universe

MDPI

Article

# On Quantum Representation of the Linear Canonical Wavelet Transform

**H. M. Srivastava [1,2,3,*], Firdous A. Shah [4] and Aajaz A. Teali [4]**

[1] Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada
[2] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
[3] Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, AZ1007 Baku, Azerbaijan
[4] Department of Mathematics, University of Kashmir, South Campus, Anantnag 192101, Jammu and Kashmir, India
* Correspondence: harimsri@math.uvic.ca

**Abstract:** For the efficient identification of quantum states, we propose the notion of linear canonical wavelet transform in the framework of quantum mechanics. Using the machinery of Dirac representation theory and integration within an ordered product of operators, we recast the linear canonical wavelet transform to a matrix element of the squeezing–displacing operator $\mathcal{U}(\mu, s)\,\mathcal{K}_M$ between analyzing vector $\langle\psi|$ and two-mode quantum state vector $|f\rangle$ to be transformed. We also derive the inner product relation and inversion formula for the linear canonical wavelet transform in the realm of quantum mechanics. Lastly, we present an explicit example for the lucid implementation of linear canonical wavelet transform in identifying the quantum states.

**Keywords:** wavelet; linear canonical transform; quantum states; Dirac representation; inversion formula

check for updates

**Citation:** Srivastava, H.M.; Shah, F.A.; Teali, A.A. On Quantum Representation of the Linear Canonical Wavelet Transform. *Universe* **2022**, *8*, 477. https://doi.org/10.3390/universe8090477

Academic Editor: Andreas Fring

Received: 27 July 2022
Accepted: 8 September 2022
Published: 12 September 2022



## 1. Introduction

The linear canonical transform (LCT) is a powerful analytical tool that embodies numerous integral transforms, such as the classical Fourier, fractional Fourier, Lorentz, and Fresnel transforms [1,2]. Mathematically, the linear canonical transform of any univariate function $f \in L^2(\mathbb{R})$ with respect to the parametric matrix $M = (A, B, C, D)$ is defined as

$$\mathscr{L}_M[f](x,p) = \int_{\mathbb{R}} f(x)\,\overline{\mathcal{K}_M(x,p)}\,dx, \tag{1}$$

where the LCT kernel $\mathcal{K}_M(x,p)$ is given by

$$\mathcal{K}_M(x,p) = \frac{1}{\sqrt{2\pi i B}}\exp\left\{\frac{-i}{2B}\left(Ax^2 - 2xp + Dp^2\right)\right\}, \quad B \neq 0. \tag{2}$$

The LCT is more flexible than other transforms due to the additional degrees of freedom and its simple geometrical manifestation, rendering it appropriate for investigating complex problems in sampling, optics, filter design, image processing, and quantum mechanics [3–5]. In quantum theory, these transforms are identified as the linear transformations that preserve the canonical commutation relations characterizing the coordinates and momenta operators as invariant [6]. They enable us to solve some classical problems and give us a clue to the quantisation of classical systems. In fact, they play a pivotal role in obtaining the solution of the Schrödinger equation or the Hamilton–Jacobi systems, which provides a bridge between classical and quantum mechanics [7]. By employing Dirac's symbolic method, a detailed correspondence between the quantum-optical and classical-optical transformations was briefly discussed in [8].

Wavelet transform is a substantial and potent time–frequency tool for analyzing non-transient signals and has been used in a number of disciplines, including signal processing, image processing, sampling theory, differential and integral equations, quantum mechanics, and medicine [9]. For any $f \in L^2(\mathbb{R})$, the wavelet transform of $f$ with respect to analyzing wavelet $\psi \in L^2(\mathbb{R})$ is defined by

$$\mathcal{W}_\psi[f](\mu, s) = \frac{1}{\sqrt{\mu}} \int_{\mathbb{R}} f(x)\, \overline{\psi\left(\frac{x-s}{\mu}\right)}\, dx, \quad \mu \in \mathbb{R}^+, s \in \mathbb{R}. \tag{3}$$

In the framework of quantum mechanical states, Fan et al. [10,11] employed Dirac's symbolic method to recast the wavelet transform (1) as a matrix element of the squeezing–translating operator between the mother wavelet state vector $\langle \psi |$ and the state vector $| f \rangle$ to be transformed. Subsequently, Hu and Fan [12] investigate the quantum wavelet transform, and established Parseval's inversion formulae by using the Dirac delta representation theory. This entangled–coherent state representation not only underlies the symplectic dilation mixed wavelet transform, but also helps in the formulation of the corresponding quantum transform operator, whose counterpart in classical optics is the lens-Fresnel mixed transform [13,14]. Song and his colleagues further developed this area in a series of articles where they presented a few novel representations of the classical wavelet transform in terms of different quantum chemical states by employing the bipartite entangled state representations and the technique of integration within an ordered product of operators [15–18].

The wavelet transform has been substantially important in capturing the local characteristics of nonstationary signals and has paved its way in diverse fields of science and engineering. However, the wavelet transform fails miserably to localize signals whose energy is weakly concentrated in the frequency domain, such as ubiquitous chirplike signals [19]. For the efficient analysis of such signals, it is both theoretically interesting and practically beneficial to intertwine the linear canonical and wavelet transforms by replacing the global kernel $\mathcal{K}_M(x, p)$ appearing in (1) with a generalized family of wavelets $\psi^M_{\mu,s}(x)$, where $M = (A, B, C, D)$ is the real unimodular matrix. Given a real unimodular matrix $M = (A, B, C, D)$, the linear canonical wavelet transform of any $f \in L^2(\mathbb{R})$ is defined as

$$\mathcal{W}^M_\psi[f](\mu, s) = \frac{1}{\sqrt{\mu}} \int_{\mathbb{R}} f(x)\, \overline{\psi\left(\frac{x-s}{\mu}\right) \mathcal{K}_M(x, p)}\, dx, \quad \mu \in \mathbb{R}^+, s \in \mathbb{R}, \tag{4}$$

where $\mathcal{K}_M(x, p)$ is given by (2). Integral Transform (4) inherits the excellent mathematical properties of the traditional wavelet and linear canonical transforms along with some of its own fascinating properties [20–22].

On the flip side, the development of quantum wavelet transforms have allowed for widespread applications in different aspects of signal and image processing, including quantum watermarking schemes and image formats that use them to extract the decompositional coefficients from a quantum image. Quantum processing tools have outperformed the conventional signal processing method, mainly due to the exorbitantly high processing and storage capacities. Keeping in view the good properties of the wavelet transform and the extra degrees of freedom of the linear canonical transforms, we are deeply motivated to study the linear canonical transform (4) in the context of quantum states. More precisely, we recast the linear canonical wavelet transform (4) in terms of the matrix elements of the squeezing–displacing operator $\mathcal{U}(\mu, s)\mathcal{K}_M$ between generalized mother wavelet vector $\langle \psi |$ and state vector $| f \rangle$ to be transformed by using the machinery of Dirac delta representation theory and integration within an ordered product of operators. Using the framework of quantum mechanics, we were able to derive the inner product relation and inversion formula for the linear canonical wavelet transform, which led us to a new orthogonal property of the linear canonical wavelets in the parameter space. We culminate our study by carrying out the numerical calculation of the linear canonical wavelet transform spectrum for the binomial state.

The rest of the article is structured as follows: Section 2 is concerned with the preliminary aspects of the linear canonical and wavelet transforms in the context of quantum mechanics. In Section 3, we present the quantum representation of the linear canonical wavelet transform and derive the corresponding inner product and inversion formulae. In Section 4, we establish the normal ordering of the linear canonical wavelet transform and carry out the numerical calculation. Some potential applications of the linear canonical wavelet transform are given in Section 5. Lastly, a conclusion is in Section 6.

## 2. Preliminaries

In this section, we give a brief overview of the conventional wavelet and linear canonical transforms in the context of quantum mechanics and their momentum representations. We first present an overview of quantum mechanics.

Quantum mechanics offers a conceptual and mathematical framework for the creation of physical theories and laws. As a result, one requires four axioms to establish a relationship between the real world and quantum mechanics, namely, state, observables, measurements, and dynamics. A "state" is a mathematical representation of the physical notion of a system's state, and "observables" represent a physical system's observable property that can theoretically be measured by self-adjoint operators. "Measurements" are described by a collection $\{M_n\}$ of measurement operators that act on the state space of the system being measured. For instance, if the state of the quantum system is $|\psi >$ immediately before the measurement, the probability that result m occurs is given by $P(n) =< \psi|M_n^\dagger M_n|\psi >$, with completeness equation $\sum_n M_n^\dagger M_n = 1$. Lastly, in "dynamics", a unitary transformation can be used to describe how a closed quantum system evolves. State $|\psi >$ of the system at time $t_1$ is related to the state $|\psi' >$ of the system at time $t_2$ by a unitary operator $U$ that depends only on the times $t_1$ and $t_2$; that is, $|\psi' >= U|\psi >$ . These axioms provide a perfectly acceptable general formulation of the quantum theory. Further reading on the foundation of quantum mechanics can be found in [23].

The Dirac representation of any function $f(x)$ can be written as $\langle x|f\rangle$, where $|f\rangle$ is the wave function of the state, and $\langle x|$ the eigenstate of coordinate $X$. In Fock space, $|x\rangle$ is expressed as

$$\big|x\big\rangle = \pi^{-1/4}\exp\left[-\frac{x^2}{2} + \sqrt{2}xa^\dagger - \frac{a^{\dagger 2}}{2}\right]|0\rangle, \quad X\big|x\big\rangle = x\big|x\big\rangle,\ X = \frac{a+a^\dagger}{\sqrt{2}}, \tag{5}$$

where $|0\rangle$ is the vacuum state annihilated by Bose annihilation operator $a$, $a|0\rangle = 0$, and $[a, a^\dagger] = 1$. Using the integration within an ordered product technique, Wavelet Transform (3) of any square integrable signal $f$ with respect to the mother wavelet $\psi$ can be recast as

$$\mathcal{W}_\psi[f](\mu, s) = \Big\langle \psi|\mathcal{U}(\mu, s)|f\Big\rangle, \tag{6}$$

where $\big\langle\psi\big|$ is the bra vector corresponding to mother wavelet $\psi$, $\big|f\big\rangle$ is the state to be transformed, and

$$\mathcal{U}(\mu, s) = \frac{1}{\sqrt{|\mu|}}\int_{\mathbb{R}} dx\left|\frac{x-s}{\mu}\right\rangle\langle x| \tag{7}$$

is the squeezing–translating operator. An application of the the normal product form of vacuum projector $|0\rangle\langle 0| =: \exp\left(-a^{\dagger 2}a\right)$ together with the technique of integration within an ordered product yields

$$\mathcal{U}(\mu, s) = K\exp\left[-\frac{a^{\dagger 2}}{2}\tanh\lambda - \frac{a^{\dagger 2}}{\sqrt{2}}\sec h\lambda\right]\ \exp\left[a^\dagger a\log\sec h\lambda\right]$$

$$\times \exp\left[\frac{a^2}{2}\tanh\lambda - \frac{as}{\sqrt{2}\mu}\sec h\lambda\right]. \tag{8}$$

For $s = 0$, Relation (7) reduces to squeezing operator

$$\mathcal{U}(\mu, 0) = \frac{1}{\sqrt{|\mu|}}\int_{\mathbb{R}} dx \left|\frac{x}{\mu}\right\rangle\langle x| = \exp\left[\frac{\lambda(a^2 - a^{\dagger 2})}{2}\right], \quad \mu = e^{\lambda}, \tag{9}$$

which maps classical dilation $x/\mu$ to the single-mode squeezing operator.

Similarly, the momentum eigenstate $|p\rangle$ of $P$ is given by

$$|p\rangle = \pi^{-1/4}\exp\left[-\frac{p^2}{2} + i\sqrt{2}pa^{\dagger} + \frac{a^{\dagger 2}}{2}\right]|0\rangle, \quad P|p\rangle = p|p\rangle, \quad P = \frac{a - a^{\dagger}}{i\sqrt{2}}. \tag{10}$$

Consequently, the momentum eigenstate of squeezing translating operator $\mathcal{U}(\mu, s)$ becomes

$$\mathcal{U}(\mu, s) = \sqrt{|\mu|}\int_{\mathbb{R}} e^{isp}|\mu p\rangle\langle p|\, dp = \exp\left[\frac{\lambda(a^2 - a^{\dagger 2})}{2}\right]e^{isP}, \tag{11}$$

where $\exp\left\{\lambda(a^2 - a^{\dagger 2})/2\right\}$ is the squeezing operator, and $e^{isP}$ is the displacing operator. Therefore, the momentum representation of Wavelet Transform (3) becomes

$$\mathcal{W}_{\psi}f(\mu, s) = \sqrt{|\mu|}\int_{\mathbb{R}} dpe^{isp}\langle\psi|\mu p\rangle\langle p|f\rangle. \tag{12}$$

Next, we express the linear canonical transform (1) in the realm of quantum mechanics. Since LCT kernel $\mathcal{K}_M(x, p)$ given by (2) is just a transition amplitude from position $x$ to position $p$, in the framework of quantum states, we can rewrite it as

$$\mathcal{K}_M(x, p) = \left\langle p|\mathcal{K}_M|x\right\rangle. \tag{13}$$

LCT operator $\mathcal{K}_M$ appearing on R.H.S (13) can be obtained by operating the completeness conditions

$$\int_{\mathbb{R}} dp|p\rangle\langle p| = 1, \quad \text{and} \quad \int_{\mathbb{R}} dx|x\rangle\langle x| = 1, \tag{14}$$

of coherent states on the both sides of LCT Kernel (13). Upon implementing the integration within ordered product and noticing that the normal ordering form of vacuum state projector $|0\rangle\langle 0| =: e^{-aa^{\dagger}}$ and $\langle p|e^{\frac{(i\pi a^{\dagger}a)}{2}} = \langle x|_{x=p}$, we can express LCT operator $\mathcal{K}_M$ appearing on R.H.S (13) as

$$\begin{aligned}
\mathcal{K}_M =: &\sqrt{\frac{2}{((A+D)+i(B-C))}}\exp\left[\frac{a^2\big((D-A)+i(B+C)\big)}{2\big((A+D)+i(B-C)\big)}\right] \\
&: \exp\left[a^{\dagger}a\Big(\frac{i\pi}{2} + \frac{2}{((A+D)+i(B-C))} - 1\Big)\right] \\
&: \exp\left[\frac{(a^{\dagger})^2\big((A-D)+i(B+C)\big)}{2\big((A+D)+i(B-C)\big)}\right] \\
= {} & C_1 \,:\, \exp\left[a^{\dagger}a\Big(\frac{i\pi}{2} + C_2\Big)\right] \,:\, C_3
\end{aligned} \tag{15}$$

where

$$C_1 = \sqrt{\frac{2}{((A+D)+i(B-C))}}, \quad C_2 = \frac{2}{((A+D)+i(B-C))} - 1 \quad \text{and}$$
$$C_3 = \exp\left[\frac{(a^\dagger)^2((A-D)+i(B+C))}{2((A+D)+i(B-C))}\right].$$

We summarize the above discussion in the following definition of the linear canonical transform in the context of quantum mechanics.

**Definition 1.** *Given a real, unimodular matrix $M = (A, B, C, D)$, the quantum linear canonical transform of any signal $\langle x|f\rangle$ is defined by*

$$\mathscr{L}_M[f](p) = \Big\langle p|\mathcal{K}_M|f\Big\rangle, \tag{16}$$

*where $\mathcal{K}_M$ is the LCT operator given by (15).*

The inversion formula corresponding to (16) is given by

$$f(x) = \mathscr{L}^{-1}\Big(\mathscr{L}_M[f(x)](p)\Big)(x) = \int_{\mathbb{R}} dp\, \langle x|\mathcal{K}_M^\dagger|p\rangle\langle p|\mathcal{K}_M|f\rangle. \tag{17}$$

## 3. Quantum Representation of the Linear Canonical Wavelet Transform

In this section, we introduce the notion of linear canonical wavelet transform in the framework of quantum states. We also establish an orthogonality formula between two signals and their respective linear canonical wavelet transforms. As a consequence of this formula, we can deduce the resolution of identity for the proposed linear canonical wavelet transform. We lastly derive the inversion formula for the proposed transform.

Having formulated the quantum representation of classical Wavelet Transform (6) and Linear Canonical Transform (16), we now introduce the formal definition of the linear canonical wavelet transform in the context of quantum states.

**Definition 2.** *Given a real, unimodular matrix $M = (A, B, C, D)$, the quantum linear canonical wavelet transform of any function $\langle x|f\rangle$ with respect to mother wavelet $\psi$ is defined by*

$$\mathcal{W}_\psi^M[f](\mu, s) = \Big\langle \psi|\,\mathcal{U}(\mu, s)\mathcal{K}_M\,|f\Big\rangle, \tag{18}$$

*where $\mathcal{U}(\mu, s)\mathcal{K}_M$ is the generalized linear canonical wavelet operator given by*

$$\mathcal{U}(\mu, s)\mathcal{K}_M = \sqrt{|\mu|}\int_{\mathbb{R}} dp\, e^{isp}|\mu p\rangle\langle p|\, C_1 \exp\left[a^\dagger a\left(\frac{i\pi}{2} + C_2\right)\right] C_3, \tag{19}$$

*and $C_1, C_2$ and $C_3$ have the usual meanings.*

Definition 2 allows for he following comments:

(i). For parametric set $M = (0, 1, -1, 0)$, Transform (18) boils down to the existing classical wavelet transforms:

$$\mathcal{W}_\psi^M[f](\mu, s) = \Big\langle \psi|\, e^{-i\pi/4}\mathcal{U}(\mu, s)|f\Big\rangle, \tag{20}$$

(ii). For parametric set $M = (\cos\theta, \sin\theta, -\sin\theta, \cos\theta)$, Linear Canonical Wavelet Transform (18) reduces to the joint fractional wavelet transform [18]:

$$\mathcal{W}_\psi^M[f](\mu, s) = \left\langle \psi|\, e^{-i\theta/2}\, e^{a^\dagger a\left(\frac{i\pi}{2} + e^{-i\theta} - 1\right)}\mathcal{U}(\mu, s)|f\right\rangle, \tag{21}$$

**Theorem 1 (Inner Product Relation).** *Let $\mathcal{W}_{\psi}^{M}[f](\mu,s)$ and $\mathcal{W}_{\psi}^{M}[g](\mu,s)$ be the linear canonical wavelet transforms of $\langle x|f\rangle$ and $\langle x|g\rangle$, respectively, with respect to the analysing wavelet $\psi$ and real uni-modular matrix $M=(A,B,C,D)$. Then, we have*

$$\int_{\mathbb{R}}\frac{d\mu}{\mu^2}\int_{\mathbb{R}}ds\,\mathcal{W}_{\psi}^{M}[f](\mu,s)\,\overline{\Big(\mathcal{W}_{\psi}^{M}[g](\mu,s)\Big)}=4\pi\,C_{\psi}\langle g|f\rangle, \tag{22}$$

*where $C_{\psi}$ is the admissibility condition given by*

$$C_{\psi}=\int_{\mathbb{R}}dp\left(\frac{|\langle p|\psi\rangle|}{p}\right)<\infty. \tag{23}$$

**Proof.** By invoking the completeness property $\int_{\mathbb{R}}dp|p\rangle\langle p|=1$ along with

$$\int_{\mathbb{R}}ds\,U^{\dagger}(\mu,s)|p'\rangle\langle p|U(\mu,s)=2\pi B\,\delta(p-p')\Big|\frac{p'}{\mu}\Big\rangle\Big\langle\frac{p}{\mu}\Big|,$$

where $\delta(p-p')=\int_{\mathbb{R}}dx\exp\{ix(p-p')\}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}}\frac{d\mu}{\mu^2}\int_{\mathbb{R}}ds\,\mathcal{W}_{\psi}^{M}[f](\mu,s)\Big(\mathcal{W}_{\psi}^{M}[g](\mu,s)\Big)^{*}\\
&=\int_{\mathbb{R}}\frac{d\mu}{\mu^2}\int_{\mathbb{R}}ds\Big\langle\psi|\mathcal{U}(\mu,s)\mathcal{K}_M|f\Big\rangle\Big\langle g|\mathcal{K}_M^{\dagger}\mathcal{U}^{\dagger}(\mu,s)|\psi\Big\rangle\\
&=\int_{\mathbb{R}}\frac{d\mu}{\mu^2}\int_{\mathbb{R}}ds\int_{\mathbb{R}}dp\int_{\mathbb{R}}dp'\langle\psi|p\rangle\Big\langle p|\mathcal{U}(\mu,s)\mathcal{K}_M|f\Big\rangle\Big\langle g|\mathcal{K}_M^{\dagger}\mathcal{U}^{\dagger}(\mu,s)|p'\Big\rangle\langle p'|\psi\rangle\\
&=\int_{\mathbb{R}}\frac{d\mu}{\mu^2}\int_{\mathbb{R}}dp\int_{\mathbb{R}}dp'\langle\psi|p\rangle\langle p'|\psi\rangle|f\rangle\langle g|\int_{\mathbb{R}}ds\,\mathcal{U}(\mu,s)\mathcal{K}_M|p'\Big\rangle\Big\langle p|\mathcal{K}_M^{\dagger}\mathcal{U}^{\dagger}(\mu,s)\\
&=2\pi B\int_{\mathbb{R}}\frac{d\mu}{\mu^2}\int_{\mathbb{R}}dp\int_{\mathbb{R}}dp'\langle\psi|p\rangle\langle p'|\psi\rangle\delta(p-p')|f\rangle\langle g\Big|\frac{p'}{\mu}\Big\rangle\Big\langle\frac{p}{\mu}\Big|\\
&=2\pi B\int_{\mathbb{R}}\frac{d\mu}{\mu^2}\int_{\mathbb{R}}dp\langle\psi\Big|\frac{p}{\mu}\Big\rangle\Big\langle\frac{p}{\mu}\Big|\psi\rangle\langle g|p\rangle\langle p|f\rangle\\
&=2\pi B\int_{\mathbb{R}}dp\int_{-\infty}^{0}\frac{d\mu}{\mu^2}\langle\psi\Big|\frac{p}{\mu}\Big\rangle\Big(\langle\psi\Big|\frac{p}{\mu}\Big\rangle\Big)^{*}\langle g|p\rangle\langle p|f\rangle\\
&\qquad+2\pi B\int_{\mathbb{R}}dp\int_{0}^{\infty}\frac{d\mu}{\mu^2}\langle\psi\Big|\frac{p}{\mu}\Big\rangle\Big(\langle\psi\Big|\frac{p}{\mu}\Big\rangle\Big)^{*}\langle g|p\rangle\langle p|f\rangle\\
&=2\pi B\int_{\mathbb{R}}dp\int_{0}^{\infty}\frac{d\mu}{\mu}\Big|\langle(-\mu p)|\psi\rangle\Big|^{2}\langle g|p\rangle\langle p|f\rangle\\
&\qquad+2\pi B\int_{\mathbb{R}}dp\int_{0}^{\infty}\frac{d\mu}{\mu}\Big|\langle(\mu p)|\psi\rangle\Big|^{2}\langle g|p\rangle\langle p|f\rangle\\
&=2\pi B\int_{\mathbb{R}}dp\,C_{\psi}\langle g|p\rangle\langle p|f\rangle+2\pi B\int_{\mathbb{R}}dp\,C_{\psi}\langle g|p\rangle\langle p|f\rangle\\
&=4\pi BC_{\psi}\int_{\mathbb{R}}dp|p\rangle\langle p|\langle g|f\rangle\\
&=4\pi B\,C_{\psi}\langle g|f\rangle,
\end{aligned}$$

where $C_{\psi}=\int_{0}^{\infty}\frac{d\mu}{\mu}\Big|\langle(-\mu p)|\psi\rangle\Big|^{2}\equiv\int_{0}^{\infty}\frac{dp}{p}\Big|\langle p|\psi\rangle\Big|^{2}<\infty$ is admissibility condition.

This completes the proof of Theorem 1. □

**Corollary 1.** *For $|f\rangle = |g\rangle$, Orthogonality Relation (22) yields*

$$\int_{\mathbb{R}} \frac{d\mu}{\mu^2} \int_{\mathbb{R}} ds \Big| \mathcal{W}_\psi^M[f](\mu,s) \Big|^2 = 4\pi B\, C_\psi \langle f|f\rangle. \tag{24}$$

*Remarks:* (i). Identity (24) is called the energy-preserving relation; thus, Linear Canonical Wavelet Transform (18) is an isometry from the space of signals to the space of transforms with $C_\psi = 1$.

(ii). For $|f\rangle = |x\rangle$ and $|g\rangle = |x'\rangle$, Relation (22) implies that

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{d\mu}{\mu^2} \int_{\mathbb{R}} ds\, \mathcal{W}_\psi^M[x](\mu,s) \Big(\mathcal{W}_\psi^M[x'](\mu,s)\Big)^* \\
&\qquad = \int_{\mathbb{R}} \frac{d\mu}{\mu^2} \int_{\mathbb{R}} ds \Big\langle \psi|\mathcal{U}(\mu,s)\mathcal{K}_M|x\Big\rangle \Big\langle x'|\mathcal{K}_M^\dagger \mathcal{U}^\dagger(\mu,s)|\psi\Big\rangle \\
&\qquad = 2\pi B \int_{\mathbb{R}} \frac{d\mu}{\mu^2} \int_{\mathbb{R}} ds \Big\langle \psi|\mathcal{U}(\mu,s)|x\Big\rangle \Big\langle x'|\mathcal{U}^\dagger(\mu,s)|\psi\Big\rangle \\
&\qquad = 4\pi B\, C_\psi \langle x|x'\rangle \\
&\qquad = 4\pi B\, C_\psi\, \delta(x-x').
\end{aligned} \tag{25}$$

(ii). If $|f\rangle = |g\rangle = |n\rangle$, a number state with $\langle n|n\rangle = 1$ then (22) becomes

$$\int_{\mathbb{R}} \frac{d\mu}{\mu^2} \int_{\mathbb{R}} ds \Big| \mathcal{W}_\psi^M[n](\mu,s) \Big|^2 = 4\pi B\, C_\psi. \tag{26}$$

This implies that constant $C_\psi$ is state-independent.

The next theorem guarantees the reconstruction of the input signal from the corresponding quantum Linear Canonical Wavelet Transform (18).

**Theorem 2 (Inversion Formula).** *Any state $\langle f|$ can be reconstructed from the corresponding linear canonical wavelet transform $\mathcal{W}_\psi^M[f](\mu,s)$ via the following formula:*

$$\langle x|f\rangle = \frac{1}{4\pi B\, C_\psi} \int_{\mathbb{R}} \frac{d\mu}{\mu^2\sqrt{\mu}} \int_{\mathbb{R}} ds \left\langle \frac{x-s}{\mu}\Big|\psi\right\rangle \mathcal{K}_M^\dagger\, \mathcal{W}_\psi^M[f](\mu,s). \tag{27}$$

**Proof.** We have

$$\Big\langle x|\mathcal{U}^\dagger(\mu,s)\mathcal{K}_M^\dagger = \Big\langle x\Big| \frac{1}{\sqrt{|\mu|}} \int_{\mathbb{R}} dx' \Big| x'\Big\rangle \left\langle \frac{x'-s}{\mu}\right|\mathcal{K}_M^\dagger = \frac{1}{\sqrt{|\mu|}} \left\langle \frac{x-s}{\mu}\right|\mathcal{K}_M^\dagger. \tag{28}$$

An application of Orthogonality Relation (22) along with (28) and taking $\langle g| = \langle x|$, we obtain

$$\begin{aligned}
4\pi B\, C_\psi \langle x|f\rangle &= \int_{\mathbb{R}} \frac{d\mu}{\mu^2} \int_{\mathbb{R}} ds\, \mathcal{W}_\psi^M[f](\mu,s) \Big(\langle \psi|\mathcal{U}(\mu,s)\mathcal{K}_M|x\rangle\Big)^* \\
&= \int_{\mathbb{R}} \frac{d\mu}{\mu^2} \int_{\mathbb{R}} ds\, \mathcal{W}_\psi^M[f](\mu,s)|\psi\rangle\langle x|\mathcal{U}^\dagger(\mu,s)\mathcal{K}_M^\dagger \\
&= \int_{\mathbb{R}} \frac{d\mu}{\mu^2} \int_{\mathbb{R}} ds\, \mathcal{W}_\psi^M[f](\mu,s)|\psi\rangle \frac{1}{\sqrt{|\mu|}} \left\langle \frac{x-s}{\mu}\right|\mathcal{K}_M^\dagger \\
&= \int_{\mathbb{R}} \frac{d\mu}{\mu^2\sqrt{|\mu|}} \int_{\mathbb{R}} ds\, \mathcal{W}_\psi^M[f](\mu,s)|\psi\rangle \left\langle \frac{x-s}{\mu}\right|\mathcal{K}_M^\dagger.
\end{aligned}$$

Equivalently,

$$\langle x|f\rangle = \frac{1}{4\pi B\, C_\psi} \int_{\mathbb{R}} \frac{d\mu}{\mu^2\sqrt{\mu}} \int_{\mathbb{R}} ds \left\langle \frac{x-s}{\mu}\Big|\psi\right\rangle \mathcal{K}_M^\dagger\, \mathcal{W}_\psi^M[f](\mu,s).$$

This completes the proof of Theorem 2. □

## 4. An Example

In this section, we visualize how Linear Canonical Wavelet Transform (18) can be implemented for better analyzing and identifying quantum states. To facilitate this, we derive the normal ordering form of Linear Canonical Operator (19). We observe that

$$\langle p|\exp\left[a^\dagger a(\frac{i\pi}{2}+C_2)\right]=\pi^{-1/4}\langle 0|\exp\left[-\frac{p^2}{2}+\sqrt{2}pa\,e^{\overline{C_2}}-\frac{a^2}{2}\,e^{\overline{C_2}}\right].$$

The normal ordering of Linear Canonical Wavelet Operator (19) can be obtained by employing the technique of integration within an ordered product of operators as

$$\begin{aligned}
&\mathcal{U}(\mu,s)\exp\left[a^\dagger a\left(\frac{i\pi}{2}+C_2\right)\right]\\
&=\sqrt{|\mu|}\int_{\mathbb{R}}dp\,e^{isp}|\mu p\rangle\langle p|\exp\left[a^\dagger a\left(\frac{i\pi}{2}+C_2\right)\right]\\
&=\sqrt{\frac{|\mu|}{\pi}}\int_{\mathbb{R}}dp\,e^{isp}\exp\left[-\frac{\mu^2p^2}{2}+i\sqrt{2}\mu p a^\dagger+\frac{a^{\dagger 2}}{2}\right]|0\rangle\\
&\qquad\times\langle 0|\exp\left[-\frac{p^2}{2}+\sqrt{2}pa\,e^{\overline{C_2}}-\frac{a^2}{2}e^{\overline{C_2}}\right]\\
&=\sqrt{\frac{|\mu|}{\pi}}\int_{\mathbb{R}}dp\,\exp\left[-p^2\frac{(\mu^2+1)}{2}+p\left(is+i\sqrt{2}\mu a^\dagger-\sqrt{2}ae^{2\overline{C_2}}\right)\right]\\
&\qquad\times\exp\left[-a^\dagger a+\frac{a^{\dagger 2}}{2}-\frac{a^2e^{\overline{C_2}}}{2}\right]\\
&=\sqrt{\frac{|\mu|}{\pi}}\exp\left[-a^\dagger a+\frac{a^{\dagger 2}}{2}-\frac{a^2e^{\overline{C_2}}}{2}\right]\sqrt{\frac{2\pi}{(\mu^2+1)}}\exp\left[\frac{(is+i\sqrt{2}\mu a^\dagger-\sqrt{2}ae^{2\overline{C_2}})^2}{2(\mu^2+1)}\right]\\
&=\sqrt{\frac{2|\mu|}{(\mu^2+1)}}\exp\left[\frac{a^{\dagger 2}(1-\mu^2)-2\sqrt{2}s\mu a^\dagger}{2(\mu^2+1)}+\frac{(1-\mu^2)a^2e^{2\overline{C_2}}-s^2+2\sqrt{2}isae^{\overline{C_2}}}{2(1+\mu^2)}\right]\\
&\qquad\times\exp\left[a^\dagger a\left(\frac{2i\mu e^{2\overline{C_2}}}{(\mu^2+1)}-1\right)\right].
\end{aligned}$$

Setting $A=\cosh\xi+\cos\theta\sinh\xi$, $B=\sin\theta$, $C=\sin\theta\sinh\xi$, $D=\cosh\xi-\cos\theta\sinh\xi$, $\mu=e^\lambda$, $\operatorname{sech}\lambda=\frac{2\mu}{1+\mu^2}$, and $\tanh\xi=\frac{\mu^2+1}{\mu^2-1}$, we obtain

$$C_1=\sqrt{\operatorname{sech}\xi}\exp\left[\frac{a^2}{2}(-e^{-i\theta}\tanh\xi)\right],\ C_2=\operatorname{sech}\xi,\ \text{and}\ C_3=\exp\left[\frac{a^\dagger}{2}(e^{i\theta}\tanh\xi)\right].$$

Therefore, the normal ordering form of Unitary Operator (19) follows

$$\begin{aligned}
\mathcal{U}(\mu,s)\mathcal{K}_M=&\sqrt{\operatorname{sech}\xi}\exp\left[\frac{a^2}{2}(-e^{-i\theta}\tanh\xi)\right]\\
&\times\sqrt{\lambda}\exp\left[a^\dagger a(\lambda e^{2\overline{\operatorname{sech}\xi}}-1)+\frac{a^{\dagger 2}}{2\tanh\lambda}-\frac{\lambda s a^\dagger}{\sqrt{2}}\right]\\
&\times\exp\left[\frac{a^2e^{2\overline{\operatorname{sech}\xi}}}{2\tanh\xi}-\frac{s^2}{2(1+e^{2\lambda})}+\frac{\sqrt{2}isae^{\overline{\operatorname{sech}\xi}}}{1+e^{2\lambda}}\right]\exp\left[\frac{a^\dagger}{2}(e^{i\theta}\tanh\xi)\right].
\end{aligned}\tag{29}$$

For the demonstration of the quantum Linear Canonical Wavelet Transform (18) and its implementation, we present an illustrative example.

**Example 1.** *Consider the second derivative of Gaussian function*

$$\langle x|\psi\rangle = (1-x^2)\,e^{-x^2/2}, \tag{30}$$

*which satisfies admissibility condition* $C_\psi = \int_0^\infty dp\,\frac{|\psi(p)|^2}{p} = \frac{1}{2}$. *Then, the corresponding state vector in Fock space is given by*

$$|\psi\rangle = \frac{\pi^{1/4}}{2}\Big(1-a^{\dagger^2}\Big)\big|0\big\rangle. \tag{31}$$

Using the normal Ordering (29) of Unitary Operator (19), the vacuum state $|0\rangle$ of the linear canonical wavelet transform can be obtained as

$$\begin{aligned}
&\langle\psi|\mathcal{U}(\mu,s)\mathcal{K}_M|0\rangle \\
&=: \langle\psi|\sqrt{\operatorname{sech}\xi}\exp\left[\frac{a^2}{2}\big(-e^{-i\theta}\tanh\xi\big)\right]\sqrt{\lambda}\exp\left[a^\dagger a\big(\lambda e^{2\,\overline{\operatorname{sech}\xi}}-1\big)+\frac{a^{\dagger 2}}{2\tanh\lambda}-\frac{\lambda s a^\dagger}{\sqrt{2}}\right] \\
&\quad\times\exp\left[\frac{a^2e^{2\,\overline{\operatorname{sech}\xi}}}{2\tanh\xi}-\frac{s^2}{2(1+e^{2\lambda})}+\frac{\sqrt{2}isae^{\overline{\operatorname{sech}\xi}}}{1+e^{2\lambda}}\right]\exp\left\{\frac{a^\dagger}{2}\big(e^{i\theta}\tanh\xi\big)\right\}|0\rangle \\
&= \frac{\pi^{1/4}}{2}(1-a^2)\langle 0|\sqrt{\lambda\operatorname{sech}\xi} \\
&\quad\times\exp\left[\frac{a^2}{2}\big(-e^{-i\theta}\tanh\xi\big)\right]\exp\left\{a^\dagger a\big(\lambda e^{2\,\overline{\operatorname{sech}\xi}}-1\big)+\frac{a^{\dagger 2}}{2\tanh\lambda}-\frac{\lambda s a^\dagger}{\sqrt{2}}\right\} \\
&\quad\times\exp\left[\frac{a^2e^{2\,\overline{\operatorname{sech}\xi}}}{2\tanh\xi}-\frac{s^2}{2(1+e^{2\lambda})}+\frac{\sqrt{2}isae^{\overline{\operatorname{sech}\xi}}}{1+e^{2\lambda}}\right]\exp\left[\frac{a^\dagger}{2}\big(e^{i\theta}\tanh\xi\big)\right]|0\rangle \\
&= \frac{\pi^{1/4}}{2}(1-a^2)\sqrt{\lambda\operatorname{sech}\xi}\exp\left[\frac{a^2}{2}\big(-e^{-i\theta}\tanh\xi\big)\right] \\
&\quad\times\exp\left[a^\dagger a\big(\lambda e^{2\,\overline{\operatorname{sech}\xi}}-2\big)+\frac{a^{\dagger 2}}{2\tanh\lambda}-\frac{\lambda s a^\dagger}{\sqrt{2}}\right] \\
&\quad\times\exp\left[\frac{a^2e^{2\,\overline{\operatorname{sech}\xi}}}{2\tanh\xi}-\frac{s^2}{2(1+e^{2\lambda})}+\frac{\sqrt{2}isae^{\overline{\operatorname{sech}\xi}}}{1+e^{2\lambda}}\right]\exp\left[\frac{a^\dagger}{2}\big(e^{i\theta}\tanh\xi\big)\right].
\end{aligned} \tag{32}$$

Similarly, single-particle state $|1\rangle$ of the linear canonical wavelet transform is given by

$$\begin{aligned}
&\Big\langle\psi|\mathcal{U}(\mu,s)\mathcal{K}_M|1\Big\rangle \\
&=: \frac{\pi^{1/4}}{2}(1-a^2)\Big\langle 0\Big|\sqrt{\lambda\operatorname{sech}\xi}\exp\left[\frac{a^2}{2}\big(-e^{-i\theta}\tanh\xi\big)\right] \\
&\quad\times\exp\left[a^\dagger a\big(\lambda e^{2\,\overline{\operatorname{sech}\xi}}-1\big)+\frac{a^{\dagger 2}}{2\tanh\lambda}-\frac{\lambda s a^\dagger}{\sqrt{2}}\right] \\
&\quad\times\exp\left[\frac{a^2e^{2\,\overline{\operatorname{sech}\xi}}}{2\tanh\xi}-\frac{s^2}{2(1+e^{2\lambda})}+\frac{\sqrt{2}isae^{\overline{\operatorname{sech}\xi}}}{1+e^{2\lambda}}\right]\exp\left[\frac{a^\dagger}{2}\big(e^{i\theta}\tanh\xi\big)\right]\Big|1\Big\rangle.
\end{aligned} \tag{33}$$

## 5. Potential Application

The wavelet transforms play a vital tool in many fields, including information retrieval, signal coding, watermarking, compression, and encryption; however, it is not suitable for lossless applications and is often computationally complex. In contrast to this, at the quantum level, the wavelet transform leads to greatly improved computational efficiency for both univariate and multivariate signals. As such, the development of the quantum wavelet transform has also allowed for diversified applications across different fields, including quantum watermarking schemes and image formats that use them to extract decomposition coefficients from a quantum image. Through simulations, these schemes outperform some traditional watermarking ideas. Additionally, quantum encryption, data compression, and image denoising based on quantum wavelet transform show better visual quality than some classical methods do. In view of the fact that the linear canonical wavelet transform is much more reliable than the classical wavelet transform, the underlying quantum mechanical variant is of utmost importance in different aspects of signal processing, such as quantum image processing, quantum data compression, quantum encryption, denoising, and information retrieval. The present study also stimulates interest in future developments, including the formulation of fast algorithms and efficient circuits for the linear canonical quantum wavelet transform. This can be achieved by assuming an efficient quantum circuit for a given wavelet kernel and starting with a high-level description of the packet and pyramid algorithms to analyze the feasibility and efficiency of the implementation of the packet and pyramid algorithms by using the given wavelet kernel. Other areas of interest include the development of quantum algorithms for signal decomposition and lossless compression.

## 6. Conclusions

In the present article, we introduced the notion of a kernel-based linear canonical wavelet transform in the framework of quantum mechanics besides recasting the proposed transform in terms of the matrix element of squeezing translating operator $\mathcal{U}(\mu, s)\mathcal{K}_M$ between generalized vector $\langle\psi|$ and state vector $|f\rangle$ to be transformed. Moreover, we established the inner product and inversion formula for the linear canonical wavelet transform by virtue of Dirac representation theory and integration within an ordered product of operators. Lastly, an example was presented for identifying the quantum states, and some potential applications were given.

**Author Contributions:** Writing—review and editing, H.M.S.; conceptualization, methodology, supervision, F.A.S.; writing—original draft, investigation and editing, A.A.T.; methodology and software, F.A.S. and A.A.T. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

1. Moshinsky, M.; Quesne, C. Linear canonical transformations and their unitary representations. *J. Math. Phys.* **1971**, *12*, 1772–1780. [CrossRef]
2. Healy, J.J.; Kutay, M.A.; Ozaktas, H.M.; Sheridan, J.T. *Linear Canonical Transforms*; Springer: New York, NY, USA, 2016.
3. Xu, T.Z.; Li, B.Z. *Linear Canonical Transform and Its Applications*; Science Press: Beijing, China, 2013.
4. Barshan, B.; Kutay, M.A.; Ozaktas, H.M. Optimal filtering with linear canonical transformations. *Opt. Commun.* **1997**, *135*, 32–36. [CrossRef]
5. Pei, S.C.; Ding, J.J. Eigen functions of linear canonical transform. *IEEE Trans. Signal Process.* **2002**, *50*, 11–26.
6. Yeon, K.H.; Walls, D.F.; Um, C.I.; George, T.F.; Pandey, L.N. Quantum correspondence for linear canonical transformations on general Hamiltonian systems. *Phys. Rev. A* **1998**, *58*, 1765–1774. [CrossRef]
7. Ranaivoson, R.T.; Andriamblololona, R.; Rakotoson, H.; Raboanary, R. Linear canonical transformations in relativistic quantum physics. *Phys. Scr.* **2021**, *96*, 065204. [CrossRef]
8. Fan, H.Y.; Hu, L.Y. Correspondence between quantum-optical transform and classical-optical transform explored by developing Dirac's symbolic method. *Front. Phys.* **2012**, *7*, 261–310. [CrossRef]
9. Debnath, L.; Shah, F.A. *Wavelet Transforms and Their Applications*; Birkhäuser: New York, NY, USA, 2015.

10. Fan, H.; Lu, H.L.; Fan, Y. Newton-Leibniz integration for ket–bra operators in quantum mechanics and derivation of entangled state representations. *Ann. Phys.* **2006**, *321*, 480–494. [CrossRef]
11. Fan, H.Y.; Lu, H.L. Mother wavelets for complex wavelet transform derived by Einstein-Podolsky-Rosen entangled state representation. *Opt. Lett.* **2007**, *32*, 554–556. [CrossRef] [PubMed]
12. Hu, L.Y.; Fan, H.Y. Wavelet transform in the context of quantum mechanics and new orthogonal property of mother wavelets in parameter space. *J. Mod. Opt.* **2008**, *55*, 1835–1844. [CrossRef]
13. Fan, H.Y.; Liu, S.G. Entangled symplectic wavelet transformation. *Opt. Lett.* **2007**, *32*, 1507–1509. [CrossRef] [PubMed]
14. Fan, H.; Liu, S.; Hu, H. Symplectic-dilation mixed wavelet transform and its correspondence in quantum optics. *Opt. Lett.* **2009**, *34*, 551–553. [CrossRef] [PubMed]
15. Song, J.; Fan, H.; Xu, Y.; Zhou, J. Complex wavelet transform of the Bell states. *Int. J. Theor. Phys.* **2011**, *50*, 1946–1952. [CrossRef]
16. Song, J.; fan, H.; Yaun, H. Wavelet transform of quantum chemical states. *Int. J. Quantum Chem.* **2012**, *112*, 2343–2347. [CrossRef]
17. Song, J.; Zhou, J.; Yuan, H.; He, R.; Fan, H. Complex wavelet transform of the two-mode quantum states. *Int. J. Theor. Phys.* **2016**, *55*, 3597–3606. [CrossRef]
18. Song, J.; He, R.; Yuan, H.; Zhou, J.; Fan, H.Y. Joint wavelet-fractional Fourier transform. *Chin. Phys. Lett.* **2016**, *33*, 110302 [CrossRef]
19. Debnath, L.; Shah, F.A. *Lecture Notes on Wavelet Transforms*; Birkhäuser: Boston, MA, USA, 2017.
20. Srivastava, H.M.; Shah, F.A.; Garg, T.K.; Lone, W.Z.; Qadri, H.L. Non-separable linear canonical wavelet transform. *Symmetry* **2021**, *13*, 2182. [CrossRef]
21. Shah, F.A.; Teali, A.A.; Tantary, A.Y. Linear canonical wavelet transforms in quaternion domains. *Adv. Appl. Clifford Algebr.* **2021**, *31*, 42. [CrossRef]
22. Shah, F.A.; Teali, A.A.; Tantary, A.Y. Special affine wavelet transform and the corresponding Poisson summation formula. *Int. J. Wavelets Multiresol. Informat. Process.* **2021**, *19*, 2050086. [CrossRef]
23. Isham, C.J. *Lectures on Quantum Theory: Mathematical and Structural Foundations*; Imperial College Press: London, UK, 1995.