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A Proposal to Solve Finite N Matrix Theory: Reduced Model Related to Quantum Cosmology

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Abstract: The $SU(N)$ invariant model of matrix theory that emerges as the regularization of the 11-dimensional super membrane is studied. This matrix model is identified with M theory in the limit $N \to \infty$. It has been conjectured that matrix models are also relevant for finite N where several examples and arguments have been given in the literature. By the use of a Dirac-like formulation usually developed in finding solutions in Supersymmetric Quantum Cosmology, we exhibit a method that could solve, in principle, any finite N model. As an example of our procedure, we choose a reduced SU(2) model and also show that this matrix model contains relevant supersymmetric quantum cosmological models as solutions. By these means, our solutions constitute an example in order to consider why the finite N matrix models are also relevant. Since the degrees of freedom of matrix models are, in some limit, identified with those of Super Yang Mills Theory SYM with a finite number of supercharges, our methodology offers the possibility, through some but yet unspecified identification, to relate the quantization presented here with that of SYM theory for any finite N.

Keywords: quantum cosmology; supersymmetry; matrix theory

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1. Introduction

In the realm of String theory, which is considered a unified theory of fields including gravity, we find a formalism known as M Theory intended to relate a single theory with all five string theories and 11 dimensional supergravity, all of these connected with dualities [1]. There exist particular matrix models with finite degrees of freedom particularly related to M Theory [1–11]. These matrix models are intrinsically supersymmetric and they also have an additional $SU(N)$ symmetry as a remanent symmetry which has its origin in area preserving diffeomorphisms (APD). It has been conjectured that the $N \to \infty$ limit of this matrix models is related to M theory. This statement was also conjectured for finite N, where it was shown that the discrete light cone quantization of M theory offers a consistent finite N matrix model where important symmetries and dualities are preserved [2–4]. Compactified M-theory through the matrix models includes the dynamics of D_p branes of different dimensionalities $p = 1, 2, \ldots$ etc. It seems that by considering the dynamics of some of these degrees of freedom, it is possible to access at least to some part of the physics of M-Theory.

The search for the ground state solution of matrix models in the general case, exactly and with approximation methods, has been studied in [2,8,12–19]. We propose here another method of approach to find the ground state solution to this matrix model. One of the novel characteristics of our method is the particular matrix representation for the fermionic degrees of freedom which changes the wave function of superspace variables [20–24] to a multicomponent wave eigenvector state function of spacetime variables [25,26]. We will expose our procedure for a particular reduced SU(2) model that could be further applied to any other invariant symmetry with finite N. The degrees of freedom of compactified M theory are related to finite N matrix models as the one we attempt to solve here [2]. Also, finite N matrix theories are related to SYM Theory with finite degrees of freedom. Such relation makes us to suspect that it will be possible to relate the degrees of freedom of the solutions obtained by our procedure and the ones of SYM theory. This will be however a matter of future work and as a first step, we will attempt here to find an analytic way to solve a reduced model.

We will show also, as a byproduct of the methodology used, that relevant supersymmetric quantum cosmological solutions arise in the $SU(2)$ invariant matrix model. It has been found that the Hamiltonian for the bosonic sector of matrix theory contains, by the use of a particular Ansatz, the Hamiltonian of some classical non-supersymmetric cosmological models [27,28]. Since the supermembrane has a close relation with supergravity, the question whether the quantum matrix model is also related to Supersymmetric quantum cosmology arises naturally. Since the matrix models are inherently supersymmetric, it could exist a non-trivial connection between matrix models and supersymmetric quantum cosmology. We will obtain, in the particular finite N matrix model we solve with our formalism, relevant supersymmetric quantum cosmological solutions, hence our results strengthen the relation between matrix models and cosmology [27], extend it to the quantum supersymmetric realm, and show that solutions to this reduced matrix model contain relevant physical information for finite N as it has been previously conjectured [3,4]. More importantly, our procedure offers the possibility to solve the matrix model for any finite N. An important difference between the type of truncations previously implemented on the matrix model [2,5] and our work, is that we make additional assumptions on the bosonic variables, by fixing classically some of them and solving for the remaining Hamiltonians and their corresponding supercharges. These simplifications allow to have non-trivial and normalizable solutions. The type of truncations made are similar to those used to quantize cosmological models by imposing symmetries [29]. The solutions to our simplified models do not allow to infer any other information about the existence of non-trivial and normalizable solutions to more general models, but otherwise they only show that finite N simplified models could have non-trivial and normalizable solutions. It will happen also that these will be physically relevant.

The work is organized as follows. First in Section 2 we describe the elements of the SU(N) supersymmetric matrix model, in Section 3 we define the relevant elements in a 2-dimensional reduced model and for the SU(2) group. We justify the fact that a simultaneous solution to all the supercharges of the model is the vacuum solution to the Hamiltonian and use this fact to search for ground state solutions. In Section 4 we solve the supercharges by making further reductions in the bosonic degrees of freedom and solve for the ground state wave function and at the same time, we show how supersymmetric quantum cosmology solutions emerge. Section 5 is devoted to conclusions and discussions about the results of the work.

2. Matrix Model

The Hamiltonian of the matrix model we will use in this work was constructed in detail in [12] and is given by

$$\begin{aligned} H = \quad & \frac{1}{2}\pi_a^m\pi_a^m + \frac{g^2}{4}g_{abc}\phi_b^m\phi_c^n g_{ade}\phi_d^m\phi_e^n - \frac{ig\hbar}{2}g_{abc}\Lambda_{a\alpha}(\Gamma^m)_{\alpha\beta}\phi_b^m\Lambda_{c\beta}, \\ & m,n = \{\,1,2,\ldots,9\}\,, \quad a,b,c = \{\,1,2,\ldots,N^2-1\}\,, \\ & \alpha,\beta = \{\,1,2\ldots,\mathcal{N}\}\,. \end{aligned} \tag{1}$$

In these expressions ϕ_a^m are the bosonic degrees of freedom and π_a^m are their related momenta. g_{abc} correspond to the structure constants of the SU(N) symmetry group. For $N = 2$ and $N = 3$ we have for instance that $f_{abc} = \epsilon_{abc}$ the antisymmetric symbol and $g_{abc} = f_{abc}$ the structure constants of the Lie algebra of the Gell-Mann matrices respectively. The fermionic degrees of freedom are represented by the $\Lambda_{a\alpha}$ variables and Γ^m are m Dirac matrices obeying the Clifford algebra $\{\Gamma^m,\Gamma^n\} = 2\delta_{mn}$. It also should be understood from the expression (1) that m, n are dimension indexes, hence this Hamiltonian can be constructed up to $d = 9$ dimensions. As a matter of an example, it will be constructed for a reduced $d = 2$ model. a, b, c are group indexes and α, β are fermionic indexes.

The Hamiltonian in (1) can be expressed as function of its supercharges and constraint operators through the supersymmetric algebra

$$\begin{aligned} \{ Q_\alpha, Q_\beta \} &= 2\delta_{\alpha\beta} H + 2g(\Gamma^n)_{\alpha\beta}\phi^n_a G_a, \\ Q_\alpha &= (\Gamma^m \Lambda_a)_\alpha \pi^m_a + i g f_{abc} (\Sigma^{mn}\Lambda_a)_\alpha \phi^m_b \phi^n_c. \end{aligned} \tag{2}$$

The operators G_a are the constraint operators related to the SU(N) invariance and in the explicit definition of Q_α, $\Sigma^{mn} = -\frac{i}{4}[\Gamma^m, \Gamma^n]$. We are interested in the ground state wave function $|\Psi\rangle$ obeying $H|\Psi\rangle = 0$. In supersymmetric models, the wave function satisfying this is automatically the ground state as the Hamiltonian is positive definite. We notice in Equation (2) that $H|\Psi\rangle = 0$ can be in general satisfied if $Q_\alpha|\Psi\rangle = 0$ and $G_a|\Psi\rangle = 0$ and in the process of construction of bosonic and fermionic operators it should be taken care that the commutator and anticommutator relations for bosonic and fermionic degrees of freedom will be given by

$$[\phi^m_a, \pi^n_b] = i\hbar\delta_{ab}\delta_{mn}, \qquad \{ \Lambda_{a\alpha}, \Lambda_{b\beta} \} = \delta_{ab}\delta_{\alpha\beta}. \tag{3}$$

It should be noticed that the election of the SU(N) group and the dimension d, will be related to the dimensional representation of the fermionic variables $\Lambda_{a\alpha}$, supercharges and Hamiltonian. The bigger the group, the dimensional representation of those operators will also be larger. This will define the dimensions of the wave vector solution.

In order to exemplify the methodology of solution of this quantum system we will use the SU(2) group and a $d = 2$ dimensional model. Even though this system will be in principle easier to solve, the general methodology does not depend on the election of N and the procedure to solve a bigger system will follow basically the same kind of calculation. For the SU(2) invariant model, the values of the group indexes a, b, c and dimension indexes m, n are; $a, b, c = 1, 2, 3$. $m, n = 1, 2$. In two dimensions the supersymmetry indexes α, β are fixed to the values $\alpha, \beta = 1, 2$. Once we have defined the group symmetry and dimensionality in which we will attempt to solve for the ground state, it should be defined the particular representation for bosonic and fermionic variables. These definition and the corresponding solutions will be worked out in the next section.

3. Ground State Wave Functions

The representation of the canonical momenta corresponding to ϕ^m_a is, as usual, $\pi^m_a = -i\hbar\frac{\partial}{\partial\phi^m_a}$. One key step in the procedure to find ground state solutions is the use of a matrix representation for the fermionic variables $\Lambda_{a\alpha}$ [25]. These matrices must obey the algebra (3) and by these means, $|\Psi\rangle$ becomes a multicomponent wave function. As we mentioned above, we will work in the SU(2) invariant 2-dimensional model and in this case the algebra (3) for $\Lambda_{\alpha\beta}$ can be achieved by six (8×8)-dimensional matrices. These matrices are given by

$$\begin{aligned} \Lambda_{11} &= \frac{\pm 1}{\sqrt{2}}(\gamma^0 \otimes \sigma^1), \quad \Lambda_{12} = \frac{\pm i}{\sqrt{2}}(\gamma^1 \otimes \sigma^1), \\ \Lambda_{21} &= \frac{\pm i}{\sqrt{2}}(\gamma^2 \otimes \sigma^1), \quad \Lambda_{22} = \frac{\pm i}{\sqrt{2}}(\gamma^3 \otimes \sigma^1), \\ \Lambda_{31} &= \frac{\pm i}{\sqrt{2}}(I \otimes i\sigma^2), \quad \Lambda_{32} = \frac{\pm i}{\sqrt{2}}(I \otimes i\sigma^3), \end{aligned} \tag{4}$$

where σ^i ($i = 1, 2, 3$) are the Pauli Matrices, γ^μ ($\mu = 0, 1, 2, 3$) are the Dirac matrices in a Majorana representation and I is the 4×4 identity matrix. The explicit form of the two supercharges Q_1 and Q_2 is given by

$$
\begin{aligned}
Q_1 =& \Lambda_{11}\pi_1^2 + \Lambda_{21}\pi_2^2 + \Lambda_{31}\pi_3^2 + \Lambda_{12}\pi_1^1 + \Lambda_{22}\pi_2^1 + \Lambda_{32}\pi_3^1 \\
&- g\Lambda_{12}(\phi_2^1\phi_3^2 - \phi_3^1\phi_2^2) - g\Lambda_{22}(\phi_3^1\phi_1^2 - \phi_1^1\phi_3^2) - g\Lambda_{32}(\phi_1^1\phi_2^2 - \phi_2^1\phi_1^2)\,, \\
Q_2 =& \Lambda_{11}\pi_1^1 + \Lambda_{21}\pi_2^1 + \Lambda_{31}\pi_3^1 - \Lambda_{12}\pi_1^2 - \Lambda_{22}\pi_2^2 - \Lambda_{32}\pi_3^2 \\
&+ g\Lambda_{11}(\phi_2^1\phi_3^2 - \phi_3^1\phi_2^2) + g\Lambda_{21}(\phi_3^1\phi_1^2 - \phi_1^1\pi_3^2) + g\Lambda_{31}(\phi_1^1\phi_2^2 - \phi_2^1\phi_1^2).
\end{aligned}
\tag{5}
$$

The Hamiltonian (1) is also explicitly given by

$$
\begin{aligned}
H =& \frac{1}{2}[(\pi_1^1)^2 + (\pi_2^1)^2 + (\pi_3^1)^2 + (\pi_1^2)^2 + (\pi_2^2)^2 + (\pi_3^2)^2] \\
&+ \frac{g^2}{2}[(\phi_2^1)^2(\phi_3^2)^2 + (\phi_3^1)^2(\phi_2^2)^2 + (\phi_1^1)^2(\phi_3^2)^2 + (\phi_1^2)^2(\phi_3^1)^2 + (\phi_1^1)^2(\phi_2^2)^2 + (\phi_1^2)^2(\phi_2^1)^2] \\
&- \frac{g^2}{2}[2\phi_2^1\phi_3^1\phi_2^2\phi_3^2 + 2\phi_1^1\phi_3^2\phi_3^1\phi_1^2 + 2\phi_1^1\phi_2^2\phi_2^1\phi_1^2] \\
&- \frac{ig\hbar}{2}\{\, \phi_2^1([\Lambda_{11}, \Lambda_{32}] + [\Lambda_{12}, \Lambda_{31}]) + \phi_2^2([\Lambda_{11}, \Lambda_{31}] + [\Lambda_{32}, \Lambda_{12}])\} \\
&- \frac{ig\hbar}{2}\{\, \phi_3^1([\Lambda_{22}, \Lambda_{11}] + [\Lambda_{21}, \Lambda_{12}]) + \phi_3^2([\Lambda_{21}, \Lambda_{11}] + [\Lambda_{12}, \Lambda_{22}])\} \\
&- \frac{ig\hbar}{2}\{\, \phi_1^1([\Lambda_{32}, \Lambda_{21}] + [\Lambda_{31}, \Lambda_{22}]) + \phi_1^2([\Lambda_{31}, \Lambda_{21}] + [\Lambda_{22}, \Lambda_{32}])\}\,.
\end{aligned}
\tag{6}
$$

In the matrix representation used Equation (4), the eigenstates of this Hamiltonian have eight components [20,25]. We will use the following representation for the Dirac matrices Γ^m, $m = 1, 2$

$$
\Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7}
$$

this representation will play a definite role in the following. Notice that these matrices appear as multiplicative factors of G_a in the algebra (2), if we take for instance $\alpha = 1$ and using (7) we have

$$
2Q_1^2 = 2H + 2g\phi_1^2 G_1 + 2g\phi_2^2 G_2 + 2g\phi_3^2 G_3, \tag{8}
$$

and we could solve $H|\Psi\rangle = 0$ by solving $Q_1|\Psi\rangle = 0$, $G_1|\Psi\rangle = 0$, $G_2|\Psi\rangle$ and $G_3|\Psi\rangle = 0$. Instead of doing this, we will use the following fact, first notice that for $\alpha = 2$ and using (7) we have

$$
2Q_2^2 = 2H - 2g\phi_1^2 G_1 - 2g\phi_2^2 G_2 - 2g\phi_3^2 G_3, \tag{9}
$$

and the sum of (8) and (9) gives $Q_1^2 + Q_2^2 = 2H$. We can alternatively use the explicit definition of Q_1 and Q_2 from (5) and after a long calculation we find

$$
Q_1^2 + Q_2^2 = 2H, \tag{10}
$$

with H given by (6). This result shows us that we can find the ground state function by solving simultaneously $Q_1|\Psi\rangle = 0$ and $Q_2|\Psi\rangle = 0$. From now on we will use this methodology to find the ground state wave function. Motivated by the results given in [27,28] related to the relation between cosmology and matrix models, since we are also considering the supersymmetric sector, it is our purpose to search for supersymmetric cosmological solutions as solutions of this matrix model.

4. Wave Functions

In this section we search for common solutions to $Q_1|\Psi\rangle = 0$ and $Q_2|\Psi\rangle = 0$ for some specific models. The fixing of bosonic degrees of freedom classically allows to simplify the equations as we will see. The next set of reduced solutions will be relevant as they will be related to supersymmetric quantum wave functions

4.1. First Solution

Let us begin with the following assumption

$$\phi_1^1 = a,\ \phi_2^2 = b,\ \phi_2^1 = d,\ \phi_1^2 = x,\ \phi_3^1 = \phi_3^2 = 0. \tag{11}$$

where a, b, d are constants. We are making a reduction to only one bosonic degree of freedom, $\phi_1^2 = x$, therefore we will only have one canonical momentum $\pi_1^2 = -i\hbar\frac{d}{dx}$. The set of variables are fixed classicaly, consequently there will be a canonical momentum associated only to true variables. Once some configuration of variables is specified, it has been verified that the canonical commutation relations (3) are satisfied. The operator Q_1 and Q_2 (see Equation (5)) take the simpler form

$$Q_1 = \Lambda_{11}\pi_1^2 - g\Lambda_{32}(ab - dx),\ \ Q_2 = -\Lambda_{12}\pi_1^2 + g\Lambda_{31}(ab - dx). \tag{12}$$

Using the representation of the $\Lambda_{a\alpha}$ variables and applying the operator Q_1 to the wavefunction $|\Psi\rangle = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8)$, the operator equation $Q_1|\Psi\rangle = 0$ yields to the eight coupled differential equations

$$\begin{array}{ll} \frac{d\psi_8}{dx} - \frac{g}{\hbar}(ab - dx)\psi_1 = 0, & \frac{d\psi_7}{dx} + \frac{g}{\hbar}(ab - dx)\psi_2 = 0, \\ \frac{d\psi_6}{dx} + \frac{g}{\hbar}(ab - dx)\psi_3 = 0, & \frac{d\psi_5}{dx} - \frac{g}{\hbar}(ab - dx)\psi_4 = 0, \\ \frac{d\psi_4}{dx} - \frac{g}{\hbar}(ab - dx)\psi_5 = 0, & \frac{d\psi_3}{dx} + \frac{g}{\hbar}(ab - dx)\psi_6 = 0, \\ \frac{d\psi_2}{dx} + \frac{g}{\hbar}(ab - dx)\psi_7 = 0, & \frac{d\psi_1}{dx} - \frac{g}{\hbar}(ab - dx)\psi_8 = 0. \end{array} \tag{13}$$

This particular system can be written in the matrix form

$$\frac{d|\Psi\rangle}{dx} = f(x)\mathcal{A}|\Psi\rangle, \tag{14}$$

where $f(x) = \frac{g}{\hbar}(ab - dx)$ and $\mathcal{A}$ is a matrix with the explicit form

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

This fact has an operational advantage, let us suppose that the solution has the following form

$$|\Psi\rangle = C\exp\left(\lambda\int f(x)dx\right)|v\rangle, \tag{16}$$

where λ is a constant and $|v\rangle$ is a constant vector. Upon substitution in Equation (14) we have that the following eigenvalue equation has to be satisfied

$$\mathcal{A}|v\rangle = \lambda|v\rangle, \tag{17}$$

so our problem has been reduced to an eigenvalue problem and the general solution of Equation (14) will be

$$|\Psi\rangle = \sum_{i=1}^{8} C_i \exp\left[\lambda_i \int f(x)dx\right]|v\rangle_i, \tag{18}$$

where C_i are complex constants, λ_i are the corresponding eigenvalues of the matrix $\mathcal{A}$ and $|v\rangle_i$ its corresponding eigenvectors. Hence, following this procedure we find that the general solution of Equation (14) is given by

$$|\Psi\rangle = \begin{pmatrix} -C_1 \exp\left(-\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) + C_5 \exp\left(\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) \\ C_2 \exp\left(-\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) - C_6 \exp\left(\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) \\ C_3 \exp\left(-\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) - C_7 \exp\left(\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) \\ -C_4 \exp\left(-\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) + C_8 \exp\left(\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) \\ C_4 \exp\left(-\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) + C_8 \exp\left(\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) \\ C_3 \exp\left(-\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) + C_7 \exp\left(\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) \\ C_2 \exp\left(-\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) + C_6 \exp\left(\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) \\ C_1 \exp\left(-\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) + C_5 \exp\left(\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) \end{pmatrix}, \tag{19}$$

and it can be easily shown that this wave function satisfies Equation (13). Now we need to find the solution to $Q_2|\Psi\rangle = 0$. The differential equations arising from this equation can also be written in the form $\frac{d|\Psi\rangle}{dx} = f(x)\mathcal{B}|\Psi\rangle$ with $\mathcal{B}$ given explicitly by

$$\mathcal{B} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \tag{20}$$

and the general solution to $Q_2|\Psi\rangle = 0$ is

$$|\Psi\rangle = \begin{pmatrix} D_4 \exp\left(-\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) \\ D_8 \exp\left(\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) \\ D_7 \exp\left(\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) \\ D_3 \exp\left(-\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) \\ D_2 \exp\left(-\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) \\ D_6 \exp\left(\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) \\ D_5 \exp\left(\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) \\ D_1 \exp\left(-\frac{g}{\hbar}[abx - \frac{dx^2}{2}]\right) \end{pmatrix}. \tag{21}$$

Now that we have both solutions (19) and (21) we can see that in this case we can choose particular values for the constants C_i and D_j in order to find a common solution that satisfy $Q_1|\Psi\rangle = 0$, and $Q_2|\Psi\rangle = 0$. For instance we can choose; $C_5 = C_2 = C_3 = C_8 = 0$, $D_4 = -C_1, D_8 = -C_6, D_7 = -C_7, D_3 = -C_4, D_2 = C_4, D_6 = C_7, D_5 = C_6, D_1 = C_1$ and the solution (21) satisfies $Q_1|\Psi\rangle = 0$ and $Q_2|\Psi\rangle = 0$ at the same time, therefore it represents the ground state of the Hamiltonian for the particular model. More important is the fact that under the following identifications

$$ab = \frac{Mc}{3}, \quad d = \frac{\sqrt{\kappa}c^3}{3G}, \quad \phi_1^2 = x(t) = R(t), \tag{22}$$

where c is the constant speed of light, G is the gravitational constant and k is the curvature in the FRW metric of a cosmological model, the common solution becomes the ground state of te Hamiltonian of a supersymmetric quantum cosmological model [30]

$$|\Psi\rangle = \begin{pmatrix} -D_1 \exp\left(-[\frac{Mc}{\hbar}R - \frac{\sqrt{\kappa}c^3}{2G\hbar}R^2]\right) \\ -D_5 \exp\left([\frac{Mc}{\hbar}R - \frac{\sqrt{\kappa}c^3}{2G\hbar}R^2]\right) \\ -D_6 \exp\left([\frac{Mc}{\hbar}R - \frac{\sqrt{\kappa}c^3}{2G\hbar}R^2]\right) \\ -D_2 \exp\left(-[\frac{Mc}{\hbar}R - \frac{\sqrt{\kappa}c^3}{2G\hbar}R^2]\right) \\ D_2 \exp\left(-[\frac{Mc}{\hbar}R - \frac{\sqrt{\kappa}c^3}{2G\hbar}R^2]\right) \\ D_6 \exp\left([\frac{Mc}{\hbar}R - \frac{\sqrt{\kappa}c^3}{2G\hbar}R^2]\right) \\ D_5 \exp\left([\frac{Mc}{\hbar}R - \frac{\sqrt{\kappa}c^3}{2G\hbar}R^2]\right) \\ D_1 \exp\left(-[\frac{Mc}{\hbar}R - \frac{\sqrt{\kappa}c^3}{2G\hbar}R^2]\right). \end{pmatrix}. \tag{23}$$

In the superfield description of supersymmetric quantum cosmology it is customary to define a weighted inner product in order to guarantee quick convergence [30,31]. Note that the components, 2, 3, 6, and 7 of the wave vector solution have the right behavior for large R. Under the inner product defined in the following manner [30]

$$\langle\Psi|\Psi\rangle = \int_0^{R_{sup}} \Psi^\dagger \Psi R^{1/2} dR, \tag{24}$$

one of the exponentials in the solution converges, it is then important to be able to isolate this convergent components in order to have a normalizable solution. It can be easily seen that the constants D_i (some of them equal to zero) can be chosen in such a way to achieve this. In this case $D_1 = D_2 = 0$.

We have found so far that it is possible to construct a simultaneous normalizable solution to both supercharges and the next question to answer is whether it is possible to construct solutions that coincide with other supersymmetric cosmological wave functions. This solution was found in [32] but with a less systematic methodology as the one shown here. The next models have not been reported elsewhere and will be found following the approach shown previously. They will show us that it is possible to find other relevant normalizable cosmological solutions as well.

4.2. Second Solution

The next solution will be described by the following assumptions

$$\phi_3^1 = b, \; \phi_1^2 = d, \; \phi_1^1 = ex, \; \phi_3^2 = ex, \; \phi_2^1 = \phi_2^2 = 0, \tag{25}$$

where b, d, e are constants. In order to satisfy the commutation relation (3) the canonical momentum related to $\phi_1^1 = \phi_3^2 = ex$ is given by $\pi_3^2 = -\frac{i\hbar}{e}\frac{d}{dx}$. In this case the supercharges take the form

$$\begin{aligned} Q_1 &= \Lambda_{12}\pi_1^1 + \Lambda_{31}\pi_3^2 - g\Lambda_{22}(bd - e^2x^2), \\ Q_2 &= \Lambda_{11}\pi_1^1 - \Lambda_{32}\pi_3^2 + g\Lambda_{21}(bd - e^2x^2), \end{aligned} \tag{26}$$

and the equations $Q_1|\Psi\rangle = 0$ and $Q_2|\Psi\rangle = 0$ become respectively the systems

$$\begin{aligned} \tfrac{d|\Psi\rangle}{dx} &= f(x)\mathcal{A}|\Psi\rangle, \\ \tfrac{d|\Psi\rangle}{dx} &= f(x)\mathcal{B}|\Psi\rangle, \end{aligned} \tag{27}$$

where $f(x) = eg/2\hbar(bd - e^2x^2)$ and the matrices $\mathcal{A}$ and $\mathcal{B}$ are respectively

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & -1-i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-i & 0 & 0 & 0 & 0 \\ -1+i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1-i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-i \\ 0 & 0 & 0 & 0 & -1-i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+i & 0 & 0 \end{pmatrix}, \tag{28}$$

$$\mathcal{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 1 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 1 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 1 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & -1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & -1 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & -1 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{29}$$

Following the same procedure described in the last section we can first write down the solutions Equation (18) to $Q_1|\Psi\rangle_1 = 0$ and $Q_2|\Psi\rangle_2 = 0$

$$\begin{aligned} |\Psi\rangle_1 &= \textstyle\sum_{i=1}^{8} C_{1i} \exp\left[\lambda_{1i} \int f(x)dx\right]|v\rangle_{1i}, \\ |\Psi\rangle_2 &= \textstyle\sum_{i=1}^{8} C_{2i} \exp\left[\lambda_{2i} \int f(x)dx\right]|v\rangle_{2i}, \end{aligned} \tag{30}$$

with the corresponding eigenvalues and eigenvectors of the matrices $\mathcal{A}$ and $\mathcal{B}$. In this model, it is also possible to find first, a common solution to both supercharges that will in general be a linear combination of increasing and decreasing exponentials and second, isolate the normalizable wave function. Here we show two particular nontrivial solutions

$$|\Psi\rangle = \begin{pmatrix} (i - i\sqrt{2}) \\ (-i + i\sqrt{2}) \\ 1/2(-2+\sqrt{2}-2i+i\sqrt{2}) \\ 1/2(2-\sqrt{2}-2i+i\sqrt{2}) \\ \sqrt{2}/2(1+i) \\ \sqrt{2}/2(-1+i) \\ 1 \\ 1 \end{pmatrix} \exp\left(-\frac{\sqrt{2}eg}{2\hbar}[bdx - \frac{e^2}{3}x^3]\right) \tag{31}$$

$$|\Psi\rangle = \begin{pmatrix} (i + i\sqrt{2}) \\ -(i + i\sqrt{2}) \\ (-2-\sqrt{2}-2i-i\sqrt{2}) \\ (2+\sqrt{2}-2i-i\sqrt{2}) \\ -\sqrt{2}/2(1+i) \\ \sqrt{2}/2(1-i) \\ 1 \\ 1 \end{pmatrix} \exp\left(\frac{\sqrt{2}eg}{2\hbar}[bdx - \frac{e^2}{3}x^3]\right). \tag{32}$$

These wave functions are simultaneous solutions to both supercharges and under the following identification of the constants and variables of the model

$$b = \frac{1}{e}, \quad d = \frac{2c^2 M^{1/2}}{3G^{1/2}}, \quad e^3 = \frac{\sqrt{6}c^3 \Lambda^{1/2}}{9G}, \quad x(t) = R(t), \tag{33}$$

where c is the speed of light, G is the gravitational constant, M is the mass parameter of the matter content of the Universe and Λ is the cosmological constant. Then the normalizable solution becomes

$$|\Psi\rangle = \begin{pmatrix} (i + i\sqrt{2}) \\ -(i + i\sqrt{2}) \\ (-2 - \sqrt{2} - 2i - i\sqrt{2}) \\ (2 + \sqrt{2} - 2i - i\sqrt{2}) \\ -\sqrt{2}/2(1+i) \\ \sqrt{2}/2(1-i) \\ 1 \\ 1 \end{pmatrix} \exp\left(\frac{\sqrt{2}c^2 M^{1/2}}{\hbar G^{1/2}} R - \frac{c^3 \Lambda^{1/2}}{3\sqrt{3}\hbar G} R^3 \right), \tag{34}$$

which is the solution to a supersymmetric quantum cosmological model, that is, a flat FRW model with cosmological constant and radiation ($\gamma = 1/3$) as matter [33]. The structure of the matrix model could also allow to find another cosmological solution but there is nothing that indicates a priori that the solutions to $Q_1|\Psi\rangle = 0$ and $Q_2|\Psi\rangle = 0$ could have common solutions and even worse, that these solutions could allow a linear combination that results in a normalizable solution.

4.3. Third Solution

The next solution considers the following

$$\phi_3^1 = ex, \ \phi_1^2 = b, \ \phi_1^1 = ex, \ \phi_3^2 = ex, \ \phi_2^1 = \phi_2^2 = 0. \tag{35}$$

where b, e are constants. Solving for the supercharges which in this case take the form

$$\begin{aligned} Q_1 &= \Lambda_{31}\pi_3^2 + \Lambda_{12}\pi_2^2 - g\Lambda_{22}(bex - e^2x^2), \\ Q_2 &= \Lambda_{11}\pi_1^1 + \Lambda_{31}\pi_3^1 + g\Lambda_{21}(bex - e^2x^2), \end{aligned} \tag{36}$$

the differential equations for the eight components of the wave function $|\Psi\rangle$, $Q_1|\Psi\rangle = 0$ and $Q_2|\Psi\rangle = 0$ can also be written in the form (14) and the explicit form of the matrices $\mathcal{A}$ and $\mathcal{B}$ for both systems is the following

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & -1-i & -i \\ 0 & 0 & i & 1-i \\ -1+i & -i & 0 & 0 \\ i & i+1 & 0 & 0 \end{pmatrix}, \tag{37}$$

$$\mathcal{B} = \begin{pmatrix} B & C \\ -C & -B \end{pmatrix}, \quad B = I_{4\times4}, \ C = \begin{pmatrix} 0 & 0 & -1 & -i \\ 0 & 0 & -i & 1 \\ 1 & -i & 0 & 0 \\ i & -1 & 0 & 0 \end{pmatrix}.$$

In tis case also common solutions exists and the complex constants in the linear combinations of the solutions can be chosen in such a way to have a normalizable solution. After the identification

$$b = \frac{3^{1/6}\sqrt{\kappa c}}{(G\Lambda)^{1/3}}, \quad e = -\frac{c\Lambda^{1/2}}{(3G)^{1/3}}, \quad x(t) = R(t). \tag{38}$$

the normalizable solution becomes

$$|\Psi\rangle = \begin{pmatrix} 1/2(-1+\sqrt{3}+i-i\sqrt{3}) \\ 1/2(1-i-\sqrt{3}+i\sqrt{3}) \\ 1/6(-3-9i+\sqrt{3}+3i\sqrt{3}) \\ 1/6(3-3i-\sqrt{3}+i\sqrt{3}) \\ 1/\sqrt{3}(1+2i) \\ -1/\sqrt{3} \\ 1 \\ 1 \end{pmatrix} \exp\left(-\frac{\sqrt{\kappa}c^3}{2\hbar G}R^2 - \frac{c^3\Lambda^{1/2}}{3\sqrt{3}\hbar G}R^3\right) \tag{39}$$

which can be identified with he supersymmetric quantum solution of the FRW model with curvature κ and cosmological constant [30,33,34].

The matrix solutions found so far, were compared to those arising from the general FRW superfield action ($c = 1$ and $\tilde{G} = \frac{8\pi G}{6}$)

$$S = \int \left[-\frac{1}{2\tilde{G}}\mathbb{N}^{-1}\mathbb{R}D_{\eta}\mathbb{R} + \frac{\sqrt{\kappa}}{2\tilde{G}}\mathbb{R}^2 + \frac{\Lambda^{1/2}}{3\sqrt{3}\tilde{G}}\mathbb{R}^3 - \frac{2\sqrt{2}}{\tilde{G}^{1/2}}\sum_i \frac{M_{\gamma_i}^{1/2}}{(3-3\gamma_i)}\mathbb{R}^{3-3\gamma_i/2}\right] d\eta d\overline{\eta} dt, \tag{40}$$

where $\mathbb{N}(\eta,\overline{\eta},t)$ and $\mathbb{R}(\eta,\overline{\eta},t)$ are the relevant superfields. This action allows contributions of curvature κ, and several matter terms like the cosmological constant Λ and fluids following the state equation $p = \gamma_i\rho$.

By the use of the Taylor expansion of these superfields, which involves fermionic gravitino fields $(\lambda, \overline{\lambda})$, the corresponding Hamiltonian H and Wheeler-deWit equation $H\Psi = 0$ is constructed following a systematic procedure [30,31,33–35]. In this approach also, a matrix representation for the fermionic fileds is chosen and the resulting wave functions are two component vectors

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \tag{41}$$

whose general solutions have the form

$$\begin{aligned} \Psi_1(R) &= C_1 \exp\left[-\frac{\sqrt{\kappa}}{2G\hbar}R^2 + \frac{1}{\sqrt{6}}\left(\frac{\rho_\Lambda}{\rho_{pl}}\right)\left(\frac{R}{l_{pl}}\right)^3 - \frac{\sqrt{18}}{\sqrt{6\pi}}\frac{1}{\rho_{pl}^{1/2}}\left(\frac{R}{l_{pl}}\right)^3 \sum_i \frac{\rho_i^{1/2}}{(3-3\gamma_i)} \right], \\ \Psi_2(R) &= C_1 \exp\left[\frac{\sqrt{\kappa}}{2G\hbar}R^2 - \frac{1}{\sqrt{6}}\left(\frac{\rho_\Lambda}{\rho_{pl}}\right)\left(\frac{R}{l_{pl}}\right)^3 + \frac{\sqrt{18}}{\sqrt{6\pi}}\frac{1}{\rho_{pl}^{1/2}}\left(\frac{R}{l_{pl}}\right)^3 \sum_i \frac{\rho_i^{1/2}}{(3-3\gamma_i)} \right], \end{aligned} \tag{42}$$

where the mass parameter is related to the scale factor and its corresponding density as $M_{\gamma_i} = \frac{1}{2}\rho_{\gamma_i}R^{3(1+\gamma_i)}$, $\rho_\Lambda = \frac{\Lambda}{8\pi G}$ and $\rho_{pl} = G^{-2}$. Certain conditions might be imposed in order to get normalizable functions [33].

Our eight component matrix solutions are particular cases compared to these Equation (42) which are just two component ones but as it can be seen, both sets of solutions have only two independent components and only one of them can be allowed to be normalizable. Consequently they can be regarded as being the same. The comparison between our solutions and those of Supersymmetric Quantum Cosmology (SUSYQC) models was made at the level of the wave vector functions, however it is possible to compare Hamiltonians in order to see that the reduced Hamiltonians and the ones in SUSYQC also coincide [30,33].

5. Conclusions

We have developed a method that allows to solve the ground state solution for the complete sector of a SU(N) invariant supersymmetric matrix model [2,5,8]. Such method was solved here for a reduced and somehow simplified model, but it is important to note that it could be in principle solved for any finite N. One relevant characteristic of the procedure followed is the use of a Dirac-like gamma matrix representation for the fermionic matrices $\Lambda_{a\alpha}$. This method allows to find the ground state wave function solutions, for the matrix model considered here, in terms of multicomponent vectors instead of scalar functions of real and Grassman variables. The additional reductions we made, compared to those earlier implemented to solve the matrix model [2,5,8], allow to find non-trivial and normalizable solutions. This could not be the case in general for non-truncated models [5].

It is known that the Discrete Light Cone quantization provides a way to define a consistent finite N matrix theory [2] and it has been found that the degrees of freedom of finite N matrix models as the one solved here correspond to those of a Super Yang Mills Theory (SYM). We have shown in this work that we are able to find the quantum solution of one particular matrix model for any finite N [8]. We suspect that we could be able to identify the degrees of freedom arising from our finite N solutions with the ones coming from SYM theory. If such identification is possible, quantization of SYM theory will be a matter of quantizing the finite N matrix model shown here. This is however a matter of a future analysis and will not be discussed in detail here. Moreover, it will be of interest the connection between the finite N matrix theory and some low energy models associated with specific symmetry groups of the Standard Model, and beyond like, SU(N) with N = 5, 3, 2, etc. This is still a more elaborated procedure which remains to be analyzed in the future.

We have found also that the particular SU(2) matrix model solved here contains supersymmetric quantum cosmological solutions as vacuum solutions. Those wave functions have been found independently in the context of supersymmetric quantum cosmology [30,33,34]. Those solutions give relevance to our method of solution and contributes to consider the importance of solutions of finite N matrix models [3,4,36–39].

If we attempt to solve this matrix model for a bigger group and dimension, the size of the vector solution is larger and powerful computational methods will be required in order to solve the systems of differential equations when solving for the corresponding supercharges. By solving a modest and simple reduced model, our intention here was in principle to expose the methodology of solution on analytic grounds where even for the reduced model, we found our method to be particularly productive. More complete models will be solved and the possible relation with SYM theories will be presented in a future work.

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References

1. Polchinski, J. String Duality. *Rev. Mod. Phys.* **1996**, *68*, 1245–1258. [CrossRef]
2. de Wit, B.; Hoppe, J.; Nicolai, H. On the quantum mechanics of supermembranes. *Nucl. Phys. B* **1988**, *305*, 545–581. [CrossRef]
3. Susskind, L. Another conjecture about M(atrix) theory. *arXiv* **1997**, arXiv:hep-th/9704080.
4. Seiberg, N. Why Is the Matrix Model Correct? *Phys. Rev. Lett.* **1997**, *79*, 3577–3580. [CrossRef]
5. Jürg, F.; Jens, H. On Zero-Mass Ground States in Super-Membrane Matrix Models. *Commun. Math. Phys.* **1998**, *191*, 613–626.
6. de Wit, B.; Luscher, M.; Nicolai, H. The Supemembrane is Unstable. *Nucl. Phys. B* **1989**, *320*, 135–159. [CrossRef]
7. Nicolai, H.; Helling, R. Supermembranes and M(atrix) Theory. *arXiv* **1998**, arXiv:hep-th/9809103.
8. Halpern, M.B.; Schwartz, C. Asymptotic search for ground states of SU(2) matrix theory. *Int. J. Mod. Phys. A* **1998**, *13*, 4367–4408. [CrossRef]

9. Banks, T.; Fischler, W.; Shenker, S.H.; Susskind, L. *M* Theory as a Matrix Model: A Conjeture. *Phys. Rev. D* **1997**, *55*, 5112–5128. [CrossRef]
10. Duff, M.J.; Howe, P.; Inami, T.; Stelle, K.S. Superstrings in D = 10 from supermembranes in D = 11. *Phys. Lett. B* **1987**, *191*, 70–74. [CrossRef]
11. Taylor, W. M(atrix) theory: Matrix quantum mechanics as a fundamental theory. *Rev. Mod. Phys.* **2001**, *73*, 419–462. [CrossRef]
12. Claudson, M.; Halpern, M.B. Supersymmetric ground State wave functions. *Nucl. Phys. B* **1985**, *250*, 689–715. [CrossRef]
13. Samuel, S. Solutions of extended supersymmetric matrix models for arbitrary gauge groups. *Phys. Lett. B* **1997**, *411*, 268–273. [CrossRef]
14. Sethi, S.; Stern, M. D-brane bound state redux. *Commun. Math. Phys.* **1998**, *194*, 675–705. [CrossRef]
15. Sethi, S.; Stern, M. The structure of the D0-D4 bound state. *Nucl. Phys. B* **2000**, *578*, 163–198. [CrossRef]
16. Fröhlich, J.; Graf, G.M.; Hasler, D.; Hoppe, J.; Yau, S.T. Asymptotic form of zero energy wave functions in supersymmetric matrix models. *Nucl. Phys. B* **2000**, *567*, 231–248. [CrossRef]
17. Hoppe, J. Zero energy states in supersymmetric matrix models. *Class. Quantum Gravity* **2000**, *17*, 1101–1105. [CrossRef]
18. Michishita, Y.; Trzetrzelewski, M. Towards the ground state of the supermembrane. *Nucl. Phys. B* **2013**, *868*, 539–553. [CrossRef]
19. Trzetrzelewski, M. Large *N* behavior of two dimensional supersymmetric Yang-Mills quantum mechanics. *J. Math. Phys.* **2007**, *48*, 012302. [CrossRef]
20. D'Eath, P.D.; Hawking, S.W.; Obregón, O. Supersymmetric Bianchi models and the square root of the Wheeler-DeWitt equation. *Phys. Lett. B* **1993**, *300*, 44–48. [CrossRef]
21. Vargas Moniz, P. *N* = 2 supersymmetric FRW quantum cosmology from a D-p-brane gas. *J. Phys. A Math. Gen.* **2004**, *37*, 10445–10458. [CrossRef]
22. Cheng, A.D.Y.; D'Eath, P.D.; Moniz, P.V. Quantization of the Bianchi type IX model in supergravity with a cosmological constant. *Phys. Rev. D* **1994**, *49*, 5246–5251. [CrossRef] [PubMed]
23. D'Eath, P.D. *Supersymmetric Quantum Cosmology*; Cambridge University Press: Cambridge, UK, 1996.
24. Vargas Moniz, P. *Quantum Cosmology: The Supersymmetric Perspective*; Lecture Notes in Physics; Springer: Berlin, Germany, 2010; Volume 1.
25. Obregón, O.; Ramírez, C. Dirac like formulation of quantum supersymmetric cosmology. *Phys. Rev. D* **1998**, *57*, 1015–1026. [CrossRef]
26. Tkach, V.I.; Rosales, J.J.; Obregón, O. Supersymmetric action for Bianchi type models. *Class. Quantum Gravity* **1996**, *13*, 2349–2356. [CrossRef]
27. Freedman, D.Z.; Gibbons, G.; Schnabl, M. Matrix cosmology. *AIP Conf. Proc.* **2004**, *743*, 286–297.
28. Alvarez, E.; Meessen, P. Newtonian M(atrix) cosmology. *Phys. Lett. B* **1998**, *426*, 282–286. [CrossRef]
29. Obregón, O.; Capovilla, R. No quantum Minisuperspace with $\Lambda \neq 0$. *Phys. Rev. D* **1994**, *49*, 6562–6565.
30. Ortiz, C.; Rosales, J.J.; Socorro, J.; Torres-arenas, J.; Tkach, V.I. Wave functions in Susy cosmological models with matter. *Phys. Lett. A* **2005**, *340*, 51–58. [CrossRef]
31. Obregón, O.; Rosales, J.J.; Socorro, J.; Tkach, V.I. Supersymmetry breaking and a normalizable wavefunction for the FRW ($k = 0$) cosmological model. *Class. Quantum Gravity* **1999**, *16*, 2861–2870. [CrossRef]
32. Lopez, J.L.; Obregón, O. Supersymmetric quantum matrix cosmology. *Class. Quantum Gravity* **2015**, *32*, 235014. [CrossRef]
33. Rosales, J.J.; Tkach, V.I. Supersymmetric cosmological FRW model and dark energy. *Phys. Rev. D* **2010**, *82*, 107502. [CrossRef]
34. Obregón, O; Rosales, J.J.; Tkach, V.I. Superfield description of the FRW universe. *Phys. Rev. D* **1996**, *53*, R1750–R1753. [CrossRef] [PubMed]
35. Macías, A.; Obregón, O.; Ryan, M.P., Jr. Quantum cosmology: The supersymmetric square root. *Class. Quantum Gravity* **1987**, *4*, 1477–1486. [CrossRef]
36. Seiberg, N. New theories in six dimensions and matrix description of M-theory on T^5 and $T^5 Z_2$. *Phys. Lett. B* **1997**, *408*, 99–104. [CrossRef]
37. Berkooz, M.; Rozali, M.; Seiberg, N. Matrix description of M-theory on T^4 and T^5. *Phys. Lett. B* **1997**, *408*, 105–110. [CrossRef]
38. Cosmas, Z.; David, F.; Thomas, C. *Sypersymmetry and Integrable Models: Matrix Membranes and Integrability*; Springer: Berlin, Germany, 1998.
39. Boulton, L.; García del Moral, M.P.; Restuccia, A. Massless ground state for a compact $SU(2)$ matrix model in 4D. *Nucl. Phys. B* **2015**, *898*, 448–455. [CrossRef]