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Selected Results in Heavy-Quark Fragmentation

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Abstract: I review a few selected topics concerning heavy-quark fragmentation, taking particular care about bottom- and charm-quark production in e^+e^- annihilation and the inclusion of non-perturbative corrections. In particular, I discuss the recent developments of calculations carried out in the framework of perturbative fragmentation functions and the perspective to extend them to other processes and higher accuracy. Special attention is paid to the use of an effective strong coupling constant to model hadronization effects.

Keywords: QCD; heavy quarks; fragmentation

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Citation: Corcella, G. Selected Results in Heavy-Quark Fragmentation. *Universe* **2022**, *8*, 490. https://doi.org/10.3390/universe8090490

Academic Editors: Francesca De Mori and Mario Merola

Received: 17 June 2022
Accepted: 10 September 2022
Published: 18 September 2022

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1. Introduction

Heavy-quark phenomenology (top, bottom, and charm) is one of the most interesting research topics in particle physics, since it allows tests of the Standard Model in both strong and electroweak sectors and often plays a role in several searches for new physics phenomena. As for QCD, on which the present contribution will be mostly concentrated, processes with heavy quarks offer the opportunity to test the parton model, factorization, and power corrections.

In principle, as the mass of the heavy quarks regulates the collinear singularity, one should trust perturbative QCD to describe heavy-quark production. However, it turns out that fixed-order calculations, e.g., at next- (NLO) or next-to-next-to-leading order (NNLO) in the strong coupling constant α_S, are reliable to predict inclusive observables, namely total cross-sections or widths, while differential distributions, such as the energy or transverse momentum spectrum of the heavy quark, exhibit large logarithms which need to be resummed to all orders to obtain meaningful results.

For this purpose, the approach of perturbative fragmentation functions [1] stands out as a powerful tool. Regarding up to power corrections $\sim (m/Q)^p$, with m being the heavy-quark mass and Q the process hard scale, e.g., the centre-of-mass energy in e^+e^- annihilation, the heavy-quark energy spectrum can be obtained as a convolution of a massless coefficient function and a process-independent perturbative fragmentation function, which describes the transition of the massless parton into the heavy quark. This factorizazion, as long as $m \ll Q$, allows one to predict the spectrum of a massive quark just by performing a massless computation. Heavy-hadron energy distributions are then determined by convoluting the parton-level ones with hadronization models which contain some tunable parameters, such as, e.g., the non-perturbative fragmentation functions in [2,3][1].

This approach has been applied to bottom/charm production in e^+e^- collisions [1,6,7], b-quark fragmentation in top [8,9] and Higgs [10] decays, and charm production in charged-current Deep Inelastic Scattering [11]. All these papers compute the energy spectrum at NLO and evolve the perturbative fragmentation function from the hard scale down to the heavy-quark mass by using the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) evolution equations [12–14]. One can easily demonstrate that, once the initial condition of the perturbative fragmentation function is given, solving the DGLAP equations allows one to resum large logarithms $\ln(m^2/Q^2)$ (collinear resummation). In particular, in these papers, collinear resummation was carried out in the next-to-leading logarithmic (NLL)

approximation. The initial condition was computed at NLO in [1] and its process independence was proved on general grounds in [6]. More recently, it was calculated in [15] and [16] at NNLO for heavy quarks and gluons, respectively. Such results, along with the NNLO computation of the Altarelli–Parisi splitting functions in [17], would permit DGLAP evolution and collinear resummation up to NNLL. Furthermore, the NLO initial condition of the perturbative fragmentation function and the coefficient functions of processes [1,7,10] exhibit terms which are enhanced whenever the heavy-quark energy fraction becomes close to 1, which corresponds to soft or collinear emissions. Such terms can be resummed using standard techniques [18,19], achieving what is called large-x, soft, or threshold resummation. The authors of [6,7,9,10] accounted for threshold resummation in the NLL approximation in both coefficient functions and initial conditions.

Regarding the inclusion of hadronization effects, an alternative approach to the use of a model with tunable parameters involves incorporating non-perturbative corrections into a frozen [20] or analytic [21] strong coupling constant. For the purpose of heavy-quark fragmentation, the model in [21] was used in [22,23] to predict B- and D-hadron production in electron–positron annihilation, in conjunction with a NLO coefficient function, NLL DGLAP evolution, and NNLL large-x resummation.

Before concluding this section, I wish to point out that, although the present manuscript will not deal with heavy-quarkonium production, relevant work on the fragmentation to heavy quarkonium can be found, e.g., in [24–26].

The present paper is structured as follows. In Section 2, the approach of perturbative fragmentation will be reviewed. In Section 3, the implementation of hadronization effects will be discussed, while a few phenomenological results will be presented in Section 4. Finally, Section 5 will contain some concluding remarks.

2. Perturbative Calculations for Heavy-Quark Fragmentation

As discussed in the introduction, the framework of perturbative fragmentation functions represents a powerful tool to compute the heavy-quark energy spectrum: as long as the quark mass is negligible with respect to the hard scale, the heavy-quark energy distribution can be obtained through a convolution of a massless coefficient function and a process-independent perturbative fragmentation function. Concerning the computation of the energy spectrum in the massless approximation, it is well known that it is divergent because of the collinear singularity, which has to be subtracted in order to obtain a finite result and consistently define the coefficient function. At NLO, the calculation is typically carried out in dimensional regularization, with the collinear singularity subtracted off in the $\overline{\text{MS}}$ scheme.[2] Referring to e^+e^- annihilation into bottom-quark pairs at the Z pole at $\mathcal{O}(\alpha_S)$ for simplicity, i.e.,

$$e^+e^- \to Z(Q) \to b(p_b)\bar{b}(p_{\bar{b}})(g(p_g)), \tag{1}$$

the energy distribution can be factorized as follows:

$$\begin{aligned} \frac{1}{\sigma_0}\frac{d\sigma}{dx_b}(x_b, m_Z, m_b) &= \sum_i \int_{x_b}^1 \frac{dz}{z}\left[\frac{1}{\sigma_0}\frac{d\hat{\sigma}_i}{dz}(z, Q, \mu_R, \mu_F)\right]^{\overline{\text{MS}}} D_i^{\overline{\text{MS}}}\left(\frac{x_b}{z}, \mu_F, m_b\right) \\ &+ \mathcal{O}((m_b/Q)^p), \end{aligned} \tag{2}$$

where $x_b = 2p_b \cdot Q/m_Z^2$ is the bottom energy fraction in the Z rest frame; σ_0 is the Born LO cross-section; σ_i the cross-section for the production of a massless parton i; D_i the perturbative fragmentation function expressing the transition of i into a heavy b; μ_R and μ_F are the renormalization and factorization scales, respectively, and p is an integer $p \geq 1$. Most analyses have so far assumed that, in Equation (2), i is a light quark, which corresponds to the so-called non-singlet approximation, while gluon splitting $g \to b\bar{b}$ is neglected. This splitting, as well as $g \to c\bar{c}$ for charm production, was accounted for in [7], but the authors found very little impact on the phenomenological results.

The perturbative fragmentation functions follow the DGLAP evolution equations and their value at any scale μ_F can be obtained after an initial condition at μ_{0F} is given. The initial condition was first computed at NLO in [1] for e^+e^- annihilation into heavy quarks, and then rederived in [6], demonstrating its process independence on more general grounds. As pointed out above, the NNLO initial conditions for quarks and gluons can be found in [15,16], respectively. Moreover, one can prove, in the same manner as for parton distribution functions, that solving the DGLAP equations allows one to resum the large heavy-quark mass logarithms, i.e., $\ln(m^2/Q^2)$, which appear in the massive NLO spectrum (collinear resummation). In particular, using both NLO initial condition and splitting functions yields next-to-leading logarithmic (NLL) collinear resummation, i.e., for an evolution from $\mu_{0F} \simeq m$ to $\mu_F \simeq Q$, terms $\alpha_S^n \ln^n(m^2/Q^2)$ (LL) and $\alpha_S^n \ln^{n-1}(m^2/Q^2)$ (NLL) are resummed. Including NNLO corrections to the splitting functions [17] and the initial condition [15,16] would potentially extend collinear resummation to NNLL. The DGLAP equation is typically solved in Mellin moment space[3], where convolutions are turned into ordinary products, and then x-space results are recovered after an inverse Mellin transform, which usually follows the minimal prescription [27].

Both the initial condition and coefficient functions of the main heavy-quark production processes exhibit terms behaving like $\sim \alpha_S[\ln(1-x_b)/(1-x_b)]_+$ and $\sim \alpha_S/(1-x_b)_+$, which become large for soft or collinear radiation, i.e., $x_b \to 1$, which corresponds to $N \to \infty$ in moment space. The authors of Refs. [6,7,9,10] performed threshold resummation in the NLL approximation in Mellin space, i.e., terms $\alpha_S \ln^{n+1} N$ and $\alpha_S^n \ln^n N$ in the Sudakov exponent are summed to all orders, and inverted the result back to x-space. The authors of Refs. [22,23] instead implemented NNLL large-x resummation in both the initial condition and the coefficient function, though matching it to NLO fixed-order results for bottom and charm fragmentation in e^+e^- collisions.

Before discussing hadronization corrections, it has to be pointed out that, although all such heavy-quark calculations resum threshold contributions with high accuracy, as discussed, f or instance, in [6] at NLL and in [28] at NNLL or NNNLL, they are still not reliable at very large x, since the Sudakov exponent exhibits a branch point, related to the Landau pole of the strong coupling constant. This effect is especially relevant in the initial condition, where renormalization and factorization scales vary around the heavy-quark mass. The branch point is found when the Mellin variable $N \sim m/\Lambda$ or $x \sim 1 - \Lambda/m$, with Λ being the QCD scale in the $\overline{\text{MS}}$ renormalization scheme. Due to this issue, one can already envisage that convoluting parton-level calculations with simple non-perturbative models or implementing an effective coupling in some given approximation will not be enough to obtain reliable predictions for very high x values.

As a whole, while all perturbative calculations to extend heavy-quark fragmentation to NNLO+NNLL have been available for a while in a number of processes, the state of the art is generally NLO+NLL. A remarkable study was carried out in [29], which calculates heavy-quark fragmentation in e^+e^- annihilation in the NNLO approximation, with NNLL DGLAP evolution and NNNLL threshold resummation, within the framework of the soft collinear effective theory (SCET). Work towards a NNLO+NNLL calculation in the perturbative fragmentation framework for e^+e^- collisions is currently underway [30]. Beyond NLO+NLL, the work in [31] describes, in the top-quark narrow-width approximation, $t\bar{t}$ production and bottom fragmentation in top decays at NNLO in the framework of perturbative fragmentation functions, with NNLL DGLAP evolution and NNLL threshold resummation in the initial condition, though with no large-x resummation in the top-to-bottom coefficient function.

3. Non-Perturbative Corrections to Heavy-Quark Fragmentation

For the sake of describing experimental data on heavy-hadron production, it is necessary to convolute the perturbative spectrum with a non-perturbative fragmentation function, which typically contains parameters which are to be tuned to experimental data. In particular, simple power laws are often used as non-perturbative fragmentation functions

and a well-known example is the model proposed in [2], which has one tunable parameter β, and x is the heavy-hadron energy fraction with respect to the fragmenting quark:

$$D_{np} = (1+\beta)(2+\beta)(1-x)x^{\beta}. \tag{3}$$

By relying on the universality of the hadronization transition, one typically tunes models such as Equation (3) to the most precise data available, such as those from e^+e^- machines, and then uses the best-fit parameters in other environments, such as hadron colliders. For the sake of consistency, the same accuracy and perturbative parameter settings are to be used in calculating both the perturbative process, which is used for hadronization tuning, and the one to which the best fit is applied. However, a drawback of this procedure is that, as long as one computes the parton-level process to a finite accuracy, albeit resummed in a given logarithmic approximation, there are missing corrections which are process-dependent, which makes the tuning method and hadronization model not really universal. Nevertheless, while any improvement in perturbative calculations as well as in modelling non-perturbative corrections for processes involving heavy quarks would be certainly desirable, for the time being, fitting a non-perturbative fragmentation function to precise data from e^+e^- experiments, e.g., at LEP [32–34] or SLD [35], and consistently applying the results to other processes still represents the best way to approach heavy-hadron production in the perturbative fragmentation approach. Within the standard resummation formalism, this procedure was carried out in [6,7,29] for heavy-quark production in e^+e^- collisions, and in [9,10] for B-hadron production in top ($t \to bW$) and Standard Model Higgs ($H \to b\bar{b}$) decays, respectively. Strictly speaking, the fits carried out in the literature should be used as long as the same hadron species are involved; in particular, while OPAL [33], DELPHI [34], and SLD [35] reconstructed both mesons and baryons, such as the Λ_b, the ALEPH sample [32] contained only mesons. As a matter of fact, since the baryon fraction is estimated to be of the order of 10%, B-hadron data are often taken together when tuning non-perturbative models. For example, Refs. [36,37], which compared Monte Carlo event generators with resummed computations for b-quark fragmentation, fitted the hadronization models to all LEP and SLD data as if they came from one single experiment. Within the soft collinear effective theory formalism, the authors of [29] fit a hadronization model with two free parameters [38] to LEP and SLD data, either altogether or discarding the OPAL ones. This model, along with its best-fit parameters, was then implemented in the Monte Carlo code developed in [31] to describe $t\bar{t}$ production and decay at NNLO.

In the rest of this section, I wish to review the alternative method, based on an effective strong coupling constant, proposed in [21] and employed in [22,23] for bottom and charm fragmentation, respectively. Above all, the pioneering work in [39] showed that, in the framework of resummations, for the sake of summing up subleading soft/collinear contributions, the momentum-independent coupling constant α_S is to be replaced by the following integral over the discontinuity of the gluon propagator ($1/s$), so that the argument of α_S is roughly the transverse momentum of the emitted parton with respect to the parent one:

$$\alpha_S \longrightarrow \frac{i}{2\pi}\int_0^{k_T^2} ds\, \mathrm{Disc}_s \frac{\alpha_S(-s)}{s} \simeq \alpha_S(k_T^2) \quad ; \quad \mathrm{Disc}\, f(s) = \lim_{\epsilon\to 0}[f(s+i\epsilon) - f(s-i\epsilon)]. \tag{4}$$

It is well known that in resummations, as well as parton showers, the argument of the strong coupling is the parton transverse momentum. The integral (4) is typically performed, neglecting the imaginary part $\sim i\pi$ in the denominator of $\alpha_S(-s)$, which reads, e.g., at leading order:

$$\alpha_{S,\mathrm{LO}}(-s) = \frac{1}{\beta_0 \ln[(-s-i\epsilon)/\Lambda^2]} = \frac{1}{\beta_0 \ln[(|s|/\Lambda^2] - i\pi\Theta(s)]}, \tag{5}$$

where β_0 and Λ are the first terms of the beta function and the QCD scale in the $\overline{\text{MS}}$ renormalization scheme, respectively, and $\Theta(x)$ is the Heaviside step function. In other words, the approximate equality in Equation (4) assumes $\ln(|s|/\Lambda^2) \gg \pi$ in the denominator Equation (5), which is clearly questionable, since one sums soft and collinear parton radiation to all orders, namely partons with small virtualities s. In fact, as discussed before, when using the standard coupling, resummed calculations are not fully reliable for very soft or collinear emissions, and even including extra non-perturbative models, such as the power law in (3), they fail to describe heavy-hadron energy spectra for very large values of x. Following [21], the work in [22,23] first defines an analytic coupling which is free from the Landau pole:

$$\bar{\alpha}_S(Q^2) = \frac{1}{2\pi i}\int_0^\infty \frac{ds}{s+Q^2}\text{Disc}[\alpha_S(-s)] \tag{6}$$

and then constructs an effective coupling constant by inserting Equation (6) in (4):

$$\tilde{\alpha}_S(k_T^2) = \frac{i}{2\pi}\int_0^{k_T^2} ds\ \text{Disc}\frac{\bar{\alpha}_S(-s)}{s}. \tag{7}$$

At LO, for simplicity, one obtains

$$\bar{\alpha}_{S,\text{LO}}(Q^2) = \frac{1}{\beta_0}\left[\frac{1}{\ln(Q^2/\Lambda^2)} - \frac{\Lambda^2}{Q^2-\Lambda^2}\right], \tag{8}$$

which clearly shows that the integrand function in (7) is free from the Landau pole, and

$$\tilde{\alpha}_{S,\text{LO}}(Q^2) = \frac{1}{2\pi i\beta_0}\left[\ln\left(\frac{Q^2}{\Lambda^2}+i\pi\right) - \ln\left(\frac{Q^2}{\Lambda^2}-i\pi\right)\right]. \tag{9}$$

One can determine the relation between standard and effective coupling constants, showing that they start to differ from $\mathcal{O}(\alpha_S^3)$ as follows:

$$\tilde{\alpha}_S(Q^2) = \alpha_S(Q^2) - \frac{(\pi\beta_0)^2}{3}\alpha_S^3(Q^2) + \mathcal{O}(\alpha_S^4). \tag{10}$$

In Figure 1, we present the standard (dashes), analytic (dots), and effective (solid) coupling constants as a function of the scale Q, and observe that for $Q > 5$ GeV, the three couplings agree. At lower energies, the standard one significantly deviates, to the point of diverging when Q becomes close to the Landau pole, while the effective and analytic couplings are close to each other, differ by about 10% for $Q > 0.5$ GeV, and roughly agree again at very low Q.

The assumption adopted in [22,23] states that it is enough to replace the standard with the effective coupling in the perturbative calculation, i.e., $\alpha_S(Q) \to \tilde{\alpha}_S(Q)$, in such a way that a prediction obtained for a heavy quark, such as the b quark, can be applied to a heavy hadron, i.e., a B meson or baryon. In the simple formulation adopted in [22,23], based on [21], there is indeed no distinction between mesons and baryons, as well as spin-0 and spin-1 hadrons. Of course, one could add extra parameters to distinguish between hadron species, but in this way the effective coupling model would lose its peculiar feature of being free from non-perturbative parameters.

As discussed in [22,23], the effective coupling constant defined through Equation (7) was implemented in the NNLO approximation, along with a calculation for e^+e^- annihilation which uses NLO coefficient functions and an initial condition of the perturbative fragmentation function, NLL DGLAP evolution in the non-singlet approximation, and NNLL large-x resummation in the initial condition. Such an accuracy in the perturbative calculation should be seen as a part of the model and, as explained in [22], it was dictated by the available precision of computations at that time and, above all, by the fact that it led to an overall reasonable comparison with the data. Before presenting the results yielded by the model, I wish to stress that another difference between the method which uses a

non-perturbative fragmentation and the one based on an effective coupling constant. While in the first case one fixes the perturbative parameters when tuning the model and only accounts for the uncertainty in the fitted parameters, in the second one there is no tuning and therefore the dependence of the prediction on the perturbative entries, such as scales and heavy-quark masses, is to be estimated.

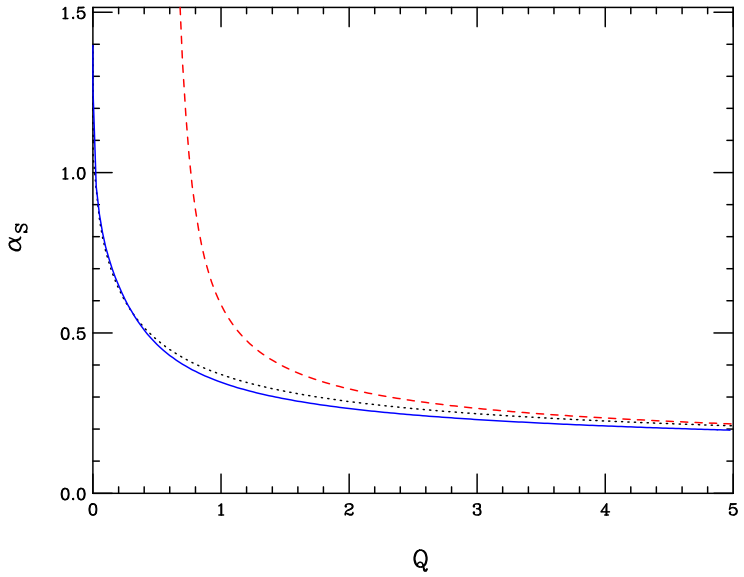


Figure 1. Standard (dashes line), analytic (dots), and effective (solid line) strong coupling constant at NNLO, as a function of the scale Q.

4. Results—*B* Production

In this section, I shall present some results on B-hadron production using models such as the one in (3) or the effective strong coupling constant (7). The results will be expressed in terms of x_B, namely the B-hadron energy fraction in the centre-of-mass frame, which is the hadron-level counterpart of x_b introduced in Section 2:

$$x_B = \frac{2p_B \cdot Q}{m_Z^2}, \tag{11}$$

where p_B is the B-hadron four-momentum. Starting from e^+e^- annihilation into bottom pairs, the NLO+NLL calculation in [6] can be convoluted with the model (3) and its free parameter β tuned to ALEPH [32], OPAL [33], and SLD [35] data. The authors of [36,37] fitted such data as if they came from one single experiment in the range $0.18 < x_B < 0.94$ and neglected the correlations among data points. Discarding a few data points at low and large x_B is necessary to obtain a reasonable fit and is justified by the fact that the coefficient function exhibits terms behaving $\sim \ln x_b$ and both the coefficient function and the initial condition present contributions $\sim \ln(1 - x_b)$, which are enhanced for $x_b \to 0$ and $x_b \to 1$ and have not been resummed. Furthermore, at very large x, one is mostly sensitive to the non-perturbative regime and resumming threshold contributions at NLL or NNLL, as well as modelling hadronization effects via an effective coupling constant or the model (3), is not enough to be reliable for $x_B \to 1$ or $N \to \infty$ in moment space. In fact, the authors of [7] improved the large-N behaviour of the heavy-quark spectra, and consequently the large-x_b one as well, by rescaling the N variable, according to

$$N \to N \frac{1 + f/N'}{1 + fN/N'}, \tag{12}$$

where $N = \exp[1/(b_0\alpha_S(\mu^2)]$, with $\mu = \mu_R$ in the coefficient function and $\mu = \mu_{0R}$ in the initial condition. The reason of the rescaling (12) is that, as anticipated before, the Sudakov exponent, resumming threshold contributions, exhibits branch points $\sim \ln(1-\lambda)$, with $\lambda = b_0\alpha_S(\mu^2)\ln N$, which are divergent for $\lambda \to 1$, namely for $N \simeq \mu^2/\Lambda$, if one uses the LO expression for $\alpha_S(\mu)$. The very fact that the divergence involves the QCD scale Λ, and hence the Landau pole of the strong coupling, means that it is unphysical and stresses the unreliability of perturbative calculations in this regime. If one roughly sets $N \sim 1/(1-x_b)$, one obtains that $x_b \simeq 1 - \Lambda/\mu$ is the equivalent relation in x_b-space where the resummed computation can be trusted, i.e., roughly $x_b < 0.96$ for bottom production. Thanks to the replacement (12) and fitting the parameter f, the authors of [7] managed to improve the large-x_b behaviour of the spectrum. In this paper, however, we follow the approach [6,22] and do not perform the replacement (12), without adding any extra parameter, such as f. By setting the perturbative parameters to values

$$\mu_R = \mu_F = m_Z,\ \mu_{0R} = \mu_{0F} = m_b,\ m_Z = 91.19\ \text{GeV},\ m_b = 5\ \text{GeV},\ \Lambda = 200\ \text{MeV}, \quad (13)$$

we obtain:

$$\beta \simeq 17.18 \pm 0.30, \quad (14)$$

with $\chi^2/\text{dof} \simeq 46/53$, from the fit. The uncertainty on β in Equation (14) is only due to the experimental errors on the data, while no perturbative uncertainty is accounted for. Regarding the effective coupling model, in [22] the authors varied all perturbative parameters in Equation (13) in typical ranges, i.e., the scales were allowed to change between half and twice the central values, and found a reasonable overall agreement in the same range as the model (3), i.e., $0.18 < x_B < 0.94$. As for the bottom-quark mass, since the model [21] assumes that, whenever the effective strong coupling is used, the b quark behaves as a B-hadron, it is not uniquely determined whether m_b should be the quark or hadron mass. Therefore, [22] made a conservative choice and varied m_b in a range which was sufficiently large to include the estimations of both quark and hadron masses. Furthermore, [22] found that the parameters which have the largest impact are m_b itself and $\mu_{F,0}$, and that the best overall fit, i.e., $\chi^2/\text{dof} \simeq 103.0/54$, is obtained for $m_b = 5$ GeV and $\mu_{0F} = m_b/2$. Moreover, the inclusion of NNLL large-x resummation in the coefficient function and the initial condition of the perturbative fragmentation function turned out to play a major role in the comparison with the data. Several comparisons among B-hadron data and calculations of $e^+e^- \to b\bar{b}$ processes were presented in [6,7,22,36,37]. Figure 2 presents an example of such a comparison, namely the NLO+NLL calculation with the hadronization model (3) and the NLO+(N)NLL computation which uses the effective coupling constant confronted with OPAL, ALEPH and SLD data altogether. Overall, the agreement of two calculations with the data is acceptable; the two theoretical predictions roughly agree for large and average x_B values, while the spectra yielded by the model (3) are above the one relying on the effective coupling at small x_B and above around the peak.

An alternative method to the fits in x_B-space involves working in Mellin space, which is feasible as long as the experimental collaborations provide the moments of the $b\bar{b}$ cross-section. As discussed in [40], an advantage of this procedure is that one does not have to rely on any hadronization model and perform any inversion from N- to x_B-space. In fact, the moments of the non-perturbative fragmentation function can be obtained by dividing the experimental ones by the theoretical N-space cross-section. As for the effective coupling model, no non-perturbative fragmentation function is fitted, so that one is provided directly with the moments of the B-hadron production cross-section, as a function of the entries in the perturbative computation, which can be directly compared with the data.

As an example of this procedure, in Table 1, the first four moments of the B-hadron cross-section are presented, as measured by DELPHI [41], along with the prediction of [22] with an effective coupling constant, as well as the NLO+NLL parton-level result computed in [6] in N-space and the extracted moments of the non-perturbative fragmentation function. The prediction of [22] accounts for the errors due to the variation of the perturbative

parameters summed in quadrature and, within the uncertainties, agrees with the moments in [41]. For the NLO+NLL calculation, the quoted moments are computed using the parameter values in Equation (13).

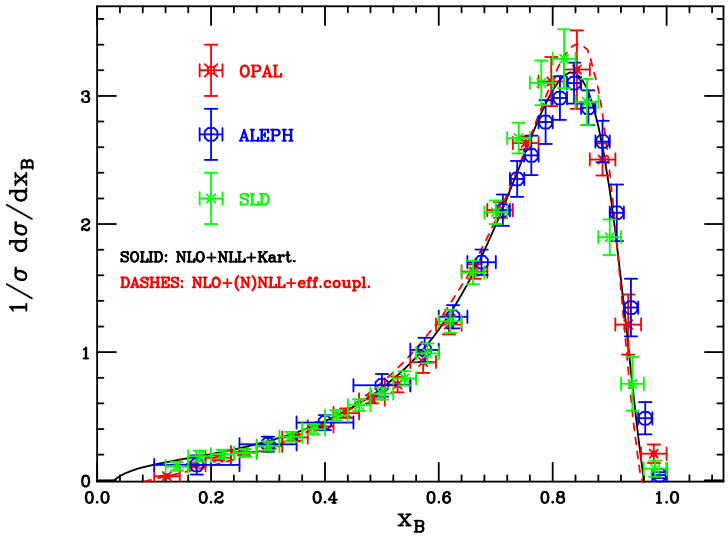


Figure 2. Experimental data on B-hadron production at SLD, OPAL, and ALEPH, compared with theoretical calculations carried out in the perturbative fragmentation framework and using either a tuned hadronization model (solid) or an effective coupling constant (dashes) to model non-perturbative corrections.

Table 1. Experimental Mellin moments of the B-hadron cross-section from the DELPHI collaboration, along with the moments yielded by the calculation in [22], based on an effective coupling constant, the perturbative moments at NLO+NLL as in [6], and the extracted N-space non-perturbative fragmentation function.

	$\langle x \rangle$	$\langle x^2 \rangle$	$\langle x^3 \rangle$	$\langle x^4 \rangle$
e^+e^- data σ_N^B	0.7153 ± 0.0052	0.5401 ± 0.0064	0.4236 ± 0.0065	0.3406 ± 0.0064
$[\sigma_N^B]_{\tilde{\alpha}_S}$	0.6867 ± 0.0403	0.5019 ± 0.0472	0.3815 ± 0.0465	0.2976 ± 0.0462
$[\sigma_N^b]_{\mathrm{NLO+NLL}}$	0.7801	0.6436	0.5479	0.4755
D_N^B	0.9169	0.8392	0.7731	0.7163

Regarding B-hadron production in top decays, as discussed before, the NLO+NLL b-quark spectrum was computed in [8,9], while [10] performed a companion calculation for standard model Higgs decay $H \to b\bar{b}$. Both computations were undertaken in the narrow-width approximation for top quarks and Higgs bosons, and convoluted with a non-perturbative fragmentation such as (3), tuned to e^+e^- data from LEP and SLD, as discussed above. Although it would be a straightforward extension, the model based on the effective strong coupling constant has not been yet applied to $t \to bW$ and $H \to b\bar{b}$ processes. In Figure 3, the x_B spectrum in top and Higgs decays is displayed, namely the B-hadron energy distribution in the top or Higgs rest frame. The difference is mostly due to the coefficient functions, which, unlike the fragmentation function, are process-dependent, with b-flavoured hadrons harder in top decays with respect to $H \to b\bar{b}$ events. In the narrow-width approximation, the B-hadron spectra in the top and Higgs decays in Figure 3 are independent of the production process; hence, they are roughly valid for both lepton and hadron colliders, regardless of the centre-of-mass energy. However, the comparison with experimental data is not straightforward, since, unlike e^+e^- annihilation at the Z pole, the top and Higgs rest frames are not experimentally accessible.

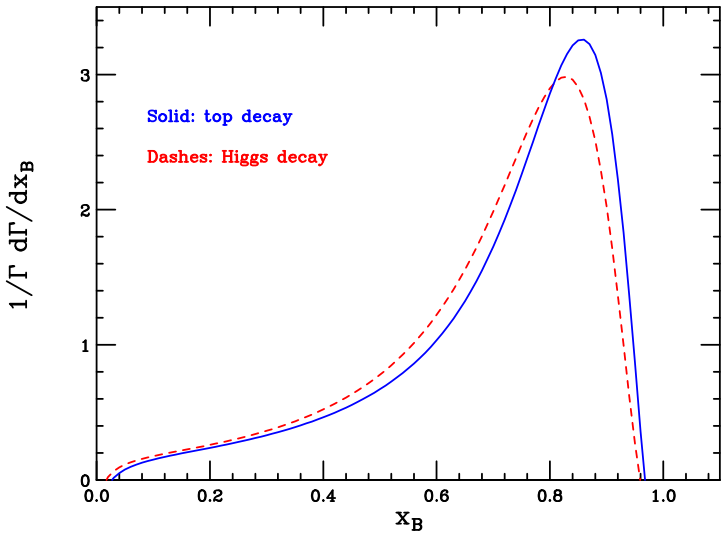


Figure 3. *B*-hadron spectrum in top (solid) and Higgs (dashes) decays, according to a NLO+NLL perturvative calculation and modelling non-perturbative corrections by means of the model (3), tuned to e^+e^- data.

While calculations such as those in [9,10] are quite inclusive and limited to the prediction of the x_B distributions, as discussed in the introduction, [30,31] provide a Monte Carlo code for top production and decay, in such a way that one is able to study other observables. At the moment, [31] included non-perturbative corrections, as those obtained the SCET analysis in [29], while [30] aims at a self-consistent tuning of non-perturbative fragmentation functions in the NNLO approximation for top decays with NNLL DGLAP evolution and possibly NNLL large-x resummation in the initial condition of the perturbative fragmentation function. Unlike [8,9], which worked out the evolution equations in the non-singlet-approximation, [30,31] also include the singlet contribution, namely gluon splitting. In particular, the work in [30] will provide the experimental community with a consistent set of NNLO non-perturbative fragmentation functions, which one will be able to use even in different contexts.

5. Results—*D* Production

In this section, we shall present a few results on charm-quark fragmentation and *D*-hadron production in the perturbative fragmentation framework. In fact, while the formalism is the same as for bottom-quark and *B*-hadron production, the scales involved are pretty different; therefore, a separate investigation is mandatory. Moreover, as far as e^+e^- annihilation is concerned, we have precise data both at *B*-factories taken by the CLEO [42] and BELLE [43] collaborations, as well at the *Z* pole, such as the ALEPH data in [44].

A thorough analysis of *D*-meson production was carried out in [7] at NLO+NLL, rescaling the Mellin variable N, as in (12), and obtaining good agreement with the data [42–44]. In this paper, I highlight the results in [23], where charmed mesons were investigated using a NLO perturbative calculation with NLL DGLAP evolution and NNLL threshold resummation, and modelling non-perturbative corrections via an effective coupling in the NNLO approximation, as in [22]. All perturbative parameters were varied in suitable ranges around the central values in Equation (13), but with the *b*-quark mass replaced by the charm one.

The work in [23] presents several predictions for *D*-meson production in e^+e^- annihilation with the effective coupling constant, varying the parameters in the perturbative calculation; hereafter, I present the results which yield the best comparison with the experimental data. In Figure 4, the ALEPH data [44] on D^{*+} production at LEP are shown, along with the purely perturbative calculation (dashed line) and the effective coupling

prediction obtained setting $\mu_{0F} = 2m_c$ and $m_c = 1.8$ GeV in the parton-level computation, which means a value of m_c close to the charmed-hadron mass. Figure 4 displays the strong impact of non-perturbative corrections and proves that the calculation relying on the effective coupling constant is capable of giving a good description of the data for $x_D < 0.85$ ($\chi^2/\text{dof} \simeq 27.18/17$, neglecting the correlations).

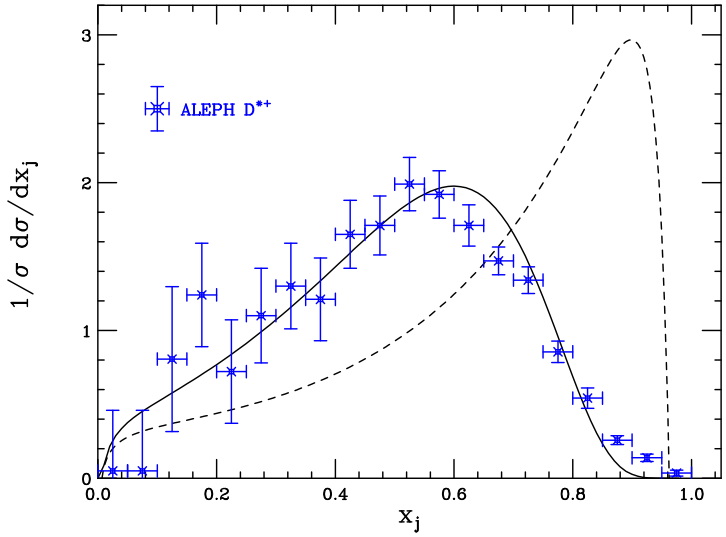


Figure 4. Experimental data on D^{*+} production at ALEPH, compared with a calculation in the NLO+(N)NLL approximation, as in [23], using an effective coupling constant (solid) and $\mu_{0F} = 2m_c$ and $m_c = 1.8$ GeV in the computation, with the other parameters set to the values described in the text. The dashed line corresponds to a parton-level prediction using the standard strong coupling constant and the same set of perturbative parameters.

As for the analysis at B-factories, Figure 5 presents the BELLE data on D^{*0} production at the $\Upsilon(4S)$ pole [43], with the best effective-coupling based prediction (solid line), setting $\mu_{0F} = 2m_c$ and $m_c = 1.8$ GeV, and the parton-level result (dashes). For $x_D < 0.85$, using the effective coupling constant, one obtains $\chi^2/\text{dof} \simeq 32.10/36$, neglecting the correlations between data points. As a whole, one can learn that at B-factories, the partonic spectrum exhibits an even sharper peak at large x with respect to LEP energies, which is due to the fact that at the $\Upsilon(4S)$ pole charm quarks are almost at rest and have very little phase space to radiate, so that they are mostly produced at $x \simeq 1$. The impact of the use of the effective coupling is extremely important, and fundamental to obtain a reasonable description of the data, after discarding a few data points at very large energies.

Still on the comparison with B-factories, [23] shows that serious discrepancies are instead present when comparing with CLEO data on D^0, D^{*0}, and D^{*+} production and with the D^{*+} spectrum measured by BELLE as well, which clearly indicates that the effective coupling model has to be improved, especially in the charm sector. Nevertheless, one finds that, within the errors, one is able of reproducing the Mellin moments of all considered data sets, as can be noticed in Figures 6–8. This result is remarkable, especially for the comparison with the BELLE and CLEO D^0 samples or the CLEO D^{*+} data, which was unsatisfactory in x-space. In fact, when computing the integrals for the Mellin moments, we have a compensation between regions where the theory prediction is higher, i.e., at middle values of x_D or around the peak, or below the data, i.e., at very large x_D. Furthermore, as thoroughly discussed in [23], the rough agreement in Figures 6–8 is clearly biased by the very large theory uncertainties and only a NNLO+NNLL investigation on $c\bar{c}$ production, possibly with the effective coupling constant as well, could decrease the theoretical errors and shed light on the comparison with the experimental moments. We just point out that a similar result was obtained in [36] for the purpose of the HERWIG [45] generator, which was not capable of reproducing the B-spectrum data in x-space, while its moments agreed with the experimental ones.

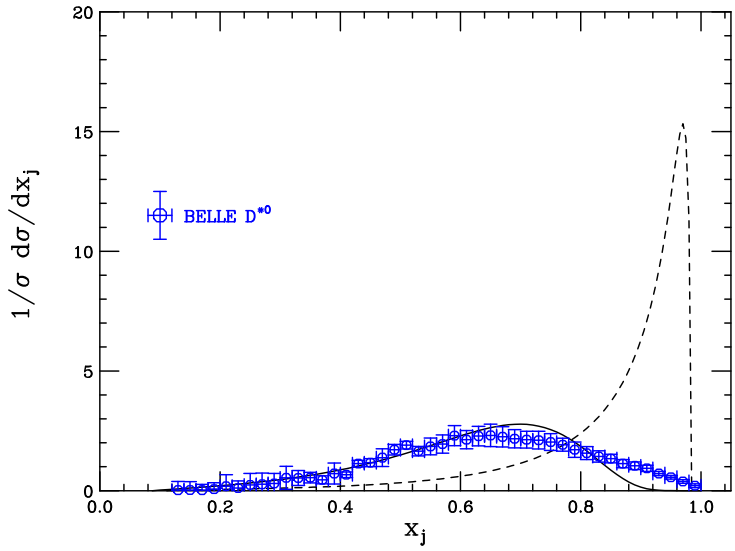


Figure 5. BELLE data on D^{*0} production at the $\Upsilon(4S)$ pole, compared with the perturbative calculation described in the text and in [23], using an effective (solid) and standard coupling constant (dashes). Perturbative parameters are set as in the results shown in Figure 4.

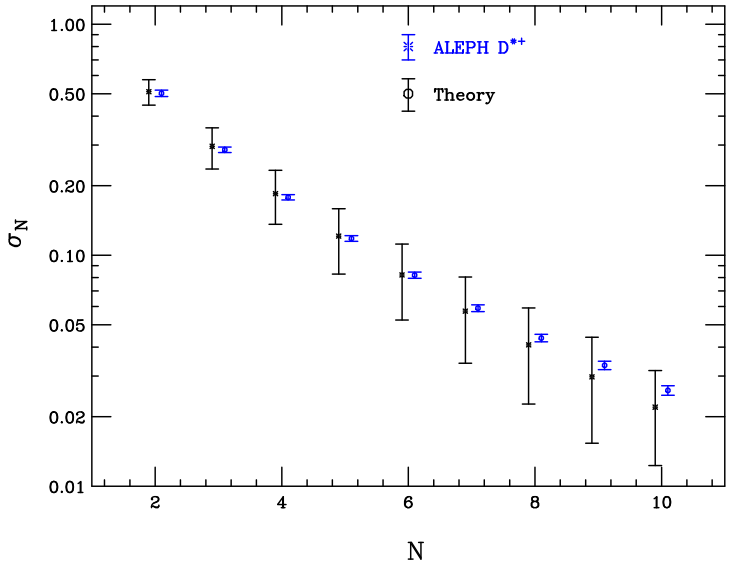


Figure 6. Charmed-hadron production cross-section in N-space according to the effective coupling model (denoted by 'Theory') described in the text, compared with the ALEPH moments for D^{*+} production. The theory errors are computed as discussed in [23].

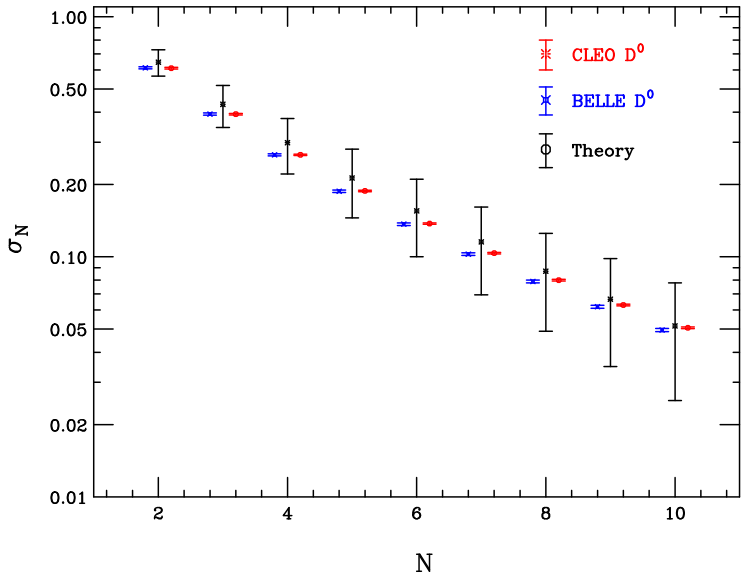


Figure 7. As in Figure 6, but comparing the effective coupling prediction with the BELLE and CLEO moments on D^0 production.

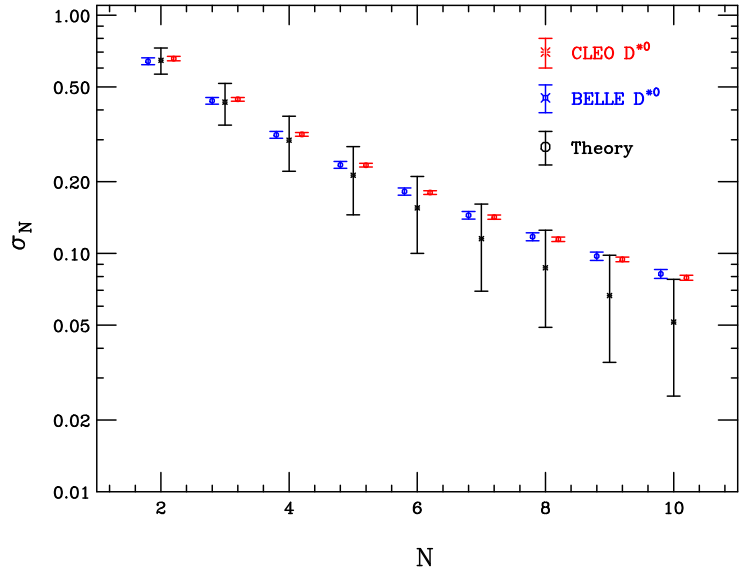


Figure 8. As in Figures 6 and 7, but with the theory moments faring against BELLE and CLEO data on D^{*+} production.

6. Conclusions

I presented an overview of a few selected results on heavy-quark fragmentation, mostly based on the perturbative fragmentation approach, which allows one to resum large logarithms of collinear origin. The investigation mainly dealt with bottom and charm production in e^+e^- annihilation and the possibility of including non-perturbative corrections through either a parameter-dependent hadronization model or an effective coupling constant, constructed in a given approximation.

Overall, I found that, using either a tuned hadronization model or an effective coupling, the data could be described with an acceptable precision in both x- and N-spaces. However, as far as the x-space comparison is concerned, unless one wishes to add extra tunable parameters, one has to discard data at very small or large x to achieve a reasonable fit. In fact, the perturbative calculation exhibits either terms which have not been resummed or

contributions for which a NLL or NNLL resummation is not sufficient to obtain a reliable prediction. An interesting feature of the comparison with charm-production data is that the prediction based on the effective coupling constant is not completely satisfactory in x-space, while it fares quite well against all experimental Mellin moments, which was interpreted as due to the compensation of discrepancies in different phase-space regions when performing x-space integrals.

In summary, this paper highlights a few relevant results in the field of heavy-quark fragmentation, but also stresses some points which deserve further investigation and extension. Above all, all ingredients to extend the parton-level calculation from NLO+NLL to NNLO+NNLL, where the logarithmic accuracy refers to both DGLAP evolution and threshold resummation, are available and should yield results pretty stable and reliable. In particular, it will be very interesting to understand how the $x \to 1$ part of the spectrum behaves at NNLO+NNLL and whether one still needs to perform the rescaling in (12) or discard a few data points to agree with the experimental measurements. Furthermore, there are small-x logarithms which affect the low-energy spectra and need to be resummed to have a sensitive prediction for $x \to 0$ too. Regarding the inclusion of non-perturbative power corrections, one can certainly use the model (3) or the effective strong coupling constant in conjunction with a NNLO+NNLL perturbative calculation. Moreover, one can even think of adding more parameters to the non-perturbative fragmentation function, as done, for example, in the context of parton distributions, ideally aiming at $\chi^2/\text{dof} \to 0$. Such an approach will be followed in the work in progress on top-pair production and decay at NNLO+NNLL in [30].

Funding: This research received no external funding.

Acknowledgments: The presented results were obtained in collaboration with Ugo Aglietti, Matteo Cacciari, Volker Drollinger, Giancarlo Ferrera, Federico Mescia, and Alexander Mitov.

Conflicts of Interest: The author declares no conflict of interest.

Notes

1 See also [4,5] for pioneering work on non-perturbative effects in Quantum Chromodynamics.

2 After performing a calculation in $d = 4 - 2\epsilon$ dimensions, the $\overline{\text{MS}}$ scheme is defined in such a way that the term $\sim (-1/\epsilon + \gamma_E - \ln(4\pi))$ is subtracted off.

3 The Mellin transform of a function $f(x)$, with $0 < x < 1$, is defined as $f_N = \int_0^1 dx x^{N-1} f(x)$.

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