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Generalized Helical Hypersurface with Space-like Axis in Minkowski 5-Space

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Abstract: We introduce the generalized helical hypersurface having a space-like axis in five-dimensional Minkowski space. We compute the first and second fundamental form matrices, Gauss map, and shape operator matrix of the hypersurface. Additionally, we compute the curvatures of the hypersurface by using the Cayley–Hamilton theorem. Moreover, we give some relations for the mean and the Gauss–Kronecker curvatures of the hypersurface. Finally, we obtain the Laplace–Beltrami operator of the hypersurface.

Keywords: Minkowski 5-space; Lorentzian inner product; Lorentzian quadruple vector product; helical hypersurface; Gauss map; curvature

1. Introduction

Geometers have succeeded by applying differential geometry, which is a branch of mathematics, on engineering, physics, architecture, biology, chemistry, and also astrophysics, working the theory of hyper-surfaces for hundreds of years.

Helical or helicoidal hyper-surfaces and related characters such as rotational, minimal, ruled hyper-surfaces have been researched by geometers for almost 500 years. Let us see some works on hyper-surfaces.

The relation for a manifold isometric to a sphere was given by Obata [1]; a Euclidean submanifold is 1-type if and only if it is minimal or minimal of a hypersphere of m-dimensional Euclidean space $\mathbb{E}^m$ was served by Takahashi [2]; the minimal submanifolds of a sphere were studied by Chern et al. [3]; the hypersurfaces having constant curvature were studied by Cheng and Yau [4]; the minimal submanifolds with the Laplace–Beltrami operator were investigated by Lawson [5].

The submanifolds of finite-type in m-dimensional Euclidean space $\mathbb{E}^m$ or m-dimensional semi-Euclidean space $\mathbb{E}^m_\nu$ having index ν were researched by Chen [6–9]. Spherical 2-type submanifolds were studied by [7,10,11]; Garay [12] focused Takahashi's theorem in $\mathbb{E}^m$. The submanifolds with the finite-type Gauss map in $\mathbb{E}^m$ were researched by Chen and Piccinni [13]. The forty years of differential geometry of 1-type submanifolds and submanifolds having a 1-type Gauss map in space forms were served by Chen et al. [14].

In three-dimensional Euclidean space $\mathbb{E}^3$, isometries of the helical and rotational surfaces were described by Bour's theorem [15], and also the helical and rotational surfaces were studied by Do Carmo and Dajczer [16]. The minimal surfaces and spheres satisfying $\Delta r = \lambda r$, $\lambda \in \mathbb{R}$ were presented by Takahashi [2]; the surfaces holding $\Delta H = AH$, $A \in Mat(3,3)$ were focused on by Ferrandez et al. [17]; the minimal helicoid was studied by Choi and Kim [18]; surfaces of revolution were researched by Garay [19]; the surfaces having $\Delta r = Ar + B$, where A is 3×3, and B is a 3×1 matrix, were introduced by Dillen et al. [20]; the surfaces of revolution having $\Delta^{III} x = Ax$ were considered by Stamatakis and Zoubi [21]; the helicoidal surfaces having $\Delta^J r = Ar$, $J = I, II, III$, were studied by Senoussi and Bekkar [22]; the Cheng–Yau operator of the surfaces of revolution was studied by Kim et al. [23].

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In three-dimensional Minkowski space $\mathbb{E}_1^3$, helical surfaces were studied by by Beneki et al. [24]; the Bour's theorem was presented by Güler and Turgut Vanlı [25]; helical surfaces having light-like profile curves were investigated via Bour's theorem by Güler [26]; helical maximal surfaces were studied by Mira and Pastor [27]; ruled and rotation surfaces were focused on by Kim and Yoon [28–30]. See also [2,25,31,32] for details of the topic.

In four-dimensional Euclidean space $\mathbb{E}^4$, general rotational surfaces were investigated by Moore [33,34]; hypersurfaces having harmonic mean curvature were given by Hasanis and Vlachos [35]; the complete hypersurfaces having CMC were considered by Cheng and Wan [36]; the Vranceanu surfaces with Gauss map were introduced by Arslan et al. [37]; the generalized rotational surfaces were studied by Arslan et al. [38]; the affine umbilical surfaces were focused on by Magid et al. [39]; the affine geometry of surfaces and hypersurfaces were studied by Scharlach [40]; the hypersurfaces having Weyl pseudosymmetric were introduced by Arslan et al. [41]; meridian surfaces were focused on by Arslan et al. [42]. Rotation surfaces having a finite-type Gauss map were considered by Yoon [43]. Helical hypersurfaces were introduced by Güler et al. [44]; the $III-$Laplacian of rotational hypersurface was considered by Güler et al. [45]; the Cheng–Yau operator of the rotational hypersurfaces was investigated by Güler and Turgay [46]; the rotational hypersurfaces having $\Delta R = AR$, where A is 4×4 matrix, were studied by Güler [47]. The curvatures of hypersphere were revealed by Güler [48].

In four-dimensional Minkowski space $\mathbb{E}_1^4$, the similar surfaces of [33,34] were described by Ganchev and Milousheva [49]; the equation $\Delta H = \alpha H$ (H is a mean curvature, α is a constant) was considered by Arvanitoyeorgos et al. [50]; meridian surfaces having elliptic or hyperbolic type were studied by Arslan and Milousheva [51]; three types of the helical hypersurfaces were given by Güler [52]; the fuzzy algebraic modeling of spatiotemporal time series paradoxes in cosmic-scale kinematics was considered by Iliadis [53]; the emergence of Minkowski spacetime by simple deterministic graph rewriting was introduced by Leuenberger [54]; generalized helical hypersurfaces including a time-like axis in Minkowski spacetime were studied by Güler [55].

In this work, a generalized helical hypersurface $\mathbf{x} = \mathbf{x}(r, \theta_1, \theta_2, \theta_3)$ with a space-like axis in Minkowski 5-space $\mathbb{E}_1^5$ is considered. Some facts of five-dimensional Minkowski geometry are given in Section 2. The fundamental form matrices, Gauss map $\mathbf{G}$, and shape operator matrix $\mathbf{S}$ of any hypersurface in $\mathbb{E}_1^5$ are revealed. The definition of a helical hypersurface $\mathbf{x}$ in $\mathbb{E}_1^5$ is described in Section 3.

Moreover, by using the Cayley–Hamilton theorem, the curvature formulas of a hypersurface are obtained, and the curvatures of the helical hypersurface $\mathbf{x}$ are computed. Some facts for the curvatures of the mean $\mathcal{K}_1$ and Gauss–Kronecker $\mathcal{K}_4$ of $\mathbf{x}$ are given. In Section 4, umbilical conditions of hypersurfaces are presented.

Additionally, in $\mathbb{E}_1^5$, the relation $\Delta\mathbf{x} = \mathcal{M}\mathbf{x}$, where $\mathcal{M}$ is the 5×5 matrix, is obtained in Section 5. Then, some examples that are appropriate to all the findings are served. Finally, a summary is presented in the last section.

2. Preliminaries

In this section, some fundamental facts and the notations of the differential geometry are described.

Let $\mathbb{E}_1^m$ denote the Minkowski (or semi-Euclidean) m-space with its metric tensor described by

$$\widetilde{g} = \langle\ ,\ \rangle = \sum_{i=1}^{m-1} dx_i^2 - dx_m^2,$$

where x_i is the Minkowski coordinates of type $(m-1, 1)$. Consider an m-dimensional semi-Riemannian submanifold $\mathbf{M}$ of the space $\mathbb{E}_1^m$. The Levi–Civita connections [56] of the manifold $\widetilde{\mathbf{M}}$ and its submanifold $\mathbf{M}$ of $\mathbb{E}_1^m$ are indicated by $\widetilde{\nabla}$, ∇, respectively. Describing the vector field tangent (respectively, normal) to $\mathbf{M}$, the letters $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{N}$ (respectively, ζ, η) are used.

The Gauss and Weingarten formulas are given, respectively, by

$$\begin{aligned}\widetilde{\nabla}_{\mathcal{X}}\mathcal{Y} &= \nabla_{\mathcal{X}}\mathcal{Y} + h(\mathcal{X},\mathcal{Y}),\\ \widetilde{\nabla}_{\mathcal{X}}\zeta &= -A_{\zeta}(\mathcal{X}) + \mathcal{D}_{\mathcal{X}}\zeta,\end{aligned}$$

where $\mathbf{h}$, $\mathcal{D}$, and $\mathcal{A}$ are the second fundamental form, the normal connection, and the shape operator of $\mathbf{M}$, respectively.

For each $\zeta \in \mathcal{T}_p^{\perp}\mathbf{M}$, the shape operator $\mathcal{A}_{\zeta}$ is a symmetric endomorphism of the tangent space $\mathcal{T}_p\mathbf{M}$ at $p \in \mathbf{M}$. The shape operator and the second fundamental form are related by

$$\langle h(\mathcal{X},\mathcal{Y}),\zeta\rangle = \langle \mathcal{A}_{\zeta}\mathcal{X},\mathcal{Y}\rangle.$$

The Gauss and Codazzi equations are given, respectively, by

$$\begin{aligned}\langle \mathfrak{R}(\mathcal{X},\mathcal{Y})\mathcal{Z},\mathcal{N}\rangle &= \langle \mathbf{h}(\mathcal{Y},\mathcal{Z}),\mathbf{h}(\mathcal{X},\mathcal{N})\rangle - \langle \mathbf{h}(\mathcal{X},\mathcal{Z}),\mathbf{h}(\mathcal{Y},\mathcal{N})\rangle,\\ (\widetilde{\nabla}_{\mathcal{X}}\mathbf{h})(\mathcal{Y},\mathcal{Z}) &= (\widetilde{\nabla}_{\mathcal{Y}}\mathbf{h})(\mathcal{X},\mathcal{Z}),\end{aligned}$$

where $\mathfrak{R}, \mathfrak{R}^{\mathcal{D}}$ are the curvature tensors matched with connections ∇ and $\mathcal{D}$, respectively, and $\widetilde{\nabla}\mathbf{h}$ is defined by

$$(\widetilde{\nabla}_{\mathcal{X}}\mathbf{h})(\mathcal{Y},\mathcal{Z}) = \mathcal{D}_{\mathcal{X}}\mathbf{h}(\mathcal{Y},\mathcal{Z}) - \mathbf{h}(\nabla_{\mathcal{X}}\mathcal{Y},\mathcal{Z}) - \mathbf{h}(\mathcal{Y},\nabla_{\mathcal{X}}\mathcal{Z}).$$

Hypersurface of Minkowski Space

Now, let $\mathbf{M}$ be an oriented hypersurface in Minkowski space $\mathbb{E}_1^{n+1}$, $\mathbf{S}$ its shape operator (i.e., the Weingarten map), and x its position vector. Note the local orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ consisting of the principal directions of $\mathbf{M}$ matching with the principal curvature k_i for $i = 1, 2, \ldots n$. Let the dual basis of this frame field be $\{\varsigma_1, \varsigma_2, \ldots, \varsigma_n\}$. Then, the first Cartan structural equation is

$$d\varsigma_i = \sum_{i=1}^{n} \varsigma_j \wedge \psi_{ij}, \quad i, j = 1, 2, \ldots, n,$$

where ψ_{ij} indicates the connection forms matching with the selected frame field. Determine the Levi–Civita connection of $\mathbf{M}$ in $\mathbb{E}_1^{n+1}$ by ∇. Hence, from the Codazzi equation, the following occurs:

$$\begin{aligned}e_i(k_j) &= \psi_{ij}(e_j)(k_i - k_j),\\ \psi_{ij}(e_l)(k_i - k_j) &= \psi_{il}(e_j)(k_i - k_l)\end{aligned}$$

for distinct $i, j, l = 1, 2, \ldots, n$.

Put $s_j = \tau_j(k_1, k_2, \ldots, k_n)$, where τ_j is the j-th elementary symmetric function given by

$$\tau_j(q_1, q_2, \ldots, q_n) = \sum_{1 \le i_1 < i_2 < \ldots < i_j \le n} q_{i_1} q_{i_2} \cdots q_{i_j}.$$

The following notation is run:

$$\mathbf{r}_i^j = \tau_j(k_1, k_2, \ldots, k_{i-1}, k_{i+1}, k_{i+2} \ldots, k_n).$$

By the definition, $\mathbf{r}_i^0 = 1$ and $s_{n+1} = s_{n+2} = \cdots = 0$. The function s_k is called the k-th mean curvature of $\mathbf{M}$. The functions $H = \frac{1}{n}s_1$ and $K = s_n$ are called the mean curvature and Gauss–Kronecker curvature of $\mathbf{M}$, respectively. When $s_j \equiv 0$ on $\mathbf{M}$, then $\mathbf{M}$ is called j-minimal. See Alias and Gürbüz [57] and also Kühnel [58].

In $\mathbb{E}_1^{n+1}$, the characteristic polynomial equation of $\mathbf{S}$ is obtained by

$$P_{\mathbf{S}}(\lambda) = 0 = \det(\mathbf{S} - \lambda\mathcal{I}_n) = \sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k}, \tag{1}$$

where $i = 0, \dots, n$, $\mathcal{I}_n$ denotes the identity matrix of order n. Then, the curvature formulas are determined by $\binom{n}{i}\mathcal{K}_i = s_i$. Here, $\binom{n}{0}\mathcal{K}_0 = s_0 = 1$ (by definition), $\binom{n}{1}\mathcal{K}_1 = s_1, \dots, \binom{n}{n}\mathcal{K}_n = s_n$.

The k-th fundamental form of $\mathbf{M}$ is given by $\mathbf{I}\left(\mathbf{S}^{k-1}(\mathcal{X}), \mathcal{Y}\right) = \left\langle \mathbf{S}^{k-1}(\mathcal{X}), \mathcal{Y} \right\rangle$. Therefore,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathcal{K}_i \mathbf{I}\left(\mathbf{S}^{n-i}(\mathcal{X}), \mathcal{Y}\right) = 0.$$

On the other side, we compute the fundamental forms, Gauss map $\mathbf{G}$, the shape operator matrix $\mathbf{S}$, i-th curvature formulas $\mathcal{K}_i$, the mean curvature $\mathcal{K}_1$, and the Gauss–Kronecker curvature $\mathcal{K}_4$ of a hypersurface $\mathbf{x} = \mathbf{x}(r, \theta_1, \theta_2, \theta_3)$ in Minkowski 5-space $\mathbb{E}^5_1$.

We identify a vector $\overrightarrow{\alpha}$ with its transpose in this work. We assume $\mathbf{x} = \mathbf{x}(r, \theta_1, \theta_2, \theta_3)$ to be an immersion from $M^4 \subset \mathbb{E}^4$ to $\mathbb{E}^5_1$.

Next, we give some definitions, notions, etc., about semi-Riemannian geometry. The readers can refer to O'Neill [59] for details.

Definition 1. *A Lorentzian inner product of* $\overrightarrow{x^1} = (x^1_1, \dots, x^1_5)$, $\overrightarrow{x^2} = (x^2_1, \dots, x^2_5)$ *of* $\mathbb{E}^5_1$ *is given by*

$$\overrightarrow{x^1} \cdot \overrightarrow{x^2} = x^1_1 x^2_1 + x^1_2 x^2_2 + x^1_3 x^2_3 + x^1_4 x^2_4 - x^1_5 x^2_5.$$

From here to the end, we will use notation " $\cdot$ " other than $\langle .,. \rangle$.

Definition 2. *A Lorentzian quadruple vector product of* $\overrightarrow{x^1}, \dots, \overrightarrow{x^4}$ *of* $\mathbb{E}^5_1$ *is defined by*

$$\overrightarrow{x^1} \times \overrightarrow{x^2} \times \overrightarrow{x^3} \times \overrightarrow{x^4} = \det \begin{pmatrix} e_1 & e_2 & e_3 & e_4 & -e_5 \\ x^1_1 & x^1_2 & x^1_3 & x^1_4 & x^1_5 \\ x^2_1 & x^2_2 & x^2_3 & x^2_4 & x^2_5 \\ x^3_1 & x^3_2 & x^3_3 & x^3_4 & x^3_5 \\ x^4_1 & x^4_2 & x^4_3 & x^4_4 & x^4_5 \end{pmatrix},$$

where e_i, $i = 1, \dots, 5$ *are the base elements of* $\mathbb{E}^5_1$.

Definition 3. *For a hypersurface given by four parameters* $\mathbf{x} = \mathbf{x}(r, \theta_1, \theta_2, \theta_3)$ *in* $\mathbb{E}^5_1$*, the first and second fundamental form matrices are defined by*

$$\mathbf{I} = \begin{pmatrix} E & F & A & D \\ F & G & B & J \\ A & B & C & Q \\ D & J & Q & S \end{pmatrix}, \quad \mathbf{II} = \begin{pmatrix} L & M & P & X \\ M & N & T & Y \\ P & T & V & Z \\ X & Y & Z & U \end{pmatrix}.$$

Here, the components of the above matrices are described by

$$\begin{array}{lllll} E = \mathbf{x}_r \cdot \mathbf{x}_r, & F = \mathbf{x}_r \cdot \mathbf{x}_{\theta_1}, & A = \mathbf{x}_r \cdot \mathbf{x}_{\theta_2}, & D = \mathbf{x}_r \cdot \mathbf{x}_{\theta_3}, & G = \mathbf{x}_{\theta_1} \cdot \mathbf{x}_{\theta_1}, \\ B = \mathbf{x}_{\theta_1} \cdot \mathbf{x}_{\theta_2}, & J = \mathbf{x}_{\theta_1} \cdot \mathbf{x}_{\theta_3}, & C = \mathbf{x}_{\theta_2} \cdot \mathbf{x}_{\theta_2}, & Q = \mathbf{x}_{\theta_2} \cdot \mathbf{x}_{\theta_3}, & S = \mathbf{x}_{\theta_3} \cdot \mathbf{x}_{\theta_3}, \\ L = \mathbf{x}_{rr} \cdot \mathbf{G}, & M = \mathbf{x}_{r\theta_1} \cdot \mathbf{G}, & P = \mathbf{x}_{r\theta_2} \cdot \mathbf{G}, & X = \mathbf{x}_{r\theta_3} \cdot \mathbf{G}, & N = \mathbf{x}_{\theta_1\theta_1} \cdot \mathbf{G}, \\ T = \mathbf{x}_{\theta_1\theta_2} \cdot \mathbf{G}, & Y = \mathbf{x}_{\theta_1\theta_3} \cdot \mathbf{G}, & V = \mathbf{x}_{\theta_2\theta_2} \cdot \mathbf{G}, & Z = \mathbf{x}_{\theta_2\theta_3} \cdot \mathbf{G}, & U = \mathbf{x}_{\theta_3\theta_3} \cdot \mathbf{G}, \end{array}$$

$\mathbf{x}_r = \frac{\partial \mathbf{x}}{\partial r}$, $\mathbf{x}_{r\theta_1} = \frac{\partial \mathbf{x}}{\partial r \partial \theta_1}$, $\mathbf{x}_{\theta_3\theta_3} = \frac{\partial^2 \mathbf{x}}{\partial \theta_3^2}$, *etc., and the Gauss map of the hypersurface* $\mathbf{x}$ *is determined by the following formula:*

$$\mathbf{G} = \frac{\mathbf{x}_r \times \mathbf{x}_{\theta_1} \times \mathbf{x}_{\theta_2} \times \mathbf{x}_{\theta_3}}{\left\| \mathbf{x}_r \times \mathbf{x}_{\theta_1} \times \mathbf{x}_{\theta_2} \times \mathbf{x}_{\theta_3} \right\|}.$$

Definition 4. *The product matrix* $\mathbf{I}^{-1}.\mathbf{II}$ *is called the shape operator matrix* $\mathbf{S}$ *of the hypersurface* $\mathbf{x} = \mathbf{x}(r, \theta_1, \theta_2, \theta_3)$. *In addition, det*$(\mathbf{S})$ *gives the mean curvature* $\mathcal{K}_1$, *trace*$(\mathbf{S})/4$ *gives the Gauss–Kronecker curvature* $\mathcal{K}_4$ *of* $\mathbf{x}$.

Definition 5. *For a hypersurface* $\mathbf{x} = \mathbf{x}(r, \theta_1, \theta_2, \theta_3)$ *in* $\mathbb{E}^5_1$, *the following relations come out*

$$\mathbf{II} = \mathbf{I}.\mathbf{S}, \mathbf{III} = \mathbf{II}.\mathbf{S}, \mathbf{IV} = \mathbf{III}.\mathbf{S}, \mathbf{V} = \mathbf{IV}.\mathbf{S}.$$

Here, $\mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}, \mathbf{V}$ *are the first, second, third, fourth, and the fifth fundamental form matrices having order* 4×4 *of the hypersurface.*

Definition 6. *In* $\mathbb{E}^5_1$, *the characteristic polynomial of* $\mathbf{S}$ *is determined by*

$$P_{\mathbf{S}}(\lambda) = \sum_{k=0}^{4} (-1)^k s_k \lambda^{4-k} = \det(\mathbf{S} - \lambda \mathcal{I}_4) = 0,$$

where $\mathcal{I}_4$ *indicates the identity matrix of order* 4. *Hence, the curvature formulas are* $\binom{4}{i}\mathcal{K}_i = s_i$, *where* $\binom{4}{0}\mathcal{K}_0 = s_0 = 1$ *(by definition),* $\binom{4}{1}\mathcal{K}_1 = s_1, \ldots, \binom{4}{4}\mathcal{K}_4 = s_4$, $\mathcal{K}_1$ *is the mean curvature, and* $\mathcal{K}_4$ *is the Gauss–Kronecker curvature, and* $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Definition 7. *A hypersurface* $\mathbf{x}$ *is called j-minimal if* $\mathcal{K}_j = 0$, $j = 0, \ldots, 4$ *on* $\mathbf{x}$, *identically.*

See [58] for details of $\mathcal{K}_j$, and also [44,45] for details of dimension four.

Next, we determine the explicit formulas of the mean curvature $\mathcal{K}_1$ and the Gauss–Kronecker curvature $\mathcal{K}_4$ of hypersurface $\mathbf{x}$.

Theorem 1. *For a hypersurface in* $\mathbb{E}^5_1$, *the general formulas of the mean curvature, and the Gauss–Kronecker curvature are defined, respectively, by*

$$\begin{aligned} \mathcal{K}_1 \;=\; & [(EN + GL - 2FM)(CS - Q^2) + (EG - F^2)(SV + UC) - (GU + NS)A^2 \\ & -(LS + EU)B^2 - (CN + GV)D^2 - (EV + CL)J^2 + 2(A^2JY + B^2XD \\ & +D^2BT + J^2AP + F^2QZ + CJMD - ABYD - BJPD + ANQD \\ & -AJTD - BMQD + AGZD - BFZD + CFYD - AGPS - CGXD \\ & +FJVD + GQPD + BJZE - CJYE + BFPS - BSTE - FQTD + BQYE \\ & +JQTE + AGQX - BFQX - GQZE + ABFU - FJPQ + AFST \\ & -AJMQ - AFQY + ABMS - ABJX + BJLQ + CFJX - AFJZ)]/(4\det \mathbf{I}), \end{aligned} \tag{2}$$

and

$$\begin{aligned} \mathcal{K}_4 \;=\; & [(LN - M^2)(UV - Z^2) + (Y^2 - UN)P^2 + (X^2 - UL)T^2 - (LY^2 + NX^2)V \\ & +2((VM - PT)XY + (LT - MP)YZ + (NP - MT)XZ + MUPT)]/\det \mathbf{I}, \end{aligned} \tag{3}$$

where

$$\begin{aligned} \det \mathbf{I} \;=\; & (EG - F^2)(CS - Q^2) + (J^2 - GS)A^2 + (D^2 - ES)B^2 - (EJ^2 + GD^2)C \\ & +2((CF - AB)DJ + (EB - FA)JQ + (GA - FB)DQ + FABS). \end{aligned}$$

Proof. By using the Definition 3, Definition 4, and Definition 6, we obtain the characteristic polynomial of the shape operator matrix of the hypersurface. Then, we have the curvatures $\mathcal{K}_1$ and $\mathcal{K}_4$ easily. □

3. Generalized Helical Hypersurface Having a Space-like Axis in $\mathbb{E}^5_1$

In Riemannian space forms, the rotational hypersurfaces can be seen in the work of Do Carmo and Dajczer [60].

Next, we define the generalized helical hypersurface in space forms.

Definition 8. *For an open interval $I \subset \mathbb{R}$, let $\gamma : I \longrightarrow \Pi$ be a curve in a plane Π, and ℓ be a line in Π. A rotational hypersurface is defined as a hypersurface rotating a curve γ around a line ℓ (called the profile curve and the axis, respectively). Suppose that, when a profile curve γ rotates around the axis ℓ, it simultaneously displaces parallel lines orthogonal to the axis ℓ, so that the speed of displacement is proportional to the speed of rotation. Therefore, the resulting hypersurface is called the generalized helical hypersurface having axis ℓ, and pitches $a, b, c \in \mathbb{R} - \{0\}$.*

We now determine space-like, time-like, and light-like curves (resp., hypersurfaces) in Minkowski 5-space $\mathbb{E}^5_1$.

Definition 9. *For a curve $\gamma = \gamma(r)$ and a hypersurface $\mathbf{x} = \mathbf{x}(r, \theta_1, \theta_2, \theta_3)$ in five-dimensional Minkowski space $\mathbb{E}^5_1$, the following applies:*

i. *The γ (resp., $\mathbf{x}$) is named space-like, if $\gamma' \cdot \gamma' > 0$ (resp., $\det \mathbf{I} > 0$),*
ii. *The γ (resp., $\mathbf{x}$) is named time-like, if $\gamma' \cdot \gamma' < 0$ (resp., $\det \mathbf{I} < 0$),*
iii. *The γ (resp., $\mathbf{x}$) is named light-like, if $\gamma' \cdot \gamma' = 0$ (resp., $\det \mathbf{I} = 0$),*

with $\gamma' = \frac{d\gamma}{dr}$.

The readers can see O'Neill [59] and Kühnel [58] for details.

Next, we determine the generalized helical hypersurface having a space-like axis in $\mathbb{E}^5_1$.

While the axis of rotation is ℓ, there is a Lorentzian transformation by which the axis is ℓ transformed to the x_1-axis of $\mathbb{E}^5_1$. The rotation matrix obtained by the space-like vector $(1, 0, 0, 0, 0)$ of the rotation axis ℓ in $\mathbb{E}^5_1$ is described as follows:

$$\mathcal{R} = \mathcal{R}(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{C}_3 & 0 & 0 & \mathcal{S}_3 \\ 0 & \mathcal{S}_2\mathcal{S}_3 & \mathcal{C}_2 & 0 & \mathcal{S}_2\mathcal{C}_3 \\ 0 & \mathcal{S}_1\mathcal{C}_2\mathcal{S}_3 & \mathcal{S}_1\mathcal{S}_2 & \mathcal{C}_1 & \mathcal{S}_1\mathcal{C}_2\mathcal{C}_3 \\ 0 & \mathcal{C}_1\mathcal{C}_2\mathcal{S}_3 & \mathcal{C}_1\mathcal{S}_2 & \mathcal{S}_1 & \mathcal{C}_1\mathcal{C}_2\mathcal{C}_3 \end{pmatrix}.$$

Here, $\mathcal{C}_1 = \cosh\theta_1$, $\mathcal{C}_2 = \cosh\theta_2$, $\mathcal{C}_3 = \cosh\theta_3$, $\mathcal{S}_1 = \sinh\theta_1$, $\mathcal{S}_2 = \sinh\theta_2$, $\mathcal{S}_3 = \sinh\theta_3$, $\theta_1, \theta_2, \theta_3 \in [0, 2\pi)$.

The semi-orthogonal rotation matrix $\mathcal{R}$ supplies the following relations:

$$\mathcal{R}.\ell = \ell, \ \mathcal{R}^t.\varepsilon.\mathcal{R} = \mathcal{R}.\varepsilon.\mathcal{R}^t = \varepsilon, \ \det\mathcal{R} = 1,$$

where $\varepsilon = diag(1, 1, 1, 1, -1)$, $diag$ means diagonal parts of the matrix.

Parametrization of the profile curve is given by

$$\gamma(r) = (\varphi(r), 0, 0, 0, f(r)), \tag{4}$$

where $\varphi, f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ are the differentiable functions for all parameters $r \in I$.

In $\mathbb{E}^5_1$, the helical hypersurface $\mathbf{x} = \mathbf{x}(r, \theta_1, \theta_2, \theta_3)$ spanned by the vector $(1, 0, 0, 0, 0)$ is given by $\mathbf{x} = \mathcal{R}.\gamma^t + (a\theta_1 + b\theta_2 + c\theta_3)\ell^t$, where $r \in I$, $\theta_1, \theta_2, \theta_3 \in [0, 2\pi)$, $a, b, c \in \mathbb{R} - \{0\}$. Therefore, in five-dimensional Minkowski space, the parametric representation of the helical hypersurface $\mathbf{M}$ is given by

$$\mathbf{x}(r, \theta_1, \theta_2, \theta_3) = (\varphi + a\theta_1 + b\theta_2 + c\theta_3, f\mathcal{S}_3, f\mathcal{S}_2\mathcal{C}_3, f\mathcal{S}_1\mathcal{C}_2\mathcal{C}_3, f\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3). \tag{5}$$

In lower dimensions, we determine the following different hyper-surfaces:

1. When $b = c = \theta_2 = \theta_3 = 0$, we have helical surface having a space-like axis in three-dimensional Minkowski space $\mathbb{E}_1^3$;
2. When $a = b = c = \theta_2 = \theta_3 = 0$, we obtain a rotational surface having a space-like axis in three-dimensional Minkowski space $\mathbb{E}_1^3$;
3. When $c = \theta_3 = 0$, we obtain a helical hypersurface having a space-like axis in four-dimensional Minkowski space-time $\mathbb{E}_1^4$;
4. When $a = b = c = \theta_3 = 0$, we find a rotational hypersurface having a space-like axis in four-dimensional Minkowski space-time $\mathbb{E}_1^4$.

Next, we reveal the curvature formulas for any hypersurface $\mathbf{x} = \mathbf{x}(r, \theta_1, \theta_2, \theta_3)$ in $\mathbb{E}_1^5$.

Theorem 2. *A hypersurface $\mathbf{x} = \mathbf{x}(r, \theta_1, \theta_2, \theta_3)$ in Minkowski 5-space $\mathbb{E}_1^5$ has the following curvature formulas, $\mathcal{K}_0 = 1$ by definition:*

$$4\mathcal{K}_1 = -\frac{\mathfrak{b}}{\mathfrak{a}},\ 6\mathcal{K}_2 = \frac{\mathfrak{c}}{\mathfrak{a}},\ 4\mathcal{K}_3 = -\frac{\mathfrak{d}}{\mathfrak{a}},\ \mathcal{K}_4 = \frac{\mathfrak{e}}{\mathfrak{a}},$$

where $P_{\mathbf{S}}(\lambda) = \mathfrak{a}\lambda^4 + \mathfrak{b}\lambda^3 + \mathfrak{c}\lambda^2 + \mathfrak{d}\lambda + \mathfrak{e} = 0$ is the characteristic polynomial of shape operator matrix $\mathbf{S}$, $\mathfrak{a} = \det \mathbf{I}$, $\mathfrak{e} = \det \mathbf{II}$, and $\mathbf{I}$, $\mathbf{II}$ are the first and the second fundamental form matrices, respectively.

Proof. The solution matrix $\mathbf{I}^{-1}.\mathbf{II}$ supplies the shape operator matrix $\mathbf{S}$ of the hypersurface $\mathbf{x}$ in $\mathbb{E}_1^5$. Computing the formula of curvatures $\mathcal{K}_j$, where $j = 0, 1, \ldots, 4$, we reveal the characteristic polynomial $P_{\mathbf{S}}(\lambda) = \det(\mathbf{S} - \lambda\mathcal{I}_4) = 0$ of $\mathbf{S}$. Then, we find the following:

$$\begin{aligned}
\binom{4}{0}\mathcal{K}_0 &= 1,\\
\binom{4}{1}\mathcal{K}_1 &= \sum_{i=1}^{4} k_i = -\frac{\mathfrak{b}}{\mathfrak{a}},\\
\binom{4}{2}\mathcal{K}_2 &= \sum_{1=i_1<i_2}^{4} k_{i_1}k_{i_2} = \frac{\mathfrak{c}}{\mathfrak{a}},\\
\binom{4}{3}\mathcal{K}_3 &= \sum_{1=i_1<i_2<i_3}^{4} k_{i_1}k_{i_2}k_{i_3} = -\frac{\mathfrak{d}}{\mathfrak{a}},\\
\binom{4}{4}\mathcal{K}_4 &= \prod_{i=1}^{4} k_i = \frac{\mathfrak{e}}{\mathfrak{a}}.
\end{aligned}$$

Here, k_i, $i = 1, \ldots, 4$, are the principal curvatures of the hypersurface $\mathbf{x}$. □

See [44,45,48] for the cases of four-dimensional Euclidean space $\mathbb{E}^4$.

Hence, the mean curvature and the Gauss–Kronecker curvature of the generalized helical hypersurface having a space-like axis given by Equation (5) are described, respectively, as follows.

Theorem 3. *The mean and Gauss–Kronecker curvatures of the generalized helical hypersurface having a space-like axis determined by Equation (5), respectively, are given by*

$$\mathcal{K}_1 = \frac{\left\{\begin{array}{c} (\Omega_1 f^2 + \Omega_2) f^2 f' \varphi'' - 3\Omega_1 f^3 \varphi'^3 + \Omega_3 f^2 f' \varphi'^2 \\ +(\Omega_1 f^3 f'' + 3\Omega_1 f^2 f'^2 - \Omega_2 f f'' + 4\Omega_2 f'^2) f\varphi' - \Omega_3 f^2 f'^3 + \Omega_4 f'^3 \end{array}\right\}}{4 f \mathcal{C}_3 \mathcal{W}^{3/2}},$$

$$\mathcal{K}_4 = \frac{\left\{\begin{array}{c} (\Psi_1 f^3 f' \varphi'^3 + \Psi_2 f^2 f'^2 \varphi'^2 + \Psi_3 f f'^3 \varphi' + \Psi_4 f'^4) f^2 \varphi'' - \Psi_1 f^5 f'' \varphi'^4 \\ -\Psi_2 f^4 f' f'' \varphi'^3 - (\Psi_3 f^3 f'^2 f'' + \Psi_5 f^2 f'^4)\varphi'^2 - (\Psi_4 f^2 f'^3 f'' + \Psi_6 f f'^5)\varphi' + \Psi_7 f'^6 \end{array}\right\}}{f^2 \mathcal{W}^3},$$

where

$$
\begin{aligned}
\Omega_1 &= -\mathcal{C}_2^3\mathcal{C}_3^4, \\
\Omega_2 &= -\left(a^2+b^2\mathcal{C}_2^2+c^2\mathcal{C}_2^2\mathcal{C}_3^2\right)\mathcal{C}_2\mathcal{C}_3^2, \\
\Omega_3 &= -b\mathcal{S}_2\mathcal{C}_2^2\mathcal{C}_3^2-2c\mathcal{C}_2^3\mathcal{S}_3\mathcal{C}_3^3, \\
\Omega_4 &= 2c^3\mathcal{C}_2^3\mathcal{S}_3\mathcal{C}_3^3+bc^2\mathcal{S}_2\mathcal{C}_2^2\mathcal{C}_3^2+3b^2c\mathcal{C}_2^3\mathcal{S}_3\mathcal{C}_3+3a^2c\mathcal{C}_2\mathcal{S}_3\mathcal{C}_3+b^3\mathcal{S}_2\mathcal{C}_2^2+2a^2b\mathcal{S}_2,
\end{aligned}
$$

$$
\begin{aligned}
\Psi_1 &= \mathcal{C}_2^6\mathcal{C}_3^6, \\
\Psi_2 &= -b\mathcal{S}_2\mathcal{C}_2^5\mathcal{C}_3^4-2c\mathcal{C}_2^6\mathcal{S}_3\mathcal{C}_3^5, \\
\Psi_3 &= -a^2\mathcal{C}_2^2\mathcal{C}_3^2\left(\mathcal{S}_2^2+\mathcal{C}_2^2\mathcal{S}_3^2\right)-b^2\mathcal{C}_2^6\mathcal{C}_3^2\mathcal{S}_3^2+bc\mathcal{S}_2\mathcal{C}_2^5\mathcal{C}_3^3\mathcal{S}_3+c^2\mathcal{C}_2^6\mathcal{C}_3^4\mathcal{S}_3^2, \\
\Psi_4 &= 2a^2b\mathcal{S}_2\mathcal{C}_2^3\mathcal{S}_3^2+a^2c\mathcal{C}_2^4\mathcal{C}_3\mathcal{S}_3^3+b^2c\mathcal{C}_2^6\mathcal{C}_3\mathcal{S}_3^3+b^3\mathcal{S}_2\mathcal{C}_2^5\mathcal{S}_3^2, \\
\Psi_5 &= \left(a^2+b^2\mathcal{C}_2^2+c^2\mathcal{C}_2^2\mathcal{C}_3^2\right)\mathcal{C}_2^4\mathcal{C}_3^4, \\
\Psi_6 &= -2c^3\mathcal{C}_2^6\mathcal{S}_3\mathcal{C}_3^5-3b^2c\mathcal{C}_2^6\mathcal{S}_3\mathcal{C}_3^3-bc^2\mathcal{S}_2\mathcal{C}_2^5\mathcal{C}_3^4-b^3\mathcal{S}_2\mathcal{C}_2^5\mathcal{C}_3^2-3a^2c\mathcal{C}_2^4\mathcal{S}_3\mathcal{C}_3^3-2a^2b\mathcal{S}_2\mathcal{C}_2^3\mathcal{C}_3^2=-\Omega_4\mathcal{C}_2^3\mathcal{C}_3^2 \\
\Psi_7 &= c^4\mathcal{C}_2^6\mathcal{S}_3^2\mathcal{C}_3^4+2b^2c^2\mathcal{C}_2^6\mathcal{S}_3^2\mathcal{C}_3^2+bc^3\mathcal{S}_2\mathcal{C}_2^5\mathcal{S}_3\mathcal{C}_3^3+2b^3c\mathcal{S}_2\mathcal{C}_2^5\mathcal{S}_3\mathcal{C}_3+4a^2bc\mathcal{S}_2\mathcal{C}_2^3\mathcal{S}_3\mathcal{C}_3+a^2c^2\left(2\mathcal{C}_2^2\mathcal{S}_3^2+\mathcal{S}_2^2\right)\mathcal{C}_2^2\mathcal{C}_3^2,
\end{aligned}
$$

and $\mathcal{W}=f^2(f'^2-\varphi'^2)\mathcal{C}_2^2\mathcal{C}_3^2+f'^2(a^2+b^2\mathcal{C}_2^2+c^2\mathcal{C}_2^2\mathcal{C}_3^2)$, $a,b,c\in\mathbb{R}-\{0\}$, $\varphi=\varphi(r)$, $\varphi'=\frac{d\varphi}{dr}$, $\varphi''=\frac{d^2\varphi}{dr^2}$, $f=f(r)$, $f'=\frac{df}{dr}$, $f''=\frac{d^2f}{dr^2}$, $r\in I\subset\mathbb{R}$, $\mathcal{C}_2=\cosh\theta_2$, $\mathcal{C}_3=\cosh\theta_3$, $\mathcal{S}_2=\sinh\theta_2$, $\mathcal{S}_3=\sinh\theta_3$, $\mathcal{C}_2^2=(\cosh\theta_2)^2$, $\mathcal{C}_3^2=(\cosh\theta_3)^2$, $\mathcal{S}_2^2=(\sinh\theta_2)^2$, $\mathcal{S}_3^2=(\sinh\theta_3)^2$, *etc.*, $\theta_1,\theta_2,\theta_3\in[0,2\pi)$.

Proof. By considering Definition 3, and by taking the first derivatives of hypersurface Equation (5) with respect to $r,\theta_1,\theta_2,\theta_3$, we obtain the following first fundamental form matrix:

$$
\mathbf{I}=\begin{pmatrix}
\varphi'^2-f'^2 & a\varphi' & b\varphi' & c\varphi' \\
a\varphi' & f^2\mathcal{C}_2^2\mathcal{C}_3^2+a^2 & ab & ac \\
b\varphi' & ab & f^2\mathcal{C}_3^2+b^2 & bc \\
c\varphi' & ac & bc & f^2+c^2
\end{pmatrix}. \tag{6}
$$

We then have

$$
\det\mathbf{I}=-f^4\mathcal{C}_3^2\mathcal{W}, \tag{7}
$$

where

$$
\mathcal{W}=f^2\left(f'^2-\varphi'^2\right)\mathcal{C}_2^2\mathcal{C}_3^2+f'^2\left(a^2+b^2\mathcal{C}_2^2+c^2\mathcal{C}_2^2\mathcal{C}_3^2\right).
$$

By using the Gauss map formula in Definition 3, we obtain the following Gauss map

$$
\mathbf{G}=\frac{1}{\mathcal{W}^{1/2}}\begin{pmatrix}
ff'\mathcal{C}_2\mathcal{C}_3 \\
(f\varphi'\mathcal{S}_3-cf'\mathcal{C}_3)\mathcal{C}_2\mathcal{C}_3 \\
[(f\varphi'\mathcal{C}_3-cf'\mathcal{S}_3)\mathcal{S}_2\mathcal{C}_3-bf'\mathcal{C}_2]\mathcal{C}_2 \\
[(f\varphi'\mathcal{C}_3-cf'\mathcal{S}_3)\mathcal{C}_2\mathcal{C}_3-bf'\mathcal{S}_2]\mathcal{S}_1\mathcal{C}_2-af'\mathcal{C}_1 \\
[(f\varphi'\mathcal{C}_3-cf'\mathcal{S}_3)\mathcal{C}_2\mathcal{C}_3-bf'\mathcal{S}_2]\mathcal{C}_1\mathcal{C}_2-af'\mathcal{S}_1
\end{pmatrix} \tag{8}
$$

of the helical hypersurface having a space-like axis determined by Equation (5) in five-dimensional Minkowski space. It is clear that $\mathbf{G}\cdot\mathbf{G}=1$.

Next, taking care of Definition 3, Gauss map Equation (8), and by using the second derivatives of the helical hypersurface with respect to $r, \theta_1, \theta_2, \theta_3$, we have the following second fundamental form matrix:

$$\mathbf{II} = \frac{1}{\mathcal{W}^{1/2}} \begin{pmatrix} f(f'\varphi'' - f''\varphi')\mathcal{C}_2\mathcal{C}_3 & -af'^2\mathcal{C}_2\mathcal{C}_3 & -bf'^2\mathcal{C}_2\mathcal{C}_3 & -cf'^2\mathcal{C}_2\mathcal{C}_3 \\ -af'^2\mathcal{C}_2\mathcal{C}_3 & -f((f\varphi'\mathcal{C}_3 - cf'\mathcal{S}_3)\mathcal{C}_2\mathcal{C}_3 - bf'\mathcal{S}_2)\mathcal{C}_2^2\mathcal{C}_3 & -aff'\mathcal{S}_2\mathcal{C}_3 & -aff'\mathcal{C}_2\mathcal{S}_3 \\ -bf'^2\mathcal{C}_2\mathcal{C}_3 & -aff'\mathcal{S}_2\mathcal{C}_3 & -f(f\varphi'\mathcal{C}_3 - cf'\mathcal{S}_3)\mathcal{C}_2\mathcal{C}_3^2 & -bff'\mathcal{C}_2\mathcal{S}_3 \\ -cf'^2\mathcal{C}_2\mathcal{C}_3 & -aff'\mathcal{C}_2\mathcal{S}_3 & -bff'\mathcal{C}_2\mathcal{S}_3 & -f^2\varphi'\mathcal{C}_2\mathcal{C}_3 \end{pmatrix}.$$

The product matrix $\mathbf{I}^{-1}$. $\mathbf{II}$ describes the following shape operator matrix of the helical hypersurface:

$$\mathbf{S} = \frac{1}{\mathcal{W}^{3/2}} \left(\ s_{ij} \ \right)_{4\times 4}. \tag{9}$$

The characteristic polynomial $P_{\mathbf{S}}(\lambda) = \det(\mathbf{S} - \lambda\mathcal{I}_4) = 0$ of the shape operator matrix determined by Equation (9) is as follows:

$$\lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4 = 0,$$

where

$$\begin{aligned}
\alpha_1 &= -\mathcal{W}^{-3/2}(s_{11} + s_{22} + s_{33} + s_{44}), \\
\alpha_2 &= \mathcal{W}^{-3}(s_{11}s_{22} - s_{12}s_{21} + s_{11}s_{33} - s_{13}s_{31} + s_{11}s_{44} + s_{22}s_{33} - s_{14}s_{41} - s_{23}s_{32} + s_{22}s_{44} - s_{24}s_{42} + s_{33}s_{44} - s_{34}s_{43}), \\
\alpha_3 &= \mathcal{W}^{-9/2}(s_{11}s_{23}s_{32} - s_{11}s_{22}s_{33} + s_{12}s_{21}s_{33} - s_{12}s_{31}s_{23} - s_{21}s_{13}s_{32} + s_{13}s_{22}s_{31} - s_{11}s_{22}s_{44} + s_{11}s_{24}s_{42} + s_{12}s_{21}s_{44} \\
&\quad - s_{12}s_{41}s_{24} - s_{21}s_{14}s_{42} + s_{22}s_{14}s_{41} - s_{11}s_{33}s_{44} + s_{11}s_{34}s_{43} + s_{13}s_{31}s_{44} - s_{13}s_{41}s_{34} - s_{31}s_{14}s_{43} + s_{14}s_{41}s_{33} \\
&\quad - s_{22}s_{33}s_{44} + s_{22}s_{34}s_{43} + s_{23}s_{32}s_{44} - s_{23}s_{42}s_{34} - s_{32}s_{24}s_{43} + s_{24}s_{33}s_{42}), \\
\alpha_4 &= \mathcal{W}^{-6}(s_{11}s_{22}s_{33}s_{44} - s_{11}s_{22}s_{34}s_{43} - s_{11}s_{23}s_{32}s_{44} + s_{11}s_{23}s_{42}s_{34} + s_{11}s_{32}s_{24}s_{43} - s_{11}s_{24}s_{33}s_{42} - s_{12}s_{21}s_{33}s_{44} \\
&\quad + s_{12}s_{21}s_{34}s_{43} + s_{12}s_{31}s_{23}s_{44} - s_{12}s_{31}s_{24}s_{43} - s_{12}s_{23}s_{41}s_{34} + s_{12}s_{41}s_{24}s_{33} + s_{21}s_{13}s_{32}s_{44} - s_{21}s_{13}s_{42}s_{34} \\
&\quad - s_{21}s_{14}s_{32}s_{43} + s_{21}s_{14}s_{33}s_{42} - s_{13}s_{22}s_{31}s_{44} + s_{13}s_{22}s_{41}s_{34} + s_{13}s_{31}s_{24}s_{42} - s_{13}s_{32}s_{41}s_{24} + s_{22}s_{31}s_{14}s_{43} \\
&\quad - s_{22}s_{14}s_{41}s_{33} - s_{31}s_{14}s_{23}s_{42} + s_{14}s_{23}s_{32}s_{41}).
\end{aligned}$$

Here, $\alpha_1 = -4\mathcal{K}_1$, $\alpha_2 = 6\mathcal{K}_2$, $\alpha_3 = -4\mathcal{K}_3$, $\alpha_4 = \mathcal{K}_4$. We then compute the components of $\mathbf{S}$ described by Equation (9) as follows:

$$
\begin{aligned}
s_{11} &= -\mathcal{C}_2\mathcal{C}_3\Big[\Big(a^2+b^2\mathcal{C}_2^2+\Big(c^2+f^2\Big)\mathcal{C}_2^2\mathcal{C}_3^2\Big)ff'\varphi''+\Big[\Big(a^2+b^2\mathcal{C}_2^2+c^2\mathcal{C}_2^2\mathcal{C}_3^2\Big)\Big((f')^2-ff''\Big)-\mathcal{C}_2^2\mathcal{C}_3^2f^3f''\Big]\varphi'\Big],\\
s_{12} &= a\mathcal{C}_2\mathcal{C}_3\mathcal{W},\\
s_{13} &= \mathcal{C}_3\Big[-bf^2(\varphi')^2\mathcal{C}_2^3\mathcal{C}_3^2-a^2\mathcal{S}_2ff'\varphi'+b\mathcal{C}_2\Big(a^2+b^2\mathcal{C}_2^2+\Big(c^2+f^2\Big)\mathcal{C}_2^2\mathcal{C}_3^2\Big)(f')^2\Big],\\
s_{14} &= \mathcal{C}_2\Big[-c\mathcal{C}_2^2\mathcal{C}_3^3f^2(\varphi')^2-\mathcal{S}_3\Big(a^2+b^2\mathcal{C}_2^2\Big)ff'\varphi'+c\mathcal{C}_3\Big(a^2+b^2\mathcal{C}_2^2+\Big(c^2+f^2\Big)\mathcal{C}_2^2\mathcal{C}_3^2\Big)(f')^2\Big],\\
s_{21} &= a\mathcal{C}_2\mathcal{C}_3\Big[ff'\varphi'\varphi''+\Big((f')^2-ff''\Big)(\varphi')^2-(f')^4\Big],\\
s_{22} &= \frac{1}{f\mathcal{C}_3}\Big((b\mathcal{S}_2+c\mathcal{C}_2\mathcal{C}_3\mathcal{S}_3)f'-\mathcal{C}_2\mathcal{C}_3^2f\varphi'\Big)\mathcal{W},\\
s_{23} &= \frac{af'\mathcal{S}_2}{f\mathcal{C}_3}\Big[\mathcal{C}_3^2f^2(\varphi')^2-\Big(b^2+c^2\mathcal{C}_3^2+\mathcal{C}_3^2f^2\Big)(f')^2\Big],\\
s_{24} &= \frac{af'\mathcal{C}_2\mathcal{S}_3}{f}\Big[f^2(\varphi')^2-\Big(c^2+f^2\Big)(f')^2\Big],\\
s_{31} &= b\mathcal{C}_2^3\mathcal{C}_3\Big[ff'\varphi'\varphi''+\Big((f')^2-ff''\Big)(\varphi')^2-(f')^4\Big],\\
s_{32} &= -\frac{af'\mathcal{S}_2}{f\mathcal{C}_3}\mathcal{W},\\
s_{33} &= \frac{1}{f\mathcal{C}_3}\left\{\begin{array}{c}\mathcal{C}_2^3\mathcal{C}_3^4f^3(\varphi')^3-c\mathcal{C}_2^3\mathcal{C}_3^3\mathcal{S}_3f^2f'(\varphi')^2-\mathcal{C}_2\mathcal{C}_3^2[a^2+b^2\mathcal{C}_2^2+(c^2+f^2)\mathcal{C}_2^2\mathcal{C}_3^2]f(f')^2\varphi'\\ +[a^2b\mathcal{S}_2+c\mathcal{C}_2\mathcal{C}_3\mathcal{S}_3(a^2+b^2\mathcal{C}_2^2)+c\mathcal{C}_2^3\mathcal{C}_3^3\mathcal{S}_3(c^2+f^2)](f')^3\end{array}\right\},\\
s_{34} &= \frac{b}{f}\mathcal{C}_2^3\mathcal{S}_3\Big[f^2(\varphi')^2-\Big(c^2+f^2\Big)(f')^2\Big],\\
s_{41} &= c\mathcal{C}_2^3\mathcal{C}_3^3\Big[ff'\varphi'\varphi''+\Big((f')^2-ff''\Big)(\varphi')^2-(f')^4\Big],\\
s_{42} &= -\frac{af'\mathcal{C}_2\mathcal{S}_3}{f}\mathcal{W},\\
s_{43} &= \frac{f'}{f}\Big[b\mathcal{C}_2^3\mathcal{C}_3^2\mathcal{S}_3f^2(\varphi')^2+\Big[a^2c\mathcal{C}_3\mathcal{S}_2-b\mathcal{C}_2\mathcal{S}_3\Big(a^2+b^2\mathcal{C}_2^2+\mathcal{C}_2^2\mathcal{C}_3^2\Big(c^2+f^2\Big)\Big)\Big](f')^2\Big],\\
s_{44} &= \frac{\mathcal{C}_2}{f}\Big[\mathcal{C}_2^2\mathcal{C}_3^3f^3(\varphi')^3-\mathcal{C}_3\Big(a^2+b^2\mathcal{C}_2^2+\Big(c^2+f^2\Big)\mathcal{C}_2^2\mathcal{C}_3^2\Big)f(f')^2\varphi'+c\mathcal{S}_3\Big(b^2\mathcal{C}_2^2+a^2\Big)(f')^3\Big].
\end{aligned}
$$

Therefore, also from Definition 6, $det(\mathbf{S})$ gives the mean curvature $\mathcal{K}_1$, $trace(\mathbf{S})/4$ gives the Gauss–Kronecker curvature $\mathcal{K}_4$ of the helical hypersurface having a space-like axis described by Equation (5) in five-dimensional Minkowski space. □

By taking care of Definition 9 with the determinant of Equation (6), we conclude the following:

Corollary 1. *The profile curve $\gamma(r)=(\varphi(r)r,0,0,0,f(r))$ of the helical hypersurface (5) having a space-like axis has the following relation $\gamma'\cdot\gamma'=\varphi'^2-f'^2>0$ (resp., $<0,=0$), i.e., it is a space-like (resp., time-like, light-like) curve. Hence, taking care of Equation (7), the following holds:*

1. *If $\varphi'^2-f'^2>0$, i.e, γ is a space-like curve, and*

$$f^2\Big(\varphi'^2-f'^2\Big)C_2^2C_3^2+f'^2\Big(a^2+b^2C_2^2+c^2C_2^2C_3^2\Big)>0$$

(resp., $<0,=0$), i.e., $\mathbf{x}$ is a space-like (resp., time-like, light-like) helical hypersurface.

2. *If $\varphi'^2-f'^2<0$, i.e, γ is a time-like curve, and $\det\mathbf{I}<0$, i.e.,*

$$f^2\Big(\varphi'^2-f'^2\Big)C_2^2C_3^2>f'^2\Big(a^2+b^2C_2^2+c^2C_2^2C_3^2\Big),$$

then $\mathbf{x}$ is a time-like helical hypersurface.

3. *If $\varphi'^2 - f'^2 = 0$ (that is, $0 \neq \varphi = \pm f$), i.e, γ is a light-like line $(1,0,0,0,1)$ or $(1,0,0,0,-1)$, and $a^2 + b^2 C_2^2 + c^2 C_2^2 C_3^2 = 0$. That is, $a = b = c = 0$, $\mathcal{C}_2 \neq 0$, $\mathcal{C}_3 \neq 0$. Hence, $\mathbf{x}$ is a light-like rotational hypersurface.*

Next, we give a relation among the curvatures described by Theorem 2, and the fundamental forms given by Definition 5 of the hypersurface in five-dimensional Minkowski space.

Theorem 4. *Among its curvatures $\mathcal{K}_j$ and its fundamental forms, a hypersurface $\mathbf{x} = \mathbf{x}(r, \theta_1, \theta_2, \theta_3)$ in Minkowski space $\mathbb{E}_1^5$ has the following relation:*

$$\mathcal{K}_0 \mathbf{V} - 4\mathcal{K}_1 \mathbf{IV} + 6\mathcal{K}_2 \mathbf{III} - 4\mathcal{K}_3 \mathbf{II} + \mathcal{K}_4 \mathbf{I} = \mathbf{O}. \tag{10}$$

Here, $\mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}, \mathbf{V}$ are the fundamental form matrices having order 4×4, and $\mathbf{O}$ is the zero matrix of order 4 of the hypersurface.

Proof. We use the Cayley–Hamilton theorem, and we obtain $P_{\mathbf{S}}(\lambda) = \sum_{k=0}^{4} (-1)^k s_k \lambda^{4-k} = \det(\mathbf{S} - \lambda \mathcal{I}_4) = 0$. Then, we reveal the following characteristic polynomial of $\mathbf{S}$:

$$\mathcal{K}_0 \lambda^4 - 4\mathcal{K}_1 \lambda^3 + 6\mathcal{K}_2 \lambda^2 - 4\mathcal{K}_3 \lambda + \mathcal{K}_4 = 0.$$

Hence, it is clear. □

Note that three-dimensional cases of Theorem 4 are known by

$$\mathcal{K}_0 \mathbf{III} - 2\mathcal{K}_1 \mathbf{II} + \mathcal{K}_2 \mathbf{I} = \mathbf{O}$$

and

$$\mathcal{K}_0 \lambda^2 - 2\mathcal{K}_1 \lambda + \mathcal{K}_2 \lambda = 0.$$

Here, $\mathcal{K}_0 = 1$, $\mathbf{O}$ is the zero matrix of order 2, $\mathcal{K}_1 = H$ describes the mean curvature, and $\mathcal{K}_2 = K$ determines the Gaussian curvature of the surface in three-dimensional space forms.

In addition, four-dimensional cases of Theorem 4 are determined by

$$\mathcal{K}_0 \mathbf{IV} - 3\mathcal{K}_1 \mathbf{III} + 3\mathcal{K}_2 \mathbf{II} - \mathcal{K}_3 \mathbf{I} = \mathbf{O}$$

and

$$\mathcal{K}_0 \lambda^3 - 3\mathcal{K}_1 \lambda^3 + 3\mathcal{K}_2 \lambda^2 - \mathcal{K}_3 \lambda = 0.$$

Here, $\mathcal{K}_0 = 1$, $\mathbf{O}$ is the zero matrix of order 3, $\mathcal{K}_1$ indicates the mean curvature, and $\mathcal{K}_3$ describes the Gauss–Kronecker curvature of the hypersurface in four-dimensional space forms.

4. The Umbilical Hypersurfaces in Minkowski Five-Space

In this section, we give some umbilical facts of the hypersurfaces in five-dimensional Minkowski space $\mathbb{E}_1^5$.

From Theorem 2, the relations among the curvatures $\mathcal{K}_{i=0,\dots,4}$ and the principal curvatures $k_{j=1,\dots,4}$ of any hypersurface in five-dimensional Minkowski space are explicitly described by

$$\begin{aligned}
\mathcal{K}_0 &= 1, \\
4\mathcal{K}_1 &= k_1 + k_2 + k_3 + k_4, \\
6\mathcal{K}_2 &= k_1 k_2 + k_1 k_3 + k_1 k_4 + k_2 k_3 + k_2 k_4 + k_3 k_4, \\
4\mathcal{K}_3 &= k_1 k_2 k_3 + k_1 k_2 k_4 + k_1 k_3 k_4 + k_2 k_3 k_4, \\
\mathcal{K}_4 &= k_1 k_2 k_3 k_4.
\end{aligned}$$

Then, we obtain the following.

Corollary 2. *For a hypersurface in five-dimensional Minkowski space $\mathbb{E}_1^5$, the following occurs:*

$$k_1 = k_2 = k_3 = k_4 \quad \Leftrightarrow \quad (\mathcal{K}_1)^2 = \mathcal{K}_2, \mathcal{K}_1\mathcal{K}_2 = \mathcal{K}_3, \mathcal{K}_1\mathcal{K}_3 = (\mathcal{K}_2)^2 = (\mathcal{K}_1)^4 = \mathcal{K}_4.$$

Remark 1. *The umbilical hypersurfaces of five-dimensional Minkowski space $\mathbb{E}_1^5$ are only (open) hyperplanes and hyperspheres.*

An umbilical point is a geometric notion depends on the lines of curvature, which is a singularity of a line of curvature. That is, a line of curvature will end at that point.

Lemma 1. *A point is an umbilical point on the hypersurface in $\mathbb{E}_1^5$ if and only if $(\mathcal{K}_1)^2 = \mathcal{K}_2$, $\mathcal{K}_1\mathcal{K}_2 = \mathcal{K}_3$, $\mathcal{K}_1\mathcal{K}_3 = (\mathcal{K}_2)^2 = (\mathcal{K}_1)^4 = \mathcal{K}_4$.*

Theorem 5. *The generalized helical hypersurface with a space-like axis given by Equation (5) has an umbilical point if and only if the following differential equation holds:*

$$\left\{ \begin{array}{l} (\Omega_1 f^2 + \Omega_2) f^2 f' \varphi'' - 3\Omega_1 f^3 \varphi'^3 + \Omega_3 f^2 f' \varphi'^2 \\ + \left[(f^2\Omega_1 - \Omega_2) f f'' + (3\Omega_1 f^2 + 4\Omega_2) f'^2 \right] f \varphi' - (\Omega_3 f^2 - \Omega_4) f'^3 \end{array} \right\}^4$$
$$-4^4 C_3^4 f^2 \left\{ \begin{array}{l} (a^2 + b^2 C_2^2 + c^2 C_2^2 C_3^2) f'^2 \\ -C_2^2 C_3^2 f^2 (\varphi'^2 - f'^2) \end{array} \right\}^3 \cdot \left\{ \begin{array}{l} (\Psi_1 f^3 f' \varphi'^3 + \Psi_2 f^2 f'^2 \varphi'^2 + \Psi_3 f f'^3 \varphi' + \Psi_4 f'^4) f^2 \varphi'' \\ -\Psi_1 f^5 f'' \varphi'^4 - \Psi_2 f^4 f' f'' \varphi'^3 - (\Psi_3 f f'' + \Psi_5 f'^2) f^2 f'^2 \varphi'^2 \\ -(\Psi_4 f f'' + \Psi_6 f'^2) f f'^3 \varphi' + \Psi_7 f'^6 \end{array} \right\} = 0.$$

Proof. A generalized helical hypersurface $\mathbf{x}$ having a space-like axis has an umbilical point in $\mathbb{E}_1^5$; then, $(\mathcal{K}_1)^4 = \mathcal{K}_4$. □

Open Problem 1. Find the $\varphi = \varphi(r)$ solutions of the 2nd order differential equation determined by Theorem 5.

Now, we state minimality conditions, determined by Definition 7, of the generalized helical hypersurface having a space-like axis given by Equation (5).

Corollary 3. *Let $\mathbf{x} : M^4 \subset \mathbb{E}^4 \longrightarrow \mathbb{E}_1^5$ be an immersion given by Equation (5). $\mathbf{x}$ has zero mean curvature, i.e., 1-minimal, if and only if the following differential equation reveals*

$$(\Omega_1 f^2 + \Omega_2) f^2 f' \varphi'' - 3\Omega_1 f^3 \varphi'^3 + \Omega_3 f^2 f' \varphi'^2$$
$$+ \left[(f^2\Omega_1 - \Omega_2) f f'' + (3\Omega_1 f^2 + 4\Omega_2) f'^2 \right] f \varphi' - (\Omega_3 f^2 - \Omega_4) f'^3 = 0.$$

Open Problem 2. Find the $\varphi = \varphi(r)$ solutions of the 2nd order differential equation described by Corollary 3.

Corollary 4. *Let $\mathbf{x} : M^4 \subset \mathbb{E}^4 \longrightarrow \mathbb{E}_1^5$ be an immersion given by Equation (5). $\mathbf{x}$ has zero Gauss–Kronecker curvature, i.e., 4-minimal, if and only if the following differential equation holds:*

$$(\Psi_1 f^3 f' \varphi'^3 + \Psi_2 f^2 f'^2 \varphi'^2 + \Psi_3 f f'^3 \varphi' + \Psi_4 f'^4) f^2 \varphi''$$
$$-\Psi_1 f^5 f'' \varphi'^4 - \Psi_2 f^4 f' f'' \varphi'^3 - (\Psi_3 f f'' + \Psi_5 f'^2) f^2 f'^2 \varphi'^2$$
$$-(\Psi_4 f f'' + \Psi_6 f'^2) f f'^3 \varphi' + \Psi_7 f'^6 = 0.$$

Open Problem 3. Find the $\varphi = \varphi(r)$ solutions of the 2nd order differential equation obtained by Corollary 4.

5. Generalized Helical Hypersurface with a Space-like Axis Supplying $\Delta \mathbf{x} = \mathcal{M}\mathbf{x}$

In this section, we define the Laplace–Beltrami operator with respect to the first fundamental form of a smooth function in $\mathbb{E}_1^5$. Then, we calculate the Laplace–Beltrami operator of the generalized helical hypersurface having a space-like axis given by Equation (5).

Firstly, we give the definition of the Laplace–Beltrami operator with respect to first fundamental form of any smooth function in five-dimensional Minkowski space.

Definition 10. *In five-dimensional Minkowski space, the Laplace–Beltrami operator depends on the first fundamental form of a smooth function $\phi = \phi(x^1, x^2, x^3, x^4)\ |_{\mathbf{D}}\ (\mathbf{D} \subset \mathbb{R}^4)$ of class C^4 described by*

$$\Delta\phi = \frac{1}{\mathbf{g}^{1/2}} \sum_{i,j=1}^{4} \frac{\partial}{\partial x^i}\left(\mathbf{g}^{1/2}\mathbf{g}^{ij}\frac{\partial \phi}{\partial x^j}\right), \tag{11}$$

where $\left(\mathbf{g}^{ij}\right) = \left(\mathbf{g}_{kl}\right)^{-1}$ and $\mathbf{g} = \det\left(\mathbf{g}_{ij}\right)$.

To apply the above definition for the generalized helical hypersurface having a space-like axis determined by Equation (5), we consider the inverse matrix of the first fundamental form matrix. Then, the components of the inverse matrix $\left(\mathbf{g}^{ij}\right) = \mathbf{I}^{-1}$ of $\mathbf{I}$, described by Definition 3, are given by

$$\begin{aligned}
\mathbf{g}^{11} &= \left(-CJ^2 - B^2S - GQ^2 + 2BJQ + CGS\right) / \det \mathbf{I}, \\
\mathbf{g}^{12} &= \left(FQ^2 + CJD - BQD + ABS - AJQ - CFS\right) / \det \mathbf{I} = \mathbf{g}^{21}, \\
\mathbf{g}^{13} &= \left(AJ^2 - BJD + GQD - AGS + BFS - FJQ\right) / \det \mathbf{I} = \mathbf{g}^{31}, \\
\mathbf{g}^{14} &= \left(B^2D - CGD - ABJ + CFJ + AGQ - BFQ\right) / \det \mathbf{I} = \mathbf{g}^{41}, \\
\mathbf{g}^{22} &= \left(-A^2S - CD^2 - Q^2E + 2AQD + CSE\right) / \det \mathbf{I} \\
\mathbf{g}^{23} &= \left(BD^2 - AJD - BSE - FQD + JQE + AFS\right) / \det \mathbf{I} = \mathbf{g}^{32}, \\
\mathbf{g}^{24} &= \left(A^2J - ABD + CFD - CJE + BQE - AFQ\right) / \det \mathbf{I} = \mathbf{g}^{42}, \\
\mathbf{g}^{33} &= \left(-F^2S - GD^2 - J^2E + 2FJD + GSE\right) / \det \mathbf{I}, \\
\mathbf{g}^{34} &= \left(F^2Q + AGD - BFD + BJE - GQE - AFJ\right) / \det \mathbf{I} = \mathbf{g}^{43}, \\
\mathbf{g}^{44} &= \left(-A^2G - CF^2 - B^2E + CGE + 2ABF\right) / \det \mathbf{I},
\end{aligned}$$

where

$$\begin{aligned}
\det \mathbf{I} &= (EG - F^2)(CS - Q^2) + (J^2 - GS)A^2 + (D^2 - ES)B^2 - (EJ^2 + GD^2)C \\
&\quad + 2((CF - AB)DJ + (EB - FA)JQ + (GA - FB)DQ + FABS).
\end{aligned}$$

We replace $\phi = \phi(x^1, x^2, x^3, x^4)$ with $\mathbf{x} = \mathbf{x}(r, \theta_1, \theta_2, \theta_3)$ in Equation (11). Therefore, by using the following inverse matrix of Equation (6):

$$\mathbf{I}^{-1} = \begin{pmatrix}
-\frac{a^2+\left(b^2+\left(c^2+f^2\right)\mathcal{C}_3^2\right)\mathcal{C}_2^2}{\mathcal{W}} & \frac{a\varphi'}{\mathcal{W}} & \frac{b\varphi'\mathcal{C}_2^2}{\mathcal{W}} & \frac{c\varphi'\mathcal{C}_2^2\mathcal{C}_3^2}{\mathcal{W}} \\
\frac{a\varphi'}{\mathcal{W}} & \frac{\left(b^2+\left(c^2+f^2\right)\mathcal{C}_3^2\right)f'^2 - \mathcal{C}_3^2 f^2\varphi'^2}{f^2\mathcal{C}_3^2\mathcal{W}} & -\frac{abf'^2}{f^2\mathcal{C}_3^2\mathcal{W}} & -\frac{acf'^2}{f^2\mathcal{W}} \\
\frac{b\varphi'\mathcal{C}_2^2}{\mathcal{W}} & -\frac{abf'^2}{f^2\mathcal{C}_3^2\mathcal{W}} & \frac{\left(a^2+\left(c^2+f^2\right)\mathcal{C}_2^2\mathcal{C}_3^2\right)f'^2 - \mathcal{C}_2^2\mathcal{C}_3^2 f^2\varphi'^2}{f^2\mathcal{C}_3^2\mathcal{W}} & -\frac{bc\mathcal{C}_2^2}{f^2\mathcal{W}} \\
\frac{c\varphi'\mathcal{C}_2^2\mathcal{C}_3^2}{\mathcal{W}} & -\frac{acf'^2}{f^2\mathcal{W}} & -\frac{bc\mathcal{C}_2^2}{f^2\mathcal{W}} & \frac{\left(a^2+\left(b^2+f^2\mathcal{C}_3^2\right)\mathcal{C}_2^2\right)f'^2 - \mathcal{C}_2^2\mathcal{C}_3^2 f^2\varphi'^2}{f^2\mathcal{W}}
\end{pmatrix},$$

and by differentiating the functions in Equation (11) with respect to $r, \theta_1, \theta_2, \theta_3$, respectively, we obtain the following.

Theorem 6. *The Laplace–Beltrami operator of the generalized helical hypersurface Equation (5) having a space-like axis given by Equation (5) supplies the following relation:*

$$\Delta \mathbf{x} = 4\mathcal{K}_1 \mathbf{G},$$

where $\mathcal{K}_1$ is the mean curvature determined by Theorem 3, and $\mathbf{G}$ is the Gauss map given by Equation (8) of the hypersurface.

Proof. By direct computing (5) with the help of Equation (11), we obtain the relation $\Delta \mathbf{x} = 4\mathcal{K}_1 \mathbf{G}$. □

On the other hand, we serve the following theorem about the Laplace–Beltrami operator and the mean curvature of the generalized helical hypersurface having a space-like axis determined by Equation (5).

Theorem 7. *Let $\mathbf{x} : M^4 \subset \mathbb{E}^4 \longrightarrow \mathbb{E}_1^5$ be an immersion described by Equation (5). $\Delta \mathbf{x} = \mathcal{M}\mathbf{x}$, where $\mathcal{M} = \left(m_{ij}\right)$ is a square matrix of order 5 if and only if $\mathcal{K}_1 = 0$, i.e., generalized helical hypersurface $\mathbf{x}$ having a space-like axis has zero mean curvature.*

Proof. We use $4\mathcal{K}_1 \mathbf{G} = \mathcal{M}\mathbf{x}$, and then obtain the following equations:

$$\begin{aligned} & (\varphi + a\theta_1 + b\theta_2 + c\theta_3)m_{11} + f\mathcal{S}_3 m_{12} + f\mathcal{S}_2\mathcal{C}_3 m_{13} + f\mathcal{S}_1\mathcal{C}_2\mathcal{C}_3 m_{14} + f\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3 m_{15} \\ = \; & \Phi f f' \mathcal{C}_2\mathcal{C}_3, \end{aligned}$$

$$\begin{aligned} & (\varphi + a\theta_1 + b\theta_2 + c\theta_3)m_{21} + f\mathcal{S}_3 m_{22} + f\mathcal{S}_2\mathcal{C}_3 m_{23} + f\mathcal{S}_1\mathcal{C}_2\mathcal{C}_3 m_{24} + f\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3 m_{25} \\ = \; & \Phi(f\varphi'\mathcal{S}_3 - cf'\mathcal{C}_3)\mathcal{C}_2\mathcal{C}_3, \end{aligned}$$

$$\begin{aligned} & (\varphi + a\theta_1 + b\theta_2 + c\theta_3)m_{31} + f\mathcal{S}_3 m_{32} + f\mathcal{S}_2\mathcal{C}_3 m_{33} + f\mathcal{S}_1\mathcal{C}_2\mathcal{C}_3 m_{34} + f\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3 m_{35} \\ = \; & \Phi\big[(f\varphi'\mathcal{C}_3 - cf'\mathcal{S}_3)\mathcal{S}_2\mathcal{C}_3 - bf'\mathcal{C}_2\big]\mathcal{C}_2, \end{aligned}$$

$$\begin{aligned} & (\varphi + a\theta_1 + b\theta_2 + c\theta_3)m_{41} + f\mathcal{S}_3 m_{42} + r\mathcal{S}_2\mathcal{C}_3 m_{43} + f\mathcal{S}_1\mathcal{C}_2\mathcal{C}_3 m_{44} + f\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3 m_{45} \\ = \; & \Phi\big[(f\varphi'\mathcal{C}_3 - cf'\mathcal{S}_3)\mathcal{C}_2\mathcal{C}_3 - bf'\mathcal{S}_2\big]\mathcal{S}_1\mathcal{C}_2 - af'\mathcal{C}_1, \end{aligned}$$

$$\begin{aligned} & (\varphi + a\theta_1 + b\theta_2 + c\theta_3)m_{51} + f\mathcal{S}_3 m_{52} + f\mathcal{S}_2\mathcal{C}_3 m_{53} + f\mathcal{S}_1\mathcal{C}_2\mathcal{C}_3 m_{54} + f\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3 m_{55} \\ = \; & \Phi\big[(f\varphi'\mathcal{C}_3 - cf'\mathcal{S}_3)\mathcal{C}_2\mathcal{C}_3 - bf'\mathcal{S}_2\big]\mathcal{C}_1\mathcal{C}_2 - af'\mathcal{S}_1, \end{aligned}$$

where $\mathcal{M}$ is the 5×5 matrix, $\Phi = 4\mathcal{K}_1\mathcal{W}^{-1/2}$. Differentiating above ODEs twice with respect to θ_1, we obtain the following:

$$m_{11} = m_{21} = m_{31} = m_{41} = m_{51} = 0, \; \Phi = 0.$$

Therefore, the following relation occurs:

$$f(\mathcal{S}_1\mathcal{C}_2\mathcal{C}_3 m_{i4} + \mathcal{C}_1\mathcal{C}_2\mathcal{C}_3 m_{i5}) = 0,$$

where $f \neq 0$, $i = 1, \ldots, 5$. Considering that the functions sin and cos are linearly independent on θ_1, all the components of the matrix $\mathcal{M}$ are 0. Since $\Phi = 4\mathcal{K}_1\mathcal{W}^{-1/2}$, then $\mathcal{K}_1 = 0$. This means hypersurface $\mathbf{x}$ determined by Equation (5) is a 1-minimal (from Definition 7) generalized helical hypersurface with a space-like axis. □

Finally, we present the following examples for all findings in this work. Firstly, we consider the pseudo-hypersphere having a space-like axis in the following examination.

Example 1. *In $\mathbb{E}_1^5$, by taking $\varphi(r) = \cosh r = \mathcal{C}_r$, $f(r) = \sinh r = \mathcal{S}_r$ in the parametric curve γ determined by Equation (4), we state the following pseudo-rotational surface, i.e., pseudo-hypersphere having a space-like axis $\mathbf{x} = \mathcal{R}.\gamma^t$:*

$$\mathbf{x} = \mathbf{x}(r, \theta_1, \theta_2, \theta_3) = (\mathcal{C}_r, \mathcal{S}_r\mathcal{S}_3, \mathcal{S}_r\mathcal{S}_2\mathcal{C}_3, \mathcal{S}_r\mathcal{S}_1\mathcal{C}_2\mathcal{C}_3, \mathcal{S}_r\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3),$$

where $a = b = c = 0$. Therefore, we obtain the following differential geometric objects of the pseudo-hypersphere having a space-like axis in five-dimensional Minkowski space:

$$\begin{aligned}
\mathbf{G} &= \mathbf{x}, \\
\mathbf{I} &= diag\left(-1, \mathcal{S}_r^2\mathcal{C}_2^2\mathcal{C}_3^2, \mathcal{S}_r^2\mathcal{C}_3^2, \mathcal{S}_r^2\right) = -\mathbf{II} = \mathbf{III} = -\,\mathbf{IV} = \mathbf{V}, \\
\mathbf{S} &= -\mathcal{I}_4, \\
\mathcal{K}_j &= (-1)^j, \\
\Delta\mathbf{x} &= 4\mathbf{x},
\end{aligned}$$

where $\mathcal{I}_4$ is the identity matrix of order 4, diag means diagonal parts of the matrix, and $j = 0, 1, \ldots, 4$. We check that the pseudo-hypersphere having a space-like axis supplies the relation given by Equation (10).

Secondly, we consider the rational pseudo-rotational surface having a space-like axis in the following examination.

Example 2. *Substituting the rational functions $\varphi(r) = \frac{r^2+1}{r^2-1} = \mathcal{C}_r$, $f(r) = \frac{2r}{r^2-1} = \mathcal{S}_r$, $r \neq \pm 1$, into the parametric curve γ described by Equation (4), we then construct the following rational pseudo-rotational hypersurface having a space-like axis:*

$$\mathbf{x} = \mathbf{x}(r, \theta_1, \theta_2, \theta_3) = (\mathcal{C}_r, \mathcal{S}_r\mathcal{S}_3, \mathcal{S}_r\mathcal{S}_2\mathcal{C}_3, \mathcal{S}_r\mathcal{S}_1\mathcal{C}_2\mathcal{C}_3, \mathcal{S}_r\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3),$$

where $a = b = c = 0$ in five-dimensional Minkowski space. Hence, we find the following:

$$\begin{aligned}
\mathbf{G} &= -\mathbf{x}, \\
\mathbf{I} &= diag\left(-\frac{4}{(r^2-1)^2}, \mathcal{S}_r^2\mathcal{C}_2^2\mathcal{C}_3^2, \mathcal{S}_r^2\mathcal{C}_3^2, \mathcal{S}_r^2\right) = \mathbf{II} = \mathbf{III} = \mathbf{IV} = \mathbf{V}, \\
\mathbf{S} &= \mathcal{I}_4, \\
\mathcal{K}_j &= 1, \\
\Delta\mathbf{x} &= -4\mathbf{x},
\end{aligned}$$

where $\mathcal{I}_4$ is the identity matrix of order 4, diag means diagonal parts of the matrix, and $j = 0, 1, \ldots, 4$. Here, the rational pseudo-hypersphere having a space-like axis holds the relation determined by Equation (10).

6. Conclusions

In this work, we consider the generalized helical hypersurface having a space-like axis in five-dimensional Minkowski space. We compute the first and second fundamental form matrices, Gauss map, shape operator matrix, and curvatures of the hypersurface. We also describe the umbilical relations of the hypersurface. We determine the Laplace–Beltrami operator of the generalized helical hypersurface having a space-like axis.

Finally, we present some examples that are relevant to all the findings. The obtained findings can be useful in future research.

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