote $T_\tau T_{\mathbf{x}} \alpha$ as $\alpha(\tau, \mathbf{x})$. As in [5,7], we define (quasi)particles as elementary excitations of a translation-invariant stationary state.

Applying the general definition of the geometric approach, we can say that an elementary excitation of stationary translation-invariant state ω is a quadratic or Hermitian map σ of the "elementary space" $\mathfrak{h}$ into the cone of states $\mathcal{C}$. This map should commute with spatial and time translations. In addition, one should fix a map of $\mathfrak{h}$ into the space $End(\mathcal{L})$

of endomorphisms of $\mathcal{L}$ such that $\sigma(f) = L(f)\omega$ [7]. In the framework of Jordan algebras, ω is an element of the positive cone $\mathcal{B}_+$ or of the dual cone $\mathcal{B}_+^\vee$. For definiteness, we assume that ω is an element of the dual cone; then $\mathcal{L}$ should be identified with $\mathcal{B}^\vee$. (Recall that the elementary space $\mathfrak{h}$ is defined as a pre-Hilbert space of smooth fast decreasing functions depending on $\mathbf{x} \in \mathbb{R}^d$ and discrete variable $i \in \mathcal{I}$. The spatial translations act as shifts by $\mathbf{a} \in \mathbb{R}^d$, the time translations commute with spatial translations. We can consider real-valued or complex-valued functions; and we should consider quadratic maps in the first case and Hermitian maps in the second case).

Let us start with linear map $\rho : \mathfrak{h} \to \mathcal{B}$ commuting with translations. Let us fix translation-invariant stationary state $\omega \in \mathcal{B}_+^\vee$ obeying additional condition $T^t\omega = \omega$ for all $T \in \mathcal{T}$. (Here, $T \to T^t$ stands for the involution in the structure group entering (2)). Then, *the map* $L : \mathfrak{h} \to End(\mathcal{B}^\vee)$ *transforming* $x \in \mathfrak{h}$ *into* $Q_{\rho(x)}$ *specifies an elementary excitation of* ω *as the map* $\sigma : x \to L(x)\omega$. To verify this statement, we notice that the map σ is quadratic or Hermitian because L is quadratic (if $\mathcal{B}$ is a Jordan algebra over $\mathbb{R}$) or Hermitian (if $\mathcal{B}$ is a complex Jordan algebra with involution), the Formula (2) implies that σ commutes with translations. In what follows, we consider mostly elementary excitations constructed this way. Notice, however, that we can start with arbitrary linear map $\rho : \mathfrak{h} \to \mathcal{B}$ and impose a weaker condition that the map $\sigma : x \to L(x)\omega$ commutes with translations. Then, σ can be regarded as an elementary excitation. (Here again, L transforms $x \in \mathfrak{h}$ into $Q_{\rho(x)}$).

Let us consider as an example a Jordan algebra defined as a set of self-adjoint elements of Weyl algebra. We define Weyl algebra corresponding to real pre-Hilbert space $\mathfrak{h}$ as a unital $*$- algebra generated by elements $a(f), a^+(g)$ depending linearly of $f, g \in \mathfrak{h}$ and obeying canonical commutation relations

$$[a(f), a(g)] = [a^+(f), a^+(g)] = 0, [a(f), a^+(g)] = \langle f, g \rangle.$$

We define a map ρ of $\mathfrak{h}$ into this Jordan algebra as a map sending $f \in \mathfrak{h}$ to $a(f) + a^+(f)$.

Let us assume that $\mathfrak{h}$ is an elementary space. Then, the translations in $\mathfrak{h}$ induce translations in Weyl algebra and in the corresponding Jordan algebra. The map ρ commutes with translations. Taking as ω the state corresponding to the Fock vacuum in a positive cone or a dual cone, we obtain an example of an elementary excitation of this state.

The same construction works for Clifford algebra specified by canonical anticommutation relations.

Notice that the Jordan algebra corresponding to Clifford algebra can be regarded as JB-algebra, but starting with Weyl algebra, we obtain a topological Jordan algebra, more precisely a Fréchet Jordan algebra. (Fréchet vector space is a complete topological vector space where the topology is specified by a countable family of seminorms. In what follows, we are talking about JB-algebras, but our results can be generalized to Fréchet analogs of these algebras.)

Let us discuss the relation of the above constructions to the construction of elementary excitations in the approach based on the consideration of associative algebra $\mathcal{A}$ with involution ($*$-algebra). The set of self-adjoint elements of such an algebra can be regarded as Jordan algebra $\mathcal{B}$ over $\mathbb{R}$; the complexification $\mathbb{C}\mathcal{B}$ of this Jordan algebra can be identified with $\mathcal{A}$ considered as a complex Jordan algebra with involution. The elements of dual cones of these Jordan algebras can be identified with not necessarily normalized states of $\mathcal{A}$. Let us assume that spatial and time translations act on $\mathcal{A}$ as automorphisms; this action generates an action of translations on the cones of Jordan algebras. Let us fix a translation invariant stationary state ω of algebra $\mathcal{A}$; the corresponding elements of dual cones of Jordan algebras are denoted by the same symbol. Excitations of ω can be regarded as elements of pre-Hilbert space $\mathcal{H}$ obtained by means of GNS construction applied to ω, and an elementary excitation is an isometric embedding $\Phi(f)$ of elementary space $\mathfrak{h}$ into $\mathcal{H}$. Following [7], we represent $\Phi(f)$ in the form $B(f)\theta$ where B is a linear map $\mathfrak{h} \to \mathcal{A}$ and $\theta \in \mathcal{H}$ is a vector corresponding to the state ω. If elements $B(f)$ are self-adjoint, we can apply the above construction of elementary excitation of ω considered as an element of a dual cone of the Jordan algebra $\mathcal{B}$ of self-adjoint elements of $\mathcal{A}$ taking $\rho = B$. Then, the state

$Q_{\rho(f)}\omega$ corresponds to the vector $B(f)\theta$. If $B^*(f) = 0$, a similar statement can be proved for $\mathbb{C}\mathcal{B}$ (for Jordan algebra with involution obtained by complexification of $\mathcal{B}$).

4. Scattering

To analyze scattering in the framework of Jordan algebras, we are starting with linear map $\rho : \mathfrak{h} \to \mathcal{B}$ (not necessarily commuting with translations). We fix translation-invariant stationary state $\omega \in \mathcal{B}_+^\vee$. The map $L : \mathfrak{h} \to End(\mathcal{B}^\vee)$ transforming $x \in \mathfrak{h}$ into $Q_{\rho(x)}$ specifies an elementary excitation of ω if the map $\sigma : x \to L(x)\omega$ commutes with translations.

We define (following the general theory of [7]) the operator

$$L(f,\tau) = T_\tau(L(T_{-\tau}f)) = T_\tau L(T_{-\tau}f)T_{-\tau}.$$

where $f \in \mathfrak{h}$. This operator is quadratic or Hermitian with respect to f, therefore we can also consider the operator $L(\tilde{f}, f, \tau)$ that is linear with respect to $\tilde{f}$ and linear or antilinear with respect to f; it coincides with $L(f,\tau)$ for $\tilde{f} = f^*$. (Notice that in the case of real vector spaces $f^* = f$.) Using the notation $Q_{\tilde{a},a}x = \{\tilde{a}, x, a^*\}$, we can write

$$L(\tilde{g}, g, \tau) = T_\tau Q_{T_{-\tau}\rho(\tilde{g}),T_{-\tau}\rho(g)}T_{-\tau} = L(\tilde{g}, g)T_{-\tau}^t$$

where $L(\tilde{g}, g) = Q_{\rho(\tilde{g}),\rho(g)}$ is a bilinear (or sesquilinear) form corresponding to the quadratic (or Hermitian) form $L(g) = Q_{\rho(g)}$.

The state

$$\Lambda(f_1, \cdots, f_n| - \infty) = \lim_{\tau_1\to-\infty,\cdots,\tau_n\to-\infty} L(f_1,\tau_1),\dots L(f_n,\tau_n)\omega \tag{3}$$

describes the collision of (quasi)particles with wave functions $f_1, \dots, f_n$. We say that (3) is a scattering state (or, more precisely, an *in*-state).

The result of the collision can be characterized by the number

$$\lim_{\tau'\to+\infty,\tau\to-\infty} \langle a|L(g_1,\tau')\dots L(g_m,\tau')L(f_1,\tau)\dots L(f_n,\tau)|\omega\rangle \tag{4}$$

where a is a stationary translation-invariant point of the cone $\mathcal{B}_+$ or of the larger cone $\mathcal{B}_+^{\vee\vee}$ (we use bra-ket notations). Comparing with the formulas of [7], we see that this number can be interpreted as a generalization of the inclusive scattering matrix. More generally, we can consider a functional

$$\sigma(\tilde{g}'_1, g'_1, \dots, \tilde{g}'_{n'}, g'_{n'}, \tilde{g}_1, g_1, \dots, \tilde{g}_n, g_n) =$$

$$\langle a| \lim_{\tau'_i\to+\infty,\tau_j\to-\infty} L(\tilde{g}'_1, g'_1, \tau'_1)\dots L(\tilde{g}'_{n'}, g'_{n'}, \tau'_{n'})L(\tilde{g}_1, g_1, \tau_1)\dots L(\tilde{g}_n, g_n, \tau_n)|\omega\rangle \tag{5}$$

that is linear or antilinear with respect to all of its arguments. It can also be regarded as an inclusive scattering matrix.

It was proven in [7] that the limit (3) exists for $f_1, \cdots, f_n$ in a dense open subset of $\mathfrak{h} \times \cdots \times \mathfrak{h}$ if

$$||[T_\alpha(L(\phi)), L(\psi)]|| \le \int d\mathbf{x}d\mathbf{x}' D^{ab}(\mathbf{x} - \mathbf{x}')|\phi_a(\mathbf{x})| \cdot |\psi_b(\mathbf{x}')| \tag{6}$$

where $D^{ab}(\mathbf{x})$ *tends to zero faster than any power as* $\mathbf{x} \to \infty$ *and* α *run over a finite interval.*

Less formally, we can formulate this condition as the requirement that the commutator $[T_\alpha(L(\phi)), L(\psi)]$ is small if the essential supports of functions ϕ and ψ in coordinate representation are far away.

Similar conditions can be formulated for the existence of limits (4) and (5) (for the existence of an inclusive scattering matrix). Later, we will formulate more concrete conditions for the existence of the above limits.

Notice that the linear map $\rho : \mathfrak{h} \to \mathcal{B}$ can be regarded as a multicomponent generalized function $\rho(\mathbf{x})$ in coordinate representation or $\rho(\mathbf{k})$ in momentum representation. (This means that we formally represent $\rho(\phi)$ as $\int d\mathbf{x}\phi(\mathbf{x})\rho(\mathbf{x})$ or as $\int d\mathbf{k}\phi(\mathbf{k})\rho(\mathbf{k})$. Discrete indices are omitted in these formulas and in what follows.) We assume that the generalized functions $\rho(\mathbf{x}), \rho(\mathbf{k})$ correspond to continuous functions denoted by the same symbols.

The expression (5) is linear with respect to its arguments, therefore it can be regarded as a generalized function

$$\sigma(\tilde{\mathbf{x}}'_1, \mathbf{x}'_1, \ldots, \tilde{\mathbf{x}}'_{n'}, \mathbf{x}'_{n'}, \tilde{\mathbf{x}}_1, \mathbf{x}_1, \ldots, \tilde{\mathbf{x}}_n, \mathbf{x}_n) =$$

$$\lim_{\tau'_i \to +\infty, \tau_j \to -\infty} \langle \alpha | L(\tilde{\mathbf{x}}'_1, \mathbf{x}'_1, \tau'_1) \ldots L(\tilde{\mathbf{x}}'_{n'}, \mathbf{x}'_{n'}, \tau'_{n'}) L(\tilde{\mathbf{x}}_1, \mathbf{x}_1, \tau_1) \ldots L(\tilde{\mathbf{x}}_n, \mathbf{x}_n, \tau_n) | \omega \rangle \tag{7}$$

or, in momentum representation,

$$\sigma(\tilde{\mathbf{k}}'_1, \mathbf{k}'_1, \ldots, \tilde{\mathbf{k}}'_{n'}, \mathbf{k}'_{n'}, \tilde{\mathbf{k}}_1, \mathbf{k}_1, \ldots, \tilde{\mathbf{k}}_n, \mathbf{k}_n) =$$

$$\lim_{\tau'_i \to +\infty, \tau_j \to -\infty} \langle \alpha | L(\tilde{\mathbf{k}}'_1, \mathbf{k}'_1, \tau'_1) \ldots L(\tilde{\mathbf{k}}'_{n'}, \mathbf{k}'_{n'}, \tau'_{n'}) L(\tilde{\mathbf{k}}_1, \mathbf{k}_1, \tau_1) \ldots L(\tilde{\mathbf{k}}_n, \mathbf{k}_n, \tau_n) | \omega \rangle. \tag{8}$$

We use the notations

$$L(\tilde{g}, g, \tau) = \int d\tilde{\mathbf{x}} d\mathbf{x} \tilde{g}(\tilde{\mathbf{x}}) g(\mathbf{x}) L(\tilde{\mathbf{x}}, \mathbf{x}, \tau) = \int d\tilde{\mathbf{k}} d\mathbf{k} \tilde{g}(\tilde{\mathbf{k}}) g(\mathbf{k}) L(\tilde{\mathbf{k}}, \mathbf{k}, \tau)$$

If ρ commutes with translations, we can say that

$$T_\tau \rho(\phi) = \rho(T_\tau \phi) = \int d\mathbf{k} e^{-i\tau E(\mathbf{k})} \rho(\mathbf{k}) \phi(\mathbf{k})$$

hence

$$L(\tilde{\mathbf{k}}, \mathbf{k}, \tau) = T_\tau (e^{-i\tau[E(\tilde{\mathbf{k}}) + E(\mathbf{k})]} Q_{\rho(\tilde{\mathbf{k}}), \rho(\mathbf{k})}) T_{-\tau} =$$

$$e^{-i\tau[E(\tilde{\mathbf{k}}) + E(\mathbf{k})]} T_\tau Q_{\rho(\tilde{\mathbf{k}}), \rho(\mathbf{k})} T_{-\tau}$$

5. Existence of Inclusive Scattering Matrix

Let us consider first of all the case when the Jordan algebra $\mathcal{B}$ is obtained as a set of self-adjoint elements of associative Banach algebra $\mathcal{A}$ with respect to the involution *. We impose the condition of asymptotic commutativity or anticommutativity on the multicomponent function $\rho(\mathbf{x})$:

$$||[\rho(\mathbf{x} + \mathbf{a}), \rho(\mathbf{a})]_\mp|| < \frac{C_n}{1 + ||\mathbf{x}||^n} \tag{9}$$

for every natural number n. It follows from this condition that

$$||[\rho(\phi), \rho(\psi)]_\mp|| < \int d\mathbf{x} d\mathbf{x}' D^{ab}(\mathbf{x} - \mathbf{x}') |\phi_a(\mathbf{x})| \cdot |\psi_b(\mathbf{x}')| \tag{10}$$

where $D^{ab}(\mathbf{x})$ tends to zero faster than any power as $\mathbf{x} \to \infty$.

The operators $L_\phi = Q_{\rho(\phi)}$ transform $b \in \mathcal{B}$ into $\rho(\phi) b \rho(\phi)$. It is easy to check that these operators obey (6), hence the limits we are interested in exist and we can consider the scattering of particles. In the case of asymptotic commutativity (anticommutativity), we are dealing with bosons (fermions).

For Jordan algebra $\mathcal{B}$ coming from associative algebra $\mathcal{A}$ with involution, it is easy to formulate sufficient conditions for commutativity of operators Q_a and Q_b. It is obvious that in the case when a and b commute in $\mathcal{A}$ or a and b anticommute in $\mathcal{A}$ (equivalently $a \circ b = 0$ in $\mathcal{B}$), we have $Q_a Q_b = Q_b Q_a$. Similar statements are correct in any JB-algebra $\mathcal{B}$: *if operators R_a and R_b commute or $a \circ b = 0$ then the operators Q_a, Q_b commute* [24–26]. (Here $a, b \in \mathcal{B}$, R_a stands for the operator of multiplication by a in $\mathcal{B}$. If R_a and R_b commute one says that a and b operator commute).

It is natural to conjecture that these statements can be generalized in the following way:
If $a \circ b$ is small, then the operators Q_a and Q_b almost commute.
If the operators R_a and R_b almost commute, then the operators Q_a and Q_b almost commute.
The first of these conjectures is proven in Appendix A. More precisely,
If the norm of the Jordan product $a \circ b$ of two elements of JB-algebra $\mathcal{B}$ is $\leq \epsilon$, then

$$||[Q_a, Q_b]|| \leq k(||a||, ||b||)\sqrt{\epsilon}. \tag{11}$$

Here, $\epsilon \geq 0$ and k is a polynomial function.
This statement allows us to give conditions for the existence of limits (3)–(5).
We impose the condition

$$||\rho(\mathbf{x}+\mathbf{a}) \circ \rho(\mathbf{a})|| < \frac{C_n}{1+||\mathbf{x}||^n} \tag{12}$$

for every natural number n. Then,

$$||\rho(\phi) \circ \rho(\psi)|| < \int d\mathbf{x} d\mathbf{x}' D^{ab}(\mathbf{x}-\mathbf{x}')|\phi_a(\mathbf{x})| \cdot |\psi_b(\mathbf{x}')| \tag{13}$$

where $D^{ab}(\mathbf{x})$ tends to zero faster than any power as $\mathbf{x} \to \infty$. Applying (A3), we obtain (6), which implies the existence of limits (3)–(5) for dense sets of families of functions in the arguments of these expressions.

It is not clear whether the second conjecture is true. However, the identity $Q_a = 2R_a^2 - R_{a^2}$ immediately implies the following weaker statement: if a and a^2 almost operator commute with b and b^2 then Q_a and Q_b almost commute:

$$||[Q_a, Q_b]|| \leq 8||a|| \cdot ||b|| \cdot |||[R_a, R_b]||| + 4||a|| \cdot |||[R_a, R_{b^2}|| + 4||b|| \cdot |||[R_b, R_{a^2}]||| + |||[R_{a^2}, R_{b^2}]|||.$$

Using the identity $Q_{e^a} = e^{2R_a}$ one can conclude that operators Q_{e^a} and Q_{e^b} almost commute if a and b almost operator commute.

One can use these statements to give conditions for the existence of scattering states and the inclusive scattering matrix.

6. Green Functions

Let us fix translation-invariant elements $\alpha \in \mathcal{B}_+, \omega \in \mathcal{B}_+^\vee$, where $\mathcal{B}_+$ is a positive cone in Jordan algebra $\mathcal{B}$ and $\mathcal{B}_+^\vee$ denotes the dual cone. The quadratic operators Q_A act in both cones, the bilinear operators $Q_{\tilde{A},A}$ act in $\mathcal{B}$ and in the dual space $\mathcal{B}^\vee$. (Here, $A, \tilde{A}$ are elements of $\mathcal{B}$.)

Let us fix elements $\tilde{A}_1, A_1, \ldots, \tilde{A}_n, A_n \in \mathcal{B}$. We introduce the notation $Q_i(\tilde{\mathbf{x}}, \tilde{\tau}, \mathbf{x}, \tau)$ for $Q_{\tilde{A}_i(\tilde{\mathbf{x}},\tilde{\tau}),A_i(\mathbf{x},\tau)}$.

We define (generalized) Green functions by the formula

$$G_n(\tilde{\mathbf{x}}_1, \tilde{\tau}_1, \mathbf{x}_1, \tau_1, \ldots, \tilde{\mathbf{x}}_n, \tilde{\tau}_n, \mathbf{x}_n, \tau_n) =$$

$$\langle \alpha | T(Q_1(\tilde{\mathbf{x}}_1, \tilde{\tau}_1, \mathbf{x}_1, \tau_1) \ldots Q_n(\tilde{\mathbf{x}}_n, \tilde{\tau}_n, \mathbf{x}_n, \tau_n) | \omega \rangle$$

where T stands for the chronological ordering with respect to $\tau_i' = \frac{1}{2}(\tilde{\tau}_i + \tau_i)$.

Omitting chronological ordering in this formula, we obtain a definition of correlation functions. Notice that in the case when Jordan algebra $\mathcal{B}$ is constructed as a set of self-adjoint elements of associative algebra with involution $\mathcal{A}$, we have

$$Q(\tilde{\mathbf{x}}, \tilde{\tau}, \mathbf{x}, \tau)a = \frac{1}{2}(A(\mathbf{x}, \tau)a\tilde{A}(\tilde{\mathbf{x}}, \tilde{\tau}) + \tilde{A}(\tilde{\mathbf{x}}, \tilde{\tau})aA(\mathbf{x}, \tau)).$$

Using this remark, we can express correlation functions for Jordan algebra $\mathcal{B}$ in terms of correlation functions for associative algebra $\mathcal{A}$.

We defined Green functions in $(\mathbf{x}, \tau)$-representation; as always taking Fourier transforms we obtain Green functions in $(\mathbf{k}, \tau)$- and $(\mathbf{k}, \varepsilon)$-representations.

One can show (under some conditions) that the inclusive scattering matrix can be calculated in terms of the asymptotic behavior of Green functions in $(\mathbf{k}, \tau)$-representation or in terms of poles and residues in $(\mathbf{k}, \varepsilon)$-representation. The proof is similar to the proof of the analogous statement in [6]. It is based on Formula (8).

7. Generalizations

Let us consider a $\mathbb{Z}_2$-graded algebra $\mathcal{A}$. We denote by $\mathcal{A}_\Lambda$ the set of even elements of tensor product $\mathcal{A} \otimes \Lambda$ where Λ is a Grassmann algebra. This set can be considered as an algebra; if for every Λ the algebra $\mathcal{A}_\Lambda$ is a Jordan algebra, we say that $\mathcal{A}$ is a Jordan superalgebra. (See, for example, [27] for a more standard definition of Jordan superalgebra and for the main facts of the theory of Jordan superalgebras.) Similarly, if the algebra $\mathcal{A}_\Lambda$ is a Lie algebra one says that $\mathcal{A}$ is a Lie superalgebra. One can say that a Jordan superalgebra $\mathcal{A}$ specifies a functor defined in the category of Grassmann algebras and taking values in the category of Jordan algebras. Analogously, a Lie superalgebra specifies a functor with values in Lie algebras, and a supergroup can be regarded as a functor with values in groups (all functors are defined on Grassmann algebras).

Starting with a differential algebra (= $\mathbb{Z}_2$-graded algebra equipped with an od derivation d obeying $d^2 = 0$), we can define a differential Jordan superalgebra. The simplest way to construct a differential Jordan superalgebra is to take the tensor product of Jordan algebra $\mathcal{B}$ by a differential supercommutative algebra $\mathcal{E}$ (for example, one can take as $\mathcal{E}$ a free Grassmann algebra with the differential that calculates the cohomology of Lie algebra).

Using these definitions, one can generalize the statements above to Jordan superalgebras and (in the framework of BRST-formalism) to differential Jordan superalgebras.

These generalizations can be used to analyze interesting examples.

In particular, one can construct Poincaré invariant theories starting with simple exceptional Jordan algebra (Albert algebra $\mathfrak{h}_3(\mathbb{O})$). Namely, one should notice that the structure group of this algebra contains a subgroup isomorphic to $SO(1,9)$ (this subgroup was used in [8,9]). Assuming that translations act trivially, we obtain an action of the ten-dimensional Poincaré group on Albert algebra. Taking a tensor product of Albert algebra and (differental) supecommutative algebra where the Poincaré group acts by automorphisms, we obtain a (differential) Jordan superalgebra with the action of the Poincaré group (and in general non-trivial action of translations).

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Appendix A

To develop a scattering theory in the framework of Jordan algebras, we need asymptotic commutativity of quadratic operators $Q_a = 2R_a^2 - R_{a^2}$ where R_a stands for the multiplication operator: $R_a(b) = b \circ a$. (See Section 5). We prove some results in this direction.

Let us start with some remarks in the simple case when Jordan algebra $\mathcal{B}$ is obtained as a set of self-adjoint elements of associative algebra $\mathcal{A}$ with involution * and the operation $a \circ b = \frac{1}{2}(ab + ba)$. We assume that $\mathcal{A}$ is a C*-algebra, then one says that $\mathcal{B}$ is a JC-algebra. It is easy to check that in the case at hand we have $Q_a = l(a)r(a)$, where $l(a)$ and $r(a)$ are the operators of multiplication by a from the left and from the right in $\mathcal{A}$. Using the relations $[l(a), l(b)] = l([a,b]), [r(a), r(b)] = r([b,a])$ and $Q_aQ_b = l(ab)r(ba), Q_bQ_a = l(ba)r(ab)$, we obtain the estimate

$$|||[Q_a, Q_b]||| \leq 2||a|| \cdot ||b|| \cdot |||[a,b]|||. \tag{A1}$$

Similarly, one can estimate the commutator $[Q_a, Q_b]$ in terms of the anticommutator of a, b (or equivalently in terms of Jordan product $a \circ b$):

$$||[Q_a, Q_b]|| \leq 2||a|| \cdot ||b|| \cdot ||[a, b]_+|| = 4||a|| \cdot ||b|| \cdot ||a \circ b||. \tag{A2}$$

The proof remains almost the same: we should use the relations $l(ab) = -l(ba) + l([a, b]_+)$, $r(ba) = -r(ab) + r([a, b]_+)$.

Let us suppose that a commutative Lie group (translation group) acts on $\mathcal{A}$ by means of involution-preserving automorphisms $\alpha(\mathbf{x}, \tau)$; we use the notation $\alpha(\mathbf{x}, \tau)a = a(\mathbf{x}, \tau)$. Then, the same group acts on $\mathcal{B}$; it transforms the operators Q_a into the operators $Q_{a(\mathbf{x},\tau)}$.

We say that the elements $a, b \in \mathcal{A}$ asymptotically commute if

$$[a(\mathbf{x}, \tau), b] \leq \rho(\mathbf{x}, \tau)$$

where ρ is small for large $\mathbf{x}$ (for definiteness, we assume that ρ tends to zero faster than any power of $||\mathbf{x}||$ and has at most polynomial growth with respect to τ).

The elements $a, b \in \mathcal{A}$ asymptotically anticommute if a similar relation is valid for anticommutator

$$[a(\mathbf{x}, \tau), b]_+ \leq \rho(\mathbf{x}, \tau).$$

It follows from (A1) and (A2) that in both cases operators Q_a, Q_b asymptotically commute:

$$||[Q_a(\mathbf{x}, \tau), Q_b]|| \leq 2||a|| \cdot ||b||\rho(\mathbf{x}, \tau).$$

We would like to generalize the above relations to the case of Jordan Banach algebras (JB-algebras). Let us suppose that the commutative Lie group (translation group) acts on $\mathcal{B}$ by means of automorphisms $\alpha(\mathbf{x}, \tau)$. The same group acts on the operators R_a and Q_a, transforming them into the operators $R_{\alpha(\mathbf{x},\tau)a}$ and $Q_{\alpha(\mathbf{x},\tau)a}$. We say that $a, b \in \mathcal{B}$ asymptotically operator commute if $||[R_{\alpha(\mathbf{x},\tau)a}, R_b]|| < $. It is natural to conjecture that in this case the operators Q_a, Q_b also asymptotically commute, i.e., $||[Q_{\alpha(\mathbf{x},\tau)a}, Q_b]|| < \rho(\mathbf{x}, \tau)$. We were not able to prove this conjecture. However, *if the pairs* $(a, b), (a^2, b), (a, b^2), (a^2, b^2)$ *asymptotically operator commute, it is obvious that* Q_a, Q_b *asymptotically commute*. This statement is sufficient for the applications we have in mind.

The same statements are true in the more general case when the translation group acts by structural transformations: $\alpha(\mathbf{x}, \tau) \in Strg(\mathcal{B})$, where $Strg(\mathcal{B})$ denotes the structure group (the group generated by automorphisms and invertible operators Q_a).

Let us assume that the norm of the Jordan product $a \circ b$ *of two elements of JB-algebra* $\mathcal{B}$ *is* $\leq \epsilon$. *Then,*

$$||[Q_a, Q_b]|| \leq k(||a||, ||b||)\sqrt{\epsilon} \tag{A3}$$

where $\epsilon \geq 0$ *and* k *is a polynomial function.*

To prove this statement, we use the theorem that every JB-algebra can be considered as a subalgebra of a direct sum of JC-algebra and purely exceptional JB algebra. (This follows from the fact that a JB-algebra $\mathcal{B}$ is a subalgebra of JBW-algebra $\mathcal{B}^{**}$ and every JBW-algebra is a direct sum of JW-algebra and purely exceptional JBW-algebra; see [26] for definitions and formulations and [3] for details and proofs). It is obvious that our statement is correct for a subalgebra if it is correct for the algebra. The statement is proven already for JC-algebra (A2). It remains to prove it for simple exceptional JB-algebra (then it is true for any purely exceptional algebra). To give the proof, we slightly modify the considerations given in [25] for $\epsilon = 0$.

The only information we need about simple exceptional JB-algebra (Albert algebra) is the equation

$$a^3 = t(a)a^2 - s(a)a + n(a) \tag{A4}$$

satisfied by any element a of Albert algebra. (Here, t, s, n are linear, quadratic, and cubic functions on Albert algebra).

Notice first of all that for any JB-algebra $\mathcal{B}$

$$[Q_a, Q_b] = [R_{a^2}, R_{b^2}] + o_1$$

where $a, b \in \mathcal{B}$, $||o_1|| \leq \epsilon f(||a||, ||b||)$. (In [25], this formula without the o_1 term is derived for $a \circ b = 0$. We use the same calculations, but instead of omitting terms containing $a \circ b$ we estimate them using $||a \circ b|| \leq \epsilon$. We apply the same procedure to the derivation of other formulas). The next step is the formula

$$[Q_a, Q_b]c = [R_{a^2}, R_{b^2}]c + o_1 c = (a^2, c, b^2) + o_1 c = -2(a^2 \circ b, c, b) + o_1 c$$

where $c \in \mathcal{B}$ and (x, y, z) stands for the associator $(x \circ y) \circ z - x \circ (y \circ z)$.

It follows from (A4) that

$$(a^2 \circ b, c, b) = -s(a, b)(a, c, b) + o_2$$

where $s(a, b)$ denotes the bilinear form, corresponding to the quadratic form $s(a)$ and $||o_2|| \leq \epsilon g(||a||, ||b||)||c||$. (See [25] for the derivation in the case $\epsilon = 0$.) Substituting $c = a$, we obtain

$$s(a, b)(a^2 \circ b) = o_3$$

where $||o_3|| \leq \epsilon h(||a||, ||b||)$. We see that either $|s(a, b)| \leq ||o_3||^{\frac{1}{2}}$ or $||a^2 \circ b|| \leq ||o_3||^{\frac{1}{2}}$. In the first case,

$$||[Q_a, Q_b]|| \leq 2||o_3||^{\frac{1}{2}}||a||||b|| + \epsilon(g(||a||, ||b||) + ||o_1||.$$

In the second case,

$$||[Q_a, Q_b]|| \leq 4||o_3||^{\frac{1}{2}}||b|| + ||o_1||.$$

In both cases, we obtain the estimate we need.

Let us consider again a commutative Lie group (translation group) acting on JB-algebra $\mathcal{B}$ by means of automorphisms or, more generally, structural transformations $\alpha(\mathbf{x}, \tau)$. We assume $(\alpha(\mathbf{x}, \tau)a) \circ b$ is small for large $\mathbf{x}$. (If $\mathcal{B}$ is a JC-algebra corresponding to C*-algebra $\mathcal{A}$, this is equivalent to the asymptotic anticommutativity of elements a, b in $\mathcal{A}$.) Then, it follows from (A3) that Q_a *and* Q_b *asymptotically commute:*

if $||(\alpha(\mathbf{x}, \tau)a) \circ b|| \leq \rho(\mathbf{x}, \tau)$ *then* $|||[Q_{\alpha(\mathbf{x},\tau)a}, Q_b]|| \leq k\sqrt{\rho(\mathbf{x}, \tau)}$.

Note

[1] We define a cone in topological vector space as a closed convex subset, which is invariant with respect to dilations $x \to \lambda x$ where $\lambda > 0$. We do not impose any further restrictions, hence in our terminology a vector space is a cone.

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