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Bandhead Energies of npp/pnn Three-Quasiparticle Quadruplets

Sushil Kumar [1,*], Manpreet Kaur [1], Sukhjeet Singh [1], Jagjit Singh [1,2,3] and A. K. Jain [4]

[1] Department of Physics, Akal University Talwandi Sabo, Bathinda 151302, India
[2] Theoretical Physics Group, Department of Physics and Astronomy, The University of Manchester, Manchester M13 9PL, UK
[3] Research Centre for Nuclear Physics (RCNP), Osaka University, Ibaraki 567-0047, Japan
[4] Amity Institute of Nuclear Science and Technology, Noida 201313, India
* Correspondence: sushil.rathi179@gmail.com

Abstract: Semi-empirical frameworks are widely used in calculating the bandhead energies of three-quasiparticle (3qp) configurations observed in well-deformed odd-A nuclei. In the present study, our aim is to improve the previous version of the semi-empirical model [Physical Review C. 1992, 45(6), 3013]. This is achieved by incorporating the ignored vital contributions owing to the irrotational motion of valance protons/neutrons, diagonal components of particle–particle coupling (ppc), and rotor–particle coupling (rpc) terms. We tested the validity of the improved version of the model by calculating the bandhead energies of twelve 3qp npp/pnn quadruplets observed in ^{163}Er, ^{171,175,177}Lu, ^{177}Ta, and ^{183}Re nuclides. Our new results show better agreement with the experimental data indicating the importance of newly added terms. We strongly expect that the present version of the model will provide support to future experimental campaigns for making configuration assignments to the newly observed 3qp bands and also in the identification of exact Nilsson's configurations of 3qp quadruplets where experimental data that differentiate among the competing configuration are scarce.

Keywords: three-quasiparticle configuration; bandhead energy; rotational band; rotor–particle coupling; particle–particle coupling

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1. Introduction

The excitation spectra of deformed odd-A nuclei generally composed of one quasi-particle (1qp), three-quasiparticle (3qp) and multi quasi-particle (mqp) states, along with other collective excitation states [1]. The majority of experimental data pertain to 1qp states and lie below the excitation energy of the order of 1 MeV, which is approximately the proton/neutron paring gap in the rare-earth mass region [2]. At higher excitation energies, i.e., $\geq$ 1MeV, either a proton or a neutron pair breaks down which leads to the observation of 3qp configurations [2]. In particular deformed nucleus, a npp/pnn 3qp quadruplet can be formed from the coupling of projections of intrinsic angular momenta of three valance particles on symmetry axis (K) which leads to the formation of a quadruplet having four possible K values, i.e., $K = |-K_1 + K_2 + K_3|$; $K = |K_1 - K_2 + K_3$; $K = |K_1 + K_2 - K_3|$ and $K = |K_1 + K_2 + K_3|$ [2]. In order to calculate the bandhead energies of different members of the npp/pnn 3qp quadruplet, we propose an improved version of the semi-empirical model which employs (i) unperturbed 1qp excitation energies observed in neighboring nuclei, (ii) residual interaction, i.e., Gallagher–Moszkowski (GM) splitting [3,4], and Newby shift energies [5] extracted from the experimental data. The present model also includes diagonal contributions which appeared due to the irrotational motion of valance particles, particle–particle coupling (ppc), and rotor–particle coupling (rpc) interactions [6,7] which were ignored in the earlier version of the model [2]. To test the validity of the improved version of the model, we calculate the bandhead energies of 12 three-quasiparticle npp/pnn quadruplets observed in six different nuclides namely, ^{163}Er, ^{171,175,177}Lu, ^{177}Ta, and ^{183}Re.

The paper is organized as follows: Section 2 briefly describes the formulation of the model. Section 3 presents our detailed discussion of the results obtained for various nuclides. Finally, the main inferences drawn from the present study are summarized in Section 4.

2. The Model

The present model formulation is an improvement over a previously developed empirical model [2] for the calculations of bandhead energies of 3qp quadruplets. The present version of model includes (i) the contribution of diagonal terms that appeared from rotor–particle and particle–particle coupling interactions and (ii) the contribution due to the irrotational motion of valence protons and/or neutrons. These contributions were ignored in the earlier formulation proposed by Jain et al. [2]. The formulation of present semi-empirical model is based on following assumptions:

1. The excitation energy of a particular 3qp quadruplet can be estimated by summing up the 1qp excitation energies of each valance nucleon observed in the neighboring odd-A nuclei. The unperturbed energy of 1qp state of each valance nucleon can be estimated as:

$$E_I = E_{qp} + (\frac{\hbar^2}{2\Im})[I(I+1) - K^2 + \delta_{K,\frac{1}{2}} a(-1)^{(I+\frac{1}{2})}(I+\frac{1}{2})] \tag{1}$$

where E_{qp}, $\hbar^2/2\Im$, I, and a are the quasi-particle energy, inertia parameter, total angular momentum, and the decoupling parameter pertaining to one quasi-particle state, respectively.

2. The effective moment of inertia used in the calculation of given 3qp bandhead energy can either be estimated using difference of the first two experimentally observed energy levels (i.e., $\Im_{3qp} = \hbar^2/2\Im = E_\gamma(I \to I-1)/2I$) of given 3qp rotational band or by using the experimental data of inertia parameters of valance nucleons of neighboring odd-A and even–even nuclei as:

$$\Im_{3qp} = \Im_{(1)} + \Im_{(2)} + \Im_{(3)} - \Im_{even-even} \tag{2}$$

where $\Im_{(1)}$, $\Im_{(2)}$, $\Im_{(3)}$ are odd nucleon moments of inertia and $\Im_{even-even}$ is the moment of inertia of the even-even core.

3. The contribution from the residual interactions (Gallagher-Moszkowski (GM) splitting [3,4]) can be taken as a sum of the neutron–proton/proton–proton/neutron–neutron interaction energies and can be obtained from the two-quasiparticle (2qp) states observed in the neighboring even–even and odd–odd nuclei.

In the framework of present model, the excitation energy of a given member of a 3qp quadruplet is expressed as:

$$E(K) = E_{pairing} + E_{qp} + E_{rot} + E_{irrot} + E_{resi} + E_{rpc} + E_{ppc} \tag{3}$$

$E_{pairing}$ is the proton/neutron pairing energy required to break a proton or neutron pair and is calculated by using the following four-point formulae [8]:

$$\Delta_p = \frac{1}{4}\{B(N,Z-2) - 3B(N,Z-1) + 3B(N,Z) - B(N,Z+1)\} \tag{4}$$

$$\Delta_n = \frac{1}{4}\{B(Z,N-2) - 3B(Z,N-1) + 3B(Z,N) - B(Z,N+1)\} \tag{5}$$

where N, Z and B is the proton number, neutron number and experimental binding energy of a nuclide. The experimental binding energies are adopted from Wang et al. [9].

E_{qp} is the total quasi-particle energy which is the sum of the three one-quasiparticle energies as:

$$E_{qp} = \sum_{i=1}^{3} E_{q_i}^{(i)} \tag{6}$$

E_{rot} is the rotational energy and estimated as [2]:

$$E_{rot} = \frac{\hbar^2}{2\Im}(I(I+1) - K^2) \tag{7}$$

E_{irrot} is the contribution appeared from irrotational motion of valance nucleons and calculated as [6,7]:

$$E_{irrot} = \frac{\hbar^2}{2\Im}\left[\left(\sum_{j_1}\left|C_{k_1}^{j_1}\right|^2 j_1(j_1+1) - k_1^2\right) + \left(\sum_{j_2}\left|C_{k_2}^{j_2}\right|^2 j_2(j_2+1) - k_2^2\right) + \left(\sum_{j_3}\left|C_{k_3}^{j_3}\right|^2 j_3(j_3+1) - k_3^2\right)\right] \tag{8}$$

where j_1, j_2 and j_3 are the total angular momenta of individual valance particles. The Nilsson's coefficients ($\left|C_{k_1}^{j_1}\right|^2$, $\left|C_{k_2}^{j_2}\right|^2$ and $\left|C_{k_3}^{j_3}\right|^2$) are calculated using Nilsson model [10] with potential parameters k_p, μ_p for protons, k_n, μ_n for neutrons adopted from Jain et al. [1] and deformation parameters (ε_2, ε_4) from Moller et al. [11].

E_{res} is the energy corresponding to the residual interactions and is calculated as [6,7]:

$$\begin{aligned} E_{res} = \sigma_{(1,2)}\left\{\left[\tfrac{1}{2} - \delta_{\Sigma(1,2),0}\right]E_{(1,2)}^{split} - \delta_{K(1,2),0}E_{(1,2)}^{N}\Pi_{(1,2)}\right\} + \\ \sigma_{(2,3)}\left\{\left[\tfrac{1}{2} - \delta_{\Sigma(2,3),0}\right]E_{(2,3)}^{split} - \delta_{K(2,3),0}E_{(2,3)}^{N}\Pi_{(2,3)}\right\} + \\ \sigma_{(1,3)}\left\{\left[\tfrac{1}{2} - \delta_{\Sigma(1,3),0}\right]E_{(1,3)}^{split} - \delta_{K(1,3),0}E_{(1,3)}^{N}\Pi_{(1,3)}\right\} \end{aligned} \tag{9}$$

where $\delta_{\Sigma(x,y),0}$= 1, if intrinsic spins of two particles are anti-parallel and $\delta_{\Sigma(x,y),0} = 0$, if intrinsic spins of two particles are parallel. The term $\sigma_{(x,y)} = 1$ for like particles and $\sigma_{(x,y)}$= -1 for unlike particles. The term $\Pi_{(x,y)} = +1$ or -1 for positive or negative parity of (x, y) combination. The $E_{(i,j)}^{split}$ $(i, j = 1, 2, 3; i \neq j)$ is the GM splitting energy [3,4] among the triplet ($\uparrow\uparrow$ or $\downarrow\downarrow$) and the singlet ($\downarrow\uparrow$ or $\uparrow\downarrow$) states of a given 2qp combination, and $E_{(i,j)}^{N}$ is the Newby shift energy [5].

The diagonal contribution of rotor-particle coupling (E_{rpc}) and particle-particle coupling (E_{ppc}) terms to the bandhead energies of given 3qp quadruplet is calculated as [6,7]:

$$\begin{aligned} E_{rpc} = \delta_{K,\frac{1}{2}}\frac{\hbar^2}{2\Im}(-1)^{I+\frac{1}{2}}\left(I+\frac{1}{2}\right)\{(\delta_{\sigma_{++-}} + \delta_{\sigma_{+-+}})\left(\langle k_1\rho_1|j_{1^+}|-k_1\rho_1\rangle\delta_{k_1,\frac{1}{2}}\right) + \\ (\delta_{\sigma_{-++}} + \delta_{\sigma_{++-}})\left(\langle k_2\rho_2|j_{2^+}|-k_2\rho_2\rangle\delta_{k_2,\frac{1}{2}}\right) + (\delta_{\sigma_{+-+}}\delta_{\sigma_{-++}})\left(\langle k_3\rho_3|j_{3^+}|-k_3\rho_3\rangle\delta_{k_3,\frac{1}{2}}\right)\} \end{aligned} \tag{10}$$

$$\begin{aligned} E_{ppc} = \delta_{K,\frac{1}{2}}\frac{\hbar^2}{2\Im}\{(\delta_{\sigma_{+-+}} + \delta_{\sigma_{-++}})\left(\langle k_1\rho_1|j_{1^+}|-k_1\rho_1\rangle\langle k_2\rho_2|j_{2^+}|-k_2\rho_2\rangle\,\delta_{k_1,\frac{1}{2}}\delta_{k_2,\frac{1}{2}}\delta_{k_3,\frac{1}{2}}\right) + \\ (\delta_{\sigma_{-++}} + \delta_{\sigma_{++-}})\left(\langle k_1\rho_1|j_{1^+}|-k_1\rho_1\rangle\langle k_3\rho_3|j_{3^+}|-k_3\rho_3\rangle\,\delta_{k_1,\frac{1}{2}}\delta_{k_3,\frac{1}{2}}\delta_{k_2,\frac{1}{2}}\right) + (\delta_{\sigma_{+-+}} + \\ \delta_{\sigma_{++-}})\left(\langle k_2\rho_2|j_{2^+}|-k_2\rho_2\rangle\langle k_3\rho_3|j_{3^+}|-k_3\rho_3\rangle\,\delta_{k_2,\frac{1}{2}}\delta_{k_3,\frac{1}{2}}\delta_{k_1,\frac{1}{2}}\right)\} \end{aligned} \tag{11}$$

where, $\langle k_1\rho_1|j_{1^+}|-k_1\rho_1\rangle$, $\langle k_2\rho_2|j_{2^+}|-k_2\rho_2\rangle$ and $\langle k_3\rho_3|j_{3^+}|-k_3\rho_3\rangle$ are the matrix elements and are calculated using the Nilsson model [10].

3. Results and Discussion

In present study, we calculated the bandhead energies of 12 three-quasiparticle quadruplets (npp/pnn) observed in six different rare-earth nuclides namely, ^{163}Er [12], ^{171}Lu [13], ^{175}Lu [14–16], ^{177}Lu [17–22], ^{177}Ta [23,24] and ^{183}Re [25,26] by using the model formulation described in Section 2. The excitation energy of a particular member of a given 3qp quadruplet is calculated by making use of Equation (1). The proton pairing (Δ_p) and neutron pairing (Δ_n) energies are calculated using four-point formulae, where experimental binding energies are taken from Wang et al. [9]. The quasi-particle energies ($E^{(1)}_{qp}$, $E^{(2)}_{qp}$, $E^{(3)}_{qp}$) and inertia parameters ($\Im_{(1)}$, $\Im_{(2)}$, $\Im_{(3)}$) of odd protons/neutrons and even–even core ($\Im_{even-even}$) are calculated by averaging of respective experimental data pertaining to neighboring odd-A/even–even nuclei [27]. The quasi-particle energies, proton/neutron pairing energies and effective moments of inertia used in the calculation of bandhead energies of 3qp quadruplets observed in ^{163}Er, $^{171,\,175,\,177}$Lu, ^{177}Ta and ^{183}Re nuclides are presented in Table 1.

Table 1. Estimated values of quasi-particle energies, proton/neutron pairing energies and effective moments of inertia observed in ^{163}Er, ^{171,175,177}Lu, ^{177}Ta and ^{183}Re nuclides.

Nuclide	Configuration	Quasi-Particle Energies (keV)			(Δ_p/Δ_n) (keV)	Inertia Parameter ($\Im_{3qp}$) (keV)
		E_{Q1}	E_{Q2}	E_{Q3}		
^{163}Er	$5/2[523]_\nu \otimes 1/2[411]_\pi \otimes 7/2[523]_\pi$	0.0	105.5	80.23	882.62	2.92
^{171}Lu	$7/2[404]_\pi \otimes 7/2[633]_\nu \otimes 1/2[521]_\nu$	0.0	222.99	199.45	730.02	3.17
^{175}Lu	$7/2[404]_\pi \otimes 7/2[514]_\nu \otimes 5/2[512]_\nu$	0.0	318.06	254.06	571.80	10.19
	$7/2[404]_\pi \otimes 7/2[514]_\nu \otimes 1/2[521]_\nu$	0.0	318.06	479.15		10.77
^{177}Lu	$7/2[404]_\pi \otimes 7/2[514]_\nu \otimes 1/2[510]_\nu$	0.0	107.16	444.94	458.92	9.75
	$7/2[404]_\pi \otimes 7/2[514]_\nu \otimes 1/2[521]_\nu$	0.0	107.16	767.11		10.70
	$1/2[411]_\pi \otimes 7/2[514]_\nu \otimes 1/2[510]_\nu$	569.70	107.16	444.94		10.45
	$9/2[514]_\pi \otimes 7/2[514]_\nu \otimes 9/2[624]_\nu$	150.39	107.16	133.77		7.33
	$7/2[404]_\pi \otimes 7/2[514]_\nu \otimes 9/2[624]_\nu$	0.0	107.16	133.77		8.11
^{177}Ta	$9/2[514]_\pi \otimes 7/2[514]_\nu \otimes 1/2[521]_\nu$	73.36	174.09	173.90	704.07	8.98
^{183}Re	$5/2[402]_\pi \otimes 9/2[624]_\nu \otimes 11/2[615]_\nu$	0.0	201.20	292.50	851.82	5.62
	$5/2[402]_\pi \otimes 9/2[624]_\nu \otimes 1/2[510]_\nu$	0.0	201.20	227.42		7.04

The energy ordering among different members of given 3qp quadruplets mainly depends on the residual interactions, i.e., GM splitting [3,4] and Newby shift energies [5]. The GM splitting energies are extracted from the difference between unperturbed bandhead energies (E^{unp}) of singlet (↓↑ or ↑↓) and triplet (↑↑ or ↓↓) 2qp states. Special care is exercised to estimate unperturbed bandhead energies of singlet (↓↑ or ↑↓) and triplet (↑↑ or ↓↓) states observed in neighboring even–even/odd–odd nuclei using the following corrections: (i) energy shift due to collective rotational motion (ii) energy shifts due to Coriolis interactions mainly for configurations containing either $i_{13/2}$ neutrons or $h_{9/2}$ protons. For the cases where experimental data are not available, we have used the empirical facts that the GM splitting energies are generally of the order of 400–500 keV for even–even nuclei and less than 100 keV for odd–odd nuclei. The GM splitting and Newby shift energies extracted for proton-neutron, neutron-neutron and proton-proton 2qp configurations used in the present calculations are presented in Table 2.

Table 2. GM splitting and Newby shift energies (in keV) extracted for proton–neutron, neutron–neutron and proton–proton 2qp configurations observed in neighboring even–even and odd–odd nuclei.

Nilsson Configuration K^{π} [$Nn_z\Lambda$]	**GM Splitting Energies (in keV) for Proton–Neutron Configurations**				
	1/2[411]$_{\pi}$	**7/2[523]$_{\pi}$**	**9/2[514]$_{\pi}$**	**7/2[404]$_{\pi}$**	**5/2[402]$_{\pi}$**
5/2[523]$_{\nu}$	53	146			
9/2[624]$_{\nu}$					207
7/2[514]$_{\nu}$	120		100	255	
1/2[510]$_{\nu}$	113			101	110
5/2[512]$_{\nu}$				112	
1/2[521]$_{\nu}$			171	77	
7/2[633]$_{\nu}$				126	
11/2[615]$_{\nu}$					249
GM splitting energies (in keV) for neutron–neutron configurations					
K^{π} [$Nn_z\Lambda$]	7/2[514]$_{\nu}$	7/2[633]$_{\nu}$	1/2[521]$_{\nu}$	9/2[624]$_{\nu}$	11/2[615]$_{\nu}$
9/2[624]$_{\nu}$	186				400
5/2[512]$_{\nu}$	256				
1/2[521]$_{\nu}$	295	428			
1/2[510]$_{\nu}$	376			300	
7/2[514]$_{\nu}$			295		
GM splitting energies (in keV) for proton–proton configurations					
K^{π} [$Nn_z\Lambda$]	7/2[523]$_{\pi}$	9/2[514]$_{\pi}$			
1/2[411] $_{\pi}$	345.6	400			
Newby shift energies (in keV) for neutron–neutron and proton–neutron configurations					
K^{π} [$Nn_z\Lambda$]	7/2[514]$_{\nu}$	9/2[624]$_{\nu}$	1/2[510]$_{\nu}$	7/2[633]$_{\nu}$	
7/2[404] $_{\pi}$	−69			−40	
9/2[514] $_{\pi}$		−2			
1/2[411] $_{\pi}$			1		

In order to confirm the validity of the present semi-empirical model, we have successfully calculated the bandhead energies of 12 npp/pnn three-quasiparticle quadruplets observed in ^{163}Er [12], ^{171}Lu [13], ^{175}Lu [14–16], ^{177}Lu [17–22], ^{177}Ta [23,24], and ^{183}Re [25,26] nuclides. The calculated bandhead energies of all the members of 3qp quadruplets observed in above mentioned nuclides are presented in Table 3. In this table, we have also listed the contributions that appeared due to the irrotational motion of valance nucleons, residual interactions, rotor–particle coupling (rpc) and, particle–particle coupling (ppc) terms. It is clear from Table 3 that, the calculated bandhead energies using the present model are in better agreement with experimental data as compared with the earlier model calculations [2]. It was also observed that the irrotational term contributed substantially to bandhead energies indicating its imperative role. The only exception in present model calculation is $\mathbf{K}^{\pi}$ = 17/2^{-}: 7/2[514]$_{\nu}\otimes$1/2[521]$_{\nu}\otimes$9/2[514]$_{\pi}$ 3qp configuration observed in the ^{177}Ta nuclide. The calculated bandhead energy of $\mathbf{K}^{\pi}$ = 17/2^{-} member of the quadruplet based on 7/2[514]$_{\nu}\otimes$1/2[521]$_{\nu}\otimes$9/2[514]$_{\pi}$ 3qp configuration observed in ^{177}Ta is 1914.8 keV as compared with the experimental value 1475.9 keV. We believe that this disagreement might be due to the highly tentative nature of spin (parity not known) of K = 17/2 bandhead [24], and this finding calls for experimental confirmation. It should be noted that this exception is not included in Table 3.

Table 3. GM splitting and Newby shift energies (in keV) extracted for proton–neutron, neutron–neutron and proton–proton 2qp configurations observed in neighboring even–even and odd–odd nuclei.

Nuclide	Configuration	K^{π}	Irrotational Correction (keV)	Residual Interactions (keV)	RPC/PPC	Bandhead Energies (keV)		
					(keV)	Expt. [27]	Present Model	Earlier Model
^{163}Er	A⊗B⊗C	$11/2^{+}$	366.06	272.3			1722.8	1356.8
		$1/2^{+}$		73.3	−2.49		1506.7	1143.2
		$13/2^{+}$		−126.3			1327.1	961.1
		$3/2^{+}$		−219.3		1538.80	1219.5	853.5
^{171}Lu	F⊗G⊗H	$13/2^{-}$	431.12	275.5			1801.3	1448.6
		$1/2^{-}$		72.5	2.75		1582.1	1226.6
		$15/2^{-}$		−229.5		1241.55	1299.5	946.7
		$1/2^{-}$		−278.5	2.75		1231.1	875.6
^{175}Lu	F⊗D⊗I	$5/2^{+}$	312.60	472.0			1954.00	1641.4
		$5/2^{+}$		−79.0			1403.00	1090.4
		$9/2^{+}$		−39.0		1511.0	1563.38	1250.7
		$19/2^{+}$		−78.0		1392.2	1475.33	1162.7
	F⊗D⊗J	$1/2^{+}$	290.83	382.5			2077.7	1756.9
		$15/2^{+}$		50.0		1732.0	1791.1	1500.3
		$1/2^{+}$		10.5			1705.7	1384.9
		$13/2^{+}$		−167.5		1590.0	1562.4	1271.5
^{177}Lu	F⊗D⊗K	$1/2^{+}$	261.46	435.0	1.78		1714.14	1450.1
		$13/2^{+}$		79.0		1502.6	1414.9	1153.4
		$1/2^{+}$		−42.0	1.78		1237.14	973.9
		$15/2^{+}$		−196.0		1356.5	1149.6	888.2
	F⊗D⊗J	$1/2^{+}$	306.37	382.5	−35.9		1991.5	1721.0
		$15/2^{+}$		10.5		1632.8	1770.3	1463.9
		$1/2^{+}$		50.5	−35.9		1619.5	1349.0
		$13/2^{+}$		−167.5		1453.9	1541.6	1235.2
	B⊗D⊗K	$5/2^{+}$	302.71	303.5			2213.1	1910.4
		$7/2^{+}$		70.5			1990.5	1687.8
		$9/2^{+}$		−192.5		1717.5	1737.9	1435.2
		$7/2^{+}$		−185.5		1617.0	1734.5	1431.2
	F⊗D⊗L	$9/2^{-}$	431.71	477.5			1645.56	1213.8
		$9/2^{-}$		−84.5		1049.46	1083.56	651.8
		$23/2^{-}$		−153.5		970.17	1071.33	639.6
		$5/2^{-}$		36.5			1188.34	756.6
	M⊗D⊗L	$7/2^{+}$	566.25	216.0		1336.5	1658.2	1091.9
		$11/2^{+}$		−26.0		1230.4	1430.8	864.6
		$25/2^{+}$		−112.0		1325.0	1396.1	829.9
		$7/2^{+}$		−70.0		1241.5	1372.2	805.1
^{183}Re	N⊗L⊗O	$15/2^{+}$	671.10	428.0			2486.8	1815.7
		$25/2^{+}$		−28.0		1906.7	2058.9	1387.8
		$3/2^{+}$		−179.0			1846.1	1174.9
		$7/2^{+}$		−221.0			1815.3	1144.2
	N⊗L⊗K	$5/2^{-}$	583.09	308.5			2189.6	1606.5
		$15/2^{-}$		−8.50		1628.3	1907.8	1324.8
		$3/2^{-}$		−101.50			1772.6	1189.5
		$13/2^{-}$		−198.5			1710.8	1127.7

A: $5/2[523]_{\nu}$, B: $1/2[411]_{\pi}$, C:$7/2[523]_{\pi}$, D: $7/2[514]_{\nu}$, E: $9/2[514]_{\pi}$, F: $7/2[404]_{\pi}$, G: $7/2[633]_{\nu}$, H: $1/2[521]_{\nu}$, I: $5/2[512]_{\nu}$, J: $1/2[521]_{\nu}$, K: $1/2[510]_{\nu}$, L: $9/2[624]_{\nu}$, M: $9/2[514]_{\pi}$, N: $5/2[402]_{\pi}$, O: $11/2[615]_{\nu}$.

In the present study, we have considered all of the three-quasiparticle quadruplets formed due to the coupling of intrinsic angular momenta of unlike particles (npp/pnn) observed in odd-A rare-earth nuclides to date. The calculations of bandhead energies of other three-quasiparticle quadruplets formed based on the coupling of intrinsic angular momenta of like particles (ppp/nnn) will require additional investigation and will be reported elsewhere. We believe that the present model calculations will be useful in predicting configuration assignment to various members of 3qp quadruplets observed in well-deformed odd-A rare-earth nuclides.

4. Conclusions

We reported the detailed formulation of the improved version of a semi-empirical model for calculations of the bandhead energies of three-quasiparticle (3qp) configurations observed in well-deformed odd-A nuclei. This improved version of the model includes the important contributions appearing from the irrotational motion of valance protons/neutrons, diagonal components of particle-particle coupling (ppc), and rotor-particle coupling (rpc) terms, which were ignored in the earlier version. The validity of the present version of the model is tested by calculating the bandhead energies of 12 three-quasiparticle npp/pnn quadruplets observed in ^{163}Er, ^{171,175,177}Lu, ^{177}Ta, and ^{183}Re nuclides. The results obtained in present calculations of bandhead energies show better agreement with the experimental data pointing towards the significant contribution of irrotational correction term. We strongly expect that our efforts will provide support to the various ongoing and upcoming experimental studies in making configuration assignments to particular member of a given 3qp quadruplet and also in the identification of correct Nilsson's configuration where more than one competing configuration exists for a particular 3qp quadruplet. As a future perspective, it is very interesting to extend our study to the calculation of bandhead energies of other three-quasiparticle quadruplets based on the coupling of intrinsic angular momenta of like particles (*ppp/nnn*), which we intend to report elsewhere in the near future.

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