

From Soft Dirac Monopoles to the Dirac Equation

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Abstract: In the model of topological particles we have four types of topologically stable dual Dirac monopoles with soft cores and finite mass. We discuss the steps for getting a Dirac equation for these particles. We show for the free and the interacting case that we arrive at the Dirac equation in the limit, where the soft solitons approach singular dual Dirac monopoles.

Keywords: solitons; topological particles; electric monopoles; spin; quantum mechanics; Dirac equation; topological constant; inflation

1. Introduction

According to Maxwell's electrodynamics, the electric self-energy of point-like electrons is infinite. Schrödinger was originally [1] inclined to a standing wave interpretation of electrons in their equation, whereas Bohr, Heisenberg, Pauli and Born [2] insisted on the corpuscular nature of electrons, where electrons have to be seen as point-like objects and the square of the wave function describes the probability of its position only.

Schrödinger and Dirac equations describe very successfully the electromagnetic interaction between electron and proton without taking explicitly into account the electromagnetic self-energy of point-like or extended electrons. This self-energy should exist according to Maxwell. It is absorbed in the observable mass m_e of electrons at rest. Although Schrödinger and Dirac equations do not treat the electromagnetic field self-consistently, neglecting the $F_{\mu\nu}F^{\mu\nu}$ contribution, they predict the properties of the hydrogen atom very well and even shaped our understanding of atomic and condensed matter physics. Only the tiny Lamb shift is not described in the Dirac equation.

The Lagrangian of quantum electrodynamics (QED) takes the energy density of the electromagnetic field into account. The divergences of point-like charges are cured, as Kramers suggested 1947 at the Shelter Island Conference, by subtracting compensating infinities from the naive QED Lagrangian. Finally, the theory gets consistent only by respecting the running of the couplings, a feature which is in excellent agreement with high energy scattering experiments.

On the other hand, for a classical relativistic model with stable extended solitons of finite mass and quantised charges, as discussed in [3,4], there is no need to absorb the self-energy of elementary charges in a mass term adjusted to the experimental electron mass. Charges and their fields are described by the same degrees of freedom and distinguished by topological quantum numbers only. Therefore, the Hamiltonian contains the self-energy of charges, the energy of the electromagnetic field and the interaction energy of charges and electromagnetic fields.

Until now, MTP was only formulated as a classical model. Thus, there is an urgent need to harmonise it with quantum physics. A classical model misses the quantum properties, well described by the Schrödinger and the Dirac equation. Quantum effects for extended solitons can also be formulated by a path integral. All these descriptions require the electric interaction energy. This can be directly determined within the soliton model. Due to the finite size of solitons, we expect at distances of several femtometers deviations from the potential between point-like charges. Inserting these potential modifications into the



Citation: Faber, M. From Soft Dirac Monopoles to the Dirac Equation. *Universe* **2022**, *8*, 387. <https://doi.org/10.3390/universe8080387>

Academic Editors: Ralf Hofmann and Gerald B. Cleaver

Received: 7 June 2022

Accepted: 20 July 2022

Published: 22 July 2022

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Schrödinger or Dirac equation should lead to a shift in energy levels, especially for s-states. It will be very interesting to compare these shifts with the experimental spectra, especially the Lamb shift.

There are essential differences from the soliton formulations of hydrogen and positronium to the conventional Schrödinger equation. Here we have to especially point out the differences in the field degrees of freedom (dofs). The soliton model describes charges and electromagnetic fields by the same dofs, by an SO(3)-valued scalar field, in strong distinction from the conventional description with Grassmann-valued ψ -fields and gauge fields A_μ . Moreover, the vector fields of the soliton model describe dual photons. The attempt to describe essential properties of nature by our soliton model implies immediately the question: how one can understand within this picture the impressive success of the conventional determination of the properties of atomic systems with the Schrödinger and Dirac equation? Therefore, we try to analyse in this article by which steps we can arrive at the conventional Dirac equation when starting from our soliton model. Moreover, this investigation sheds light on the above-suggested insertion of the soliton–antisoliton potential energy into the Schrödinger and Dirac equation.

In Section 2, we present a very short formulation of the soliton model and some of its properties. Extended versions of this model were published in references [3,4]. The attempt to get the conventional Dirac equation, we realise in two steps. First, we concentrate on the free Dirac equation in Section 3 using the relativistic properties of solitons. Then we introduce the interaction with an external field by artificially separating the dofs of charged solitons and their fields in Section 4. Since the conventional Dirac equation uses a pure $1/r$ -potential, it is sufficient to formulate this separation in the limit of point-like charges, where this separation can be formulated more easily. Due to charge quantisation, the generalisation to extended charges is obvious if the distance between solitons is large compared to their size. Finally, we discuss the question of why the conventional Dirac equation can be formulated with gauge fields A_μ , despite the fact that the soliton model is formulated with dual vector fields.

2. Model of Topological Particles

The Model of Topological Particles (MTP) [3,4] formulates electric monopoles and their fields as topologically stable solitons of finite mass and is characterised by two topological quantum numbers which can be related to spin and charge. There exist four classes of such solitons without any singularities with the quantum numbers of the components of Dirac fermions.

For the description of electromagnetic phenomena and especially of solitons in 3 + 1D with long-range coulombic interaction the MTP uses the three degrees of freedom of an SO(3)-field only. However, the calculations are simpler using the 2×2 -matrices

$$Q(x) = q_0(x) - i\vec{\sigma}\vec{q}(x) \in \text{SU}(2), \quad q_0^2 + \vec{q}^2 = 1, \tag{1}$$

matrices in the double covering group of SO(3). In the model of topological particles (MTP), see reference [4], the Lagrangian reads

$$\mathcal{L}_{\text{MTP}} := -\frac{\alpha_f \hbar c_0}{4\pi} \left(\frac{1}{4} \vec{R}_{\mu\nu} \vec{R}^{\mu\nu} + \Lambda(q_0) \right), \quad \Lambda(q_0) := \frac{q_0^6}{r_0^4}, \quad \alpha_f := \frac{e_0^2}{4\pi\epsilon_0 \hbar c_0} \tag{2}$$

where the curvature field $\vec{R}_{\mu\nu}$,

$$\vec{R}_{\mu\nu} := \vec{\Gamma}_\mu \times \vec{\Gamma}_\nu \tag{3}$$

is the area density on SU(2), the Jacobian for the map of areas on M^4 to areas on S^3 . It is algebra valued as indicated by the vector symbol. The vector field¹

$$\partial_\mu Q(x) =: -i\vec{\Gamma}_\mu(x) \vec{\sigma} Q(x) \tag{4}$$

is the affine connection in the tangential space of SU(2).

With the generalised velocity $\vec{\Gamma}_\mu$ we get the energy momentum tensor

$$\Theta^\mu{}_\nu(x) := \frac{\partial \mathcal{L}(x)}{\partial \vec{\Gamma}_\mu} \vec{\Gamma}_\nu - \mathcal{L}(x) \delta_\nu^\mu = \stackrel{(2)}{=} -\frac{\alpha_f \hbar c_0}{4\pi} \left\{ (\vec{\Gamma}_\nu \times \vec{\Gamma}_\sigma) (\vec{\Gamma}^\mu \times \vec{\Gamma}^\sigma) \right\} - \mathcal{L}(x) \delta_\nu^\mu, \tag{5}$$

which is automatically symmetric.

This model is a generalisation of the Sine–Gordon model [5] from 1 + 1D and 1 dof to 3 + 1D and 3 dofs. It can be considered as a modification of the Skyrme model [6–8] describing solitons with long-range forces in 3 + 1D space-time with Minkowski metric. Further, it can be seen as a model for soft dual Dirac monopoles [9,10]; this means Dirac monopoles without any singularities, without a Dirac string [10,11] and without a singularity in the centre. Another model with monopoles without singularities is the Georgi–Glashow model [12–14]. There the forces between monopoles are influenced not only by their structure, but also by the strength of the Higgs field.

The minimum of the potential energy in Equation (2) is at the equatorial S^2 of $S^3 \cong \text{SU}(2)$. Solitonic solutions are therefore characterised by two topological quantum numbers counting the number of coverings, S^2 and S^3 . They allow us to classify stable and static solitons.

With the hedgehog ansatz

$$\vec{n}(x) := \frac{\vec{x}}{r}, \quad \vec{q}(x) := \vec{n}(x) \sin \alpha(x), \quad q_0 := \cos \alpha(x) \tag{6}$$

we can solve the minimisation procedure for a static soliton, which leads to the non-linear differential equation

$$\frac{(1 - \cos^2 \alpha) \cos \alpha}{\rho^2} + \partial_\rho^2 \cos \alpha - 3\rho^2 \cos^5 \alpha = 0. \tag{7}$$

There are four homotopy classes of configurations. In every class there is one solution, with respect to the hedgehog ansatz

$$q_0 = \pm \frac{r_0}{\sqrt{r^2 + r_0^2}}, \quad q_i = \pm \frac{x_i}{\sqrt{r^2 + r_0^2}} \quad \text{with} \quad r^2 := \vec{x}^2. \tag{8}$$

The vectors \vec{q} of these configurations in a plane through the origin are depicted in four diagrams in Table 1. There, \vec{q} -vectors are drawn in full-red for $q_0 > 0$, dashed-green for $q_0 < 0$ and nearly black for $q_0 \approx 0$. The four classes can be transformed to each other by centre transformations $Q(\vec{r}) \xrightarrow{\mathbb{Z}} -Q(\vec{r})$ and parity transformations $Q(\vec{r}) \xrightarrow{\Pi} Q(-\vec{r})$. The configurations within every class differ by global rotations and/or translations.

The four classes of solutions are distinguished in the second row of Table 1 by the number Z of electron charges $-e_0$ and in the third row by the chirality χ , the sign of the topological charge \mathcal{Q} . These four soliton types have an identical mass [3]:

$$m_0 = \frac{\alpha_f \hbar c_0 \pi}{r_0 4}. \tag{9}$$

It can be adjusted to the mass m_e of electrons and positrons by choosing

$$r_0 = \alpha_f \frac{\hbar}{m_e c_0} \frac{\pi}{4} = 2.21 \text{ fm}. \tag{10}$$

Table 1. The diagrams show the imaginary components \vec{q} of the field of single solitons, in full red for the hemisphere with $q_0 > 0$, in dashed green for $q_0 < 0$ and in nearly black for $q_0 \rightarrow 0$. The transformations \mathcal{T} of the hedgehog configuration in the first column modify the fields \vec{q} and q_0 and the topological quantum numbers Z and \mathcal{Q} .

$\mathcal{T} = 1$	$\mathcal{T} = z\Pi$	$\mathcal{T} = z$	$\mathcal{T} = \Pi$
$Z = 1$	$Z = 1$	$Z = -1$	$Z = -1$
$\mathcal{Q} = \frac{1}{2}$	$\mathcal{Q} = -\frac{1}{2}$	$\mathcal{Q} = \frac{1}{2}$	$\mathcal{Q} = -\frac{1}{2}$

Thus, we can describe the four components of Dirac spinors, electrons e_- and positrons e_+ , spinup \uparrow and down \downarrow . According to the $SO(3)$ description, single isolated soliton configurations differing by a non-trivial centre transformation z are identical. In multiple soliton configurations, the $SU(2)$ -configuration depends on the other solitons.

It is nice to see that the four classes of solutions agree with the well-known components of Dirac spinors in the standard representation: $e_-^\uparrow, e_-^\downarrow, e_+^\uparrow, e_+^\downarrow$. The four configurations differ in centre transformations $\mathcal{Q} \xrightarrow{z} -\mathcal{Q}$ and in parity transformations $\vec{n} \xrightarrow{\Pi} -\vec{n}$. In Table 1, we observe that a parity transformation corresponds to the product of charge transformation and spin reversal.

Now we would like to clarify which assumptions are necessary to get the Dirac equation for solitons.

3. Free Dirac Equation

Within relativistic quantum theory, the Dirac equation is the basic equation for the understanding of the properties of spin-1/2 fermions. It is also the basis for their understanding within quantum field theory. The solutions ψ of the Dirac equations have the form of spinors consisting of four complex functions for spin-up and spin-down components of fermions and anti-fermions. The outstanding result of the Dirac equation was the explanation of the fine-structure of the hydrogen spectrum and its degeneracy [15], a result later improved only by QED [16].

In relativistic quantum mechanics we can attribute to particles an intrinsic parity and use $\gamma_0 = \text{diag}(1, 1, -1, -1)$ as the parity operator. The Dirac equation has two pairs of solutions of opposite parity. This abstract parity property has a figurative realisation in the diagrams of Table 1 with two pairs of configurations which differ by an application of the parity operator Π .

According to quantum mechanics, we assign to the four configurations in Table 1 unit vectors in a four component Hilbert space, and arrange them in four-component spinors ψ_0 which fulfil the equation

$$H_0\psi_0 := m_0c_0^2\gamma_0\psi_0, \tag{11}$$

for the Hamiltonian H_0 , if we attribute to positrons a negative mass.

Starting from the properties of solitons, we will now proceed in detail to a derivation of the Dirac equation for moving solitons. From our knowledge about gamma matrices, the Lorentz transformation of momentum four-vectors and of Dirac spinors, we know the result which we expect. However, it may be interesting to see in detail how we can derive the Dirac equation for free particles from the relativistic properties of the soliton four momentum only. In the next section we will then proceed to the interacting case.

As discussed in reference [3], the mass of solitons moving with classical momentum \vec{p} increases with the well-known factor $\gamma(\vec{p}) := 1/\sqrt{1-\vec{p}^2/(m_0c_0)^2}$, as one can understand from the invariance of the MTP–Lagrangian under Lorentz transformations. Thus, we generalise the Hamiltonian in Equation (11) to

$$H := \gamma(\vec{p}) m_0 c_0^2 \gamma_0. \tag{12}$$

The relativistic four-momentum for a soliton of mass m_0 and velocity \vec{v} reads

$$p^\mu = \gamma m_0 (c_0, \vec{v})^\mu, \quad \gamma = \frac{1}{\sqrt{1-\frac{\vec{v}^2}{c_0^2}}}. \tag{13}$$

These relations can be represented graphically by the right-angled triangles in Figure 1, resembling the decomposition of a vector into orthogonal components.

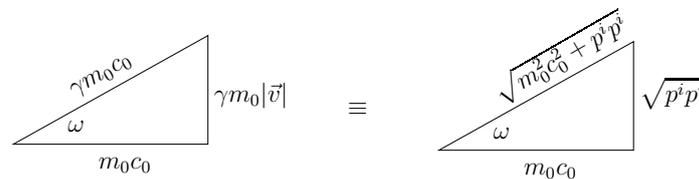


Figure 1. Graphical representation of the relations in Equation (13) between the four-momentum components and the invariant mass.

Since MTP uses the degrees of freedom of an SO(3)-field and simplifies the calculations by formulations in the defining (two dimensional) representation of SU(2), the MTP–Lagrangian Equation (2) for free particles cannot distinguish between solitons with the same Q and different Z which differ by a centre transformation only; see Table 1. Due to this indistinguishability, we can apply a unitary transformation between the first and the second pair of solitons, between the upper and the lower component of the Dirac spinor. Since the matrix

$$i := \begin{pmatrix} 0 & -\mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \tag{14}$$

also used for the definition of the symplectic group, acts like an imaginary unit, $i^2 = -\mathbb{1}_4$, we can formulate such a “rotation” with the rotational vector $\vec{\omega} := \omega \vec{e}_\omega$ as

$$U_\omega := \exp\left\{i \vec{\omega} \frac{\vec{\sigma}}{2}\right\} = \cos \frac{\omega}{2} + \underbrace{\begin{pmatrix} 0 & -\mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}}_{-\vec{\gamma}} \vec{\sigma} \vec{e}_\omega \sin \frac{\omega}{2}, \tag{15}$$

where the well-known $\vec{\gamma}$ -matrices in the Dirac (or standard) representation appear. It turns out that U_ω is the expected Lorentz transformation of Dirac spinors, a reversed version of the Foldy–Wouthuysen transformation [17].

Due to the anticommutation relations between γ^0 and γ^i which appeared in the underbrace of Equation (15)

$$\gamma^0 \gamma^i + \gamma^i \gamma^0 = 0 \quad \text{with} \quad \gamma^0 := \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \tag{16}$$

we get

$$U_\omega \gamma^0 \stackrel{(15)}{=} \gamma^0 U_\omega^\dagger. \tag{17}$$

The transformation of the Hamiltonian Equation (12) reads therefore

$$H \stackrel{(12)}{=} \gamma(\vec{p}) m_0 c_0^2 \gamma_0 \xrightarrow{U_\omega} H_\omega := U_\omega H U_\omega^\dagger \stackrel{(17)}{=} H(U_\omega^\dagger)^2. \tag{18}$$

with

$$H_\omega \stackrel{(18)}{=} \gamma(\vec{p}) m_0 c_0^2 \gamma_0 \exp\{-i \vec{\omega} \vec{\sigma}\} \stackrel{(15)}{=} \gamma(\vec{p}) m_0 c_0^2 \gamma_0 (\cos \omega + \vec{\gamma} \vec{e}_\omega \sin \omega). \tag{19}$$

Since the prefactor of the last bracket is proportional to the hypotenuse p_0 of the triangle in Figure 1, there is a natural choice of ω and \vec{e}_ω given by that triangle. We read from Figure 1 $\cos \omega = 1/\gamma(\vec{p})$ and $\sin \omega = |\vec{v}|/c_0$, leading to

$$p_0 := H_\omega / c_0 \stackrel{(19)}{=} m_0 c_0 \gamma_0 + \gamma_0 \gamma^i p^i. \tag{20}$$

We multiply with γ_0 from the left, reorder the terms and get

$$\gamma^\mu p_\mu \stackrel{(20)}{=} m_0 c_0. \tag{21}$$

From Equation (12), we conclude that the rotated spinor

$$\psi := U_\omega \psi_0 \tag{22}$$

obeys the Dirac equation

$$\gamma^\mu p_\mu \psi \stackrel{(21)}{=} m_0 c_0 \psi. \tag{23}$$

With the usual quantum mechanical laws which solitons also have to obey, coordinates and momenta as canonical conjugate variables, $p_\mu = i\hbar\partial_\mu$ and Borns rule, we get the action of free Dirac particles

$$S_f := \int d^4x \bar{\psi} (\gamma^\mu p_\mu - m_0 c_0) \psi \tag{24}$$

with the free monopole currents

$$j^\mu := c_0 \bar{\psi} \gamma^\mu \psi. \tag{25}$$

We observe that we get the correct Dirac equation only by choosing ω as given by the triangle in Figure 1, as was emphasised in reference [18]. The correct non-relativistic limit is only retained if the scalar product $\vec{\sigma}\vec{p}$ is not destroyed by the transformation Equation (15).

Further, we would like to mention that the solutions of the Dirac equations are the columns of the matrix

$$\gamma^\mu p_\mu + m_0 c_0. \tag{26}$$

After getting the Dirac equation for free solitons, we will now concentrate on their interaction with electromagnetic fields. This will be done mainly in two steps. First, we have to separate the dofs of particles and fields, and then we have to make a transition to dual variables.

4. Interaction with Electromagnetic Fields

The MTP–Lagrangian (2) describes multiparticle systems and their fields $f_{\mu\nu}$ on the same footing. To get the Dirac equation considering single point-like charges moving in electromagnetic fields, we have to separate artificially the dofs of charges and fields. This is possible only in the limit of point-like solitons (dual Dirac-monopoles), see references [3,19], where the soliton field $Q(x)$ is purely imaginary. In this limit we arrive at the formulation of dual Dirac monopoles in the Wu–Yang representation [11]

$$Q(x) \rightarrow Q(x) \stackrel{(1)}{=} -i\vec{\sigma}\vec{n}(x), \quad \vec{n}^2 := 1, \quad \vec{q} \stackrel{(1)}{=} \vec{n} \tag{27}$$

where the \vec{n} -field takes values on S^2 . Thus, the Jacobian of the map from $M^4 \rightarrow S^2$ is given by the curvature field

$$\vec{R}_{\mu\nu}(x) \rightarrow \vec{R}_{\mu\nu}(x) \stackrel{(3)}{\stackrel{(27)}{=} } \partial_\mu \vec{n}(x) \times \partial_\nu \vec{n}(x). \tag{28}$$

which we relate by [4]

$${}^*f_{\mu\nu} =: -\frac{e_0}{4\pi\epsilon_0 c_0} \vec{R}_{\mu\nu} \vec{n} \stackrel{(28)}{=} -\frac{e_0}{4\pi\epsilon_0 c_0} \vec{n}(\partial_\mu \vec{n} \times \partial_\nu \vec{n}). \tag{29}$$

to the dual² Abelian electromagnetic field strength in SI units. The Lagrange density \mathcal{L}_{MTP} reduces to

$$\mathcal{L}_{\text{MTP}} \rightarrow \mathcal{L}_{\text{ED}} \stackrel{(2)}{\stackrel{(27)}{=} } -\frac{\alpha_f \hbar c_0}{4\pi} \frac{1}{4} (\vec{n} \vec{R}_{\mu\nu})(\vec{n} \vec{R}^{\mu\nu}) \stackrel{(29)}{=} -\frac{1}{4\mu_0} {}^*f_{\mu\nu}(x) {}^*f^{\mu\nu}(x). \tag{30}$$

$\mathcal{L}_{\text{dual}}$ differs in the sign only from the Lagrangian of Maxwell’s electrodynamics.

We use the singularities of the \vec{n} -field at given time t to identify the soliton currents. These singularities are located in infinitesimal space-like three dimensional volumes, the centres of hedgehog configurations of the \vec{n} -field. We describe them by N time-like world-lines of singularities

$$X_i^\mu(\tau) \quad i = 1, \dots, N, \tag{31}$$

evolving in time with velocities smaller than c_0 from $t = -\infty$ to $t = +\infty$, or meeting with lines of opposite charge, where particle pairs are created or annihilated. This makes it possible to define for each \vec{n} -field configuration a singular vector current along the above world-lines

$$\sum_{i=1}^N Z_i \int d\tau \frac{dX_i^\kappa(\tau)}{d\tau} \delta^4(x - X_i(\tau)) := \frac{1}{8\pi} \epsilon^{\kappa\lambda\mu\nu} \partial_\lambda \{ \vec{n}(\partial_\mu \vec{n} \times \partial_\nu \vec{n}) \}. \tag{32}$$

Integrals over arbitrary spatial volumes V over the left and the right side of this equation equate the number of world lines crossing the volume with the number of coverings of S^2 by the \vec{n} -field at the volume surface ∂V . For a single soliton resting at the origin with $Z = 1, \tau = t, X^\kappa(\tau) = c_0 t \delta_0^\kappa$, both sides reduce to a three-dimensional delta-function at the origin $\delta^3(\vec{x})\delta_0^\kappa$ for the time-component of the singular vector current. Multiplying the world lines of delta-functions with $-e_0 c_0$, we define the electric current density

$$q j^\kappa := -e_0 c_0 \sum_{i=1}^N Z_i \int d\tau \frac{dX_i^\kappa(\tau)}{d\tau} \delta^4(x - X_i(\tau)) \tag{33}$$

of point-like charges $q_i = -e_0 Z_i$. A further multiplication by μ_0 leads to

$$\mu_0 q j^\kappa \stackrel{(32)}{=} -\partial_\lambda \underbrace{\frac{e_0 c_0 \mu_0}{4\pi} \frac{1}{2} \epsilon^{\kappa\lambda\mu\nu} \vec{n}[\partial_\mu \vec{n} \times \partial_\nu \vec{n}]}_{f^{\kappa\lambda}} \stackrel{(29)}{=} \partial_\lambda f^{\lambda\kappa}. \tag{34}$$

The contributions of these inhomogeneous Maxwell equations are non-zero along the world lines Equation (31) only and absorb all currents in the divergencies of $f^{\mu\nu}$. After the transition of Equation (27) from extended monopoles to point-like particles, the Hodge transformation of Equation (27) with respect to ${}^{**}f^{\mu\nu} = -f^{\mu\nu}$ has simplified Equations (34). To keep the structure $L = T - V$ of Lagrangians, we flip the sign of the Lagrangian \mathcal{L}_{ED} , which after the Hodge transformation reads

$$\mathcal{L} := -\mathcal{L}_{\text{ED}} \stackrel{(30)}{=} -\frac{1}{4\mu_0} f_{\mu\nu} f^{\mu\nu}. \tag{35}$$

For the next step, the introduction of the electro-magnetic four-potential A_μ , we need the homogeneous Maxwell equations. This deserves a further argument. The idea behind MTP is that particles are topological solitons and characterised by topological quantum numbers. Topologically stable objects can not be removed from the configurations by the minimisation of the energy. They can be removed by the fusion with appropriate antiparticles only. In MTP, the electric charge is such a topological property related to the map $\Pi_2(S^2)$; see Equation (32). In the limit Equation (27), MTP has two degrees of freedom only, compared to four degrees of freedom of A_μ . This reduces the flexibility to fulfil the Abelian–Bianchi identity and may lead to non-vanishing conserved magnetic currents

$$g^\mu := c \partial_\nu {}^* f^{\nu\mu} \quad \text{with} \quad \partial_\mu g^\mu = 0. \tag{36}$$

Since they are not stabilised by topological quantum numbers, their contributions are minimised by the least action principle and the equations of motion; see references [3,19]:

$$\partial_\mu \vec{n} g^\mu = 0 \quad \Rightarrow \quad {}^* f_{\mu\nu} g^\nu = 0. \tag{37}$$

The second set of equations proves that the dual Coulomb and Lorentz forces on magnetic currents are vanishing. We show now that the magnetic currents do not contribute to Coulomb and Lorentz forces on electric currents and thus can be neglected in the formulation of their dynamics.

We start from the observation that the divergence of the total energy momentum tensor Equation (5) is vanishing. After the separation of dofs of charges and their fields, the divergence of the energy momentum tensor of the fields

$$\Theta_\nu^\mu(x) \rightarrow T_\nu^\mu(x) := -\frac{1}{\mu_0} {}^* f_{\nu\sigma}(x) {}^* f^{\mu\sigma}(x) - \mathcal{L}_{ED}(x) \delta_\nu^\mu, \tag{38}$$

leading to a reaction to the force density on charges

$$\begin{aligned} f_e^\mu &:= -\partial^\nu T_\nu^\mu \stackrel{(38)}{=} \frac{1}{\mu_0} \partial^\nu ({}^* f_{\nu\rho} {}^* f^{\mu\rho}) - \frac{1}{4\mu_0} \partial^\mu ({}^* f_{\rho\nu} {}^* f^{\rho\nu}) \\ &= \frac{1}{\mu_0} \left[\underbrace{\partial^\nu {}^* f_{\nu\rho}}_{\frac{1}{c_0} g_\rho} {}^* f^{\mu\rho} + \underbrace{{}^* f_{\nu\rho} \partial^\nu}_{-{}^* f_{\nu\rho} \partial^\rho {}^* f^{\mu\nu}} {}^* f^{\mu\rho} + \frac{1}{2} {}^* f_{\nu\rho} \partial^\mu {}^* f^{\rho\nu} \right] \\ &\stackrel{(36)}{=} \frac{1}{\mu_0 c_0} \underbrace{{}^* f^{\mu\rho} g_\rho}_0 + \frac{1}{2\mu_0} {}^* f_{\nu\rho} \underbrace{[\partial^\nu {}^* f^{\mu\rho} + \partial^\rho {}^* f^{\nu\mu} + \partial^\mu {}^* f^{\rho\nu}]}_{-\mu_0 q \epsilon^{\mu\nu\rho\sigma} j_\sigma} \\ &\stackrel{(37)}{=} -\frac{q}{2} \epsilon^{\mu\nu\rho\sigma} {}^* f_{\nu\rho} j_\sigma = f^{\mu\sigma} j_\sigma q. \end{aligned} \tag{39}$$

Electric and magnetic fields cause, therefore, the only forces on electric currents. According to these arguments, we can neglect magnetic currents in the description of the dynamics of electric currents and assume the validity of the homogeneous Maxwell equations by formulating

$$f_{\mu\nu} := \partial_\mu a_\nu - \partial_\nu a_\mu. \tag{40}$$

To complete the separation of dofs of charges and fields, we have to divide $f_{\mu\nu}$ in internal contributions $\mathcal{F}_{\mu\nu}$ related to the soliton currents j^μ and external fields $F_{\mu\nu}$

$$f_{\mu\nu} := \mathcal{F}_{\mu\nu} + F_{\mu\nu}, \quad a_\mu := \mathcal{A}_\mu + A_\mu \tag{41}$$

For the Lagrangian, this separation leads to three contributions: the action density of external electromagnetic fields (F^2), the interaction term between charges and fields ($\mathcal{F} F$) and the Lagrange density Equation (24) of free particles (\mathcal{F}^2). Since the internal contribution $\mathcal{F}_{\mu\nu}$ exhausts the field of all point-like sources, the corresponding infinite

self-energies have to be absorbed in the Lagrangian of the free particles and substituted by the finite experimental mass m_0 . This results in the equivalence, derived in Equation (24):

$$-\frac{1}{4\mu_0} \int d^4x \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \equiv \int d^4x \bar{\psi} (\gamma^\mu p_\mu - m_0 c_0) \psi \stackrel{(24)}{=} S_f. \tag{42}$$

The interaction term of the action therefore becomes

$$S_{\text{int}} := -\frac{1}{2\mu_0} \int d^4x F_{\mu\nu} \mathcal{F}^{\mu\nu} \stackrel{(40)}{=} \frac{1}{\mu_0} \int d^4x A_\nu \partial_\mu \mathcal{F}^{\mu\nu} \tag{43}$$

$$\stackrel{(34)}{=} \int d^4x q A_\nu j^\nu \stackrel{(25)}{=} \int d^4x q A_\nu \bar{\psi} \gamma^\nu \psi.$$

Collecting the three action contributions, we get from the Lagrangian Equation (35) the total action for a Dirac particle in an external vector field A_μ :

$$S \stackrel{(42)}{=} \int d^4x \left\{ \bar{\psi} [\gamma^\mu (p_\mu - q A_\mu) - m_0 c_0] \psi - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} \right\}. \tag{44}$$

5. Conclusions

The presently accepted theory, the standard model of particle physics, one can see from two vantage points:

- It provides excellent fits to scattering data;
- Embarrassingly, its prediction of the vacuum energy density of the cosmos is off by a factor of 10^{50} – 10^{120} [20] and has nothing to say about the classical self-energy of its fundamental massive fermions.

The idea of MTP is to arrive at a geometric formulation of particle physics. If successful, this would

- Provide a geometrical basis for the standard model.
- Move particle physics closer to gravity and facilitate unification.
- Show in detail that particles and their fields are describable by the same degrees of freedom.
- Explain why algebras are so important in particle physics. They allow one to work in the tangential spaces of the corresponding group manifolds, supplying the deeper degrees of freedom for the common descriptions of particles and their fields.
- Explain gauge symmetries as basis changes in the manifolds of the corresponding groups.
- Allow one to understand the origin of cosmological phenomena like inflation, see Section 6.4 of reference [4],
- Suggest a solution for the cosmological constant problem: substituting the constant by a function, the potential energy density of solitons; see Section 6.5 of reference [4].
- Allow one to understand that dark matter does not consist of particles but of non-quantised lumps of matter, e.g., alpha waves in MTP, as suggested in Section 6.3 of reference [4].

I have presented the interesting consequences of MTP in Section 7.2 of reference [4].

To summarise, with MTP we can describe electric charges, quantised already at the classical level in units of the elementary charge, by topological solitons of finite size and self-energy, without any divergencies. In this model we get four classes of stable solitons which nicely correspond to the four components of Dirac electrons. This begs for an answer as to which conditions and approximations the Dirac equation can be formulated for these solitons. Since the Dirac equation is the basic equation of relativistic quantum mechanics, it is obvious that the basic concepts of quantum mechanics have to be respected. Hilbert space vectors have to be attributed to the soliton states, Born’s rule has to be applied to quantum mechanical probabilities and the canonical commutation relations to coordinates and momenta. To get the Dirac equation for moving non-interacting particles, it turns out

to be essential that the MTP Lagrangian can not distinguish between solitons differing by a centre transformation only.

In the Dirac equation for interacting particles, charges and electromagnetic fields are formulated by different degrees of freedom. This requests a corresponding separation of degrees of freedom in the MTP. This is achieved in the limit when solitons are approximated by dual-point-like Dirac monopoles. The problematic consequences of the singularity of these point-like charges are avoided by attributing to the free fermions the measured masses of electrons. In this limit the difference between electrons and positrons is lost, and unitary transformations between them are possible. Electrons and positrons can be distinguished by the sign of the mass term. The interaction between charges and fields is implemented in Schrödinger and Dirac equations by the covariant derivative. It is impossible to obtain this type of interaction from the MTP just by using its dual formulation in the presence of non-vanishing magnetic currents. We show that the dynamics of charges in the MTP is determined by Coulomb and the Lorentz forces only. Since it is not influenced by magnetic currents, they can be neglected. The duality transformation allows us then to formulate electromagnetic fields in terms of the usual Abelian vector potentials.

We conclude that in the limit of point-like charges, after absorbing the infinite self-energy into the experimental masses, the interaction between charges is described by a $1/r$ potential. On the other hand, the finite size of solitons of the MTP leads to a running of the charge and to a modification in the $1/r^2$ force. This supports the idea of implementing the soliton–antisoliton potential energy of the MTP in Schrödinger and Dirac equations and determining in this singularity free description the modifications of the eigenvalues of the stationary equations and to compare them to the experimental spectra, especially to the size of the Lamb shift.

A completely different construction of electrons was suggested in reference [21], and a different attempt to derive the Dirac equation was undertaken in reference [22].

Funding: Open Access Funding by TU Wien.

Acknowledgments: I want to thank to Dmitry Antonov for bringing my attention to the Foldy–Wouthuysen transformation. To the three referees and the scientific editor I am grateful for their strong but very constructive critiques, and I am grateful to Rudolf Golubich for careful reading of the manuscript.

Conflicts of Interest: The author declares no conflict of interest.

Notes

- 1 For simplicity, we omit the dot for scalar products in three dimensions, if it does not lead to ambiguities. The \times symbol acts always in three dimensional algebra and not in coordinate space.
- 2 The $*$ -operator in Equation (29) performs a Hodge duality transformation, transforming $\mathbf{E}/c_0 \rightarrow \mathbf{B}$ and $\mathbf{B} \rightarrow -\mathbf{E}/c_0$, and can be formulated with the four-dimensional ϵ -tensor; compare the underbrace in Equation (34).

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