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Spinor-Vector Duality and the Swampland

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Abstract: The Swampland Program aims to address the question, "when does an effective field theory model of quantum gravity have an ultraviolet complete embedding in string theory?", and can be regarded as a bottom-up approach for investigations of quantum gravity. An alternative top-down approach aims to explore the imprints and the constraints imposed by string-theory dualities and symmetries on the effective field theory representations of quantum gravity. The most celebrated example of this approach is mirror symmetry. Mirror symmetry was first observed in worldsheet contructions of string compactifications. It was completely unexpected from the effective field theory point of view, and its implications in that context were astounding. In terms of the moduli parameters of toroidally compactified Narain spaces, mirror symmetry can be regarded as arising from mappings of the moduli of the internal compactified space. Spinor-vector duality, which was discovered in worldsheet constructions of string vacua, is an extension of mirror symmetry that arises from mappings of the Wilson line moduli and provide a probe to constrain and explore the moduli spaces of (2, 0) string compactifications. Mirror symmetry and spinor-vector duality are mere two examples of a much wider symmetry structure, whose implications have yet to be unravelled. A mapping between supersymmetric and non-supersymmetric vacua is briefly discussed. T-duality is another important property of string theory and can be thought of as phase-space duality in compact space. I propose that manifest phase-space duality and the related equivalence postulate of quantum mechanics provide the background independent overarching principles underlying quantum gravity.

Keywords: string phenomenology; string dualities; phase-space duality

Citation: Faraggi, A.E. Spinor-Vector Duality and the Swampland. *Universe* **2022**, *8*, 426. https://doi.org/10.3390/universe8080426

Academic Editors: Ignatios Antoniadis and Kostas Skenderis

Received: 15 June 2022
Accepted: 11 August 2022
Published: 18 August 2022

1. Introduction

Physics is first and foremost an experimental science. Be that as it may, the language which is used to encode experimental observations is mathematics. It therefore makes sense to construct mathematical models that aim to describe physical reality, where successful mathematical models are those that are able to account for a wider range of observational data. Over the past century, the mathematical modelling of physical observations cumulated in two fundamental theories. In the subatomic domain, quantum mechanics and its incarnation in the form of Quantum Field Theory (QFT) account for all available data, whereas in the celestial, galactical and cosmological realms, Einstein's general relativity theory of gravity is used to parametrise the observations. However, these two basic theories are fundamentally incompatible. The incompatibility is particularly glaring when it comes to the vacuum. Whereas gravitationally based observations mandate that the vacuum energy is very small, quantum field theory models that are used to describe subatomic data predict vacuum energy that by far exceed physical observations. There are further basic incompatibilities between the two theories that are related with their distinct mathematical formulations, e.g., the black hole information paradox and the unrenormalisability of quantum field theory formulations of general relativity.

Attempts to construct consistent mathematical formulations of quantum gravity, therefore, occupy and motivate much of the research in fundamental physics. There are numerous approaches to this problem that include string theory (for recent reviews see e.g., [1,2])

and loop quantum gravity [3]. String theory is a mundane extension of point-particle quantum mechanics that provides a perturbatively self-consistent framework for quantum gravity. The consistency requirements of string theory mandate the existence of the gauge and matter structures that are the bedrock of the Standard Model of particle physics. String theory, therefore, provides an effective framework to develop a phenomenological approach to quantum gravity. The consistency conditions of string theory require additional degrees of freedom beyond those observed in the Standard Models of particle physics and cosmology. In some guises, these can be interpreted as additional bosonic spacetime dimensions, with 26 extra dimensions required in the bosonic string and 10 in the fermionic, whereas the heterotic-string is a hybrid of a left-moving fermionic sector and right-moving bosonic sector.

The number of string theories in higher dimensions is relatively scarce and includes five theories that are supersymmetric and eight that are not. Moreover, these ten-dimensional theories are connected in a lower dimension by perturbative interpolations or orbifolds or by some non-perturbative transformations. The extra dimensions are compactified on an internal space such that they are hidden from contemporary experimental observations, resulting in a plethora of vacua in lower dimensions. Nevertheless, there may exist symmetries that underlie the entire space of vacua in lower dimensions, akin, perhaps, to the symmetries that underlie the vacua in higher dimensions. Mirror symmetry is an example of such a symmetry between vacua in lower dimensions [4–6].

String vacua in four dimensions are in general studied by using exact worldsheet constructions, as well as effective field theory target space tools that explore the low-energy spectrum of string compactifications. Among the exact worldsheet tools, we may list toroidal orbifolds [7], the fermionic formulation [8–10] and the interacting Conformal Field Theory constructions [11]. The effective field theory models typically are obtained as compactifications of ten or eleven-dimensional supergravity on a complex Ricci flat internal manifold [12,13].

A fundamental issue in this regard is the relation between the exact string solutions and their low-energy effective field theory and vice versa. This question motivates much of the contemporary interest in string phenomenology, in the context of the "so-called" swampland program [14], which aims to address the question when can an effective field-theory model of quantum gravity be completed as a fully fledged string theory and when it cannot. The swampland program further aims to uncover the fundamental principles that underlie quantum gravity. The swampland program [15] can be viewed as a bottom-up approach for the exploration of the synthesis of quantum mechanics with gravity.

These questions are vital because, at the end of the day, it is likely that the correlation of a string vacuum with experimental data is performed by using its effective field-theory limit, whereas the string construction may produce the boundary data in the form of the gauge and Yukawa couplings. To date, the relation between exact string solutions and their effective field-theory smooth limit is only well understood in limited cases [16] and entail mostly the analysis of various supergravity theories that are EFT limits of the corresponding string theories.

An alternative approach to the swampland program is a top-down approach that aims to explore the symmetries of exact string solutions and their imprints in the effective field-theory limit. The questions then are two fold: 1. what is the complete set of symmetries that underlie the moduli spaces of string vacua? 2. Can we guess from this set of symmetries the general principles that underlie quantum gravity?

Mirror symmetry is an example of a symmetry that was observed initially via the exact worldsheet Conformal Field Theory (CFT) constructions with profound implications for the geometrical spaces that are utilised in the effective field-theory limits [17]. The exact worldsheet string theories have a rich symmetry structure that arises due to the exchange of massless and massive modes. Mirror symmetry is believed to be related to T-duality [18]. In toroidal orbifold compactifications T-duality arises due to the exchange of the moduli of the internal six dimensional compactified manifold [19]. The toroidal

orbifold compactifications of the heterotic-string have additional moduli that correspond to Wilson line moduli.

Spinor-vector Duality (SVD) [20,21] is a map between dual string vacua that can be understood as a result of exchanging Wilson line moduli [22]. In this sense, the spinor-vector duality is an extension of mirror symmetry. The SVD operates under the exchange of the total number of spinorial plus anti-spinorial representations and the total number of vectorial representations of the underlying GUT symmetry group, where the GUT group is $SO(12)$, or $SO(10)$, in the case of models with $N = 2$, or $N = 1$, spacetime supersymmetry, respectively. The spinor-vector duality which is observed in $Z_2 \times Z_2$ orbifold compactifications generalises to exact string solutions with interacting internal CFTs [23].

Similarly to mirror symmetry, we can seek to explore the imprint of SVD in the effective field-theory limit of string compactifications. This program was initiated over the past couple of years by studying resolutions of dual orbifold models. In $Z_2 \times Z_2$ orbifolds, SVD is realised plane by plane and, hence, arises in vacua with a single Z_2 of the internal coordinates, whereas a second Z_2 action corresponds to the Wilson line [24]. These cases were studied in the effective field-theory limits in five [25] and six dimensions [26]. The SVD in the full $Z_2 \times Z_2$ orbifold requires some refinement of the orbifold singularity resolution tools that was developed in [27]. The exploration of SVD in the effective field-theory limit entails the analysis of complex manifolds with vector bundles. The SVD, therefore, provides a useful tool to explore and constrain the moduli spaces of Calabi–Yau manifolds with vector bundles. The next question that one may entertain is what is the complete set of symmetries that underlie the string vacua and whether symmetries such as mirror symmetry, or SVD, provide such a complete set; i.e., does every string vacuum have a mirror dual; does every Z_2 string vacuum with a number of spinorial plus anti-spinorial respresentations and vectorial representations of the GUT group have a dual vacuum in which the numbers are interchanged. We may conjecture that this is indeed the case and that an effective field-theory model that does not have such a dual is necessarily in the swampland.

We may envision that mirror symmetry and SVD are mere reflections of a much wider symmetry structure that underlies the space of string vacua. This much wider space can be glimpsed by compactifying the string models to two dimensions. Compactifications to 2 dimensions give rise to 24 dimensional lattices that exhibit a very rich symmetry structure. We may explore the possibility that mirror symmetry and the SVD are reflections of this wider symmetry structure and whether this symmetry structure corresponds to a finite and complete space. The fact that string theory predicts that a finite number of degrees of freedom is required to obtain a perturbatively consistent theory of quantum gravity gives hope that this is indeed the case.

We have learned that SVD is a mere extension of the duality symmetries that underlie the space of string vacua under exchanges of Wilson line moduli. In the context of toroidal compactifications, the full set of symmetries correspond to exchanges of the parameters of the background fields that include the metric, the anti-symmetric tensor field and the Wilson line moduli. Exchanges of the parameters of the metric and antisymmetric tensor field correspond to generalised T-duality symmetries [19]. We may interpret T-duality as phase-space duality in compact space; i.e., T-duality exchanges momentum modes with winding modes that we may regard as the phase-space of the compact space. Requiring manifest phase-space duality has been the starting point in the development of the equivalence postulate approach to quantum mechanics. In the EPOQM, quantum mechanics is derived from a geometrical principle. We may speculate that the requirements of manifest phase-space duality and the equivalence postulate of quantum mechanics provide the overarching principles that underlie quantum gravity.

2. SVD in Four-Dimensional Worldsheet Constructions

The spinor-vector duality was first observed in free fermionic constructions of the heterotic-string in four dimensions. The SVD was initially observed by simple counting [20], using the classification tools developed in [28] for type II string, and in [20,21,29] for heterotic-strings with unbroken $SO(10)$ GUT symmetry and $N = 1$ spacetime supersymmetry.

This is illustrated in Table 1, where s, $\bar{s}$ and v refer to the total number of spinorial 16, anti-spinorial $\overline{16}$ and vectorial 10 representations, respectively. The SVD operates plane by plane of the $Z_2 \times Z_2$ orbifold. It is readily checked that the total number of models with two spinorial, two anti-spinorial, and one spinorial plus one anti-spinorial representations of the $SO(10)$ GUT group is the same as the total number of models with two vectorial representations. This is easily checked by adding the numbers in the first three rows, which is equal to the number in the last row.

Table 1. Number of models with a total number of 2 representations in the first plane.

First Plane			Second Plane			Third Plane			
s	$\bar{s}$	v	s	$\bar{s}$	v	s	$\bar{s}$	v	**# of Models**
2	0	0	0	0	0	0	0	0	1325963712
0	2	0	0	0	0	0	0	0	1340075584
1	1	0	0	0	0	0	0	0	3718991872
0	0	2	0	0	0	0	0	0	6385031168

The SVD was subsequently proven to arise due to exchange of discrete GGSO phases in the free fermionic formulation [21,30] or due to discrete torsions in a toroidal orbifold representation [22,31]. The spinor-vector duality is readily understood if we consider the case in which an $SO(10) \times U(1)$ symmetry is enhanced to E_6. In this case, the string compactification possesses a (2, 2) worldsheet supersymmetry. The representation of E_6 includes the chiral 27 and anti-chiral $\overline{27}$, which decompose as $27 = 16_{+1/2} + 10_{-1} + 1_{+2}$ and $\overline{27} = \overline{16}_{-1/2} + 10_{+1} + 1_{-2}$ under $SO(10) \times U(1)$.

If one now counts the total number of $\#_1(16 + \overline{16})$ and $\#_2(10)$, it is apparent that, in this case, $\#_1 = \#_2$. That is, the point in the moduli space in which the symmetry is enhanced to E_6 is a self-dual point under SVD. This is similar to the case of T-duality on a circle, in which, at the self-dual point under T-duality, the gauge symmetry is enhanced from $U(1)$ to $SU(2)$. Away from the self-dual point, the E_6 symmetry is broken to $SO(10) \times U(1)$ and the worldsheet supersymmetry is broken from $(2, 2)$ to $(2, 0)$. In general, the E_6 symmetry is broken in the toroidal orbifold models by Wilson lines, or by some discrete phases, whereas in the fermionic language, they may appear as Generalised GGSO phases in the one-loop partition function. The SVD duality states that, for any string vacuum, in which $E_6 \to SO(10) \times U(1)$, with $\#_1(16 + \overline{16})$ and $\#_2(10)$ representations, there exist a dual vacuum in which $\#_1 \leftrightarrow \#_2$.

The spinor-vector Duality (SVD) is depicted in Figure 1, which shows a distribution of the number of models with a $\#_1(16 + \overline{16})$ and $\#_2(10)$. Figure 1 is symmetric under the exchange of rows and columns reflecting the duality under the spinor-vector exchange.

As the free fermionic heterotic-string vacua correspond to the $Z_2 \times Z_2$ orbifolds, they contain three twisted sectors that preserve each an $N = 2$ spacetime supersymmetry. The SVD is realised in fact in each twisted sector separately; i.e., it can be realised in models that possess $N = 2$, rather than $N = 1$, spacetime supersymmetry [24]. In the $N = 2$ vacua, the enhanced symmetry at the self-dual point is E_7, which is broken to $SO(12) \times SU(2)$ away from the self-dual point, and the SVD is realised in terms of the relevant representations of E_7 and $SO(12) \times SU(2)$ [24].

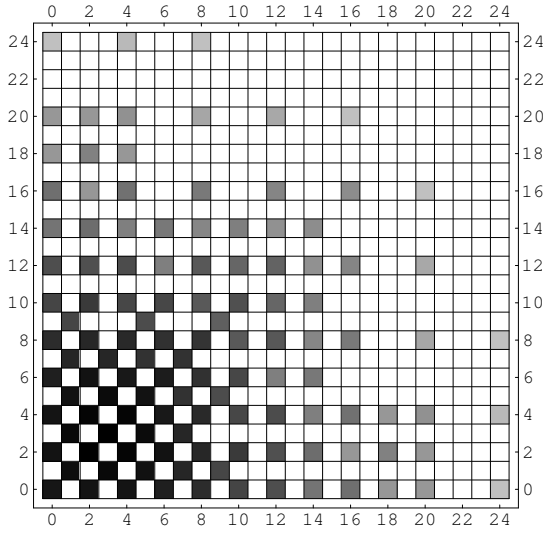

Figure 1. Density plot showing the spinor-vector duality in the space of fermionic $Z_2 \times Z_2$ heterotic-string models. The plot shows the number of vacua with a given number of $(16 + \overline{16})$ and 10 multiplets of $SO(10)$. It is invariant under exchange of rows and columns, reflecting the spinor-vector duality underlying the entire space of vacua. Models on the diagonal are self-dual under the exchange of rows and columns, i.e., $\#(16 + \overline{16}) = \#(10)$ without enhancement to E_6, which are self-dual by virtue of the enhanced symmetry.

A further understanding of the spinor-vector duality is gained by translating to the bosonic $Z_2 \times Z_2$ representation. Since the SVD operates in each of the Z_2 planes separately, we can study it in vacua with a single Z_2 twist of the compactified coordinates [24]. Using the level one $SO(2n)$ characters [32], we have the following:

$$O_{2n} = \frac{1}{2}\left(\frac{\theta_3^n}{\eta^n} + \frac{\theta_4^n}{\eta^n}\right) \quad , \qquad V_{2n} = \frac{1}{2}\left(\frac{\theta_3^n}{\eta^n} - \frac{\theta_4^n}{\eta^n}\right), \tag{1}$$

$$S_{2n} = \frac{1}{2}\left(\frac{\theta_2^n}{\eta^n} + i^{-n}\frac{\theta_1^n}{\eta^n}\right) \quad , \qquad C_{2n} = \frac{1}{2}\left(\frac{\theta_2^n}{\eta^n} - i^{-n}\frac{\theta_1^n}{\eta^n}\right), \tag{2}$$

where

$$\theta_3 \equiv Z_f\begin{pmatrix}0\\0\end{pmatrix} \quad , \quad \theta_4 \equiv Z_f\begin{pmatrix}0\\1\end{pmatrix} \quad , \quad \theta_2 \equiv Z_f\begin{pmatrix}1\\0\end{pmatrix} \quad , \quad \theta_1 \equiv Z_f\begin{pmatrix}1\\1\end{pmatrix},$$

and Z_f is the partition function of a single complex worldsheet fermion in terms of theta functions. The partition function of the $E_8 \times E_8$ heterotic-string reduced to four dimensions is given by

$$Z_+ = (V_8 - S_8)\left(\sum_{m,n}\Lambda_{m,n}\right)^{\otimes 6}(\overline{O}_{16} + \overline{S}_{16})(\overline{O}_{16} + \overline{S}_{16}), \tag{3}$$

where for each S_1, we have

$$p^i_{\mathrm{L,R}} = \frac{m_i}{R_i} \pm \frac{n_i R_i}{\alpha'} \qquad \text{and} \qquad \Lambda_{m,n} = \frac{q^{\frac{\alpha'}{4}p_{\mathrm{L}}^2}\,\bar{q}^{\frac{\alpha'}{4}p_{\mathrm{R}}^2}}{|\eta|^2}.$$

A $Z_2 \times Z_2' : g \times g'$ action on Z_+ is applied. The first Z_2 couples a fermion number in the observable and hidden sectors with a Z_2–shift in a compactified coordinate, and it is given by $g : (-1)^{(F_1+F_2)\delta}$ where the fermion numbers $F_{1,2}$ operate on the spinorial representations of the observable and hidden $SO(16)$ groups as

$$F_{1,2} : (\overline{O}^{1,2}_{16}, \overline{V}^{1,2}_{16}, \overline{S}^{1,2}_{16}, \overline{C}^{1,2}_{16}) \longrightarrow (\overline{O}^{1,2}_{16}, \overline{V}^{1,2}_{16}, -\overline{S}^{1,2}_{16}, -\overline{C}^{1,2}_{16})$$

and δ identifies points shifted by a Z_2 shift in the X_9 direction, i.e., $\delta X_9 = X_9 + \pi R_9$. The effect of the shift is to insert a factor of $(-1)^m$ into the lattice sum in Equation (3), i.e., $\delta : \Lambda^9_{m,n} \longrightarrow (-1)^m \Lambda^9_{m,n}$. The second Z_2 operates as a twist on the internal coordinates given by the following.

$$g' : (x_4, x_5, x_6, x_7, x_8, x_9) \longrightarrow (-x_4, -x_5, -x_6, -x_7, +x_8, +x_9). \tag{4}$$

Alternatively, the first Z_2 action can be interpreted as a Wilson line in X_9 [22].

$$g \ : (0^7, 1|1, 0^7) \ \rightarrow \ E_8 \times E_8 \rightarrow SO(16) \times SO(16).$$

The Z_2 space twisting breaks $N = 4 \rightarrow N = 2$ spacetime supersymmetry and $E_8 \rightarrow E_7 \times SU(2)$, or with the inclusion of the Wilson line $SO(16) \rightarrow SO(12) \times SO(4)$. The orbifold partition function is as follows.

$$Z = \left(\frac{Z_+}{Z_g \times Z_{g'}} \right) = \left[\frac{(1+g)}{2} \frac{(1+g')}{2} \right] Z_+.$$

The partition function includes an untwisted sector and three twisted sectors. Its schematic form is shown in Figure 2.

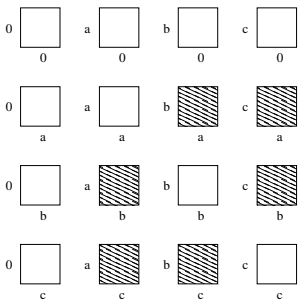

Figure 2. The $Z_2 \times Z_2'$ partition function of the g'–twist and g Wilson line with discrete torsion $\epsilon = \pm 1$.

The winding modes in the sectors twisted by g and gg' are shifted by $1/2$. These sectors therefore only contain massive states. The sector twisted by g' produces massless twisted matter states. The partition function has two modular orbits and one discrete torsion $\epsilon = \pm 1$ between the two orbits. Massless states are obtained for zero lattice modes. The terms in the sector g' contributing to the massless spectrum take the following form:

$$\Lambda_{p,q} \left\{ \frac{1}{2} \left(\left| \frac{2\eta}{\theta_4} \right|^4 + \left| \frac{2\eta}{\theta_3} \right|^4 \right) \left[P^+_\epsilon Q_s \overline{V}_{12} \overline{C}_4 \overline{O}_{16} + P^-_\epsilon Q_s \overline{S}_{12} \overline{O}_4 \overline{O}_{16} \right] + \right.$$
$$\left. \frac{1}{2} \left(\left| \frac{2\eta}{\theta_4} \right|^4 - \left| \frac{2\eta}{\theta_3} \right|^4 \right) \left[P^+_\epsilon Q_s \overline{O}_{12} \overline{S}_4 \overline{O}_{16} \right] \right\} \quad + \text{ massive} \tag{5}$$

where the following is the case.

$$P^+_\epsilon = \left(\frac{1 + \epsilon(-1)^m}{2} \right) \Lambda_{m,n} \ ; \ P^-_\epsilon = \left(\frac{1 - \epsilon(-1)^m}{2} \right) \Lambda_{m,n} \tag{6}$$

Depending on the sign of $\epsilon = \pm$, it is noted from Equation (6) that either the spinorial states or the vectorial states are massless. In the case with $\epsilon = +1$, we see from Equation (7) that, in this case, massless momentum modes from the shifted lattice arise in P^+_ϵ whereas P^-_ϵ gives rise to massive modes. Therefore, in this case, the vectorial character $\overline{V}_{12}$ in Equation (6) produces massless states, whereas the spinorial character $\overline{S}_{12}$ generates massive states. In the case with $\epsilon = -1$, Equation (8) reveals that exactly the opposite occurs.

$$\epsilon = +1 \quad \Rightarrow \quad P^+_\epsilon = \Lambda_{2m,n} \qquad P^-_\epsilon = \Lambda_{2m+1,n} \tag{7}$$

$$\epsilon = -1 \quad \Rightarrow \quad P^+_\epsilon = \Lambda_{2m+1,n} \qquad P^-_\epsilon = \Lambda_{2m,n} \tag{8}$$

It is noted that the spinor-vector duality is produced by the exchange of the discrete torsion $\epsilon = +1 \to \epsilon = -1$ in the $Z_2 \times Z_2'$ partition function. This is similar to the case of mirror symmetry in the $Z_2 \times Z_2$ orbifold of Ref. [33], where the mirror symmetry map is generated by the exchange of the discrete torsion between the two Z_2 orbifold twists.

It is important to emphasise that the example discussed here in detail is a particular example that provides insight into the inner working of the spinor-vector duality map. However, as exemplified in Figure 1 and Table 1, the SVD applies to the wider space of string vacua with $N = 2$ and $N = 1$ spacetime supersymmetry [20,21,24], which typically can be of the order of 10^{12} distinct models. The analysis in these papers utilises the free fermionic formulation, which somewhat obscures the role played by the geometrical moduli fields. It is demonstrated in [21,24] in terms of the GGSO projection coefficients of the one-loop partition function that the SVD always exists in this space of vacua. The bosonic analysis in [22,31] exposes the role of the moduli fields and shows that the SVD corresponds to an exchange between two Wilson lines. Furthermore, as demonstrated in [22] and discussed further below, the SVD can be interpreted as a map between two Wilson lines, which is induced by the spectral flow operator of the $N = 2$ worldsheet supersymmetry in the bosonic sector of the heterotic-string. The SVD can then be interpreted to arise from the breaking of the $N = 2$ worldsheet supersymmetry. At the enhanced symmetry point, with unbroken $N = 2$ worldsheet supersymmetry, the gauge symmetry is enhanced to E_6. In this case, the spectrum is self-dual under the SVD and the spectral flow operator acts as a generator of E_6, exchanging between the spinorial and vectorial representations in the decomposition of E_6 representations under the $SO(10) \times U(1)$ subgroup. This picture then generalises to string vacua with interacting internal CFTs [23], which utilise the Gepner construction of such vacua [11]. Starting with string vacua with $(2, 2)$ worldsheet supersymmetry and E_6 gauge symmetry, the $N = 2$ worldsheet supersymmetry on the bosonic side is broken with a Wilson line, and the spectral flow operator then induces the transformation which extend the SVD to these cases [23]. It is important, however, to emphasise that the bosonic representation that I discussed in detail here is crucial for seeking the imprint of the SVD in the effective field-theory limit, just as in the case of mirror symmetry.

The technical details of the relation between the discrete torsion and the Wilson line realisations of the SVD are provided in Ref. [22]. For our purpose here, it is sufficient to realise that, in terms of the toroidal background fields, there exist choices of the background fields that give rise to the spectra of the dual models. In those terms, the Z_2 twist action of the internal coordinates is provided by Equation (4), whereas the dual Wilson lines are given by

$$g = (0, 0, 0, 0, 0, 1|0, 0|1, 0, 0, 0, 0, 0, 0, 0). \tag{9}$$

and

$$g = (0, 0, 0, 0, 0, 0|1, 0|1, 0, 0, 0, 0, 0, 0, 0). \tag{10}$$

Furthermore, the map between the two can be represented in terms of spectral flow operators [22]. A representation of the duality in terms of the spectral flow operator

is discussed further below by using an alternative set of free fermion basis vectors. I emphasise, however, that to relate the worldsheet symmetries to the properties of the effective field-theory limit of the string compactifications is facilated by using the bosonic data in the form of Equations (9) and (10). The interpretation of the worldsheet data in the effective field-theory limit is often obscured, as is the case, for example, for mirror symmetry. For this purpose, the representation of the SVD in terms of the Wilson lines is particularly useful.

Mirror symmetry was initially observed in worldsheet CFT constructions of string compactifications, and the profound implications for the geometrical complex manifolds that are utilised in the effective field-theory limit of the string compactifications were subsequently understood. In particular, mirror symmetry facilitates the counting of intersections between sub-surfaces of the complex manifolds in a field known as enumerative geometry, which are otherwise notoriously difficult to perform. The calculation is facilitated by the relation of the intersection curves to the calculation of the Yukawa couplings between the string states. Thus, worldsheet constructions provide a useful tool to study the properties of the string vacua in the effective field-theory limit.

In a similar spirit, it is interesting to explore the implications of spinor-vector duality in the effective field-theory limit of the string compactifications. Similarly, to mirror symmetry, where the modular properties of the worldsheet string theory have profound implications in the effective field theory limit on the properties of complex manifolds, the SVD may have similar implications for compactifications on complex manifolds with the vector bundles that correspond to the gauge degrees of freedom in the worldsheet realisations. In the case of the $Z_2 \times Z_2$ orbifold and in the case of mirror symmetry, the discrete torsion is between two Z_2 twists of the internal compactified torus; in the case of spinor-vector duality, the discrete torsion is between the internal Z_2 twist and the Z_2 Wilson line that breaks the worldsheet supersymmetry from $(2,2)$ to $(2,0)$. In the past year, we demonstrated the realisation of the SVD in the effective field-theory limit of compactifications to five [25], and six [26] dimensions by studying resolutions of the orbifold singularities, whereas the analysis of spinor-vector duality in compactifications to four dimensions requires the further development of the resolutions tools [27]. The SVD, therefore, provides a tool to explore the effective field-theory limit of $(2,0)$ string compactifications. The SVD can serve as a probe of the moduli spaces of heterotic-string compactifications with $(2,0)$ worldsheet supersymmetry. While the moduli spaces of string compactifications with standard embedding and $(2,2)$ worldsheet supersymmetry are fairly well understood, the moduli spaces of $(2,0)$ models are obscured. Recently, we demonstrated the viability of this approach in the case of compactifications to five and six dimensions [25,26], where the effective field-theory limit is obtained by resolving orbifold singularities. In this context, the worldsheet description serves as a guide to guess how the worldsheet description should be interpreted in the effective field-theory limit. It is noted that the SVD in the worldsheet formalism generalises to string compactifications with interacting internal CFTs [23], as well as to cases with more discrete torsions [31].

Further insight into the structure underlying the SVD is revealed by breaking the untwisted NS symmetry from $SO(16) \times SO(16)$ to $SO(8)^4$ and generating the $SO(10)$ or $SO(12)$ symmetries by enhancements [24] (details of the free fermionic formalism can be found in the original literature [8] and are not repeated here. In the notation used in Equations (11) and (12), the entries in the curly brackets denote the periodic worldsheet fermions). This is obtained by defining four basis vectors with four non-overlapping sets of four periodic fermions, as shown in the set of boundary condition basis vectors given in Equation (11):

$$\begin{aligned}
v_1 = 1 &= \{\psi^\mu,\ \chi^{1,\ldots,6}, y^{1,\ldots,6}, \omega^{1,\ldots,6}|\\
&\quad \bar{y}^{1,\ldots,6}, \bar{\omega}^{1,\ldots,6}, \bar{\eta}^{1,2,3}, \bar{\psi}^{1,\ldots,5}, \bar{\phi}^{1,\ldots,8}\},\\
v_2 = S &= \{\psi^\mu, \chi^{1,\ldots,6}\},\\
v_3 = z_1 &= \{\bar{\phi}^{1,\ldots,4}\},\\
v_4 = z_2 &= \{\bar{\phi}^{5,\ldots,8}\},\\
v_5 = z_3 &= \{\bar{\psi}^{1,\ldots,4}\},\\
v_6 = z_0 &= \{\bar{\eta}^{0,1,2,3}\},\\
v_7 = b_1 &= \{\chi^{34}, \chi^{56}, y^{34}, y^{56}|\bar{y}^{34}, \bar{y}^{56}, \bar{\eta}^0, \bar{\eta}^1\},
\end{aligned} \tag{11}$$

The models generated by the basis (11) preserve $N = 2$ space–time supersymmetry, as the single supersymmetry breaking vector b_1 is included in the basis.

I refer to the basis vectors in Equation (11) as modular basis vectors. In this case, the Z_2 twist of the b_1 basis vector that acts in the internal dimensions breaks one $SO(8)$ symmetry to $SO(4)^2$. The modular basis vector z_0, which is charged under this $SO(8)$, acts as a spectral flow operator in a similar way to the S–spectral flow operator that mixes between the spacetime supersymmetric multiplets on the supersymmetric side of the heterotic-string. The GUT symmetries on the other hand are realised in the string models by enhancements. In this case, the basis vector z_3, subject to specific choices of GGSO projection coefficients, enhances the untwisted $SO(8) \times SO(4)$ symmetry to $SO(12)$ (or $SO(8) \times SO(2) \to SO(10)$ if an additional twist basis vector b_2 is included). In the (2, 2) vacua with enhanced E_6 (or E_7) symmetry, the z_0 basis vector acts as a symmetry generator of the enhanced symmetry and exchanges between the $SO(10) \times U(1)$ multiplets inside the E_6 representations. When the E_6 symmetry is broken to $SO(10) \times U(1)$, the spectral flow operator induces the map between the dual Wilson lines and, hence, between the two spinor-vector dual vacua [22,24].

The two spectral flow basis vectors, the S-vector and the z_0-vector, are mere two examples of a much richer symmetry structure. This much richer symmetry structure can be glimsped by compactifying the string models to two dimensions. In this case, the target space on the bosonic side of the heterotic-string is 24 dimensional. The $z_{0,1,2,3}$ basis vectors in Equation (11) each contains four periodic worldsheet fermions. The 24 dimensional lattice in two dimensions can, therefore, be divided into six such non-overlapping basis vectors. The symmetry structure of 24 dimensional lattices is a topic of much interest. What role this symmetry structure plays in the phenomenological and mathematical properties of string theory has yet to be unravelled. An embryonic attempt to explore this rich symmetry structure in the context of the phenomenological free fermionic models was discussed in Ref. [34]. Here, I provide a brief account of these investigations.

In the light-cone gauge, the worldsheet-free fermions of the heterotic-string in two dimensions (in the usual notation [8,34]) are: $\chi^i, y^i, \omega^i, i = 1, \ldots, 8$ (real left-moving fermions) and $\bar{y}^i, \bar{\omega}^i, i = 1, \ldots, 8$ (real right-moving fermions), $\bar{\psi}^A, A = 1, \ldots, 4$, $\bar{\eta}^B, B = 0, 1, 2, 3$, $\bar{\phi}^\alpha, \alpha = 1, \ldots, 8$ (complex right-moving fermions). The left– and right–moving real fermions are combined into complex fermions as $\rho_i = 1/\sqrt{2}(y_i + i\omega_i)$, $i = 1, \ldots, 8$, $\bar{\rho}_i = 1/\sqrt{2}(\bar{y}_i + i\bar{\omega}_i)$, $i = 1, \ldots, 4$, $\bar{\rho}_i = 1/\sqrt{2}(\bar{y}_i + i\bar{\omega}_i)$, $i = 5, \ldots, 8$.

The models of interest are generated by a maximal set V of seven basis vectors:

$$V = \{v_1, v_2, \ldots, v_7\},$$

$$\begin{aligned}
v_1 = \mathbf{1} &= \{\chi^{1,\ldots,8}, y^{1,\ldots,8}, \omega^{1,\ldots,8}|\\
&\quad \bar{y}^{1,\ldots,8}, \bar{\omega}^{1,\ldots,8}, \bar{\eta}^{0,1,2,3}, \bar{\psi}^{1,\ldots,4}, \bar{\phi}^{1,\ldots,8}\},\\
v_2 = H_L &= \{\chi^{1,\ldots,8}, y^{1,\ldots,8}, \omega^{1,\ldots,8}\},\\
v_3 = z_1 &= \{\bar{\phi}^{1,\ldots,4}\},\\
v_4 = z_2 &= \{\bar{\phi}^{5,\ldots,8}\},\\
v_5 = z_3 &= \{\bar{\psi}^{1,\ldots,4}\},\\
v_6 = z_4 &= \{\bar{\eta}^{0,1,2,3}\},\\
v_7 = z_5 &= \{\bar{y}^{1,\ldots,4}, \bar{\omega}^{1,\ldots,4}\},
\end{aligned} \tag{12}$$

and the corresponding matrix of one-loop generalised GSO projection coefficients:

$$\begin{array}{c} \\ \mathbf{1} \\ H_L \\ \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathbf{1} & H_L & z_1 & z_2 & z_3 & z_4 & z_5 \end{array} \\ \begin{pmatrix} -1 & -1 & +1 & +1 & +1 & +1 & +1 \\ -1 & -1 & \pm 1 & \pm 1 & \pm 1 & \pm 1 & \pm 1 \\ \\ +1 & \pm 1 & +1 & \pm 1 & \pm 1 & \pm 1 & \pm 1 \\ +1 & \pm 1 & \pm 1 & +1 & \pm 1 & \pm 1 & \pm 1 \\ +1 & \pm 1 & \pm 1 & \pm 1 & +1 & \pm 1 & \pm 1 \\ +1 & \pm 1 & \pm 1 & \pm 1 & \pm 1 & +1 & \pm 1 \\ +1 & \pm 1 & \pm 1 & \pm 1 & \pm 1 & \pm 1 & +1 \end{pmatrix} \end{array} \tag{13}$$

Following the usual free fermion methodology [8], the range of symmetries can be classified with varying numbers of basis vectors. Massless bosons are obtained from the Neveu–Schwarz sector: the z_i sectors and the $z_i + z_j$, $i = 1, \ldots, 6$, $i \neq, j$, where $z_6 = \mathbf{1} + H_L + z_1 + z_2 + z_3 + z_4 + z_5 = \{\bar{\rho}^{5,6,7,8}\}$. For example, with the five basis vectors $\{\mathbf{1}, H_L, z_1, z_2, z_3\}$, the configurations of the symmetry groups are summarised in the Table 2.

Table 2. Symmetry configurations in compactifications to 2 dimensions.

$c\left[\frac{z_1}{H_L}\right]$	$c\left[\frac{z_2}{H_L}\right]$	$c\left[\frac{z_3}{H_L}\right]$	$c\left[\frac{z_1}{z_2}\right]$	$c\left[\frac{z_1}{z_3}\right]$	$c\left[\frac{z_2}{z_3}\right]$	**Gauge Group G**
+	+	+	+	+	+	$SO(24) \times SO(24)$
+	+	+	+	+	−	$SO(8) \times SO(16) \times SO(24)$
+	+	−	+	+	+	$SO(16) \times SO(32)$
−	−	+	+	+	+	$SO(8) \times SO(40)$
−	−	+	−	+	+	$E_8 \times SO(8) \times SO(24)$
−	−	−	+	+	+	$SO(48)$
−	−	−	−	+	+	$E_8 \times SO(32)$

The untwisted symmetry is $SO(8)_1 \times SO(8)_2 \times SO(8)_3 \times SO(24)$. There are a total of six independent phases in this case producing 64 distinct possibilities, which give rise to seven distinct configurations shown in Table 2. The symmetry structure of the worldsheet string constructions may have profound phenomenological implications. It is apparent that we have thus far only glimpsed some of these potential implications. For example, we can consider modular maps, akin to the supersymmetry map S and the SVD map z_0 that can induce maps between supersymmetric and non-supersymmetric vacuum configurations [35–37]. This map is illustrated in the model shown in Equation (14).

	ψ^μ	χ^{12}	χ^{34}	χ^{56}	$\bar{\psi}^{1,\ldots,5}$	$\bar{\eta}^1$	$\bar{\eta}^2$	$\bar{\eta}^3$	$\bar{\phi}^{1,\ldots,8}$
ine ine $\mathbf{1}$	1	1	1	1	$1, \ldots, 1$	1	1	1	1 1 1 1 1 1 1 1
$\tilde{S}$	1	1	1	1	$0, \ldots, 0$	0	0	0	0 0 1 1 1 1 0 0
ine b_1	1	1	0	0	$1, \ldots, 1$	1	0	0	0 0 0 0 0 0 0 0
b_2	1	0	1	0	$1, \ldots, 1$	0	1	0	0 0 0 0 0 0 0 0
b_3	1	0	0	1	$1, \ldots, 1$	0	0	1	0 0 0 0 0 0 0 0

	$y^{3,\ldots,6}$	$\bar{y}^{3,\ldots,6}$	$y^{1,2}, \omega^{5,6}$	$\bar{y}^{1,2}, \bar{\omega}^{5,6}$	$\omega^{1,\ldots,4}$	$\bar{\omega}^{1,\ldots,4}$
ine ine $\mathbf{1}$	$1, \ldots, 1$	$1, \ldots, 1$	$1, \ldots, 1$	$1, \ldots, 1$	$1, \ldots, 1$	$1, \ldots, 1$
$\tilde{S}$	$0, \ldots, 0$	$0, \ldots, 0$	$0, \ldots, 0$	$0, \ldots, 0$	$0, \ldots, 0$	$0, \ldots, 0$
ine b_1	$1, \ldots, 1$	$1, \ldots, 1$	$0, \ldots, 0$	$0, \ldots, 0$	$0, \ldots, 0$	$0, \ldots, 0$
b_2	$0, \ldots, 0$	$0, \ldots, 0$	$1, \ldots, 1$	$1, \ldots, 1$	$0, \ldots, 0$	$0, \ldots, 0$
b_3	$0, \ldots, 0$	$0, \ldots, 0$	$0, \ldots, 0$	$0, \ldots, 0$	$1, \ldots, 1$	$1, \ldots, 1$

(14)

The map introduced in the model shown in Equation (14) is obtained by augmenting the SUSY generator basis vector S by four periodic hidden sector worldsheet fermions. It is referred to as the $\tilde{S}$-map. Up to the $\tilde{S}$-map, the basis vectors in Equation (14) are

identical to the NAHE-set basis vectors [38] and are refered to as the $\overline{\text{NAHE}}$-set. The effect of the $\tilde{S}$-map is to make the gravitino massive. The four-dimensional models obtained by the $\tilde{S}$–map correspond to compactifications of a tachyonic ten-dimensional vacuum, and the tachyonic states are projected out in the four-dimensional models [39]. Every model constructed with the NAHE-set can be mapped to a $\overline{\text{NAHE}}$-based model by the $\tilde{S}$-map, with the model of Ref. [40] providing a concrete example. The important point to realise is that understanding the string dynamics in the early universe mandates the understanding of the string vacua not only in the relatively safe and stable limit of the supersymmetric configurations but rather also those of non-supersymmetric and unstable configurations. Such modular maps may be particularly relevant for developing an understanding of the unstable configurations and the early dynamics of string theory.

We noted the rich symmetry structure underlying worldsheet string constructions. What are the implications for the effective field-theory target space limits? Turning back to the case of spinor-vector duality, we note that in the exact string theory solutions the entire space of compactifications is connected at the $N = 4$ level. The moduli space is the moduli space of the underlying $N = 4$ toroidal compactification. The observation made in Ref. [29] is that the number of chiral models is predetermined by the moduli space of the $N = 4$ toroidal compactification; i.e., it is determined by GGSO phases that correspond to $N = 4$ moduli. That is, the information of the chiral spectrum is predetermined by the moduli data of the background toroidal lattice. At the $N = 2$ or $N = 1$ levels, the vacua are related by continuous interpolations, or discrete mappings of the Wilson line moduli. This is because, in the $N = 1$ case, the interpolating moduli are projected out from the spectrum and the mapping between the dual models can only be discrete. The important observation, however, is that the space of models is determined by the moduli space of the underlying $N = 4$ compactifications. The additional $Z_2 \times Z_2$ projections produce different spectra depending on the point in the $N = 4$ moduli space where the projection is made. This is crucial from the point of view of the effective field-theory limit. From the worldsheet compactification, we may conjecture that every string vacua should be connected either by a continuous interpolation or discrete mapping by moduli of the underlying toroidal compactifications. For example, it was noted above that the spinor-vector duality mapping is induced by the modular map, which is induced by the basis vector z_0 that preserves the $N = 4$ spacetime time supersymmetry. This is similar to the case of the supersymmetry generator in the $N = 1$ and $N = 2$ theories that are generated by the basis vector S, which itself preserves $N = 4$ spacetime supersymmetry. We may speculate that the entire space of $(2, 0)$ compactifications is connected by interpolations or by discrete transformations, where the cases of discrete transformations correspond to the cases where the $N = 4$ moduli are projected out. This view is supported by the well-known observations that the known ten-dimensional vacua are connected via orbifolds or by interpolations in lower dimensions [41–43].

The question is as follows: What can we learn from these symmetries of the worldsheet compactifications about the effective field-theory moduli spaces? The imprint of these symmetries in the form of mirror symmetry and spinor-vector duality has already been noted. However, the connectedness of the worldsheet space of vacua has yet to leave its mark. At this stage, we can only speculate what this may entail. For example, we may question whether every Calabi–Yau manifold has its mirror, and whether every $(2, 0)$ Calabi–Yau manifold should exhibit the spinor-vector duality. An effective field-theory limit that does not exhibit these properties is necessarily in the swampland. The question is basically how do the symmetries of the worldsheet description constrain their effective field-theory limits. In this respect, the SVD and mirror symmetry provide a top-down approach to the question posed by the swampland program; i.e., what is the relation between exact string solutions and their effective field-theory limits? Whereas the swampland program aims to explore when does an effective field theory description of quantum gravity have an embedding in string theory, the duality program seeks to explore how the symmetries of the exact string solutions constrain the effective field-theory limit of string vacua. As we

observed, the space of symmetries of the worldsheet compactifications is vast, relating not only supersymmetric vacua but also to supersymmetric and non-supersymmetric vacua via the $\tilde{S}$-map. Thus, the road ahead is treacherous and one should not expect a short journey. Nevertheless, on the way there are many interesting concrete and interesting questions to explore, e.g., the moduli spaces of $(2,0)$ string compactifications, the calculation of correlators on dual manifolds and the interpretation of the parameters of the worldsheet theories in the effective field-theory limit. The ultimate questions are those of completeness. What is the complete set of symmetries of the worldsheet theories and does it completely constrain the effective field-theory limit of quantum gravity?

3. String Dualities and Fundamental Principles Underlying Quantum Gravity

The vast space of symmetries underlying the worldsheet string vacua was noted in the previous section. Shedding light on this vast space provides an ample number of questions to explore and investigate. Another approach is to try to extract from the string symmetries, the fundamental principles that may underlie quantum gravity and to use them as a starting point to try to formulate the theory. T-duality is thought to be a basic property of string theory that arises due to its extended nature. We have seen that spinor-vector duality can be thought of as an extention of T-duality to include symmetries under the mappings of Wilson line moduli rather than the internal moduli of the torus. We may continue to explore T-duality as a basic symmetry, e.g., the double geometry formalism of Ref. [44]. Another possible interpretation of T-duality is as a form of phase-space duality in compact space, where the duality exchanges position modes (winding modes) with momentum modes. We can then promote phase-space duality to a level of a fundamental principle and explore the consequences.

This is in essence the program that was pursued in the formulation of quantum mechanics from an equivalence postulate [45–47]. The starting point for the development of this approach was the requirement of manifest phase space duality that arises due to the involutive nature of Legendre transformations [45]. I focus here on the one-dimensional stationary case. The basic relation between the phase space variables is defined via generating function S_0 and its Legendre dual T_0.

$$p = \frac{\partial S_0}{\partial q} \quad \text{and} \quad q = \frac{\partial T_0}{\partial p} \tag{15}$$

Setting the condition $\tilde{S}_0(\tilde{q}) = S_0(q)$, i.e., that S_0 transforms as a scalar function under the transformations $q \to \tilde{q}(q)$, we obtain manifest $p \leftrightarrow q - S_0 \leftrightarrow T_0$ duality with [45,46]

$$\begin{aligned} p &= \frac{\partial S_0}{\partial q} & \qquad q &= \frac{\partial T_0}{\partial p} \\ S_0 &= p\frac{\partial T_0}{\partial p} - T_0 & \qquad T_0 &= q\frac{\partial S_0}{\partial q} - S_0 \end{aligned}$$

$$\left(\frac{\partial^2}{\partial S_0^2} + U(S_0)\right)\begin{pmatrix} q\sqrt{p} \\ \sqrt{p}\end{pmatrix} = 0 \qquad \left(\frac{\partial^2}{\partial T_0^2} + \mathcal{V}(T_0)\right)\begin{pmatrix} p\sqrt{q} \\ \sqrt{q}\end{pmatrix} = 0 \tag{16}$$

where dual second-order differential equations in the third line are associated with the dual Legendre transformations in the second line. The potential function $U(S_0)$ is given by $U(S_0) = \{q, S_0\}/2$, where $\{h(x), x\} = h'''/h' - (3/2)(h''/h')^2$ denotes the Schwarzian derivative.

There are several important points to note. The first is the existence of self-dual states, which are simultaneous solutions of the dual pictures. For these states, we have $S_0 = -T_0 + constant$ and the following.

$$S_0(q) = \gamma \ln \gamma_q q \qquad T_0(p) = \gamma \ln \gamma_p p \tag{17}$$

Here, $S_0 + T_0 = pq = \gamma$, with $\gamma_q \gamma_p \gamma = \mathrm{e}$ and γ_q, γ_p are constants. As the Legendre transformation is undefined for linear functions, the case of physical states with $S_0 = Aq + B$ with constants A and B is excluded. As we will see below, these cases coincide precisely with the self-dual states, and quantum mechanics rectifies this inconsistency of the classical limit. For now, we note the self-dual solutions that are given in Equation (17).

From the basic properties of the Schwarzian derivative, it is further noted that the potential function $U(S_0)$ is invariant under the $GL(2, C)$–transformations:

$$\tilde{q} = \frac{Aq + B}{Cq + D}, \quad \tilde{p} = \rho^{-1}(Cq + D)^2 p, \tag{18}$$

where $\rho = AD - BC \neq 0$. However, under arbitrary coordinate transformations $q \to \tilde{q} = v(q)$, we have $\tilde{U}(\tilde{s}) \neq U(s)$, whereas condition $\tilde{S}(\tilde{q}) = S_0(q)$ implies that the differential equation Equation (16) is covariant under the coordinate transformations. This suggests that different physical systems labelled by different potentials can be connected by coordinate transformations. This suggests the fundamental equivalence postulate [45,46]: *Given two physical systems labelled by potential functions $W^a(q^a) \in H$ and $W^b(q^b) \in H$, where H denotes the space of all possible Ws, there always exist a coordinate transformation $q^a \to q^b = v(q^a)$ such that $W^a(q^a) \to W^{av}(q^b) = W^b(q^b)$.*

It follows that there should always exist a coordinate transformation connecting any state to the trivial state $W^0(q^0) = 0$. Conversely, any nontrivial state $W \in H$ can be obtained from the states $W^0(q^0)$ by a coordinate transformation.

A natural setting to develop this approach is provided by the classical Hamilton–Jacobi formalism. I focus here on the stationary case. In the HJ formalism, the physical problem is solved by using canonical transformations that map a non-trivial Hamiltonian, with nonvanishing kinetic and potential energies, to a trivial Hamiltonian. The Classical Stationary Hamilton–Jacobi Equation (CSHJE) is as follows:

$$\frac{1}{2m}\left(\partial_q S_0\right)^2 + W(q) = 0, \tag{19}$$

where $W(q) = V(q) - E$, and it provides the solution and the functional relation between the phase-space variables extracted by the relation $p = \partial_q S_0$, where S_0 is the solution of the CSHJE equation. We can pose a similar question while imposing the functional relations $p = \partial_q S_0(q)$ on the trivialising transformation $q \to q^0(q)$ and $S_0^0(q^0) = S_0(q)$. The CSHJE is not consistent with this procedure because the free state $W^0(q^0) \equiv 0$ is a fixed point under coordinate transformations [45,46]. It is noted that this state corresponds to the self-dual state under phase-space duality. Consistency of the equivalence postulate, therefore, implies that the CSHJE has to be be modified. Focusing on the stationary case, the most general modification is given by the following.

$$\frac{1}{2m}\left(\partial_q S_0\right)^2 + W(q) + Q(q) = 0. \tag{20}$$

By the equivalence postulate, Equation (20) is covariant under general coordinate transformations. This is achieved if the combination $(W + Q)$ transforms as a quadratic differential under general coordinate transformations. Furthermore, all nontrivial states should be obtained from the state $W^0(q^0)$ by a coordinate transformation. It follows that each of the functions $W(q)$ and $Q(q)$ transform as quadratic differentials up to an additive term; i.e., under $q \to \tilde{q}(q)$, we have the following:

$$\begin{aligned} W(q) \to \tilde{W}(\tilde{q}) &= \left(\frac{\partial q}{\partial \tilde{q}}\right)^2 W(q) + (q; \tilde{q}) \\ Q(q) \to \tilde{Q}(\tilde{q}) &= \left(\frac{\partial q}{\partial \tilde{q}}\right)^2 Q(q) - (q; \tilde{q}). \end{aligned}$$

and the following.

$$(W(q)+Q(q)) \rightarrow (\tilde{W}(\tilde{q})+\tilde{Q}(\tilde{q})) = \left(\frac{\partial q}{\partial \tilde{q}}\right)^2 (W(q)+Q(q))$$

It is seen that all non-trivial potential functions $W^a(q^a)$ can be generated from the trivial state $W^0(q^0) \equiv 0$ by coordinate transformations $q^0 \rightarrow q^a$, with $W^a(q^a) \equiv (q^0;q^a)$; i.e., they arise from the inhomogeneous term. Considering the transformation $q^a \rightarrow q^b \rightarrow q^c$ versus $q^a \rightarrow q^c$ gives rise to the cocycle condition on the inhomogeneous term.

$$(q^a;q^c) = \left(\frac{\partial q^b}{\partial q^c}\right)^2 \Big[(q^a;q^b) - (q^c;q^b)\Big]. \tag{21}$$

The cocycle condition, Equation (21), underlies the equivalence postulate and embodies its underlying symmetries. In particular, it is invariant under the Möbius transformations:

$$(\gamma(q^a);q^b) = (q^a;q^b), \tag{22}$$

where

$$\gamma(q) = \frac{Aq+B}{Cq+D} \quad \text{and} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2,C). \tag{23}$$

The intimate connection between the equivalence postulate of quantum mechanics and the phase-space duality ought to be emphasised. This is elucidated by considering further the structure of the formalism in the one-dimensional case [45]. The one-dimensional case captures the symmetry structures that underlie the formalism and is amenable, rather straightforwardly, for generalisations to higher dimensions in Euclidean and Minkowski spacetimes [47]. Preserving these symmetry structures is the key to generalisations, but the basic physical features can already be gleaned from the one-dimensional structure. A basic identity in the one-dimensional stationary case takes the form of a difference between two Schwarzian derivatives:

$$\left(\frac{\partial S(q)}{\partial q}\right)^2 = \frac{\beta^2}{2}\left(\left\{e^{\frac{2i}{\beta}S},q\right\} - \{S,q\}\right) \tag{24}$$

where $\{f,q\} = f'''/f' - 3(f''/f')^2/2$ denotes the Schwarzian derivative, and β is a constant with the dimension of an action. Making the identification,

$$W(q) = V(q) - E = -\frac{\beta^2}{4m}\left\{e^{\frac{(i2S_0)}{\beta}},q\right\}, \tag{25}$$

and

$$Q(q) = \frac{\beta^2}{4m}\{S_0,q\}, \tag{26}$$

we have that S_0 is the solution of the Quantum Stationary Hamilton–Jacobi Equation (QSHJE),

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 + V(q) - E + \frac{\hbar^2}{4m}\{S_0,q\} = 0. \tag{27}$$

From Equation (25) and the properties of the Schwarzian derivative, it follows that S_0, and the solution of the QSHJE Equation (27) is given by the following: (see also [48–53]):

$$e^{\frac{2i}{\beta}S_0} = \gamma(w) = \frac{Aw+B}{Cw+D} = e^{i\alpha}\frac{w+i\bar{\ell}}{w-i\ell} \tag{28}$$

where $\ell = \ell_1 + i\ell_2$; $\{\alpha,\ell_1,\ell_2\} \in R$. Here, $w = \psi^D/\psi$ and ψ^D and ψ are two linearly independent solutions of a second-order differential equation given by the following:

$$\left(-\frac{\beta^2}{2m}\frac{\partial^2}{\partial q^2}+V(q)-E\right)\psi(q)=0 \tag{29}$$

i.e., ψ^D and ψ are the two solutions of the Schrödinger equation and we can identify $\beta \equiv \hbar$. Equation (24) follows from the requirement that the QHJE is covariant under coordinate transformations [45].

The self-dual solutions under phase-space duality coincide with the $W^0(q^0) \equiv 0$ states of the quantum Hamilton–Jacobi equation (QHJE). The important observation is that the consistency of phase-space duality, as well as the EPOQM, requires a departure from classical mechanics that mandates that $S_0 \neq \gamma(q)$, where $\gamma(q)$ is a Möbious transformation of q and, in turn, that the quantum potential $Q(q) = (\beta^2/4m)\{S_0, q\}$ never vanishes. It is further noted that the solutions of Equation (27) precisely coincide with the self-dual solutions in Equation (17).

The key features of the formalism are encoded in the Schwarzian identity (24) and cocycle condition (21). These two key elements generalise to any number of dimensions with Euclidean or Minkowski metrics [47]. The Schwarzian identity generalises to a quadratic identity given by the following:

$$\alpha^2(\nabla S_0)^2 = \frac{\Delta(Re^{\alpha S_0})}{Re^{\alpha S_0}} - \frac{\Delta R}{R} - \frac{\alpha}{R^2}\nabla\cdot(R^2\nabla S_0), \tag{30}$$

which holds for any constant α and any functions R and S_0. Then, if R satisfies the continuity equation

$$\nabla\cdot(R^2\nabla S_0)=0, \tag{31}$$

and setting $\alpha = i/\hbar$, we have the following.

$$\frac{1}{2m}(\nabla S_0)^2 = -\frac{\hbar^2}{2m}\frac{\Delta(Re^{\frac{i}{\hbar}S_0})}{Re^{\frac{i}{\hbar}S_0}} + \frac{\hbar^2}{2m}\frac{\Delta R}{R}. \tag{32}$$

In complete analogy with the one-dimensional case, we make identifications.

$$W(q) = V(q) - E = \frac{\hbar^2}{2m}\frac{\Delta(Re^{\frac{i}{\hbar}S_0})}{Re^{\frac{i}{\hbar}S_0}}, \tag{33}$$

$$Q(q) = -\frac{\hbar^2}{2m}\frac{\Delta R}{R}. \tag{34}$$

Equation (33) implies the D-dimensional Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\Delta + V(q)\right]\Psi = E\Psi. \tag{35}$$

and the general solution

$$\Psi = R(q)\left(Ae^{\frac{i}{\hbar}S_0} + Be^{-\frac{i}{\hbar}S_0}\right), \tag{36}$$

and similarly in the relativistic case, the Schwarzian identity generalises to the following:

$$\alpha^2(\partial S)^2 = \frac{\Box(Re^{\alpha S})}{Re^{\alpha S}} - \frac{\Box R}{R} - \frac{\alpha}{R^2}\partial\cdot(R^2\partial S), \tag{37}$$

which holds for any constant α and any functions R and S. Then, if R satisfies the continuity equation $\partial(R^2\cdot\partial S) = 0$ and setting $\alpha = i/\hbar$, we have the following.

$$(\partial S)^2 = -\hbar^2\frac{\Box(Re^{\frac{i}{\hbar}S})}{Re^{\frac{i}{\hbar}S}} + \hbar^2\frac{\Box R}{R}. \tag{38}$$

Setting the following:

$$W(q) = mc^2 = -\hbar^2 \frac{\Box(R\mathrm{e}^{\frac{i}{\hbar}S})}{R\mathrm{e}^{\frac{i}{\hbar}S}} \tag{39}$$

$$Q(q) = \hbar^2 \frac{\Box R}{R} \tag{40}$$

reproduces the relativistic Klein–Gordon equation

$$\left(\hbar^2\Box + mc^2\right)\Psi(q) = 0 \tag{41}$$

with the general solution.

$$\Psi = R(q)(A\mathrm{e}^{\frac{i}{\hbar}S} + B\mathrm{e}^{-\frac{i}{\hbar}S}). \tag{42}$$

Cocycle condition Equation (21) similarly generalises to any number of dimensions with Euclidean or Minkowski metrics. For example, in Minkowski spacetime setting $q \equiv (ct, q_1, \ldots, q_{D-1})$, with $q^v = v(q)$, a general transformation of the coordinates, we have the following:

$$(p^v|p) = \frac{\eta^{\mu\rho} p^v_\mu p^v_\rho}{\eta^{\mu\rho} p_\mu p_\nu} = \frac{p^t J \eta J^t p}{p^t \eta p}, \tag{43}$$

and J is the Jacobian matrix.

$$J^\mu{}_\rho = \frac{\partial q^\mu}{\partial q^v{}_\rho}. \tag{44}$$

Furthermore, we obtain the cocycle condition:

$$(q^a; q^c) = (p^c|p^b)\Big[(q^a; q^b) - (q^c; q^b)\Big], \tag{45}$$

and it is invariant under D-dimensional Möbius transformations with respect to the Minkowski metric. Cocycle condition Equation (21) similarly generalises to any number of dimensions with Euclidean metric and is given by the following:

$$(q^a; q^c) = (p^c|p^b)\Big[(q^a; q^b) - (q^c; q^b)\Big], \tag{46}$$

where

$$(p^v|p) = \frac{\sum_k (p^v_k)^2}{\sum_k p_k^2} = \frac{p^t J^t J p}{p^t p}, \tag{47}$$

and

$$J_{ki} = \frac{\partial q_i}{\partial q^v_j}, \tag{48}$$

is the Jacobian of the D-dimensional transformation, $q \to q^v = v(q)$, with $S^v_0(q^v) = S_0(q)$ and $p_k = \partial_q S_0$. It is shown [47] that cocycle condition Equation (46) is invariant under D-dimensional Möbius transformations with Euclidean or Minkowski metrics.

It is therefore seen that the key ingredients of the EPOQM formalism generalise to any number of dimensions. Furthermore, and crucially, the key symmetry property of quantum mechanics in this approach is the invariance of the cocycle condition under D-dimensional Möbius transformations. This is crucial because consistent implementation of the Möbius symmetry necessitates that spatial space is compact. The reason being that Möbius transformations include a symmetry under reflections with respect to the unit sphere [47]. This is the fundamental property of quantum mechanics in the EPOQM formulation. This feature of the formalism coincides with the property that the quantum potential is never vanishing.

The same basic structure generalises to curved space as well. The basic quadratic identity can be written in curved space in the following form:

$$\alpha^2(\partial_\mu S)(\partial^\mu S) = \frac{\frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}\partial^\mu\left(Re^{\alpha S}\right)}{Re^{\alpha S}} - \frac{\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}\partial^\mu R\right)}{R} - \frac{\alpha}{R^2}\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}R^2\partial^\mu S\right), \quad (49)$$

where S and R are scalar functions. We can similarly extend this basic structure in the case of fields. In particular, treating the spacetime metric as a field, we can utilise the Wheeler–deWitt to write a corresponding QHJE for the spatial part of the metric [54,55]. This equation is given by the following.

$$\alpha^2 G_{ijkl}\frac{\delta S}{\delta g_{ij}}\frac{\delta S}{\delta g_{kl}} = \frac{1}{Re^{\alpha S}}G_{ijkl}\frac{\delta^2\left(Re^{\alpha S}\right)}{\delta g_{ij}\delta g_{kl}} - G_{ijkl}\frac{1}{R}\frac{\delta^2(R)}{\delta g_{ij}\delta g_{kl}} - \frac{\alpha}{R^2}G_{ijkl}\frac{\delta}{\delta g_{ij}}\left(R^2\frac{\delta S}{\delta g_{ij}}\right) \quad (50)$$

Following the EPOQM structure, the WDW equation corresponding to Equation (50) is obtained by identifying the first part on the right-hand side of Equation (50) with the classical potential.

$$\frac{1}{Re^{\alpha S}}G_{ijkl}\frac{\delta^2\left(Re^{\alpha S}\right)}{\delta g_{ij}\delta g_{kl}} = -\sqrt{g}\,{}^3\mathcal{R} \quad (51)$$

In Equations (50) and (51) $g = \det g_{ij}$, ${}^3\mathcal{R}$ is the spatial intrinsic curvature, and $G_{ijkl} = \frac{1}{2}g^{-1/2}(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl})$ is the supermetric, and $\Psi\left[g_{ij}(\mathbf{x})\right] = Re^{\alpha S}$ is a wavefunctional in the superspace: the "wavefunction of the universe". The quantum version of the corresponding Hamilton–Jacobi equation is then given by the following.

$$G_{ijkl}\frac{\delta S}{\delta g_{ij}}\frac{\delta S}{\delta g_{kl}} - \sqrt{g}\,{}^3\mathcal{R} - G_{ijkl}\frac{\hbar^2}{R}\frac{\delta^2(R)}{\delta g_{ij}\delta g_{kl}} = 0. \quad (52)$$

We next turn to examine these consequences in the context of the EPOQM. The pivotal property of the EPOQM is the Möbious symmetry that underlies quantum mechanics in this formalism. The Möbius symmetry indicates that the spatial space is compact. The time coordinate cannot of course be compact and it is well known that the Möbius symmetry does not imply compactness in Minkowski space. However, the Möbius symmetry in the EPOQM underlies quantum mechanics in the nonrelativistic limit, and it can only be applied consistently if spatial space is compact. On the other hand, observations dictate that space is locally flat. These two properties are compatible with observations if spatial space is, for example, T^3. On the other hand, as elaborated above, the EPOQM dictates that $Q(R) \neq 0$. The EPOQM formalism, therefore, predicts the compactness of space as well as the non-vanishing ground-state energy associated with the quantum potential $Q(R)$. Both properties arise as a basic consequence of the Möbius symmetry that underlies quantum mechanics in the EPOQM approach. Having demonstrated the robustness of the generalisation of the Schwarzian identity, Equation (24), to any number of dimensions, as well as in curved space, we can take a leap of faith and propose that the spatial identity in Equation (50) generalises to general space in the following form:

$$\alpha^2 G_{\mu\nu\eta\rho}\frac{\delta S}{\delta g_{\mu\nu}}\frac{\delta S}{\delta g_{\eta\rho}} = \frac{1}{Re^{\alpha S}}G_{\mu\nu\eta\rho}\frac{\delta^2\left(Re^{\alpha S}\right)}{\delta g_{\mu\nu}\delta g_{\eta\rho}} - G_{\mu\nu\eta\rho}\frac{1}{R}\frac{\delta^2(R)}{\delta g_{\mu\nu}\delta g_{\eta\rho}} - \frac{\alpha}{R^2}G_{\mu\nu\eta\rho}\frac{\delta}{\delta g_{\mu\nu}}\left(R^2\frac{\delta S}{\delta g_{\eta\rho}}\right), \quad (53)$$

and that in this form, this identity can serve as a starting point for a covariant approach to quantum gravity. Note that, in this form, we have not assigned any interpretation to the terms arising in this identity, which would require further explorations into its properties and implications.

4. Conclusions

The synthesis of gravity with quantum mechanics is likely to occupy theoretical physicists well into the third millennium. It is naive to expect a quick fix. Twentieth century physicists have made deep inroads in our understanding of the material world in the very small scale and in the very large scale. More importantly, these inroads are supported by substantial observational data, which are the key to their respective success. However, they are not satisfactory. There is a fundamental dichotomy between the mathematical model that describes the observational data in the sub-atomic world versus the mathematical model that describe the observational data in the celestial, galactic and cosmological spheres. This dichotomy is particularly glaring with regards to the vacuum. While gravitationally based observations show that the vacuum energy is highly suppressed, the quantum field-theory models that are used to account for the sub-atomic data predict a vacuum energy, which is many orders of magnitude larger.

The available experimental data indicate that the Standard Model provides a viable parametrization of all observational data up to the GUT and Planck scales. The multiplet structure of the Standard Model fermions, the logarithmic evolution of the Standard Model parameters, the longevity of the proton and the suppression of the left-handed neutrino masses strongly support the embedding of the Standard Model states in multiplets of a Grand Unified Theory, which is realised at the GUT or Planck scales. This is the minimal hypothesis that one may infer from the currently available observational data. However, embedding the Standard Model in a Grand Unified Theory still leaves too many unexplained parameters. In particular, in the flavour sector of the Standard Model. The fundamental origin of these parameters, particularly in the flavour sector, can only be revealed by unifying the Standard Model with gravity. String theory is a mundane extension of the quantum field-theory framework, which is used in the Standard Model. Whereas in quantum field theories, elementary particles are idealised point particles, the augmentation of the Standard Model with gravity necessitates a departure from the view of elementary particles as idealised points. We should not be surprised. There is nothing sacred about elementary particles as idealised points. The minimal hypothesis is to assume that elementary particles are not zero dimensional but rather have one internal dimension, i.e., strings. String theory provides a perturbatively self-consistent framework for quantum gravity. Furthermore, the consistency requirements of string theory necessitate the appearance of the gauge and matter sectors that are the bedrock of the Standard Model. String theory, therefore, provides a framework for the development of a phenomenological approach to quantum gravity. Nonperturbative extensions of string theory reveal that, in that context, higher dimensional objects play a role as well. However, to confront string theory with observational data, we may use any of its perturbative limits.

Phenomenological string models have been constructed since the mid-eighties. However, the majority of these constructions merely contain some of the ingredients of the Standard Model, such as possessing chiral families that are charged under some GUT gauge group, but do not offer room for more in-depth analyses. A class of string models that exhibits realistic phenomenological properties and provides room for more in-depth analysis includes the quasi-realistic models constructed in the free fermionic formulation. These models correspond to $Z_2 \times Z_2$ orbifolds of six-dimensional toroidal compactifications and can be studied in any of the pertubative string limits, as phenomenological models as well as in cosmological scenarios. It should be understood, however, that our current understanding of string theory is rudimentary. In particular, we only truly understand string theory in its static limits, and the understanding of its time-dependent dynamics is still very much lacking. In order to explore time dependency, string dynamics is facilitated by proceeding to the effective field-theory limit of string constructions. Typically, this effective field-theory limit only involves the massless degrees of freedom of the string models.

This track has led to the "so-called" "Swampland Program". The aim of the Swampland Program is to address the question: when does an effective field-theory model of quantum gravity have an ultraviolet complete embedding in string theory. This approach

can be viewed as a bottom-up approach to the phenomenological exploration of string quantum gravity. An alternative top-down approach was advocated in this paper. The top-down approach aims to explore the imprint of the string theory dualities and symmetries in the effective field-theory limit. The most celebrated example of this approach is mirror symmetry. Mirror symmetry was first observed in the worldsheet construction of string vacua. It was entirely unexpected from the effective field-theory point of view, and its profound implications in this context were astounding. Spinor-vector duality, as described here, is an extension of mirror symmetry. While mirror symmetry corresponds to mappings of the internal moduli of the compactified space, spinor-vector duality arises from mappings of the Wilson line moduli, and they provide a probe to explore the moduli spaces of $(2,0)$ string compactifications.

It is important to note that a notable characteristic of the top-down approach is that it has access to the massive string modes, which are not seen in the effective field limit. Many of the string dualities are generated by the exchange of massless and massive string states and are, therefore, naturally gleaned in the top-down approach, but they are completely obscured in the effective field-theory limit. Moreover, mirror symmetry and the spinor-vector duality are two mere examples. String theory possess the nonperturbative dualities that generated some interest in the 1990s, but more importantly, there are many dualities that have yet been explored and understood. An example of such a map that was discussed here is the $\tilde{S}$–map, which induces a map between supersymmetric and non-supersymmetric vacua in four dimensions. Understanding such mappings in depth is of vital importance because it may have implications for the dynamics of string theory; i.e., it is a mapping between stable and unstable configurations. Even so, the space of string symmetries to be explored is vast, and while the phenomenological models that can be constructed in string theory do suggest that it is relevant to the observed experimental data, our understanding of it is rudimentary. The field is still in its infancy. It is important, however, to discern what are the important questions to ask. In this respect, the important question is not whether this or that string vacuum is the correct model of the world but rather whether any of the properties of the string vacua would be relevant to observable data. We know for certain that some of these properties are indeed relevant, e.g., the replication of the fermion families. The aim is to proceed deeper in associating the properties of the string vacua with the experimentally observed data, e.g., in the flavour data.

The next step in trying to formulate the theory of quantum gravity is to hypothesise some fundamental principles that may underlie quantum gravity and use them as a starting point to formulate the mathematical approach. I proposed here that T-duality may be interpreted as phase-space duality in compact space. Manifest phase-space duality is the starting point of the derivation of quantum gravity from phase-space duality and the equivalence postulate of quantum mechanics. It should be opined that the current string-based approaches to the fundamental formulation of quantum gravity are not satisfactory because they are background-dependent. Phase-space duality and the equivalence postulate of quantum mechanics are background independent principles. I proposed here that they provide the overarching principles that underlie quantum gravity. The next steps in this saga have yet to be written and will occupy us in the millennia to come.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This work is supported in part by a Weston visiting professorship at the Weizmann Institute of Science. I would like to thank Doron Gepner for discussions and the Department of Particle Physics and Astrophysics for hospitality.

Conflicts of Interest: The author declare no conflict of interest.

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