Article

Thermalization with Non-Zero Initial Anomalous Quantum Averages

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Abstract: We discuss the thermalization process in kinetic approximation in the presence of non-zero initial anomalous quantum expectation values on top of an initial non-Planckian (non-thermal) level population. In particular, we derive a system of “kinetic” equations for the level population and anomalous expectation values in four-dimensional massive scalar field theory with ϕ^4 self-interaction. We analytically show, in the linear approximation, that for their small initial values, the anomalous quantum averages relax down to zero. Furthermore, we show analytically that this system does not have an equilibrium solution with non-zero time independent anomalous expectation value.

Keywords: quantum field dynamics; out of equilibrium; analogs of kinetic equations with anomalous expectation values

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1. Introduction

The understanding of quantum field theory dynamics over strong field backgrounds demands the consideration of correlation functions over quantum states with non-zero anomalous quantum expectation values or anomalous averages. We use here the standard terminology from the BCS theory of superconductors [1,2], where anomalous averages play a key role: we name as the anomalous averages and the expectation values of the form $\text{Tr}[\hat{\rho} a_{\vec{q}} a_{\vec{q}'}] \equiv \langle a_{\vec{q}} a_{\vec{q}'} \rangle$, where $\hat{\rho}$ is the density matrix characterizing the state of the theory and $a_{\vec{q}}$ includes the annihilation operators in the theory; the trace is taken over the Hilbert space of the theory. For the standard Fock space states (e.g., Poincare invariant ground state), the anomalous averages are always zero. However, in a quantum field theory over strong background fields, even for zero initial values, the anomalous averages are generated dynamically in loop corrections [3]. This happens, e.g., in the expanding universe [4–8], during collapse processes [9], in the presence of strong electric fields [10,11] and in the presence of background scalar fields [12]. Refer to [13] for a general discussion.

At the same time, quantum field dynamics in flat spacetime with non-trivial anomalous expectation values are poorly studied. That is mainly because, to obtain non-zero anomalous averages, one has to consider some sort of coherent states. Meanwhile, the consideration of states with non-zero anomalous averages in flat space quantum field theory can be a suitable playground for the development of methods to work in similar situations over strong background fields. Concretely, in this paper, we consider the thermalization process in the massive scalar field theory with ϕ^4 self-interaction in four-dimensional Minkowski spacetime.

The thermalization process in the kinetic approximation with zero anomalous expectation values was considered in many places (see, e.g., textbooks [14–16]). However, to the best of our knowledge, the thermalization process in the kinetic approximation with non-zero anomalous averages was not considered.

In this paper, we consider an initial state containing a non-Planckian (non-thermal) level population, which is expressed via $\text{Tr}[\hat{\rho} a_q^+ a_{q'}] \equiv \langle a_q^+ a_{q'} \rangle$, and anomalous averages, $\text{Tr}[\hat{\rho} a_{\vec{q}} a_{\vec{q}'}] \equiv \langle a_{\vec{q}} a_{\vec{q}'} \rangle$. Our goal is to show that such a state will evolve in time towards the Planckian level population, $n_p = \frac{1}{e^{\beta \epsilon_p} - 1}$, and zero anomalous average. Namely, the final state of the thermalization process is described by the Planckian distribution of modes, n_p , over the Poincare invariant ground state.

To show this phenomenon, we derive a system of kinetic equations for the level population and anomalous averages. Then, we show that this system has a solution with zero anomalous averages and Planckian level population only for such modes, which diagonalize the free Hamiltonian. To derive the system of generalized kinetic equations, we use standard textbook methods, which can be found, e.g., in [15,16]. To have the analytic headway in finding solutions of this system, we consider spatially homogeneous states and small initial anomalous averages. However, we also derive the system of kinetic equations for n_p and χ_p without the latter assumptions.

2. Set up of the Problem

In this note, we consider the four-dimensional real massive self-interacting scalar field theory in flat spacetime.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4. \quad (1)$$

The signature of the metric is $(+ - - -)$. (We work in the units $\hbar = 1 = c$.) We are interested only in the infrared (IR) effects and assume that all coupling constants in this Lagrangian acquire their physical (UV renormalized) values.

It is well known that a gas of quanta, for which its dynamics is described by the Lagrangian (1), will thermalize for any physically reasonable initial level population $\text{Tr}[\hat{\rho} a_q^+ a_{q'}] \equiv \langle a_q^+ a_{q'} \rangle \sim n_q^0 \delta(\vec{q} - \vec{q}')$. Below, we will review how to make this observation theoretically for the situation that is close to the equilibrium. Namely, we will show that, in time, the distribution will become Planckian $n_q = \frac{1}{e^{\beta \epsilon_q} - 1}$, with temperature $1/\beta$ following from the initial data and $\epsilon_q = \sqrt{\vec{q}^2 + m^2}$ being the energy of the mode with momentum $\vec{q}$.

The goal of this paper is to observe thermalization with a non-Planckian initial level population, when there is also non-zero initial anomalous quantum expectation values, $\text{Tr}[\hat{\rho} a_{\vec{q}} a_{\vec{q}'}] \equiv \langle a_{\vec{q}} a_{\vec{q}'} \rangle \sim \chi_q^0 \delta(\vec{q} + \vec{q}')$. In this paper, we make analytic observations for small initial values of the anomalous averages, i.e., also when the state of the theory is close to equilibrium.

There are two method to introduce non-zero initial values of the anomalous averages, which are related to each other via canonical transformations. The first option is to use the following standard modes.

$$\varphi(x, t) = \int \frac{d^3 p}{(2\pi)^3} \left[a_p \frac{e^{i(\vec{p}\vec{x} - \epsilon_p t)}}{\sqrt{2\epsilon_p}} + a_p^+ \frac{e^{-i(\vec{p}\vec{x} - \epsilon_p t)}}{\sqrt{2\epsilon_p}} \right], \quad \epsilon_p = \sqrt{\vec{p}^2 + m^2}, \quad (2)$$

In other words, from the very beginning, one starts with the modes, which diagonalize the free Hamiltonian following from the Lagrangian (1). Then, one considers such an initial state in which the following is the case:

$$\langle a_q^+ a_{q'} \rangle = n_q^0 \delta(\vec{q} - \vec{q}') \quad \text{and} \quad \langle a_{\vec{q}} a_{\vec{q}'} \rangle = \chi_q^0 \delta(\vec{q} + \vec{q}'), \quad (3)$$

where both n_q^0 and χ_q^0 are not zero, and n_q^0 is not equal to the Planckian distribution. For simplicity, we begin with the consideration of spatially homogeneous states. Inhomogeneous situations will be briefly discussed below. Moreover, in this paper, we consider

only such states for which Wick's theorem does work. Furthermore, to make analytical considerations, we have to consider small initial χ_q^0 .

The second option is to consider the decomposition of the field operator as follows:

$$\varphi(x, t) = \int \frac{d^3 p}{(2\pi)^3} \left[b_p \frac{f_p(t, x)}{\sqrt{2\epsilon_p}} + b_p^\dagger \frac{f_p^*(t, x)}{\sqrt{2\epsilon_p}} \right], \quad (4)$$

where the modes have the following form.

$$f_p(t, x) = u_p e^{i(\vec{p}\vec{x} - \epsilon_p t)} + v_p e^{-i(\vec{p}\vec{x} - \epsilon_p t)}. \quad (5)$$

To obtain the canonical commutation relations for b_p with $b_p^\dagger$ and for the field operator $\varphi(x, t)$ with its conjugate momentum, constants u_p and v_p should obey relation $|u_p|^2 - |v_p|^2 = 1$.

Furthermore, to obtain the proper UV (Hadamard) behavior of the correlation functions, it is necessary to demand condition $\lim_{|\vec{p}| \rightarrow \infty} v_p = 0$. In fact, for very large momenta, modes $f_p(t, x)$ behave in the standard way, i.e., as those in (2). As a result, correlation functions have standard UV singularities. This is necessary for the proper UV renormalization of physical quantities in loops, e.g., only in such a case does beta-function of coupling constant λ depend only on the kinematic number of the degrees of freedom rather than on the values of u_p and v_p .

For a mode decomposition such as (4) and (5), one can consider, e.g., an initial state of the following form.

$$\text{Tr}[\hat{\rho} b_q^\dagger b_p] = \delta^3(\vec{p} - \vec{q}) n_p^0, \quad \text{Tr}[\hat{\rho} b_p b_q] = 0. \quad (6)$$

However, by performing the Bogoliubov transformation to the standard modes (2):

$$a_p = u_p b_p + v_p b_p^\dagger, \quad a_p^\dagger = u_p b_p^\dagger + v_p b_p, \quad (7)$$

we can reduce the problem to the one with the mode decomposition (2) and a state of form (3), where the initial level population and anomalous averages are equal to the following.

$$\bar{n}_q^0 = n_q^0 + O(v_q^2), \quad \bar{\chi}_q^0 = u_p v_p (1 + 2n_q^0). \quad (8)$$

Again, for simplicity we have assumed that v_q is small for all q .

Of course, (5) is not the most generic form of the modes, because Bogoliubov rotations are not the most generic canonical transformations. However, for the consideration of such states that are very close to spatial homogeneity, it is sufficient to consider the modes of form (5).

3. Dynamics of the Level-Population and Anomalous Averages at the Linear Order in the Coupling Constant

Thus, we restrict our attention to the first method, (2) and (3), of setting up the initial anomalous expectation values. In the theory (1), the free Hamiltonian has the standard form. Meanwhile, the normal ordered interaction term is as follows.

$$\begin{aligned} H_{int}(t) = & \frac{\lambda}{4!} \int \frac{d^3 p_1 \dots d^3 p_4}{(2\pi)^{12}} \frac{1}{4\sqrt{\epsilon_{p_1} \epsilon_{p_2} \epsilon_{p_3} \epsilon_{p_4}}} \times \\ & \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \left[a_{p_1} a_{p_2} a_{p_3} a_{p_4} e^{-i(\epsilon_{p_1} + \epsilon_{p_2} + \epsilon_{p_3} + \epsilon_{p_4})t} + \right. \\ & 4 a_{p_1}^\dagger a_{p_2} a_{p_3} a_{p_4} e^{i(\epsilon_{p_1} - \epsilon_{p_2} - \epsilon_{p_3} - \epsilon_{p_4})t} + 6 a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} e^{i(\epsilon_{p_1} + \epsilon_{p_2} - \epsilon_{p_3} - \epsilon_{p_4})t} + \\ & \left. 4 a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger a_{p_4} e^{i(\epsilon_{p_1} + \epsilon_{p_2} + \epsilon_{p_3} - \epsilon_{p_4})t} + a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger a_{p_4}^\dagger e^{i(\epsilon_{p_1} + \epsilon_{p_2} + \epsilon_{p_3} + \epsilon_{p_4})t} \right]. \quad (9) \end{aligned}$$

Note that such an interaction with the Hamiltonian does not commute with the total particle number unlike the one in the superconductor BCS theory.

To understand the thermalization process, one writes the interaction picture evolution equations for the following expectation values.

$$\frac{d}{dt} \langle a_q^+ a_{q'} \rangle = i \left\langle \left[H_{int}(t), a_q^+ a_{q'} \right] \right\rangle, \quad \frac{d}{dt} \langle a_q a_{q'} \rangle = i \left\langle \left[H_{int}(t), a_q a_{q'} \right] \right\rangle. \quad (10)$$

The novel moment that is considered in this paper in comparison with the standard situation is that now we add the equation for $\langle a_q a_{q'} \rangle$ on top of the standard equation for $\langle a_q^+ a_{q'} \rangle$. These quantities are related to the level-population and anomalous averages such as $\langle a_q^+ a_{q'} \rangle = n_q \delta(\vec{q} - \vec{q}')$ and $\langle a_q a_{q'} \rangle = \chi_q \delta(\vec{q} + \vec{q}')$, if the average is taken over the spatially homogeneous state. Throughout this paper, we always assume that n_q and χ_q are functions of the modulus of $\vec{q}$.

Consider first the linear order in λ . At this order, using Wick's theorem, one obtains the following¹:

$$\begin{aligned} \frac{dn_q}{dt} = \frac{i\lambda}{2} \int \frac{d^3p}{4(2\pi)^3 \epsilon_q \epsilon_p} & \left[\left(\chi_q \chi_p e^{-2i(\epsilon_q + \epsilon_p)t} - \chi_q^* \chi_p^* e^{2i(\epsilon_q + \epsilon_p)t} \right) + \right. \\ & \left. (1 + 2n_p) \left(\chi_q e^{-2i\epsilon_q t} - \chi_q^* e^{2i\epsilon_q t} \right) + \left(\chi_p^* \chi_q e^{2i(\epsilon_q - \epsilon_p)t} - \chi_p \chi_q^* e^{2i(\epsilon_q - \epsilon_p)t} \right) \right], \end{aligned} \quad (11)$$

and the following is the case.

$$\begin{aligned} \frac{d\chi_q}{dt} = \frac{-i\lambda}{2} \int \frac{d^3p}{4(2\pi)^3 \epsilon_q \epsilon_p} & \left[2\chi_q(1 + 2n_p) + (1 + 2n_q)\chi_p^* e^{2i(\epsilon_q + \epsilon_p)t} + (1 + 2n_q)\chi_p e^{2i(\epsilon_q - \epsilon_p)t} + \right. \\ & \left. (1 + 2n_q)(1 + 2n_p)e^{2i\epsilon_q t} + 2\chi_q \chi_p^* e^{2i\epsilon_p t} + 2\chi_q \chi_p e^{-2i\epsilon_p t} \right]. \end{aligned} \quad (12)$$

In deriving these equations, we have used $\langle a_q^+ a_{q'} \rangle = n_q \delta(\vec{q} - \vec{q}')$ and $\langle a_q a_{q'} \rangle = \chi_q \delta(\vec{q} + \vec{q}')$ for translationally invariant states. Here, n_p and χ_p are the level population and anomalous averages at the moment of time t , while n_q^0 and χ_q^0 , which were introduced above, are their values at the initial moment of time. Moreover, we assume the kinetic approximation here, i.e., that n_p and χ_q are slow functions of time t in comparison with the modes.

Oscillating terms on the right hand sides of (11) and (12) can be neglected, as they do not lead to the secular growth of n_q and χ_q after the integration of both sides of these equations over t . This means that, in limit $\lambda \rightarrow 0$, the change of n_q and χ_q in time, which is caused by such terms, is negligible. Then, neglecting the oscillating terms, one obtains the following:

$$\frac{dn_q}{dt} = 0 \quad (13)$$

and the following is the case.

$$\epsilon_q \frac{d\chi_q}{dt} = \chi_q \frac{-i\lambda}{2} \int \frac{d^3p}{4(2\pi)^3 \epsilon_p} \left[2(1 + 2n_p) \right]. \quad (14)$$

We can obtain the analytic headway assuming that the system is very close to the equilibrium; i.e., the following is the case.

$$\chi_q^0(t=0) \ll 1, \quad n_q^0(t=0) = \frac{1}{e^{\beta\epsilon_q} - 1} + \tilde{n}_q, \quad \tilde{n}_q \ll \frac{1}{e^{\beta\epsilon_q} - 1}. \quad (15)$$

The first condition here is necessary to linearize the equation for χ_q , if they were nonlinear, as will be the case below in the λ^2 order.

If one substitutes Planckian distribution for n_p into the right hand side (RHS) of Equation (14), it is not hard to recognize the term responsible for mass renormalization. Namely, the integral on the RHS of (14) is divergent and can be absorbed into the renormalization of ϵ_q . In fact, if one substitutes $\chi_q = \chi_q^{ren} e^{i2\delta\epsilon_q t}$ into Equation (14), then $\delta\epsilon_q$ can be adjusted such that (14) acquires form $d\chi_q^{ren}/dt = 0$.

Furthermore, the RHS of Equation (14) appears due to the bubble diagram of the $\lambda \varphi^4$ theory, which is responsible for self-energy renormalization. The resummation of such diagrams can be related to the renormalization of self-energy, which should be performed in the process of normal ordering of the interaction term (9) in the Hamiltonian. This is another argument why the RHS of Equation (14) can be attributed to mass renormalization. Thus, from now on, we deal with χ_q^{ren} with the renormalized mass but denote it as usual by χ_q .

4. Kinetic Equations at the Second Order in the Coupling Constant

Let us continue with the second order in λ . Essentially, here we briefly review the derivation of kinetic equations [15,16] but in the presence of anomalous averages. Again, one starts with (10) but does not truncate (with the use of the Wick contractions) RHS immediately after the calculation of the commutator. Namely, one uses commutation relations for the annihilation and creation operators to obtain, on the RHS of (10), the expectation values of the operators that contain fourth powers of a_p and a_p^+ operators. For example, the equation for $\langle a_q^+ a_{q'} \rangle$ contains such terms as follows.

$$\langle [a_{p_1}^+ a_{p_2}^+ a_{p_3}^+ a_{p_4}, a_q^+ a_{q'}] \rangle = \delta^3(\vec{q} - \vec{p}_4) \langle a_{p_1}^+ a_{p_2}^+ a_{p_3}^+ a_{q'} \rangle - 3\delta^3(\vec{q}' - \vec{p}_1) \langle a_q^+ a_{p_2}^+ a_{p_3}^+ a_{p_4} \rangle. \quad (16)$$

Such an expression stands under the integral over $d^3 p_1 \dots d^3 p_4$ on the RHS of (10), which comes from the definition of H_{int} in (9). With this method, one obtains the dynamical equations for $\langle a_q^+ a_{q'} \rangle$ and $\langle a_q a_{q'} \rangle$, which are expressed via the expectation values of the quartic in a and a^+ operators.

The next step is to derive the interaction picture evolution equations for the obtained quartic in a and a^+ operators—such operators which appear, e.g., on the RHS of (16). In particular, the following is obtained.

$$\frac{d}{dt'} \langle a_q^+ a_{p_2}^+ a_{p_3}^+ a_{p_4} \rangle = i \left\langle \left[H_{int}(t'), a_q^+ a_{p_2}^+ a_{p_3}^+ a_{p_4} \right] \right\rangle. \quad (17)$$

Continuing this procedure, one obtains the so called Bogoliubov hierarchy [15,16]. To truncate it and to close the system of the equations for the level population and anomalous averages, one uses the kinetic approximation and Wick contractions. In the previous section, we performed the contraction on the previous step.

The key difference with respect to the standard procedure [15,16] is that now we are taking into account the fact that anomalous averages are also not equal to zero. For example, from the first expectation value on the right hand side of (16), we obtain the following contribution to the RHS of (10):

$$\begin{aligned}
 \int \frac{d^3 p_1 \dots d^3 p_4}{(2\pi)^{12}} \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \delta^3(\vec{q} - \vec{p}_4) \langle [H_{int}(t'), a_{p_1}^+ a_{p_2}^+ a_{p_3}^+ a_{q'}] \rangle = \\
 \frac{\lambda}{4!} \delta^3(\vec{q} - \vec{q}') \int \frac{d^3 p_1 d^3 p_2 d^3 p_3}{(2\pi)^9} \frac{1}{4\sqrt{\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_q}} \times \\
 \left\{ 4! \left[n_q - (1 + n_q) \right] \chi_1^* \chi_2^* \chi_3^* e^{i(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_q) t'} + \right. \\
 18 \times 4 \left[n_q(1 + n_2) - (1 + n_q)n_2 \right] \chi_1^* \chi_3^* e^{i(\epsilon_q + \epsilon_1 + \epsilon_3 - \epsilon_2) t'} + \\
 12 \cdot 6 \left[(1 + n_2) - n_2 \right] \chi_q \chi_1^* \chi_3^* e^{i(\epsilon_1 + \epsilon_3 - \epsilon_q - \epsilon_2) t'} + \\
 12 \times 6 \left[n_q(1 + n_1)(1 + n_2) - (1 + n_q)n_1 n_2 \right] \chi_3^* e^{i(\epsilon_3 + \epsilon_q - \epsilon_1 - \epsilon_2) t'} + \\
 18 \times 4 \left[(1 + n_1)(1 + n_2) - n_1 n_2 \right] \chi_q \chi_3^* e^{i(\epsilon_3 - \epsilon_1 - \epsilon_2 - \epsilon_q) t'} + \\
 6 \times 4 \left[n_q(1 + n_1)(1 + n_2)(1 + n_3) - (1 + n_q)n_1 n_2 n_3 \right] e^{i(\epsilon_q - \epsilon_1 - \epsilon_2 - \epsilon_3) t'} + \\
 \left. 4! \chi_q \left[(1 + n_1)(1 + n_2)(1 + n_3) - n_1 n_2 n_3 \right] e^{i(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_q) t'} \right\}, \quad (18)
 \end{aligned}$$

where $n_i \equiv n_{p_i}$, $i = 1, 2, 3$. In deriving the last expression, we have used the Wick contractions and $\langle a_q^+ a_{q'} \rangle = n_q \delta(\vec{q} - \vec{q}')$ and $\langle a_q a_{q'} \rangle = \chi_q \delta(\vec{q} + \vec{q}')$.

On the next step from Equation (18), one finds the time evolution of such operators as $a_{p_1}^+ a_{p_2}^+ a_{p_3}^+ a_{q'}$ via equations of the following form:

$$\langle a_q^+ a_{p_2}^+ a_{p_3}^+ a_{p_4} \rangle(t) = i \int_{t_0}^t dt' \langle [H_{int}(t'), a_q^+ a_{p_2}^+ a_{p_3}^+ a_{p_4}] \rangle, \quad (19)$$

which then can be substituted into (10) and (16). Here, t_0 is the time after which the interaction term is adiabatically turned on. Please keep in mind that we eventually take limit $t - t_0 \rightarrow \infty$.

The final step is to take the integral over t' in such expressions as (19) in the kinetic approximation. In this approximation, all n 's and χ 's are assumed to be slow functions of time in comparison with the oscillating modes. That is a natural assumption in the vicinity of the equilibrium. Then, such integrals as in (19) result in delta-functions establishing energy conservation in limit $t - t_0 \rightarrow \infty$.

If we now exclude all terms that contain oscillating functions of t and such terms for which arguments of the delta-functions, which establish energy-momentum conservation, cannot be zero, and we obtain the following system of kinetic equations for n_p and χ_p :

$$\begin{aligned}
 \epsilon_q \frac{d}{dt} n_q = \frac{\lambda^2}{16} \int \frac{d^3 p_1 d^3 p_2 d^3 p_3}{(2\pi)^9 \epsilon_1 \epsilon_2 \epsilon_3} \delta^4(\underline{q} + \underline{p}_1 - \underline{p}_2 - \underline{p}_3) \times \\
 \left[(1 + n_q)(1 + n_1)n_2 n_3 - n_q n_1(1 + n_2)(1 + n_3) \right], \quad (20)
 \end{aligned}$$

and the following is the case:

$$\begin{aligned}
 \epsilon_q \frac{d}{dt} \chi_q = \frac{\lambda^2}{16} \int \frac{d^3 p_1 d^3 p_2 d^3 p_3}{(2\pi)^9 \epsilon_1 \epsilon_2 \epsilon_3} \delta^4(\underline{q} + \underline{p}_1 - \underline{p}_2 - \underline{p}_3) \times \\
 \left\{ \chi_q \left[(1 + n_1)n_2 n_3 - n_1(1 + n_2)(1 + n_3) \right] + 2 \chi_1^* \chi_2 \chi_3 \left[n_q - (1 + n_q) \right] \right\}, \quad (21)
 \end{aligned}$$

where $\underline{q} = (\epsilon_q, \vec{q})$, $\epsilon_q = \sqrt{\vec{q}^2 + m^2}$ and $\underline{p}_i = (\epsilon_i, \vec{p}_i)$, $\epsilon_i = \sqrt{\vec{p}_i^2 + m^2}$, $i = 1, 2, 3$. It is possible to show that these equations resume a subclass of IR dominant Feynman diagrams in the Schwinger–Keldysh diagrammatic technique [15,16,18].

We have derived these equations for spatially homogeneous states. The generalization to slightly spatially inhomogeneous situations is straightforward: One only has to assume that n_p and χ_p are functions of t and $\vec{x}$ simultaneously and make the following changes of the derivatives on the left hand side (LHS) of Equations (20) and (21).

$$\epsilon_q \frac{d}{dt} \rightarrow \epsilon_q \partial_t + \vec{q} \vec{\partial} = q^\mu \partial_\mu. \quad (22)$$

This concludes the derivation of the system of kinetic equations. We will use it to study the thermalization process in the kinetic approximation, i.e., when the system is close to the equilibrium.

Now one can observe that Equation (20) for n_q does not contain any terms with anomalous averages χ_k . It is only the well-known quantum Boltzmann equation [15,16]. The RHS of (20) is the collision integral that describes competing scattering processes of gaining quanta on level ϵ_p and losing them from the same level. The new feature of the system of kinetic Equations (20) and (21) is that the RHS of the second equation describes similar kinetic scattering processes in which momentum and energy can be absorbed or generated by the background quantum state of the system.

Let us stress here an important moment relating our formal derivations to real physical processes. In fact, obviously, spatially inhomogeneous variants of Equations (20) and (21) only qualitatively describe situations realized in nature. To obtain equations describing the real physical dynamics quantitatively, one just has to modify the rates of various kinetic processes in the collision integrals—on the RHS of (20) and (21). These rates can be found empirically.

Now, it is not hard to see that Planckian distribution $n_q = \frac{1}{e^{\beta\epsilon_q} - 1}$, $\beta = \text{const}$ solves (20): Such a distribution annihilates both sides. Furthermore, from Boltzmann's equation follows the second law of thermodynamics. Thus, dynamics described by this equation has a direction. Hence, Boltzmann's equation predicts thermalization for an initial non-Planckian distribution n_q^0 .

Now we are ready to observe what happens with χ_q on the way towards equilibrium. Let us assume again that at the initial moment of time $t = 0$, the system was not very far from the equilibrium, i.e., that the conditions (15) are fulfilled. Again, only in such a situation can we make analytical observations. Otherwise, one has to solve the system of kinetic equations in question numerically. The second and third conditions in (15) allow one to exclude the situation in which the initial level population generates an increase in χ_q in the linear approximation.

Under conditions (15), we obtain that n_q approximately solves Equation (20), while from (21), we obtain the following.

$$\epsilon_q \frac{d}{dt} \chi_q \approx \chi_q \frac{\lambda^2}{16} \int \frac{d^3 p_1 d^3 p_2 d^3 p_3}{(2\pi)^9 \epsilon_1 \epsilon_2 \epsilon_3} \delta^4(\underline{q} + \underline{p}_1 - \underline{p}_2 - \underline{p}_3) \left[(1 + n_1) n_2 n_3 - n_1 (1 + n_2) (1 + n_3) \right]. \quad (23)$$

Since, for Planckian distribution n_p , the following relation is true:

$$\delta^4(\underline{q} + \underline{p}_1 - \underline{p}_2 - \underline{p}_3) \left[(1 + n_q) (1 + n_1) n_2 n_3 - n_q n_1 (1 + n_2) (1 + n_3) \right] = 0,$$

one obtains the following.

$$\delta^4(\underline{q} + \underline{p}_1 - \underline{p}_2 - \underline{p}_3) \left[(1 + n_1) n_2 n_3 - n_1 (1 + n_2) (1 + n_3) \right] = \delta^4(\underline{q} + \underline{p}_1 - \underline{p}_2 - \underline{p}_3) \frac{-(1 + n_1) n_2 n_3}{n_q} < 0. \quad (24)$$

As a result, the solution of (23) has form $\chi_q(t) = C_q e^{-\Gamma_q t}$, where constant C_q is given by initial conditions and $\Gamma_q > 0$ is defined as follows.

$$\epsilon_q \Gamma_q = \frac{\lambda^2}{16} \int \frac{d^3 p_1 d^3 p_2 d^3 p_3}{(2\pi)^9 \epsilon_1 \epsilon_2 \epsilon_3} \delta^4(q + p_1 - p_2 - p_3) [n_1(1+n_2)(1+n_3) - (1+n_1)n_2 n_3], \quad (25)$$

It is not hard to see that the integral in this expression is convergent and its sign coincides with the sign of the integrand.

5. Conclusions and Discussion

So far we have considered the approach to such an equilibrium state in which the anomalous average is zero. Let us show now that, in theory (1), the equilibrium with non-zero anomalous average is impossible.

Consider an equilibrium state with non-zero anomalous average $\langle a_{\vec{p}} a_{\vec{q}} \rangle = \delta^3(\vec{p} + \vec{q}) \chi_p$. As this state is stationary, the Wightman function for it, $W(x_1, t_1, x_2, t_2) = \langle \varphi(x_1, t_1) \varphi(x_2, t_2) \rangle$, should depend only on the time difference $\tau = t_2 - t_1$, as there should be space and time translational symmetry in equilibrium. For this to be true, the anomalous average must depend on time in the following manner.

$$\chi_q(t) = e^{2i\epsilon_q t} \kappa_q, \quad t = \frac{t_1 + t_2}{2}. \quad (26)$$

This can be seen after the substitution of the mode expansion (2) of the field operator into the expression of the Wightman function, $W(x_1, t_1, x_2, t_2) = \langle \varphi(x_1, t_1) \varphi(x_2, t_2) \rangle$. In such a case as (26), the anomalous average $\chi_q(t)$ is not a slow function of time even at the equilibrium, when κ_q is time independent. However, n_p and κ_q enter into the Wightman function in combination $N_p = n_p + \text{Re } \kappa_p$ for which the kinetic equation has a standard form (20).

Let us stress, however, that there are theories in which the formation of a condensate is possible. Non-trivial condensate is directly related to the presence of non-zero anomalous averages of certain forms. One can keep in mind, e.g., the BCS theory, in which the condensate of Cooper pairs corresponds to the situation with non-zero anomalous averages or the superfluid He4 or Bose–Einstein condensates [1,2]. However, in contrast to (1), these theories are non-relativistic; in such theories, anomalous averages can be distinguished from the level distributions even in stationary situations and interaction terms in these theories with respect to the total particle number.

Thus, we have derived the system of kinetic equations for the level population and anomalous expectation values in four-dimensional massive scalar field theory with φ^4 self-interaction. Using this system in the linear approximation, we have shown analytically that for their small initial values, the anomalous quantum averages relax down to zero.

Furthermore, we have shown that this system does not have an equilibrium solution with non-zero time independent anomalous expectation values. It is instructive to observe if the last observation has any relation to the fact that the Coleman–Weinberg mechanism of the dynamical condensate generation is impossible in the real massive scalar field theory. That should be a non-trivial observation because, in this paper, we have essentially resumed the leading IR loop corrections, while the Coleman–Weinberg potential follows from the resummation of the leading UV loop contributions.

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Note

- ¹ Throughout this paper we consider only such states, with respect to which the expectation values are taken, for which the Wick’s theorem is fulfilled [17]. Also at the beginning we consider only such states which respect the spatial translational invariance. Extension to non-invariant states will be considered below.

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