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Inextensible Flows of Null Cartan Curves in Minkowski Space $\mathbb{R}^{2,1}$

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Abstract: This research focused on studying the flows of a null Cartan curve specified by the velocity and acceleration fields. We have proven that the tangential and normal velocities are influenced by the binormal velocity along the motion. The velocity fields are used to drive the time evolution equations for the Cartan frame and the torsion of the null curve. The objective of this work is to construct a family of inextensible null Cartan curves from the flows of the initial null Cartan curve. The surface formed by this family of inextensible flows of the null Cartan curve is obtained numerically and visualized. In this paper, we refer to the surface traced out by the family of the null Cartan curve as the generated or constructed surface. We present a novel model for the inextensible null Cartan curve, which moves with a constant binormal velocity to describe the process of constructing a family of null Cartan curves. Through this model, the time evolution equation for the torsion of the inextensible null Cartan curve arises as the Korteweg-de Vries (K-dV) equation, and we obtain and visualize the soliton solutions. The soliton solutions represent the torsion of the family of null Cartan curves at various time values. We construct the family of inextensible null Cartan curves and visualize the generated surface. In addition, we investigate the flows of inextensible null Cartan curves specified by acceleration fields, and we obtain the explicit relationships between the acceleration and velocity functions. Finally, we provide an application for the inextensible flows of the null Cartan curve with constant normal acceleration.

Keywords: inextensible flows; evolution of curves; motion of curves; null Cartan curves; time evolution equations

MSC: 35C08; 51B20; 53A04; 53A17; 53A35; 53B20; 53C50; 53E10; 53Z05

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1. Introduction

The term "inextensible" refers to curves that maintain their arc length during motion. A great deal of interest has recently been shown in the study of the differential geometry of inextensible flows of curves (IFC). The (IFC) is currently seen as a fascinating topic in differential geometry. It has numerous uses in physics, such as ice melting, rolling stones, vortex filaments, flame propagation, and the magnetic moment vector along a classical magnetic spin chain [1,2]. Many issues in image processing, computer animation, and computer vision require the study of the flows of curves [3–7].

Many authors investigated the (IFC): Baş and Körpınar [8], and they derived some conditions for inextensible flows (IF) of spacelike curves on oriented spacelike surfaces in M_1^3, where they were necessary and sufficient through the motion. Nasar et al. [9] studied the motion of curves in E^n. Gaber [10,11] investigated the (IFC) in spherical space S^3 and the motion of spacelike and timelike curves in de-Sitter space $\mathbb{S}^{2,1}$ and obtained the associated time evolution equations (TEEs) for curvatures as a system of (PDEs). Gürbüzü [12] studied the (IF) of spacelike and timelike curves in $\mathbb{R}^{2,1}$. Körpinar [13] constructed a novel method for (IF) of timelike curves in Minkowski space $\mathbb{R}^{3,1}$ and characterized the curvatures of a timelike curve. Uçum et al. [14] investigated the (IF) of partially null and pseudo null curves in semi-Euclidean 4-space with index 2 ($\mathbb{E}^4_2$) and obtained (TEEs) for the (IF). Yıldız [15] studied the time evolution of non-null curves in Minkowski space $\mathbb{R}^n_1$ and derived the integrability conditions for the evolution. Yoon et al. [16] studied the evolution of spacelike curves in Minkowski space and obtained the inextensible evolutions of timelike ruled surfaces generated by the timelike normal vector and spacelike binormal vector. Gaber [17] investigated the (TEEs) of curves in $\mathbb{R}^3$ according to Bishop frame of type–1 and gave some new models to explain the comparison between the motion by Frenet frame and the motion by Bishop frame of type–1.

In [18], a novel characterization of (IFC) based on the Fermi–Walker derivative and also the Fermi–Walker parallelism in three-dimensional space was constructed. Through the motion of charged particles under the action of electric and magnetic fields, the Fermi–Walker derivative was obtained. In [19], a geometrical description for timelike biharmonic particles in spacetime was presented, and the evolution of the curvatures of these particles was computed based on the Bianchi type–I cosmological model.

Magnetic curve flows in various geometric manifolds and physical spacetime structures have recently been investigated. A novel method for (IF) of spacelike curves in Minkowski space-time by using the Frenet frame of curves was established by [20], and the properties for curvatures of a spacelike curve were defined. Additionally, [21] investigated the (IF) of tangent bimagnetic particles in space. In [22], a novel representation of binormal spherical indicatrices for magnetic curves was investigated. Additionally, the $Bb-$magnetic curves were studied in terms of (IF) and some physical and geometrical properties of the moving charged particles that corresponded to the $Bb-$magnetic curves were investigated.

In [23], for magnetic n-lines due to inextensible Heisenberg antiferromagnetic flow, the fractional evolution equations were computed for constructing the soliton surface associated with the inextensible Heisenberg antiferromagnetic flow. In [24], novel and local conditions were proposed to characterize magnetic flux surfaces for inextensible Heisenberg ferromagnetic flow in the binormal direction. During the evolution of the magnetic vector fields, the accuracy of the theoretical methodology was verified. In [25], Lorentz equations with magnetic b$-$lines in the binormal direction in Minkowski space were computed. The equations of fractional flow for magnetic b$-$lines with inextensible Heisenberg optical antiferromagnetic flow were calculated, and the optical soliton surface was determined. In [26], a novel kind of spherical electromagnetic flow $S_\alpha-$density of $S_\alpha-$optical fibers was investigated. In [27], the optical Hashimoto map corresponds to a quasi-frame for timelike curves in three-dimensional Minkowski space was investigated, and the effect on the q$-$Hashimoto map of specific flow equations like the equation of a vortex filament and the Heisenberg antiferromagnetic flow was examined.

In the present paper, we study the inextensible flows of null Cartan curves (IFNCC) in Minkowski space $\mathbb{R}^{2,1}$ by the velocity and acceleration fields. We study the (TEE) for the pseudo arclength of the null Cartan curve (NCC), and we derive the necessary and sufficient conditions for the (NCC) to be inextensible. Also, we derive the (TEEs) for the Cartan frame, as well as the (TEE) for the torsion of the (NCC) in terms of velocity functions. The main purpose of this work is to construct the family of (IFNCC) from an initial (NCC). In this paper, we refer to the surface that is traced out by the family of inextensible (NCC) as the generated surface or constructed surface. In other words, the surface is defined as trajectories of evolving (NCC). We provide a novel model to explain the process of constructing this family by choosing the case of inextensible (NCC) moves by a constant binormal velocity. We obtain the soliton solutions that represent the torsion of the family of (NCC) at different time values, and we plotted the solutions. In addition, we study the (IFNCC) specified by acceleration functions, and we obtain the explicit relationships between the acceleration and velocity functions. We present an application on the motion of the (NCC) with constant normal acceleration.

The current work is structured as follows: In Section 2, we introduce the geometric concepts of the null curves in Minkowski space $\mathbb{R}^{2,1}$. In Section 3, we illustrate the main results in the present paper. In Section 4, we discuss the method of construction family of inextensible (NCC), and we present a new model. In Section 5, we give graphical interpretations of the given model. In Section 6, we give some geometric descriptions for the constructed surface by the family of (IFNCC). In Section 7, we investigate the (IFNCC) specified by acceleration fields, and we give an application. Finally, we present our conclusions.

2. The Geometric Concepts of Null Curves in Minkowski Space $\mathbb{R}^{2,1}$

Definition 1 ([28])**.** *The three-dimensional Minkowski space $\mathbb{R}^{2,1}$ is the Euclidean space provided with Lorentzian inner product: $\langle X, X\rangle = -dx_0^2 + dx_1^2 + dx_2^2$, for $\{X = (x_0, x_1, x_2) \mid x_0, x_1, x_2 \in \mathbb{R}\}$. The inner product and the vector product of the two vectors $X = (x_0, x_1, x_2), Y = (y_0, y_1, y_2) \in \mathbb{R}^{2,1}$ are defined by:*

- $\langle X, Y\rangle = -x_0y_0 + x_1y_1 + x_2y_2$.
- $X \times Y = (x_2y_1 - x_1y_2, x_2y_0 - x_0y_2, x_0y_1 - x_1y_0)$.

Let X be a vector in $\mathbb{R}^{2,1}$, the vector X is a spacelike vector if $\langle X, X\rangle > 0$, timelike if $\langle X, X\rangle < 0$, and null or lightlike vector if $\langle X, X\rangle = 0$.

Definition 2 ([28])**.** *Let $\alpha = \alpha(u)$, $\alpha : I \to \mathbb{R}^{2,1}$ be a regular parameterized curve in Minkowski space $\mathbb{R}^{2,1}$, the curve is defined as a spacelike curve if $\langle \alpha', \alpha'\rangle > 0$, timelike curve if $\langle \alpha', \alpha'\rangle < 0$, and null if $\langle \alpha', \alpha'\rangle = 0$.*

Definition 3 ([29])**.** *The regular curve $\alpha : J \in \mathbb{R} \to \mathbb{R}^{2,1}$ is defined on some interval $J \in \mathbb{R}$ is a null curve or if its tangent vector $\alpha'(u)$ is a future-directed null vector for each $t \in J$. A null Cartan curve (NCC) is a null curve whose parameterization is given by the following pseudo-arc function s:*

$$s(u) = \int_0^u \rho(u)du, \tag{1}$$

where, $\rho(u) = \sqrt{\|\alpha''(u)\|} = \langle \alpha''(u), \alpha''(u)\rangle^{\frac{1}{4}}$.

Definition 4 ([30,31])**.** *Assume that $\alpha : I \to \mathbb{R}^{2,1}$ be the (NCC), and let $\mathfrak{F}_{\mathfrak{c}} = \{T, N, B\}$ be the null Cartan frame defined at a point p along the curve, where T is a null vector, N is a spacelike vector, and B is a null vector, and they satisfy the following:*

- $\langle T, T\rangle = \langle B, B\rangle = 0$, $\langle N, N\rangle = 1$, $\langle T, B\rangle = -1$, *and* $\langle N, B\rangle = \langle N, T\rangle = 0$.
- $T \times N = -T$, $N \times B = -B$, *and* $B \times T = N$.

Definition 5 ([30])**.** *Let $\alpha : I \to \mathbb{R}^{2,1}$ be the (NCC) with the Cartan frame $\mathfrak{F}_{\mathfrak{c}} = \{T, N, B\}$. The Cartan frame satisfies the following equations:*

$$\begin{pmatrix} \alpha \\ T \\ N \\ B \end{pmatrix}_s = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & -\tau & 0 & k \\ 0 & 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ T \\ N \\ B \end{pmatrix}, \tag{2}$$

where k and τ are the curvature and the torsion of the curve. The curvature k can be 0 when the curve is a straight null line or 1 in all other cases. So, we can rewrite (2) as follows:

$$\begin{pmatrix} \alpha \\ T \\ N \\ B \end{pmatrix}_s = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\tau & 0 & 1 \\ 0 & 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ T \\ N \\ B \end{pmatrix}, \tag{3}$$

3. Main Results

Assume that $\alpha : J = [0, l] \times [0, t] \to \mathbb{R}^{2,1}$ is a one-parameter family of smooth (NCC) in $\mathbb{R}^{2,1}$, let u be the parameter of the (NCC), $0 \leq u \leq l$, and t is the time parameter. Assume that s is the pseudo-arclength of the (NCC), and it is defined by:

$$s(u,t) = \int_0^u \rho(u,t)du, \tag{4}$$

where

$$\rho(u,t) = \langle \alpha_{uu}, \alpha_{uu} \rangle^{\frac{1}{4}}. \tag{5}$$

The evolution equation for the (NCC) is governed by the law:

$$\alpha_t = \lambda\, T + \nu\, N + \mu\, B, \tag{6}$$

where, $\lambda = \lambda(s, t, \tau, \tau_s, \tau_{ss}, ...)$, $\nu = \nu(s, t, \tau, \tau_s, \tau_{ss}, ...)$ and $\mu = \mu(s, t, \tau, \tau_s, \tau_{ss}, ...)$ are the velocity fields in the direction of the tangent, normal, and binormal vectors.

Theorem 1. *Consider the motion of the (NCC) in Minkowski space $\mathbb{R}^{2,1}$, then the (TEE) of the pseudo arclength is given by:*

$$\frac{\partial s}{\partial t} = \rho_t = -\frac{\rho_s^2}{2\rho}(\mu_s + v) - \frac{\rho_s}{2}\frac{\partial}{\partial s}(\mu_s + v) + \frac{\rho}{2}(v_{ss} + 2\lambda_s - 2\tau(\mu_s + v) - \tau_s \mu) \tag{7}$$

Proof of Theorem 1. Taking the $t-$derivative of (5), then

$$2\rho^3 \rho_t = \langle \alpha_{uut}, \alpha_{uu} \rangle. \tag{8}$$

Since $\alpha_u = \rho T$, by taking the $u-$derivative, then we have:

$$\alpha_{uu} = \rho_u T + \rho^2 N. \tag{9}$$

Taking the $u-$derivative of (6), then we obtain:

$$\alpha_{tu} = (\lambda_u - \rho\nu\tau)T + (\nu_u + \lambda\rho - \rho\tau\mu)N + (\mu_u + \rho\nu)B, \tag{10}$$

for simplicity we put:

$$\begin{aligned} \Omega_1 &= \lambda_u - \rho\tau\nu \\ \Omega_2 &= \nu_u + \rho\lambda - \rho\tau\mu \\ \Omega_3 &= \mu_u + \rho\nu \end{aligned} \tag{11}$$

Then, (10) takes the form

$$\alpha_{tu} = \Omega_1 T + \Omega_2 N + \Omega_3 B. \tag{12}$$

Take the $u-$derivative of (12), then

$$\alpha_{tuu} = (\Omega_{1,u} - \tau\rho\Omega_2)T + (\Omega_{2,u} + \rho\Omega_1 - \tau\rho\Omega_3)N + (\Omega_{3,u} + \rho\Omega_2)B. \tag{13}$$

By substituting from (9), and (13) into (8), we obtain

$$2\rho^3 \rho_t = -\rho_u(\Omega_{3,u} + \rho\Omega_2) + \rho^2(\Omega_{2,u} + \rho\Omega_1 - \tau\rho\Omega_3) \tag{14}$$

Substituting from (11) into (14) and by using the commutative condition $\frac{\partial}{\partial u}() = \rho\frac{\partial}{\partial s}()$, then we explicitly have

$$\rho_t = -\frac{\rho_s^2}{2\rho}(\mu_s + \nu) - \frac{\rho_s}{2}\frac{\partial}{\partial s}(\mu_s + \nu) + \frac{\rho}{2}(\nu_{ss} + 2\lambda_s - 2\tau(\mu_s + \nu) - \tau_s \mu) \tag{15}$$

Hence, the evolution of the pseudo arclength $\frac{\partial s}{\partial t} = \rho_t$ depends on the velocities λ, ν, μ and their derivatives λ_s, ν_s, μ_s with respect to s. Then, the theorem holds. □

Definition 6. *Consider the motion of the (NCC) in Minkowski space $\mathbb{R}^{2,1}$, the (NCC) is inextensible if it preserves its arclength, so the time evolution of the pseudo arclength is vanishing ($\frac{\partial s}{\partial t} = \rho_t = 0$).*

Lemma 1. *Consider the motion of inextensible null Cartan curves (IFNCC) in Minkowski space $\mathbb{R}^{2,1}$, then the tangential and the normal velocities are dependent on the binormal velocity and the torsion by the following conditions:*

$$\begin{aligned} \nu &= -\mu_s, \\ \lambda &= \frac{1}{2}\left(\mu_{ss} + \int \tau_s \mu ds\right). \end{aligned} \tag{16}$$

Proof. Since the (TEE) of the pseudo arclength ρ_t is obtained by (15) in terms of tangential, normal, and binormal velocities. Additionally, since the (NCC) is inextensible, then according to the definition (6) of the inextensible (NCC), we have $\rho_t = 0$. By substituting in (15), and comparing the coefficients of (ρ_s^2), (ρ_s), and (ρ), we obtain

$$\nu = -\mu_s, \tag{17}$$

and

$$\nu_{ss} + 2\lambda_s - 2\tau(\mu_s + \nu) - \tau_s \mu = 0. \tag{18}$$

Substituting from (17) into (18), then

$$-\mu_{sss} + 2\lambda_s - \tau_s \mu = 0. \tag{19}$$

By integrating (19), the lemma holds. It is obvious that the values of the tangential and normal velocities depend on the torsion of the curve and also the choice of the binormal velocity. □

Theorem 2. *Consider the flows of the inextensible (NCC) defined by $\alpha : J = [0, l] \times [0, t] \to \mathbb{R}^{2,1}$, and the (TEEs) for the Cartan frame are given by:*

$$\begin{bmatrix} \alpha \\ T \\ N \\ B \end{bmatrix}_t = \begin{bmatrix} 0 & \lambda & \nu & \mu \\ 0 & \varphi_1 & \varphi_2 & 0 \\ 0 & \varphi_3 & 0 & \varphi_2 \\ 0 & 0 & \varphi_3 & -\varphi_1 \end{bmatrix} \begin{bmatrix} \alpha \\ T \\ N \\ B \end{bmatrix}, \tag{20}$$

where,

$$\begin{aligned} \varphi_1 &= \lambda_s + \tau\mu_s, \\ \varphi_2 &= -\mu_{ss} + \lambda - \tau\mu, \\ \varphi_3 &= (\lambda_s + \tau\mu_s)_s - \tau(-\mu_{ss} + \lambda - \tau\mu). \end{aligned} \tag{21}$$

Proof of Theorem 2. Since the nul Cartan curve $\alpha(s, t)$ is inextensible. Therefore, $\rho_t = 0$ and the tangential, normal, and binormal velocities satisfy the conditions (16), and then the parameters s and t commute, so we have:

$$\alpha_{st} = \alpha_{ts}. \tag{22}$$

Since the tangent vector is given by $\alpha_s = T$, by taking the $t-$derivative, we obtain:

$$\alpha_{st} = T_t. \tag{23}$$

Taking the $s-$derivative of (6), and using (3), then:

$$\alpha_{ts} = (\lambda_s - \tau\nu)T + (\nu_s + \lambda - \tau\mu)N + (\mu_s + \nu)B. \tag{24}$$

Using the velocities conditions (16) and substituting from (23) and (24) into (22), hence we obtain:

$$T_t = (\lambda_s + \tau\mu_s)T + (-\mu_{ss} + \lambda - \tau\mu)N. \tag{25}$$

Taking the $s-$derivative of (25), then

$$\begin{aligned} T_{ts} = &\big((\lambda_s + \tau\mu_s)_s - \tau(-\mu_{ss} + \lambda - \tau\mu)\big)T + \big((-\mu_{ss} + \lambda - \tau\mu)_s + (\lambda_s + \tau\mu_s)\big)N \\ &+ (-\mu_{ss} + \lambda - \tau\mu)B. \end{aligned} \tag{26}$$

Since $T_s = N$, then, by taking the $t-$derivative, we have:

$$T_{st} = N_t \tag{27}$$

Using the velocities conditions (16) and the commutative condition $T_{ts} = T_{st}$, then, we obtain:

$$N_t = \big((\lambda_s + \tau\mu_s)_s - \tau(-\mu_{ss} + \lambda - \tau\mu)\big)T + (-\mu_{ss} + \lambda - \tau\mu)B. \tag{28}$$

Assume that the evolution of the binormal null vector B is given by:

$$B_t = a_{11}T + a_{12}N + a_{13}B. \tag{29}$$

Since the binormal vector B is null, $\langle B, B\rangle = 0$, then $\langle B_t, B\rangle = 0$, hence:

$$a_{11} = 0. \tag{30}$$

Additionally, since $\langle B, N\rangle = 0$, then $\langle B_t, N\rangle = -\langle B, N_t\rangle$, hence

$$a_{12} = (\lambda_s + \tau\mu_s)_s - \tau(-\mu_{ss} + \lambda - \tau\mu). \tag{31}$$

Since$\langle B, T\rangle = -1$, then $\langle B_t, T\rangle = -\langle B, T_t\rangle$, hence

$$a_{13} = -(\lambda_s + \tau\mu_s) \tag{32}$$

Substitute from (30), (31) and (32) into (29), hence

$$B_t = \big((\lambda_s + \tau\mu_s)_s - \tau(-\mu_{ss} + \lambda - \tau\mu)\big)N - (\lambda_s + \tau\mu_s)B. \tag{33}$$

For simplicity, we choose:

$$\begin{aligned} \varphi_1 &= \lambda_s + \tau\mu_s, \\ \varphi_2 &= -\mu_{ss} + \lambda - \tau\mu, \\ \varphi_3 &= (\lambda_s + \tau\mu_s)_s - \tau(-\mu_{ss} + \lambda - \tau\mu). \end{aligned} \tag{34}$$

Hence,

$$\begin{aligned} \varphi_1 &= -\varphi_{2,s}, \\ \varphi_3 &= \varphi_{1,s} - \tau\varphi_2. \end{aligned} \tag{35}$$

By substituting from (34) and (35) into (25), (28), and (33), we obtain:

$$\begin{aligned} T_t &= \varphi_1 T + \varphi_2 N \qquad, \\ N_t &= \varphi_3 T \qquad + \varphi_2 B\,, \\ B_t &= \qquad \varphi_3 N - \varphi_1 B. \end{aligned} \tag{36}$$

Hence, the theorem holds. □

By taking the $t-$derivative of (36), then we obtain the following lemma:

Lemma 2. *Consider the inextensible flows of the null Cartan curve (IFNCC), then the second time derivative of the Cartan frame can be given in terms of velocity functions by:*

$$\begin{aligned} T_{tt} &= (\varphi_{1,t} + \varphi_2\varphi_3 + \varphi_1^2)T + (\varphi_{2,t} + \varphi_1\varphi_2)N + \varphi_2^2 B, \\ N_{tt} &= (\varphi_{3,t} + \varphi_1\varphi_3)T + (2\varphi_2\varphi_3)N + (\varphi_{2,t} - \varphi_1\varphi_2)B, \\ B_{tt} &= (\varphi_3^2)T + (\varphi_{3,t} - \varphi_1\varphi_3)N + (-\varphi_{1,t} + \varphi_2\varphi_3 + \varphi_1^2)B. \end{aligned} \tag{37}$$

Lemma 3. *Consider the (IFNCC) in $\mathbb{R}^{2,1}$, the (TEE) of the torsion τ is given by:*

$$\tau_t = -(\varphi_{3,s} + \tau\varphi_1). \tag{38}$$

Proof. Since $\tau = -\langle B_s, N\rangle$, by taking the $t-$derivative, then

$$\tau_t = -\langle B_{st}, N\rangle - \langle B_s, N_t\rangle. \tag{39}$$

Since $B_s = -\tau\ N$, and from the second equation of (36), then we have:

$$\langle B_s, N_t\rangle = -\tau\langle N, N_t\rangle = 0. \tag{40}$$

Taking the $s-$derivative of the third equation of (36), then

$$B_{ts} = -\tau\varphi_3 T + (\varphi_{3,s} + \tau\varphi_1)N + (-\varphi_{1,s} + \varphi_3)B. \tag{41}$$

Substitute from (40) and (41) into (39), then the lemma holds. □

4. The Method of Construction Family of Inextensible Null Cartan Curves in $\mathbb{R}^{2,1}$

In this section, our purpose is to obtain a family of (IFNCC). This is equivalent to constructing the surface by the motion of an inextensible null Cartan curve. The process of constructing a family of (IFNCC) can be described as follows:

Step 1. We choose specific values of the velocity functions (certain values of velocities are based on physical phenomena, such as the motion of vortex filaments, where the normal velocity equals the curvature of the curve). Then, we substitute these values of the velocity functions into (38) to obtain the general solution that represents the torsion of the (NCC).

Step 2. As soon as we determine the torsion, we substitute it into (3) and solve the system numerically with specific initial conditions for $s \in [s_0, s_{max}]$.

Step 3. Solve the (PDEs) systems (20) numerically by using specific initial conditions that are given as the numerical results obtained from Step. 2 for $t \in [t_0, t_{max}]$.

Step 4. To validate the solutions, we can use the Cartan frame properties provided by Definition 4.

Step 5. Now, we have the null Cartan curve $\alpha(s,t) = (\alpha_1, \alpha_2, \alpha_3)$ at every point (s,t), then we can graph the family of (IFNCC) and visualize the surface generated by this family.

Step 6. We use Wolfram Mathematica 12 to solve the (PDE) system (3) and (20), and we visualize the surface from the family of (IFNCC).

A Model of Construction of the Family of Inextensible Null Cartan Curves

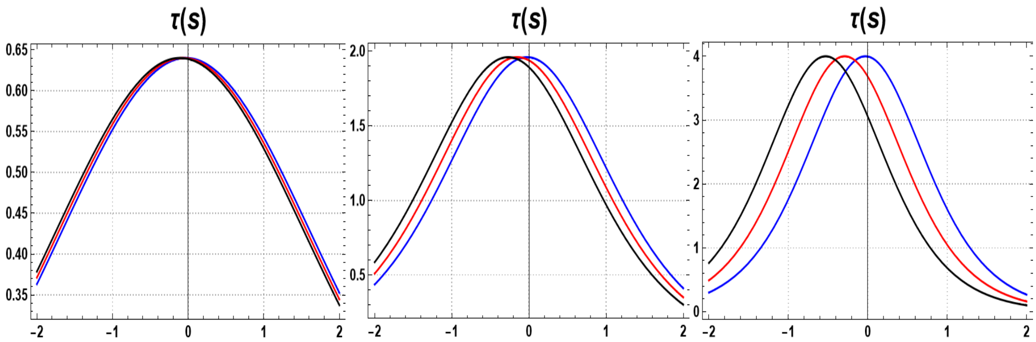
If the binormal velocity $\mu = \mu_0 = const$, by using (16), then $\nu = 0$ and $\lambda = \frac{1}{2}\tau\mu_0$. Then, the (TEE) of the torsion that is given by (38) takes the form of the Korteweg-de Vries (KdV) equation, which is a non-linear partial differential equation of the third order:

$$\tau_t = -\frac{1}{2}\mu_0(\tau_{sss} + 3\tau\tau_s). \tag{42}$$

The general solution takes the form:

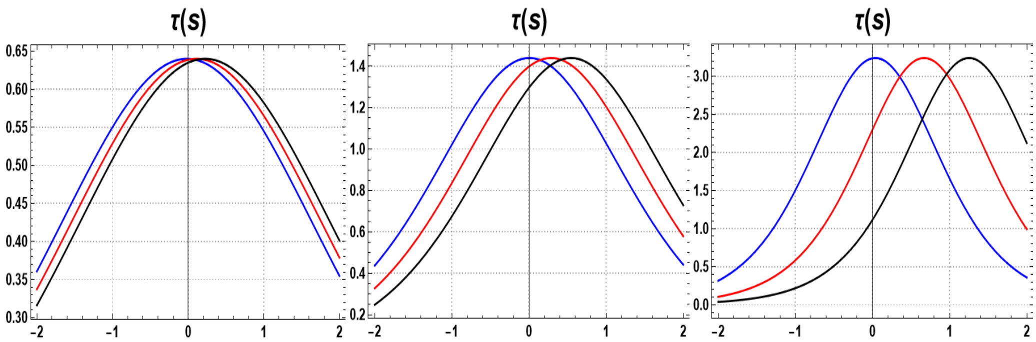
$$\tau(s,t) = 4\,c_1^2\,\mathrm{sech}^2(c_1\,s - 2c_1^3\mu_0\,t + c_2). \tag{43}$$

This solution represents the solitary wave solution. To visualize the soliton solutions (43), we choose two values of the constant binormal velocity $\mu_0 = -0.1$ and $\mu_0 = 0.3$. Figures 1 and 2 represent the soliton solution (43) with the binormal velocity $\mu = \mu_0 = -0.1$ and $\mu = \mu_0 = 0.3$, respectively, at different values of the time $t = 0.1, 1.4, 2.6$ for $s \in [-2, 2]$, $t \in [0, 3]$, and $c_2 = 0.01$.



(**a**) $c_1 = 0.4$ (**b**) $c_1 = 0.7$ (**c**) $c_1 = 1$

Figure 1. Soliton solutions (43) for $s \in [-2, 2]$, $t \in [0, 3]$, $\mu_0 = -0.1, c_2 = 0.01$. The curves with blue, red, and black colors represent the soliton solutions at $t = 0.1, 1.4, 2.6$, respectively.

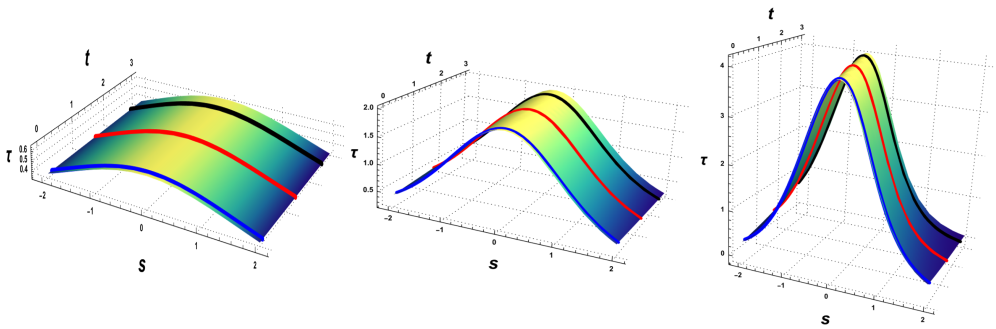


(**a**) $c_1 = 0.4$ (**b**) $c_1 = 0.6$ (**c**) $c_1 = 0.9$

Figure 2. Soliton solutions (43) for $s \in [-2, 2]$, $t \in [0, 3]$, $\mu_0 = 0.3, c_2 = 0.01$. The curves with blue, red, and black colors represent the soliton solutions at $t = 0.1, 1.4, 2.6$, respectively.

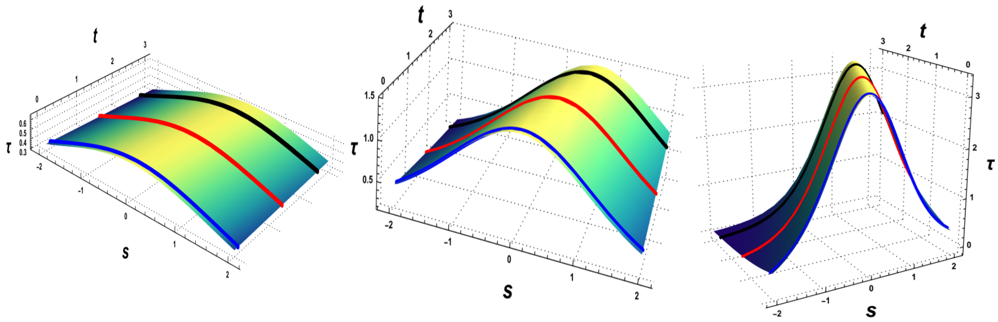
Figures 3 and 4 represent the torsion of the family of (IFNNC) with the binormal velocity $\mu_0 = -0.1$ and $\mu_0 = 0.3$, respectively. The curves with blue, red, and black colors represent the torsion of the inextensible (NCC) at $t = 0.1, 1.4, 2.6$ for $s \in [0, 2]$, $t \in [0, 3]$, and $c_2 = 0.01$.

By substituting from (43) into (3) and (20), then by solving them numerically with initial conditions $\alpha(0,t) = (0,0,0)$, $T(0,t) = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$, $N(0,t) = (0,1,0)$, and $B(0,t) = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$. Hence, we can get the family of (IFNCC) and we can construct the surface from this family.



(**a**) $c_1 = 0.4$ (**b**) $c_1 = 0.7$ (**c**) $c_1 = 1$

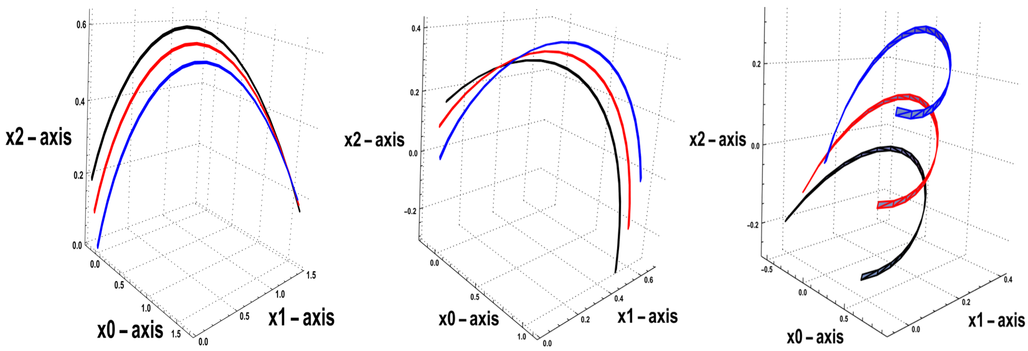
Figure 3. Torsion of (IFNCC): $\tau(s,t) = 4c_1^2\,\mathrm{sech}^2(c_1\,s - 2c_1^3\,\mu_0\,t + c_3)$ for $s \in [-2,2]$, $t \in [0,3]$, $\mu_0 = -0.1$, $c_2 = 0.01$. The curves with blue, red, and black colors represent the torsion at $t = 0.1, 1.4, 2.6$, respectively.



(**a**) $c_1 = 0.4$ (**b**) $c_1 = 0.6$ (**c**) $c_1 = 0.9$

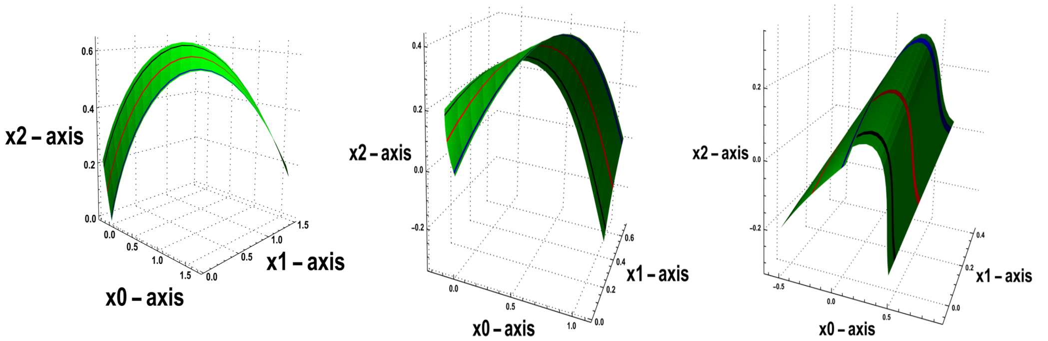
Figure 4. Torsion of (IFNCC): $\tau(s,t) = 4c_1^2\mathrm{sech}^2(c_1 s - 2c_1^3\,\mu_0\,t + c_3)$ for $s \in [-2,2]$, $t \in [0,3]$, $\mu_0 = 0.3$, $c_2 = 0.01$. The curves with blue, red, and black colors represent the torsion at $t = 0.1, 1.4, 2.6$, respectively.

At the constant binormal velocity $\mu_0 = -0.1$, we plot the family of (IFNNC) and the generated surface from the family of (IFNCC) as illustrated in Figures 5 and 6, the curves with blue, red, and black colors represent the evolution of the (NCC) at some values of the time $t = 0.1, 1.4, 2.6$ for $s \in [0,2]$, $t \in [0,3]$, and $c_2 = 0.01$.



(**a**) $c_1 = 0.4$ (**b**) $c_1 = 0.7$ (**c**) $c_1 = 1$

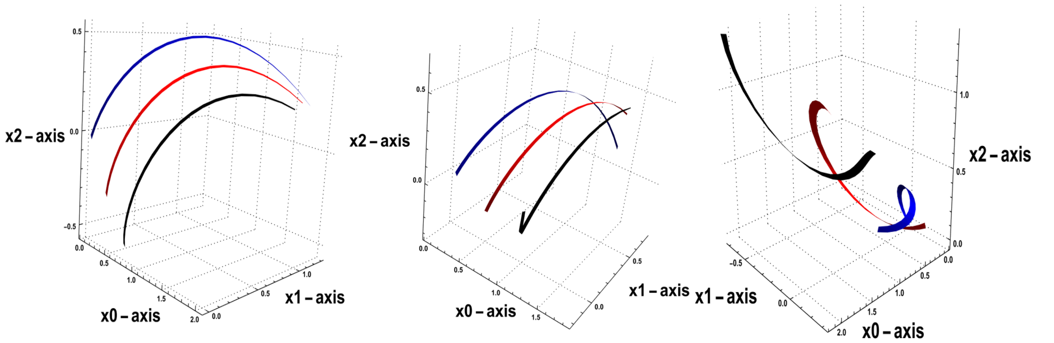
Figure 5. Curves with blue, red, and black colors represent the family of (IFNCC) at $t = 0.1, 1.4, 2.6$, respectively, for $s \in [0,2]$, $t \in [0,3]$, $\mu_0 = -0.1, c_2 = 0.01$.



(**a**) $c_1 = 0.4$ (**b**) $c_1 = 0.7$ (**c**) $c_1 = 1$

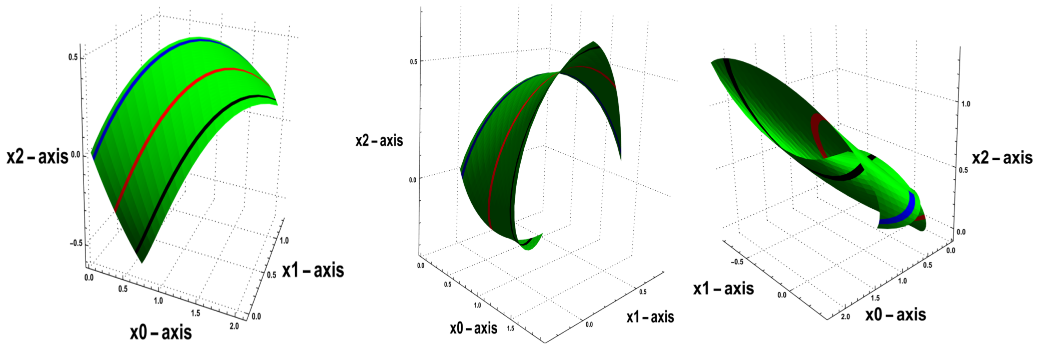
Figure 6. Surface constructed by (IFNCC) for $s \in [0, 2]$, $t \in [0, 3]$, $\mu_0 = -0.1, c_2 = 0.01$.

At the constant binormal velocity $\mu_0 = 0.3$, we plot the family of (IFNNC) and the generated surface from the family of (IFNCC), as illustrated in Figures 7 and 8. The curves with blue, red, and black colors represent the evolution of the (NCC) at $t = 0.1, 1.4, 2.6$ for $s \in [0, 2]$, $t \in [0, 3]$, and $c_2 = 0.01$.



(**a**) $c_1 = 0.4$ (**b**) $c_1 = 0.6$ (**c**) $c_1 = 0.9$

Figure 7. Curves with blue, red, and black colors represent the family of (IFNCC) at $t = 0.1, 1.4, 2.6$, respectively, for $s \in [0, 2]$, $t \in [0, 3]$, $c_2 = 0.01, \mu_0 = 0.3$.



(**a**) $c_1 = 0.4$ (**b**) $c_1 = 0.6$ (**c**) $c_1 = 0.9$

Figure 8. Surface constructed by the family of (IFNCC) for $s \in [0, 2]$, $t \in [0, 3]$, $c_2 = 0.01, \mu_0 = 0.3$.

5. Graphical Interpretations

Solitons are a particular type of non-dispersive long wave that travels in the form of packets at a constant velocity. They are additionally known as shallow-water waves with a permanent shape. The soliton solutions of hyperbolic type, also known as dark soliton or bright soliton. The soliton is known as the bright soliton when the solution is expressed in terms of the ($sech$) function, and the dark soliton when the solution is expressed in terms of the ($tanh$) function. Solitons are essential for the analysis of wave propagation for various types of physical phenomena and related fields. Solitons have many applications in pure mathematics, applied mathematics, and biology. The soliton solutions are studied by many researchers [32–36]. In this part, we describe the graphs of the previous model as follows:

The soliton solutions (43) are visualized in Figures 1 and 2 in $2-$dimensions, and in Figures 3 and 4 in the $3-$dimensions. The soliton solutions (43) arise in the form of hyperbolic (*sech*) function, so the soliton solutions of the type bright soliton. The term $4\,c_1^2$ in the solition solution (43) represents the amplitude of the wave. It has a significant effect on increasing the torsion of the curve. The shape of the soliton solutions depends on the binormal velocity μ_0 and the amplitude $4\,c_1^2$. We illustrate the effect of changing the amplitude $4\,c_1^2$ and the changing the values of the normal velocity at $\mu_0 = -0.1$ and $\mu_0 = 0.3$ on the soliton solutions in the domain $s \in [0,2]$, $t \in [0,3]$ with $c_2 = 0.01$ as follows:

1. Case 1: Consider the binormal velocity $\mu_0 = -0.1$. The shape in Figure 1a for the soliton solutions with $c_1 = 0.4$ does not vary for different values of the time at $t = 0.1, 1.4, 2.6$. The shape in Figure 1b, for the soliton solutions with $c_1 = 0.7$ does not change at different values of the time $t = 0.1, 1.4, 2.6$, but there is a little shift to the left while as time increases. The shape in Figure 1c, for the soliton solutions with for $c_1 = 1$ does not change by increasing the time $t = 0.1, 1.4, 2.6$, respectively, and there is a slight shift to the left. The soliton solutions represent the torsion of the family of (NCC) and by increasing the amplitude, the torsion will increase, and it has the following maximum values:
   (a) For $c_1 = 0.4$, the torsion has maximum value $\tau = 0.64$ at $s = -0.0250001$ and $t = 2.470 * 10^{-6}$.
   (b) For $c_1 = 0.7$, the torsion has maximum value $\tau = 1.96$ at $s = -0.0142857$ and $t = 4.75746 * 10^{-7}$.
   (c) For $c_1 = 1$, the torsion has maximum value $\tau = 4$ at $s = -0.00999962$ and $t = -1.92635 * 10^{-6}$.
2. Case 2: Consider the binormal velocity $\mu_0 = 0.3$. The shape in Figure 2a, of the soliton solutions at $c_1 = 0.4$ does not change for different values of the time at $t = 0.1, 1.4, 2.6$. The shape in Figure 2b for the soliton solutions at $c_1 = 0.6$ does not change at different values of the time $t = 0.1, 1.4, 2.6$, respectively, but there is a slight shift to the right while the time increases. In Figure 2c, for $c_1 = 0.9$, the shape does not vary with increasing the time $t = 0.1, 1.4, 2.6$, respectively, and there is a slight shift to the right. The soliton solutions represent the torsion of the family of the (NCC) and by increasing the amplitude, the torsion will increase, and it has the following maximum values:
   (a) For $c_1 = 0.4$, the torsion has maximum value $\tau = 0.64$, at $s = -0.0250001$ and $t = -8.23183 * 10^{-7}$.
   (b) For $c_1 = 0.6$, the torsion has maximum value $\tau = 1.44$ at $s = -0.0166667$ and $t = 1.62604 * 10^{-7}$.
   (c) For $c_1 = 0.9$, the torsion has maximum value $\tau = 3.24$ at $s = -0.0111111$ and $t = 2.22144 * 10^{-8}$.
3. The choice of the value of the amplitude $4\,c_1^2$ affects the properties of the vectors T, N, and B for the Cartan frame, where it can be used to verify the numerical solutions.
4. Figures 5 and 6, illustrate the flows of the family of the (NCC) at time $t = 0.1, 1.4, 2.6$ and the generated surface traced out by this family of the (NCC) for $s \in [0,2]$, $t \in [0,3]$, and $c_2 = 0.01$ with the binormal velocity $\mu_0 = -0.1$ for different values of the amplitude with $c_1 = 0.4$, $c_1 = 0.7$, and $c_1 = 1$.
5. Figures 7 and 8 illustrate the flows of the family of the (NCC) at the time $t = 0.1, 1.4, 2.6$ and the constructed surface traced by this family of the (NCC) at constant binormal velocity $\mu_0 = 0.3$ for $s \in [0,2]$, $t \in [0,3]$, and $c_2 = 0.01$ and different values of the amplitude with $c_1 = 0.4$, $c_1 = 0.6$, and $c_1 = 0.9$.

6. The Geometric Description of the Constructed Surface

Let $M = \Sigma_t = \alpha(s,t)$ be the surface traced out by the flows of (NCC).

Lemma 4. *For the surface $M = \Sigma_t = \alpha(s,t)$ in $\mathbb{R}^{2,1}$, the first fundamental quantities g_{11}, g_{12}, g_{22} are given by:*

$$\begin{aligned} g_{11} &= \langle \alpha_s, \alpha_s \rangle = 0, \\ g_{22} &= \langle \alpha_s, \alpha_t \rangle = -\mu, \\ g_{12} &= \langle \alpha_t, \alpha_t \rangle = \mu_s^2 - 2\lambda\mu. \end{aligned} \tag{44}$$

Lemma 5. *Let the vector $n(s,t)$ be a unit normal to the constructed surface M in $\mathbb{R}^{2,1}$. Then, it is given by:*

$$n(s,t) = \frac{\alpha_s \times \alpha_t}{\|\alpha_s \times \alpha_t|} = \frac{1}{\mu}(\mu_s T - \mu N). \tag{45}$$

The second fundamental quantities L_{11}, L_{12}, L_{22} can computed as follows:

$$\begin{aligned} L_{11} &= \langle \alpha_{ss}, n \rangle = -1, \\ L_{12} &= \langle \alpha_{st}, n \rangle = -\varphi_2, \\ L_{22} &= \langle \alpha_{tt}, n \rangle = (\mu_{st} - \lambda\varphi_2 - \mu\varphi_3) - \frac{\mu_s}{\mu}(\mu_t - \mu_s\varphi_2 - \mu\varphi_1). \end{aligned} \tag{46}$$

Definition 7 ([28])**.** *The Gaussian curvature G and the mean curvature H for the surface M in $\mathbb{R}^{2,1}$ are defined by*

$$G = \epsilon\frac{det(II)}{det(I)} = \epsilon\frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2} \quad , \quad H = \frac{\epsilon}{2}\frac{L_{11}g_{22} + L_{22}g_{11} - 2L_{12}g_{12}}{g_{11}g_{22} - g_{12}^2}, \quad \epsilon = \langle n, n \rangle. \tag{47}$$

Lemma 6. *Consider the surface $M = \Sigma_t = \alpha(s,t)$ in $\mathbb{R}^{2,1}$ constructed by the family of (IFNCC). The Gaussian curvature G and the mean curvature H are given in terms of velocities as follows:*

$$\begin{aligned} G &= \frac{\mu(\mu_{st} - \lambda\varphi_2 - \mu\varphi_3) - \mu_s(\mu_t - \mu_s\varphi_2 - \mu\varphi_1) + \mu\varphi_2^2}{\mu(\mu_s^2 - 2\lambda\mu)^2}, \\ H &= -\frac{\mu + 2\varphi_2(\mu_s^2 - 2\lambda\mu)}{2(\mu_s^2 - 2\lambda\mu)^2}, \end{aligned} \tag{48}$$

where $\varphi_1, \varphi_2, \varphi_3$ are given by (21).

Now, we consider the surface constructed by (IFNCC) in model 1 that is specified by velocity functions $\mu = \mu_0 = const$, $\nu = 0$, $\lambda = \frac{1}{2}\tau\mu_0$, and torsion given by (43).

Lemma 7. *The first fundamental quantities g_{11}, g_{12}, g_{22} and the second fundamental quantities L_{11}, L_{12}, L_{22} for the surface constructed by the family of the (NCC) in model* 1*, are given, respectively, by:*

$$g_{11} = 0, \quad g_{12} = -\tau\mu_0^2, \quad g_{22} = -\mu_0. \tag{49}$$

and

$$L_{11} = -1, \quad L_{12} = \frac{1}{2}\tau\mu_0, \quad L_{22} = -\frac{\mu_0^2}{4}(2\tau_{ss} + \tau^2), \tag{50}$$

Lemma 8. *The Gaussian curvature G and the mean curvature H for the surface constructed by the family of the (NCC) in model* 1 *are given respectively by:*

$$G = -\frac{\tau_{ss}}{2\tau^2\mu_0^2}, \quad H = -\frac{1 + \mu_0^2\tau^2}{2\mu_0^3\tau^2}, \tag{51}$$

where the torsion is given by (43).

7. Inextensible Flows of Null Cartan Curve Specified by Acceleration Fields

In this section, we investigate the (IFNCC) specified by the acceleration fields. Consider $\alpha : J = [0,l] \times [0,t] \to \mathbb{R}^{2,1}$ to be a one-parameter family of smooth (NCC) in Minkowski space $\mathbb{R}^{2,1}$, and, let t and s are the time and the pseudo arclength parameters of the initial (NCC), where $0 \leq s \leq l$. Let $\alpha(s,t)$ be the flows of inextensible null Cartan curves that satisfy (16). Consider the motion of the (NCC) specified by fields $\eta_1(s,t), \eta_2(s,t)$ and $\eta_3(s,t)$ in the direction of the tangent vector, normal vector, and binormal vector, respectively. The law governing the motion is given by:

$$\alpha_{tt} = \eta_1(s,t)T + \eta_2(s,t)N + \eta_3(s,t)B \tag{52}$$

Lemma 9. *Assuming that $\alpha(s,t)$ is (IFNCC), the relationships between the velocity functions that describe the motion of (NCC) by (6) and the acceleration functions that govern the motion of (NCC) by (52) are specified by:*

$$\begin{aligned} \eta_{1,s} - \tau\eta_2 &= \varphi_{1,t} + \varphi_1^2 + \varphi_2\varphi_3, \\ \eta_{2,s} + \eta_1 - \tau\eta_3 &= \varphi_{2,t} + \varphi_1\varphi_2, \\ \eta_{3,s} + \eta_2 &= \varphi_2^2. \end{aligned} \tag{53}$$

Proof. Take the $s-$derivative of (52), then

$$\alpha_{tts} = (\eta_{1,s} - \tau\eta_2)T + (\eta_{2,s} + \eta_1 - \tau\eta_3)N + (\eta_{3,s} + \eta_2)B. \tag{54}$$

Since $\alpha_{stt} = T_{tt}$, using the commutative condition $\alpha_{tts} = \alpha_{stt}$, then

$$T_{tt} = (\eta_{1,s} - \tau\eta_2)T + (\eta_{2,s} + \eta_1 - \tau\eta_3)N + (\eta_{3,s} + \eta_2)B. \tag{55}$$

Comparing (37) and (55), then the lemma holds. □

Explicitly, by taking the t-derivative of (6) and using (20), then we have the following lemma:

Lemma 10. *The acceleration fields η_1, η_2 and η_3 that describe the time evolution Equation (52) can be given explicitly in terms of the velocity fields λ, ν and μ by the following:*

$$\begin{aligned} \eta_1 &= \lambda_t + \lambda\varphi_1 - \mu_s\varphi_3, \\ \eta_2 &= -\mu_{st} + \lambda\varphi_2 + \mu\varphi_3, \\ \eta_3 &= \mu_t - \mu_s\varphi_2 - \mu\varphi_1, \end{aligned} \tag{56}$$

where, φ_1, φ_2 and φ_3 are defined in (21).

8. Application on Inextensible Flows of Null Cartan Curve Specified by Normal Acceleration

Consider the (NCC) evolves by the constant normal acceleration function $\eta_2 = 1$, and assume that the tangential, and binormal acceleration vanish $\eta_1 = 0$, and $\eta_3 = 0$, then the (TEE) governs the motion, takes the form:

$$\alpha_{tt} = N \tag{57}$$

By substituting in (53), then

$$\varphi_1 = 0\,, \quad \varphi_2 = 1\,, \varphi_3 = -\tau\,. \tag{58}$$

Substituting from (58) into (38), then the torsion satisfies the following (PDE):

$$\tau_{1,t} = \tau_{1,s} \tag{59}$$

The (PDE) (59) represents the transport equation, and it has a general solution of the form:

$$\tau(s,t) = f(s+t), \tag{60}$$

where $f(s+t)$ is an arbitrary function. Since $\alpha_{ss} = T_s = N$, then by comparing with (57), we have the following (PDE):

$$\alpha_{tt} = \alpha_{ss}. \tag{61}$$

The (PDE) (61), represents a one-dimensional wave equation. Consider the initial conditions $\alpha(s,0) = \chi(s) = (\chi_1(s), \chi_2(s), \chi_3(s))$ and $\alpha_t(s,0) = \psi(s) = (\psi_1(s), \psi_2(s)\psi_3(s))$, then the general solution takes the following form:

$$\alpha(s,t) = \frac{1}{2}\left(\chi(s+t) + \chi(s-t) + \int_{s-t}^{s+t} \psi(x)dx\right). \tag{62}$$

It is critical to validate the solution by using the features of the frame defined in Definition 4.

Remark 1. *In this application, to generate the surface $\alpha(s,t)$ from the family of (NCC), specific initial conditions are used based on a certain physical phenomenon associated with the flows of inextensible null Cartan curves.*

9. Discussion

In this work, we investigate the flows of inextensible null Cartan curves by creating the family of null Cartan curves from the initial null Cartan curve, and then we construct the surface from the tracing out of this family of inextensible (NCC). Furthermore, we describe the generated surface and provide certain geometric properties, such as determining the first and second fundamental quantities, as well as Gaussian and mean curvatures. This study differs from earlier studies since it describes not only the torsion of the family of curves, but it also well characterizes and visualizes the family of the null Cartan curves at different values of the time. This research can be used to solve several physical and engineering problems involving curve flows.

10. Conclusions

In the current paper, we investigate the inextensible flows of null Cartan curves (IFNCC) in Minkowski space $\mathbb{R}^{2,1}$ using the velocity fields (6) and the acceleration fields (52). We acquired the following new results:

1. The (TEE) for the pseudo arclength of the null Cartan curve is obtained, and the necessary and sufficient conditions for the null Cartan curve (NCC) to be inextensible are derived. These conditions show that the tangential velocity (λ) and the normal velocity (ν) are dependent on the binormal velocity (μ) and on the torsion (τ) by (16).
2. The (TEEs) (the first and the second-time derivatives) of the Cartan frame are derived in terms of the velocity fields by (20), and (37), respectively.
3. The (TEE) for the torsion τ is obtained in terms of the velocities (38).
4. The flows of inextensible (NCC) is constructed, and we present a novel model to describe the process of constructing this family of (IFNCC) with velocities $\mu = \mu_0 = const, \nu = 0$ and $\lambda = \frac{1}{2}\tau\mu_0$. In this model, the (TEE) of the torsion of the inextensible (NCC) appears in the form of the Korteweg-de Vries (K-dV) equation. We obtain the soliton solutions for the (K-dV) equation, and we graph these solitons for different time values with certain amplitudes. By using the value of the torsion, we visualize the flows of the initial (NCC), then we visualize the generated surface of these flows for different values of the constant velocity μ_0 and various amplitudes. Additionally, we compute the first and second fundamental quantities for the generated surface, as well as the Gaussian curvature G and mean curvature H (49), (50) and (51), respectively.

5. The investigation of (IFNCC) specified by acceleration functions (52) is presented. Additionally, we obtained the explicit relationships between the acceleration functions η_1, η_2 and η_3 and the velocity functions λ, ν and μ by (53) and (56).
6. We provided an application for the inextensible flows of (NCC) with constant normal acceleration. In this application, the time evolution equation of torsion arising as a first order (PDE) given by (59). It is known as the transport equation, and it has the general solution given (60). In addition, in this application, the flows $\alpha(s,t)$ of the (NCC) satisfy (PDE) (61), and it represents a one-dimensional wave equation, and it has the general solution of the form (62).

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Abbreviations

The following abbreviations are used in this manuscript:

IF	Inextensible Flows
IFC	Inextensible Flow(s) of Curve(s)
IFNCC	Inextensible Flows of Null Cartan Curve
NCC	Null Cartan Curve
PDE(s)	Partial Differential Equation(s)
TEE(s)	Time Evolution Equation(s)

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