

Article

General Thermodynamic Properties of FRW Universe and Heat Engine

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Abstract: In this work, the Friedmann–Robertson–Walker (FRW) Universe is considered a thermodynamic system, where the cosmological constant generates the thermodynamic pressure. Using a unified first law, we have determined the amount of energy dE crossing the apparent horizon. Since heat is one of the forms of thermal energy, so the heat flows δQ through the apparent horizon = amount of energy crossing the apparent horizon. Using the first law of thermodynamics, on the apparent horizon, we found $TdS = A(\rho + p)H\tilde{r}_h dt + A\rho d\tilde{r}_h$ where $T, S, A, H, \tilde{r}_h, \rho, p$ are respectively the temperature, entropy, area, Hubble parameter, horizon radius, fluid density and pressure. Since the apparent horizon is dynamical, so we have assumed that $d\tilde{r}_h$ cannot be zero in general, i.e., the second term $A\rho d\tilde{r}_h$ is non-zero on the apparent horizon. Using Friedmann equations with the unified first law, we have obtained the modified entropy-area relation on the apparent horizon. In addition, from the modified entropy-area relation, we have obtained modified Friedmann equations. From the original Friedmann equations and also from modified Friedmann equations, we have obtained the same entropy. We have derived the equations for the main thermodynamical quantities, such as temperature, volume, mass, specific heat capacity, thermal expansion, isothermal compressibility, critical temperature, critical volume, critical pressure and critical entropy. To determine the cooling/heating nature of the FRW Universe, we have obtained the coefficient of Joule–Thomson expansion. Next, we have discussed the heat engine phenomena of the thermodynamical FRW Universe. We have considered the Carnot cycle and obtained its completed work. In addition, we studied the work completed and the thermal efficiency of the new heat engine. Finally, we have obtained the thermal efficiency of the Rankine cycle.

Keywords: universe; thermodynamics; entropy; heat engine



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1. Introduction

In astrophysics, the thermodynamic properties of the black hole are extensively studied. It is related to the quantum aspects of spacetime geometry with classical thermodynamic theory. The origin of this study started from the pioneering work of Hawking and Bekenstein, who first proposed Hawking's temperature and Bekenstein–Hawking entropy [1,2], where entropy is related to area, and temperature is related to surface gravity. Hawking and Page have proposed that the first-order phase transition occurs between the Schwarzschild anti-de Sitter (AdS) black hole and the thermal AdS space [3]. Several authors [4–8] have studied the geometry of AdS black hole thermodynamics. In the black hole thermodynamic study, the thermodynamic pressure P arose from negative cosmological constant ($\Lambda < 0$). Thus, $P = -\frac{\Lambda}{8\pi}$ [9–11]. The black hole horizon entropy and cosmological horizon entropy [1,12] satisfy the same Bekenstein–Hawking entropy-area relation $S = A/4G$, where A is the horizon surface area, and G is Newton's gravitational constant. For the AdS black hole, Johnson [13–15] has introduced the holographic heat engine. For various types of AdS black holes, the heat engine phenomena have been discussed by several authors [16–19]. Heat engines for regular black holes have been studied in [20–22]. Kaburaki and Okamoto [23] have studied the Carnot engine phenomena of Kerr

black holes. Recently, for some kinds of AdS black holes [24–26], we have discussed the thermodynamic nature, P - V criticality, Joule–Thomson expansion and also the work done with efficiency for the Carnot and Rankine cycles of the heat engine.

As per the proposal of the dynamical (non-stationary) black hole thermodynamics (on the trapping horizon) [27,28], Einstein’s field equations for spherical symmetric space-time can be written in the form of “unified first law”. Similarly, the thermodynamic proposal can be applied to the non-stationary spherically symmetric Friedmann–Robertson–Walker (FRW) Universe on the apparent horizon (trapping horizon). In the black hole thermodynamics, Jacobson [29] has investigated that the Einstein’s field equations may be obtained from the entropy–area relation. Using the unified first law and entropy–area relation on the apparent horizon, Cai et al. [30,31] have derived the Friedmann equations for modified gravity theories like Gauss–Bonnet gravity, scalar-tensor theory, and Lovelock gravity. Several authors [32–37] have studied the thermodynamic phenomena and also obtained modified Friedmann equations from modified entropy–area relations in the FRW Universe.

In the Universe model, Pilot [38] has studied the thermodynamic heat engine, and Askin et al. [39] have discussed the heat engine phenomena in the Carnot cycle for polytropic gas. The Carnot cycle in general relativity has been studied in [40]. Recently, we have described the thermodynamic properties of the FRW Universe and its heat engine phenomena with efficiency for Carnot cycle [41]. Motivated by the above-mentioned works, here, we will study the unified first law and thermodynamics for the non-flat FRW Universe in the presence of the cosmological constant. With the help of Einstein’s field equations, we calculate the form of the entropy–area relation and show that the entropy–area relation can generate Einstein’s field equations. We discuss the thermodynamic quantities, Joule–Thomson expansion, and some kinds of heat engines for the FRW Universe. The organization of the work is as follows: In Section 2, we study the unified first law and, using Friedmann equations, we find the form of entropy–area relation. In addition, using the entropy–area relation, we determine the modified Friedmann equations. In Section 3, we discuss some thermodynamic quantities and the coefficient of Joule–Thomson expansion. In Section 4, we discuss the heat engine and also study the Carnot cycle as well as Rankine cycle with their work done and efficiencies for the FRW Universe. Finally, in Section 5, we conclude the whole work.

2. Construction of Entropy for the FRW Universe

In this section, we’ll study the unified first law for non-flat Friedmann–Robertson–Walker (FRW) Universe and construct the entropy using the Friedmann equations. We assume the line element for FRW Universe as

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] \quad (1)$$

where $a(t)$ is the scale factor and $k = 0, -1, +1$. The Einstein–Hilbert action can be considered in the form

$$S = \int d^4x \sqrt{-g} \left[\frac{R - 2\Lambda}{16\pi G} + \mathcal{L}_m \right] \quad (2)$$

where R is the Ricci scalar, Λ is cosmological constant, G is the Newton’s gravitational constant, \mathcal{L}_m is the matter Lagrangian and $g = \det(g_{ij})$ (choosing $c = 1$). For perfect fluid source, the stress–energy tensor is $T_{ij} = (\rho + p)u_i u_j + p g_{ij}$, where ρ and p are the energy density and pressure of perfect fluid, respectively, where u_i is the four velocity satisfying the relation $u_i u^i = -1$. Thus, the Friedmann equations are given by

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3} \quad (3)$$

and

$$\dot{H} - \frac{k}{a^2} = -4\pi G(\rho + p) \quad (4)$$

where $H = \dot{a}/a$ is the Hubble parameter. From the continuity equation, we obtain

$$\dot{\rho} + 3H(\rho + p) = 0 \quad (5)$$

which can also be found from the above Friedmann Equations (3) and (4).

Now the FRW metric (1) can be written in the form [42] $ds^2 = h_{ij}dx^i dx^j + \tilde{r}^2 d\Omega_2^2$ where $x^0 = t$, $x^1 = r$, $\tilde{r} = a(t)r$, and $h_{ij} = \text{diag}(-1, \frac{a^2}{1-kr^2})$. If the FRW universe can be considered as a thermodynamical system, then its dynamical apparent horizon can be determined by the relation $h^{ij} \frac{\partial \tilde{r}}{\partial x^i} \frac{\partial \tilde{r}}{\partial x^j} = 0$. Thus, the apparent horizon radius \tilde{r}_h and its time derivative are obtained as [28,42,43]

$$\tilde{r}_h = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}}, \quad \dot{\tilde{r}}_h = -\tilde{r}_h^3 H \left(\dot{H} - \frac{k}{a^2} \right). \quad (6)$$

The surface gravity and the temperature on the apparent horizon for the FRW Universe are obtained as [30]

$$\kappa = -\frac{1}{\tilde{r}_h} \left(1 - \frac{\dot{\tilde{r}}_h}{2\tilde{r}_h H} \right), \quad T = \frac{|\kappa|}{2\pi} = \frac{1}{2\pi\tilde{r}_h} \left(1 - \frac{\dot{\tilde{r}}_h}{2\tilde{r}_h H} \right) \quad (7)$$

where $h = \det(h_{ij})$. In particular, if the apparent horizon radius \tilde{r}_h is remaining fixed within a small time interval dt , then $\dot{\tilde{r}}_h \ll 2\tilde{r}_h H$. Thus, in this approximation, volume will not be changed in it and hence we obtain [30] $T \approx \frac{1}{2\pi\tilde{r}_h}$.

Hayward proposed the unified first law as in the form [28,31,42,43] $dE = A\Psi + WdV$, where dE is the energy change inside the apparent horizon, $A = 4\pi\tilde{r}_h^2$ is the surface area of the apparent horizon, $V = \frac{4\pi}{3}\tilde{r}_h^3$ is the volume inside the apparent horizon surface, $W = \frac{1}{2}(\rho - p)$ is the work density function and the energy flux is given by

$$\Psi = -\frac{1}{2}(\rho + p)H\tilde{r}dt + \frac{1}{2}(\rho + p)adr. \quad (8)$$

Therefore, we obtain

$$A\Psi + WdV = -A(\rho + p)H\tilde{r}dt + A\rho d\tilde{r}. \quad (9)$$

Since heat is one kind of thermal energy, the heat flow δQ through the apparent horizon = amount of energy (dE) crosses the apparent horizon. Therefore, from (9), we obtain

$$\delta Q = dE = -A(\rho + p)H\tilde{r}_h dt + A\rho d\tilde{r}_h. \quad (10)$$

It should be noted that Cai and Kim [30] have assumed $\delta Q = -dE$, and they have assumed only the first term and ignored the second term $A\rho d\tilde{r}_h$ in (10) on the apparent horizon. Till now, these assumptions have been taken into account for all thermodynamical studies. Since the apparent horizon is dynamical, thus $d\tilde{r}_h$ cannot be zero in general. Therefore, we now consider all the terms of (10) for further study on the apparent horizon. The first law of thermodynamics on the apparent horizon is $\delta Q = TdS$. Thus, from Equation (10), we obtain

$$TdS = -A(\rho + p)H\tilde{r}_h dt + A\rho d\tilde{r}_h. \quad (11)$$

- **Derivation of Entropy using Friedmann Equations:**

From Equations (3), (4) and (6), we obtain

$$\rho = \frac{3}{8\pi G \tilde{r}_h^2} - \frac{\Lambda}{8\pi G}, \quad \rho + p = \frac{\dot{\tilde{r}}_h}{4\pi G \tilde{r}_h^3 H}. \quad (12)$$

Putting the expressions (7) and (12) in (11), we obtain the entropy as

$$S = \frac{A}{4G} - \frac{\Lambda A^2}{32\pi G} + S_0 \quad (13)$$

where S_0 is an integration constant. We see that the entropy S depends on the cosmological constant Λ . Putting $\Lambda = 0$ and $S_0 = 0$, we obtain $S = A/4G$, which is similar to the form of the black hole entropy on the horizon.

• Derivation of Modified Friedmann Equations Using Entropy:

For the FRW Universe in the presence of cosmological constant Λ , we choose the entropy with the form $S = \frac{A}{4G} - \frac{\Lambda A^2}{32\pi G} + S_0$ and the expression of the temperature has the form (7); then, using the continuity Equation (5) with the Equations (6) and (11), we obtain the differential equation:

$$\left[AG \left(\rho + \frac{\Lambda}{8\pi G} \right) - \frac{1}{2} \right] d \left(H^2 + \frac{k}{a^2} \right) = \frac{8\pi G}{3} d\rho. \quad (14)$$

Integrating the above equation and after simplification, we obtain

$$\left(H^2 + \frac{k}{a^2} \right) + K \left(H^2 + \frac{k}{a^2} \right)^{3/2} = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3} \quad (15)$$

where K is an integration constant. Now, putting the expression of ρ in conservation Equation (5), we obtain

$$\left(\dot{H} - \frac{k}{a^2} \right) \left[1 + \frac{3K}{2} \left(H^2 + \frac{k}{a^2} \right)^{1/2} \right] = -4\pi G(\rho + p). \quad (16)$$

Thus, from entropy $S = \frac{A}{4G} - \frac{\Lambda A^2}{32\pi G} + S_0$, we obtain the modified Friedmann Equations (15) and (16). For $K = 0$, we can recover the original Friedmann Equations (3) and (4). However, if we consider the two Equations (15) and (16), then, from Equation (11), we may obtain the entropy in the form $S = \frac{A}{4G} - \frac{\Lambda A^2}{32\pi G} + S_0$, where S_0 is an integration constant. This is independent of K . Thus, from original Friedmann Equations (3) and (4) and also from modified Friedmann Equations (15) and (16), we obtain the same entropy $S = \frac{A}{4G} - \frac{\Lambda A^2}{32\pi G} + S_0$.

3. Thermodynamic Quantities

The usual Friedmann Equations (3) and (4) can generate the entropy in the form (13), which has also been generated from modified Friedmann Equations (15) and (16). Thus, we consider the entropy on the apparent horizon for both usual Friedmann equations and modified Friedmann equations in the FRW Universe as given in (13). In the study of black hole thermodynamics and FRW Universe [41], the cosmological constant Λ is treated as thermodynamic pressure P , where $P = -\frac{\Lambda}{8\pi G}$ and allows for a variable. Thus, using the same consideration, the entropy expression (13) can be written as

$$S = \frac{\pi \tilde{r}_h^2}{G} - 4\pi^2 \tilde{r}_h^4 P + S_0. \quad (17)$$

Thus, the apparent horizon radius \tilde{r}_h can be expressed as

$$\tilde{r}_h = \frac{1}{\sqrt{8\pi GP}} \left[1 + \left(1 - 16G^2P(S - S_0) \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \quad (18)$$

where the entropy satisfies $S < S_0 + \frac{1}{16G^2P}$. The expression of volume inside the apparent horizon can be written as

$$V = \frac{4\pi\tilde{r}_h^3}{3} = \frac{1}{6\sqrt{8\pi} G^{\frac{3}{2}} P^{\frac{3}{2}}} \left[1 + \left(1 - 16G^2P(S - S_0) \right)^{\frac{1}{2}} \right]^{\frac{3}{2}}. \quad (19)$$

In addition, the expression of temperature (7) can be written as

$$T = \sqrt{\frac{GP}{2\pi}} \left[1 + \left(1 - 16G^2P(S - S_0) \right)^{\frac{1}{2}} \right]^{-\frac{1}{2}}. \quad (20)$$

From Equation (12), we obtain the energy density and pressure of the fluid as

$$\rho = \frac{3}{8\pi G\tilde{r}_h^2} - P = -P + 3P \left[1 + \left(1 - 16G^2P(S - S_0) \right)^{\frac{1}{2}} \right]^{-1} \quad (21)$$

and

$$p = -\frac{1}{8\pi G\tilde{r}_h^2} + P = P - P \left[1 + \left(1 - 16G^2P(S - S_0) \right)^{\frac{1}{2}} \right]^{-1}. \quad (22)$$

Thus, we can write $P = \frac{1}{2}(\rho + 3p)$. The mass inside the region of apparent horizon surface is $M = \rho V$. Using Equations (3), (6) and (19), we obtain the mass

$$M = \frac{1}{6\sqrt{8\pi} G^{\frac{1}{2}} P^{\frac{3}{2}}} \left[2 - \left(1 - 16G^2P(S - S_0) \right)^{\frac{1}{2}} \right] \left[1 + \left(1 - 16G^2P(S - S_0) \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}. \quad (23)$$

The specific heat capacity for FRW Universe is obtained as [41]

$$C_P = T \left(\frac{\partial S}{\partial T} \right)_P = \frac{1}{4G^2P} \left(1 - 16G^2P(S - S_0) \right)^{\frac{1}{2}} \left[1 + \left(1 - 16G^2P(S - S_0) \right)^{\frac{1}{2}} \right]. \quad (24)$$

The coefficient of thermal expansion and isothermal compressibility are obtained as [44]

$$\alpha = \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_P = -\sqrt{\frac{18\pi}{GP}} \left[1 + \left(1 - 16G^2P(S - S_0) \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \quad (25)$$

and

$$\kappa_T = -\frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_T = -\frac{3}{2P} + \frac{12G^2(S - S_0)(1 - 16G^2P(S - S_0))^{-\frac{1}{2}}}{1 + (1 - 16G^2P(S - S_0))^{\frac{1}{2}}}. \quad (26)$$

The minus sign indicates that, when the pressure increases, the volume always reduces. Similar to the critical behavior of the AdS black hole, here we study the critical behavior of the FRW Universe. The critical points can be found from the following conditions:

$$\left(\frac{\partial P}{\partial \tilde{r}_h} \right)_{cr} = 0, \quad \left(\frac{\partial^2 P}{\partial \tilde{r}_h^2} \right)_{cr} = 0. \quad (27)$$

From these conditions, we obtain the following critical values of the thermodynamic quantities:

$$p_{cr} = c_1 - \frac{1}{8\pi G \tilde{r}_{cr}^2}, P_{cr} = c_1 = \text{constant}, \quad (28)$$

$$S_{cr} = \frac{\pi \tilde{r}_{cr}^2}{2G} - 4c_1 \pi^2 \tilde{r}_{cr}^4 + S_0, \quad (29)$$

$$V_{cr} = \frac{1}{6\sqrt{8\pi} (c_1 G)^{\frac{3}{2}}} \left[1 + \left(1 - 16c_1 G^2 (S_{cr} - S_0) \right)^{\frac{1}{2}} \right]^{\frac{3}{2}}, \quad (30)$$

$$T_{cr} = \sqrt{\frac{c_1 G}{2\pi}} \left[1 + \left(1 - 16c_1 G^2 (S_{cr} - S_0) \right)^{\frac{1}{2}} \right]^{-\frac{1}{2}}. \quad (31)$$

Now, we will study the Joule–Thomson expansion process [45,46], which describes the temperature changes from a high-pressure region to a low-pressure region, provided the enthalpy function (\mathcal{H}) remains constant throughout the expansion scenario. The Joule–Thomson coefficient can be defined as $\mu = \left(\frac{\partial T}{\partial P} \right)_{\mathcal{H}}$ which describes the slope of the isenthalpic curve [47]. The Joule–Thomson coefficient can be expressed as

$$\mu = \frac{1}{C_P} \left[T \left(\frac{\partial V}{\partial T} \right)_P - V \right] \quad (32)$$

or

$$\mu = \frac{1}{S} \left[P \left(\frac{\partial V}{\partial P} \right)_{\mathcal{H}} + 2V \right]. \quad (33)$$

The sign of μ is important to determine the active role for the cooling or heating nature of the Universe. The cooling nature occurs if $\mu > 0$, while the heating nature occurs if $\mu < 0$. For the FRW Universe, the Joule–Thomson expansion coefficient is obtained as

$$\mu = \sqrt{\frac{G}{8\pi P}} \left[1 + \left(1 - 16G^2 P (S - S_0) \right)^{\frac{1}{2}} \right]^{-\frac{3}{2}} \left[1 + \frac{(1 - 8G^2 P (S - S_0))}{(1 - 16G^2 P (S - S_0))^{\frac{1}{2}}} \right]. \quad (34)$$

From the above expression, we observe that μ is always positive, so the cooling process always occurs in the FRW Universe. If we put $\mu = 0$ in (33), we obtain the inversion pressure for the inversion curve, which is obtained as $P_{inv} = \frac{35}{578G^2(S-S_0)}$. In addition, if we put $\mu = 0$ in (33), the inversion temperature is obtained in the form

$$T_{inv} = V \left(\frac{\partial T}{\partial V} \right)_P = -\sqrt{\frac{GP_{inv}}{18\pi}} \left[1 + \left(1 - 16G^2 P_{inv} (S - S_0) \right)^{\frac{1}{2}} \right]^{-\frac{1}{2}} = -\frac{1}{12} \sqrt{\frac{7}{17\pi(S-S_0)}}. \quad (35)$$

4. Heat Engine

Now, we study the heat engine description for the FRW Universe. Physically, the heat engine is a system where thermal energy transforms into mechanical energy. Here, we will study the Carnot engine, new heat engine, and Rankine engine for the FRW Universe. Classically, the Carnot engine is a heat engine in the theoretical thermodynamic cycle, and the corresponding cycle is known as the Carnot cycle. Johnson [13] has introduced the P - V diagram for the Carnot cycle to determine the work done. The temperatures of the heat engine's hot and cold regions are denoted by T_H and T_C , respectively. The net heat that flows from stage 1 to stage 2 along the upper isotherm process is $Q_H = T_H \Delta S_{1 \rightarrow 2} = T_H (S_2 - S_1)$ and the exhausted heat that flows from stage 3 to stage 4 along the lower isothermal process is $Q_C = T_C \Delta S_{3 \rightarrow 4} = T_C (S_3 - S_4)$. Using the relation (19), the entropies S_i 's are related to the volumes V_i 's as

$$V_i = \frac{1}{6\sqrt{8\pi} G^{\frac{3}{2}} P_i^{\frac{3}{2}}} \left[1 + \left(1 - 16G^2 P_i (S_i - S_0) \right)^{\frac{1}{2}} \right]^{\frac{3}{2}}, \quad i = 1, 2, 3, 4, \quad (36)$$

where $P_i = \frac{1}{2}(\rho_i + 3p_i)$, $i = 1, 2, 3, 4$. For the Carnot heat engine, total work done is $W = Q_H - Q_C$. For the Carnot heat engine, the efficiency is $\eta_{Car} = \frac{W}{Q_H} = 1 - \frac{Q_C}{Q_H}$. Since, for the Carnot cycle, $V_4 = V_1$ and $V_3 = V_2$, thus the maximum efficiency is obtained in the form

$$\begin{aligned}
 (\eta_{Car})_{max} = & 1 + 8G^2 P_1^3 P_2^{\frac{3}{2}} P_4^{\frac{1}{2}} (S_2 - S_1) \left(1 + \left(1 - 16G^2 P_2 (S_2 - S_0) \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\
 & \times \left[\left\{ P_2^2 (P_1 - P_4) \left(1 + \left(1 - 16G^2 P_1 (S_1 - S_0) \right)^{\frac{1}{2}} \right) + 8G^2 P_1 P_2^2 P_4 (S_1 - S_0) \right. \right. \\
 & \left. \left. + P_1^2 (P_3 - P_2) \left(1 + \left(1 - 16G^2 P_2 (S_2 - S_0) \right)^{\frac{1}{2}} \right) - 8G^2 P_1^2 P_2 P_3 (S_2 - S_0) \right\} \right. \\
 & \left. \times \left\{ P_1 + \left\{ P_1^2 - 16G^2 P_1 P_4^2 (S_1 - S_0) + 2P_4 (P_4 - P_1) \right. \right. \right. \\
 & \left. \left. \left. \times \left(1 + \left(1 - 16G^2 P_1 (S_1 - S_0) \right)^{\frac{1}{2}} \right) \right\}^{\frac{1}{2}} \right\}^{\frac{1}{2}} \right]^{-1}.
 \end{aligned} \quad (37)$$

Here, we assume a new heat engine, which has two isobars and two isochores [13]. The total work done along the isobars is given as $W = \Delta P_{4 \rightarrow 1} \Delta V_{1 \rightarrow 2} = (P_1 - P_4)(V_2 - V_1)$. The net inflow of heat is given by

$$\begin{aligned}
 Q_H &= \int_{T_1}^{T_2} C_P(P_1, T) dT \\
 &= \frac{1}{8\pi G} \left[\frac{GP_1}{10\pi} (T_2^5 - T_1^5) - \frac{1}{3} (T_2^3 - T_1^3) \right].
 \end{aligned} \quad (38)$$

In the FRW Universe, the thermal efficiency for the new heat engine is given by

$$\eta_{New} = \frac{W}{Q_H} = \frac{5G(P_4 - P_1)(T_2^3 - T_1^3)}{T_1^3 T_2^3 [3GP_1(T_2^5 - T_1^5) - 10\pi(T_2^3 - T_1^3)]} \quad (39)$$

where $T_i = \sqrt{\frac{GP_i}{2\pi}} \left[1 + \left(1 - 16G^2 P_i (S_i - S_0) \right)^{\frac{1}{2}} \right]^{-\frac{1}{2}}$, $i = 1, 2$.

Another thermodynamic cycle of a heat engine is the Rankine cycle [48,49], which converts heat into mechanical work during phase transition. From the diagram of ref [49], we observe that the working material starts from A to B for increasing temperature and pressure. The working material goes from B to E , and a phase transition occurs between these states from C to D due to constant temperature. After that, due to the temperature reduction, the working material follows from E to F and returns to A by reducing its volume. We now apply the same mechanism to the FRW Universe. From the first law of thermodynamics, we obtain the change of enthalpy function $d\mathcal{H}_P = TdS$ for constant pressure (i.e., $dP = 0$). Thus, we obtain the enthalpy function $\mathcal{H}_P(S) = \int TdS$ for constant pressure. Now, according to the formalism of Wei et al. [48,49], we write the efficiency for the Rankine cycle of the heat engine for the FRW Universe as in the following form:

$$\begin{aligned}
 \eta_{Ran} &= 1 - \frac{T_A(S_F - S_A)}{\mathcal{H}_{P_B}(S_F) - \mathcal{H}_{P_B}(S_A)} \\
 &= 1 - \frac{T_1(S_3 - S_1)}{\mathcal{H}_{P_2}(S_3) - \mathcal{H}_{P_2}(S_1)}
 \end{aligned} \quad (40)$$

where the subscripts A, B, F can be changed to $1, 2, 3$, respectively. Thus, we obtain the following form of efficiency for Rankine cycle as

$$\eta_{\text{Rank}} = 1 - 12G^2 \sqrt{P_1 P_2} (S_3 - S_1) \left[1 + \left(1 - 16G^2 P_1 (S_1 - S_0) \right)^{\frac{1}{2}} \right]^{-\frac{1}{2}} \\ \times \left[\left[1 + \left(1 - 16G^2 P_2 (S_3 - S_0) \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \left[2 - \left(1 - 16G^2 P_2 (S_3 - S_0) \right)^{\frac{1}{2}} \right] \right. \\ \left. - \left[1 + \left(1 - 16G^2 P_2 (S_1 - S_0) \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \left[2 - \left(1 - 16G^2 P_2 (S_1 - S_0) \right)^{\frac{1}{2}} \right] \right]^{-1}. \quad (41)$$

The efficiency for the Rankine cycle depends on the values of P_1 , P_2 , S_0 , S_1 and S_3 .

5. Conclusions

We have considered the FRW model of the Universe, which can be treated as a thermodynamical system. On the trapping (apparent) horizon, we have determined the radius and temperature. Using the unified first law, we have determined the amount of energy dE crossing the apparent horizon. Since heat is one of the forms of thermal energy, thus the heat flow δQ through the apparent horizon = amount of energy crossing the apparent horizon, i.e., $\delta Q = dE$. Using the first law of thermodynamics, on the apparent horizon, we have found $TdS = A(\rho + p)H\tilde{r}_h dt + A\rho d\tilde{r}_h$. Cai et al. [30] have assumed $\delta Q = -dE$ instead of $\delta Q = dE$, and they have ignored the second term $A\rho d\tilde{r}_h$ on the apparent horizon. Since the apparent horizon is dynamical, we have assumed that $d\tilde{r}_h$ cannot be zero. Using this consideration and Friedmann equations with the continuity equation, we have determined the expression of entropy (on the apparent horizon) in the form $S = \frac{A}{4G} - \frac{\Lambda A^2}{32\pi G} + S_0$, which is the entropy-area relation. The second term occurs due to the non-vanishing term of $d\tilde{r}_h$. In particular, if the cosmological constant = 0, we can recover the Bekenstein–Hawking entropy. Conversely, from the entropy-area relation, we have obtained modified Friedmann equations. For $K = 0$, we can recover the original Friedmann equations. If we consider $d\tilde{r}_h = 0$ on the apparent horizon, then we can recover the Bekenstein–Hawking entropy $S = \frac{A}{4G} + S_0$. From original Friedmann equations and also from modified Friedmann equations, we may obtain the same entropy. For the thermodynamic study of the FRW Universe, the cosmological constant Λ is treated as thermodynamical pressure P . In the thermodynamic system of the FRW Universe, we have obtained the temperature and specific heat capacity. In addition, we have obtained the mass inside the apparent horizon. We have determined the coefficient of thermal expansion and isothermal compressibility. We found the critical values of entropy, temperature, volume, and pressure. We have studied the coefficient μ for Joule–Thomson expansion in the FRW Universe. We found $\mu > 0$, which indicates that the FRW Universe produced a cooling nature. The inversion pressure and inversion temperature have been obtained. Next, we investigated the heat engine phenomena of the thermodynamical FRW Universe. For the Carnot cycle, we have calculated the work done and the maximum efficiency. We have also calculated the work done and the efficiency of a new heat engine. Finally, we have determined the efficiency of the Rankine cycle in the heat engine for the FRW Universe.

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