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A Real Scalar Field Unifying the Early Inflation and the Late Accelerating Expansion of the Universe through a Quadratic Equation of State: The Vacuumon

Pierre-Henri Chavanis

Laboratoire de Physique Théorique (IRSAMC), CNRS and UPS, Université de Toulouse, 31062 Toulouse, France; chavanis@irsamc.ups-tlse.fr

Abstract: In a previous paper we introduced a cosmological model describing the early inflation, the intermediate decelerated expansion, and the late accelerating expansion of the universe in terms of a single barotropic fluid characterized by a quadratic equation of state. We obtained a scalar field representation of this fluid and determined the potential $V(\phi)$ connecting the inflaton potential in the early universe to the quintessence potential in the late universe. This scalar field has later been called the 'vacuumon' by other authors, in the context of the Running Vacuum model. In this paper, we study how the scalar field potential is modified by the presence of other cosmic components such as stiff matter, black-body radiation, baryonic matter, and dark matter. We also determine the mass m and the self-interaction constant λ of the scalar field given by the second and fourth derivatives of the potential at its extrema. We find that its mass is imaginary in the early universe with a modulus of the order of the Planck mass $M_P = (\hbar c/G)^{1/2} = 1.22 \times 10^{19}\,\mathrm{GeV/c^2}$ and real in the late universe with a value of the order of the cosmon mass $m_\Lambda = (\Lambda\hbar^2/c^4)^{1/2} = 2.08 \times 10^{-33}\,\mathrm{eV/c^2}$ predicted by string theory. Although our model is able to describe the evolution of the homogeneous background for all times, it cannot account for the spectrum of fluctuations in the early universe. Indeed, by applying the Hamilton–Jacobi formalism to our model of early inflation, we find that the Hubble hierarchy parameters and the spectral indices lead to severe discrepancies with the observations. This suggests that the vacuumon potential is just an effective classical potential that cannot be directly used to compute the fluctuations in the early universe. A fully quantum field theory may be required to achieve that goal. Finally, we discuss the connection between our model based on a quadratic equation of state and the Running Vacuum model which assumes a variation of the cosmological constant with the Hubble parameter.

Keywords: cosmology; inflation; dark matter; dark energy; equation of state; scalar field; de Sitter era; primordial fluctuations

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1. Introduction

The universe displays three main periods of evolution: an early phase of inflation during which the scale factor increases exponentially rapidly with time, an intermediate phase of decelerated expansion during which the scale factor increases algebraically, and a late phase of accelerating expansion during which the scale factor increases exponentially rapidly again.[1] The idea that a period of accelerated expansion (early inflation) may have occured in the early universe was introduced by Guth [1] in 1981 to explain the observed isotropy, homogeneity, and flatness of the universe in a natural way. The early inflation also explains the near scale-invariant spectrum of cosmological perturbations. The present acceleration of the universe was discovered at the end of the twentieth century [2–5] and was a surprise.

The intermediate phase of decelerated expansion is relatively well-understood. It corresponds to a relativistic radiation era followed by a nonrelativistic matter era. These two periods are described by a linear equation of state

$$P = \alpha\rho c^2, \tag{1}$$

where P is the pressure and ρc^2 is the energy density. The coefficient is equal to $\alpha = 1/3$ for radiation (corresponding to a gas of photons or other ultrarelativistic particles such as neutrinos) and to $\alpha = 0$ for pressureless matter (baryonic matter and dark matter). One can also consider a small value of $\alpha = k_B T/mc^2 \sim 10^{-7}$ in the matter era in order to take into account thermal effects (to make this estimate we have used $v_c \sim (k_B T/m)^{1/2} \sim 100\,\mathrm{km/s}$ obtained from the rotation curves of the galaxies).

The early inflation and the late accelerating expansion of the universe are less well understood. The fundamental constant that describes quantum mechanics is the Planck constant $\hbar = 1.05 \times 10^{-31}\,\mathrm{m^2\,g\,s^{-1}}$ [6]. From the Planck constant, and from the other fundamental constants of physics (the speed of light c and the gravitational constant G), one can construct a density

$$\rho_P = \frac{c^5}{\hbar G^2} = 5.16 \times 10^{99}\,\mathrm{g\,m^{-3}}, \tag{2}$$

called the Planck density. This density is extremely high. It is expected to play a fundamental role in the early universe (which is very dense) and be responsible for the first phase of inflation. It is usually believed that inflation corresponds to a de Sitter stage during which the density of the universe is constant and equal to the Planck density.[2] On the other hand, Einstein [7] introduced a cosmological constant Λ in the equations of general relativity in order to obtain a static universe. Although Einstein rejected this constant after the discovery of the expansion of the universe, calling it his "biggest blunder" [8], we now know that a nonzero value of the cosmological constant $\Lambda = 1.00 \times 10^{-35}\,\mathrm{s^{-2}}$ is favored by current observations. The effect of the cosmological constant is equivalent to the effect of a constant density

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} = 5.96 \times 10^{-24}\,\mathrm{g\,m^{-3}}, \tag{3}$$

called the cosmological density. This density is extremely low. It is expected to play a fundamental role in the late universe (which is very dilute) and be responsible for its observed present-day acceleration. The origin of this acceleration is generally called dark energy. It is usually believed that the late accelerating expansion of the universe corresponds to a second de Sitter stage during which the density of the universe is constant and equal to the cosmological density. The ratio between the Planck density (early universe) and the cosmological density (late universe) is

$$\frac{\rho_P}{\rho_\Lambda} \sim 10^{123}. \tag{4}$$

They differ by 123 orders of magnitude. The cosmological constant is usually interpreted in terms of the vacuum energy. The vacuum is described by an equation of state $P = -\rho c^2$ [9–11] which implies a constant energy density leading to a de Sitter exponential expansion. However, particle physics predicts that the vacuum energy should be of the order of the Planck scale. Therefore, this interpretation leads to a discrepancy of 123 orders of magnitude with the measured value of Λ. This is the so-called "cosmological constant problem" [12,13].

Various models have been proposed to describe the primordial inflation. The model of Starobinsky [14] takes into account quantum gravitational effects. It has a purely geometric nature and consists in a generalization of the Einstein–Hilbert action to contain an R^2 contribution, where R is the Ricci scalar curvature. It leads to a nonsingular de Sitter stage instead of the initial big bang singularity predicted by the classical Einstein equations. The phase of inflation in the very early universe can also be described by a hypothetical scalar field ϕ, called inflaton, running down a potential. This scalar field has its origin in the quantum fluctuations of the vacuum and is usually associated with a nonequilibrium phase transition [15]. Interestingly, the Starobinsky model can be mapped on a scalar field model with a special form of potential (see, e.g., [16] and references therein).

Various models of dark energy have also been proposed to describe the late acceleration of the universe. The simplest model is the cold dark matter model with a cosmological constant (ΛCDM model), which describes dark matter as a pressureless fluid and dark energy as a cosmological constant [17,18]. The ΛCDM model provides a very good description of the large scale structure of the universe and can account for the observations of the Planck mission [19,20]. However, the ΛCDM model suffers from the cosmological constant problem mentioned above and from the "cosmic coincidence problem" [21–23], namely why the fractions of dark matter and dark energy turn out to be of the same order of magnitude at the present epoch although they scale differently with the universe's expansion.[3] The CDM model also suffers from small-scale problems (at the galactic scale) such as the core-cusp problem of dark matter halos [24], the missing satellite problem [25–27], and the "too big to fail" problem [28]. This leads to the so-called small-scale crisis of CDM [29]. In order to solve or alleviate these difficulties, other models of dark energy have been introduced. Inspired by the models of inflation, some authors have proposed to describe the dark energy in terms of a self-interacting scalar field called quintessence [30–33] which can be interpreted as a dynamical vacuum energy. Other authors [34] have invoked an exotic fluid with a negative pressure called the Chaplygin gas [35]. This model provides a unification of dark matter and dark energy in terms of a single dark fluid. Additional models of unified dark matter and dark energy (called UDM or quartessence models [36]), such as the generalized Chaplygin gas [34,37], the modified Chaplygin gas [38], the polytropic gas [39–47], or the logotropic dark fluid [48–52], have been introduced subsequently. These models are consistent with observation data only if they are extremely close to the ΛCDM model [53] which is equivalent (in its UDM interpretation) to a single dark fluid with a constant negative pressure $P = -\rho_\Lambda c^2$ [42,53,54].

In a series of papers [39–47], we have proposed to describe the whole evolution of the universe, from its early inflation to its late accelerating expansion, by a quadratic equation of state of the form

$$P = -(\alpha+1)\frac{\rho^2}{\rho_P}c^2 + \alpha\rho c^2 - (\alpha+1)\rho_\Lambda c^2, \tag{5}$$

in which the coefficients are the Planck density ρ_P (see footnote [2]) and the cosmological density ρ_Λ. The left term describes the early inflation, the middle term describes the decelerated expansion (corresponding to a fluid with a linear equation of state $P = \alpha\rho c^2$), and the right term describes the late accelerating expansion. This equation of state can be simplified in some limits. In the early universe ($\rho \gg \rho_\Lambda$), the last term in Equation (5) is negligible and the quadratic equation of state

$$P = -(\alpha+1)\frac{\rho^2}{\rho_P}c^2 + \alpha\rho c^2 \tag{6}$$

describes the transition between the early inflation and the phase of decelerated expansion. On the other hand, in the late universe ($\rho \ll \rho_P$), the first term in Equation (5) is negligible and the affine equation of state

$$P = \alpha\rho c^2 - (\alpha+1)\rho_\Lambda c^2 \tag{7}$$

describes the transition between the phase of decelerated expansion and the late accelerating expansion (the ΛCDM model, corresponding to a constant equation of state $P = -\rho_\Lambda c^2$, is recovered for $\alpha = 0$). Interestingly, the equations of state (6) and (7) can be viewed as generalized polytropic equations of state of the form $P/c^2 = \alpha\rho + k\rho^{1+1/n}$ involving a linear term $P/c^2 = \alpha\rho$ and a polytropic term $P/c^2 = k\rho^{1+1/n}$ with a negative pressure ($k < 0$) and an index $n = +1$ (resp. $n = -1$) in the early (resp. late) universe. These generalized polytropic equations of state have been studied at a general level in [41–43]. They can be viewed as generalized (or modified) Chaplygin gas models.

Our main results can be summarized as follows:

(i) In Ref. [41], we studied in detail the equation of state (6) with $\alpha = 1/3$ describing the smooth transition between the early inflation and the radiation era. This equation of state provides a "graceful exit" to the de Sitter era. We considered more general models with an arbitrary value of α (instead of $\alpha = 1/3$) and an arbitrary positive polytropic index n_e (instead of $n_e = +1$). We showed that the results remain qualitatively the same in these more general situations.

(ii) In Ref. [42], we studied in detail the equation of state (7) with $\alpha = 0$ describing the smooth transition between the matter era and the late inflation. We showed that this equation of state returns the ΛCDM model. We considered more general models with an arbitrary value of α (instead of $\alpha = 0$) and an arbitrary negative polytropic index n_l (instead of $n_l = -1$). We showed that the results remain qualitatively the same in these more general situations.

(iii) In Refs. [39,40,42,47], we described the complete history of the universe. It involves two de Sitter eras (early and late inflation) bridged by an intermediate decelerated era. These results are reported in Figure 14 of [42]. They have been obtained by connecting the results valid in the early universe (see Equation (6)) to the results valid in the late universe (see Equation (7)). This approach describes the successive phases of early inflation, radiation, matter, and late inflation. In this manner, our model smoothly connects the primordial inflation to the ΛCDM model. In our model, the early and late evolution of the universe is remarkably symmetric. It is described by two polytropic equations of state of index $n_e = +1$ and $n_l = -1$ respectively. In addition, the cosmological density ρ_Λ in the late universe is the counterpart of the Planck density ρ_P in the early universe. As the universe expands, the density decreases from the Planck density $\rho_P = 5.16 \times 10^{99}\,\mathrm{g\,m^{-3}}$ to the cosmological density $\rho_\Lambda = 5.96 \times 10^{-24}\,\mathrm{g\,m^{-3}}$, spanning 123 orders of magnitude (see Figure 15 of [42]). The resulting model of universe is non-singular and non-phantom. There is no big bang singularity in the past, nor big rip singularity in the future. The early and late behaviors of the universe are described by two de Sitter eras with density ρ_P and ρ_Λ, respectively. The universe exists eternally in the past and in the future. There is no question such as "What happens for $t < 0$ before the big bang?". We called this nonsingular and fully symmetric model of universe, exhibiting two extreme de Sitter eras bridged by a period of decelerated expansion, the "aioniotic" universe (see Section 7.4 of [42]).

(iv) In Refs. [39–42,47], we studied the thermal history of the universe. As the Friedmann equations are dissipationless, the total entropy of the universe (including all kinds of matter and energy) is constant. It has the very large value $S/k_B = 5.04 \times 10^{87}$ [41]. We obtained a generalized Stefan–Boltzmann law valid in the early universe (see Equation (84a) of [41]). In our model, the temperature T increases exponentially rapidly during the inflation up to the Planck temperature $T_P = 1.42 \times 10^{32}\,\mathrm{K}$, then decreases algebraically during the radiation and matter eras (see Figure 16 of [42]). This is very different from other models of inflation where the temperature drops drastically during the exponential inflation and one has to invoke a phase of re-heating by various high energy processes (that are not very well-understood) in order to restore the initial temperature.

(v) In Refs. [39,40,42,47], we developed a scalar field representation of our model. We determined the "inflaton" potential (see Equation (123) of [42]) associated with the equation of state (6) which describes the smooth transition between the early inflation and the radiation era, and we determined the "quintessence" potential (see Equation (125) of [42]) associated with the equation of state (7) which describes the smooth transition between the matter era and the late inflation.

(vi) In Refs. [45–47], we considered the possibility that the cosmic history of the universe involves an additional stiff matter era after the inflation and prior to the radiation era. This stiff matter era is described by an equation of state of the form $P = \rho c^2$ where the speed of sound, given by $c_s^2 = P'(\rho)$, is equal to the speed of light ($c_s = c$) [45,55,56]. We proposed to describe the transition between the inflation and the stiff matter era in the primordial universe by an equation of state of the form of Equation (6) with $\alpha = 1$, and we derived the corresponding scalar field potential (see Equation (140) in [45] and Equation (F.42) in [46]).[4]

(vii) In Refs. [39,40,42,44], we solved the general model described by the quadratic equation of state (5). We first derived the explicit relation between the energy density and the scale factor (see Equation (86) of [44]). We then obtained an exact analytical solution of the Friedmann equations giving the complete temporal evolution of the scale factor $a(t)$ and energy density $\rho(t)$ from $t = -\infty$ to $t = +\infty$ (see Equation (106) of [44]). This solution describes the early inflation, the intermediate decelerated expansion, and the late accelerating expansion of the universe. The quadratic equation of state (5) therefore provides a unification of the early and late inflation of the universe. We determined the general scalar field potential associated with this equation of state. We obtained its exact analytical expression in terms of Jacobian Elliptic functions (see Equation (121) of [44]) and proposed a simple approximate expression obtained by using matched asymptotic expansions (see Equation (131) of [44]). Interestingly, our scalar field theory describes, with a unique potential, the whole evolution of the universe, from its early inflation to its late accelerating expansion, passing through a phase of algebraically decelerating expansion. In this sense, the scalar field potential $V(\phi)$ unifies the inflaton potential in the early universe and the quintessence potential in the late universe.

Very similar results have been obtained in parallel by J. Solà and his collaborators in the context of the Running Vacuum Model (RVM). The basic idea behind this model (see the review [57] for an exhaustive list of references) is that the cosmological constant actually depends on time. Using results of particle physics and the renormalization group approach, they argued that the cosmological constant $\Lambda(H)$ is related to the Hubble constant H by a quartic equation. When combined with radiation and matter, this model produces a phase of early inflation, a phase of decelerated expansion, and a phase of late inflation which are very similar to the ones obtained in our model (see Appendix C for a comparison between the two approaches). Recently, Basilakos et al. [58] developed a scalar field representation of the RVM. They proposed to call the scalar field that accounts for the temporal evolution of the vacuum energy the "vacuumon". As they obtained exactly the same potentials in the early and late universe as the ones obtained previously in our papers (compare Equations (4.18) and (4.42) of [58] with Equations (123) and (125) of [42]), we will also call "vacuumon" the scalar field associated with the quadratic equation of state (5) of our model [39–47].[5] Following this terminology, what we referred to as "inflaton" in the early universe (associated with the equation of state (6)) will be called "early vacuumon" and what we referred to as "quintessence" in the late universe (associated with the equation of state (7)) will be called "late vacuumon". Even more recently, these authors developed a new model that they called String-inspired Running Vacuum [59]. They managed to make a connection between the phenomenological RVM and string theory. Interestingly, their new model includes a stiff matter era in the primordial universe similar to the one introduced heuristically in our papers [45,46]. They described the transition between the inflation and the stiff matter era in a manner similar to the one described in Section XI of [45] or in Appendix F of [46]. They also derived a scalar field potential coinciding with the one previously obtained in our papers [45,46] (compare Equation (40) of [59] with Equation (140) of [45] or Equation (F.42) of [46]).[6] Therefore, the RVM [57–59] and our model [39–47] are consistent, valuable, and complementary to each other (see Appendix C).

We have argued above that the quadratic equation of state (5) describes the whole evolution of the universe from its early inflation to its late accelerating expansion. There is, however, a difficulty with this description which is connected to the value that one should ascribe to the parameter α. If we take $\alpha = 1/3$, the first two terms in Equation (5) describe the transition between the inflation and the radiation and the last term describes the late accelerating expansion of the universe. However, this equation of state does not account for the matter era. Indeed, it describes a universe undergoing early inflation, radiation era, and late inflation. Alternatively, if we take $\alpha = 0$, the first term in Equation (5) describes the early inflation and the last two terms describe the transition between the matter era and the late accelerating expansion of the universe. However, this equation of state does not account for the radiation era. Indeed, it describes a universe undergoing early inflation, matter era, and late inflation. This implies that the quadratic equation of state (5) is not able to describe the whole content of the universe. Indeed, it cannot describe simultaneously the radiation era and the matter era. At that point, we have two possibilities:

(A) The first possibility is to assume that the coefficient α that appears in the equation of state (5) depends on the density in such a way that $\alpha \to 1/3$ at high densities and $\alpha \to 0$ at low densities. In the early universe (high densities), we can neglect matter and dark energy and use Equation (6) with the coefficient $\alpha = 1/3$ (radiation). In the late universe (low densities), we can neglect inflation and radiation and use Equation (7) with the coefficient $\alpha = 0$ (matter). We then have to match these two asymptotic limits. This is the point of view adopted in [39,40,42,47]. This point of view is also consistent with the RVM (see Appendix C) provided that the density ρ is interpreted as the *total* density of the universe, i.e., the sum of the running vacuum energy density plus the energy density of radiation in the early universe or the matter energy density in the late universe (the same comment applies to the pressure P).

(B) Another possibility is to assume that the quadratic equation of state (5) with a fixed coefficient α describes only one cosmic fluid. This exotic fluid could correspond to a scalar field in its hydrodynamic representation. Then, we must consider, in addition, the contributions of other species treated as independent noninteracting fluids. These additional species correspond to standard fluids (stiff matter, radiation, and baryonic or dark matter) described by a linear equation of state. Different choices are possible. A first choice is to take $\alpha = 1/3$. In that case, the quadratic equation of state (5) characterizes a scalar field which is responsible for a phase of early inflation, a phase of radiation (it could be the standard radiation corresponding to photons or relativistic particles or a "dark radiation" different from the standard radiation), and a phase of late inflation. This scalar field provides a unification of inflation, radiation, and dark energy. Then, we have to add standard radiation ($\alpha_r = 1/3$), baryonic matter ($\alpha_b = 0$), and dark matter ($\alpha_{\rm dm} = 0$) as additional species. Another choice is to take $\alpha = 0$. In that case, the quadratic equation of state (5) characterizes a scalar field which is responsible for a phase of early inflation, a phase of pressureless dark matter, and a phase of late inflation. This scalar field provides a unification of inflation, dark matter, and dark energy. Then, we have to add standard radiation ($\alpha_r = 1/3$) and baryonic matter ($\alpha_b = 0$) as additional species. We can also choose another value of α, different from $1/3$ or 0, such as $\alpha = 1$ corresponding to stiff matter [45,46]. In that case, the quadratic equation of state (5) characterizes a scalar field which is responsible for a phase of early inflation, a stiff matter era, and a phase of late inflation. This scalar field provides a unification of inflation, stiff matter and dark energy. Then, we have to add standard radiation ($\alpha_r = 1/3$), baryonic matter ($\alpha_b = 0$) and dark matter ($\alpha_{\rm dm} = 0$) as additional species.

In conclusion, we are led to a considering a model of universe involving a scalar field described by a quadratic equation of state given by Equation (5) plus zero, one, or several X-fluids described by a linear equation of state $P = \alpha_X \rho c^2$. We assume that these different species are independent from each other and noninteracting. The case of a scalar field alone described by the quadratic equation of state (5) has been treated in our previous papers [39,40,42,44]. We found that this scalar field has a potential given by Equation (108) in the general case, and by Equations (115) and (132) in the early and late universe, respectively.

This is the potential of the "bare" vacuumon. If we now consider a scalar field described by the quadratic equation of state (5) in the presence of other species, such as X-fluids described by a linear equation of state $P = \alpha_X \rho c^2$, the potential of the scalar field will change.[7] In this paper, we explain how this potential can be calculated and we give its general expression under the form of an integral. This is the potential of the "dressed" vacuumon due to the presence of other species. Unfortunately, its general expression cannot be obtained analytically, except in particular limits.

Once the potential $V(\phi)$ has been obtained, we can determine the main characteristics of the scalar field such as its mass m and self-interaction constant λ which are given by the second and fourth derivatives of the potential at its extrema [47]. The potential of the vacuumon presents a maximum $V = \rho_P c^2$ in the early universe at $\phi = 0$ and a minimum $V = \rho_\Lambda c^2$ in the late universe at $\phi = \phi_{\rm max}$. Correspondingly, the early vacuumon (inflaton) has an imaginary mass of the order of the Planck mass $M_P = (\hbar c/G)^{1/2} = 1.22 \times 10^{19}\,{\rm GeV}/c^2$ and the late vacuumon (quintessence) has a real mass of the order of the cosmon mass $m_\Lambda = \hbar\sqrt{\Lambda}/c^2 = 2.08 \times 10^{-33}\,{\rm eV}/c^2$ predicted by string theory. We find that the mass of the vacuumon in the early universe satisfies a fundamental quantization rule while its mass in the late universe does not. Finally, we apply the Hamilton–Jacobi formalism [60] to our model of early inflation in order to obtain the Hubble hierarchy parameters and the spectral indices. We show that it leads to severe discrepancies with the observations. This suggests that the scalar field potential of the vacuumon is just an effective classical potential that cannot be directly used to compute the spectrum of fluctuations in the early universe. A fully quantum field theory may be required to achieve that goal.

The paper is organized as follows. In Section 2, we recall the basic equations that describe the cosmic evolution of a fluid with a linear equation of state and the basic equations that describe the cosmic evolution of a canonical self-interacting real scalar field. In Section 3, we briefly review our model of universe [39–47] based on the quadratic equation of state (5). In Section 4, we determine the potential of a scalar field associated with an arbitrary barotropic equation of state $P(\rho)$ in the presence of X-fluids and apply these general results to the quadratic equation of state (5). In Section 5, we show how these results can be simplified in the case of a scalar field alone in the universe and we recover the results of our previous papers [39–47]. In Section 6, we determine the parameters of the scalar field (vacuumon) such as its mass and its self-interaction constant in the early and late universe. In Section 7, we determine the potential of the scalar field in the presence of an additional fluid in the intermediate regime between the early and late inflation. In that case, the scalar field potential can be obtained analytically. In Section 8, we consider the evolution of the scalar field in the presence of one fluid in the late universe. In Section 9, we determine the Hubble hierarchy parameters corresponding to our model of early inflation (early vacuumon) and conclude that a fully quantum field theory is required to achieve a form of agreement with observations.

2. Basic Equations

In this section, we recall the basic equations determining the cosmological evolution of a fluid described by a linear equation of state (called here X-fluid) and the basic equations determining the cosmological evolution of real canonical self-interacting scalar field ϕ described by a potential $V(\phi)$.

2.1. Friedmann Equations

If we consider an expanding homogeneous background and adopt the Friedmann–Lemaître–Robertson–Walker (FLRW) metric, the Einstein field equations reduce to the Friedmann equations

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda}{3}, \tag{8}$$

$$2\dot{H} + 3H^2 = -\frac{8\pi G}{c^2}P - \frac{kc^2}{a^2} + \Lambda, \tag{9}$$

where $H = \dot{a}/a$ is the Hubble parameter, $a(t)$ is the scale factor, Λ is the cosmological constant, and k determines the curvature of space. The universe is closed if $k > 0$, flat if $k = 0$, or open if $k < 0$. Equation (9) can also be written as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda}{3}. \tag{10}$$

Combining Equations (8) and (9), we obtain the energy conservation equation

$$\frac{d\rho}{dt} + 3H\left(\rho + \frac{P}{c^2}\right) = 0. \tag{11}$$

This equation can be directly obtained from the conservation of the energy–momentum tensor $D_\mu T^{\mu\nu} = 0$ which is included in the Einstein equations through the contracted Bianchi identities. It can be rewritten as

$$\frac{d}{dt}(\rho c^2 a^3) = -P\frac{d(a^3)}{dt}. \tag{12}$$

Introducing the volume $V \propto a^3$ and the energy $E = \rho c^2 V$, Equation (12) takes the form $dE = -PdV$. It can be interpreted as the first principle of thermodynamics for an adiabatic evolution of the universe $dS = 0$ [41].

The equation of state parameter is defined by

$$w = \frac{P}{\rho c^2}. \tag{13}$$

According to Equation (11), the energy density decreases with the scale factor if $w > -1$ and increases with the scale factor if $w < -1$ (it is constant if $w = -1$). The case where the energy density increases with the scale factor corresponds to a phantom universe.

The deceleration parameter is defined by

$$q = -\frac{\ddot{a}a}{\dot{a}^2}. \tag{14}$$

The universe is decelerating if $q > 0$ and accelerating if $q < 0$.

In this paper, we consider a flat universe ($k = 0$) in agreement with the inflation paradigm [1] and the observations of the cosmic microwave background (CMB) [19,20]. On the other hand, we set the cosmological constant to zero ($\Lambda = 0$) because, in our model, the acceleration of the expansion of the universe will be taken into account in the equation of state of the scalar field. The Friedmann equations then reduce to the form

$$\frac{d\rho}{dt} + 3H\left(\rho + \frac{P}{c^2}\right) = 0, \tag{15}$$

$$H^2 = \frac{8\pi G}{3}\rho, \tag{16}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right). \tag{17}$$

In that case, the deceleration parameter is related to the equation of state parameter by

$$q = \frac{1+3w}{2}. \tag{18}$$

The universe is decelerating if $w > -1/3$ and accelerating if $w < -1/3$. When $w = -1/3$ the scale factor increases linearly with time.

We assume that the universe contains different X-fluids of density ρ_X and a scalar field of density ρ_ϕ. The total density and the total pressure are $\rho = \sum_X \rho_X + \rho_\phi$ and $P = \sum_X P_X + P_\phi$. The Friedmann Equation (16) can then be written as

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\left(\sum_X \rho_X + \rho_\phi\right). \tag{19}$$

2.2. X-Fluids

We assume that the X-fluids are described by a linear equation of state of the form

$$P_X = \alpha_X \rho_X c^2, \tag{20}$$

where α_X is constant. We also assume that these fluids are independent from each other (and from the scalar field) so that they individually satisfy the energy conservation equation

$$\frac{d\rho_X}{dt} + 3H\left(\rho_X + \frac{P_X}{c^2}\right) = 0. \tag{21}$$

Solving this equation with the equation of state (20), we find

$$\frac{\rho_X}{\rho_0} = \frac{\Omega_{X,0}}{a^{3(1+\alpha_X)}}, \tag{22}$$

where $\rho_0 c^2$ denotes the present value of the energy density of the universe and $\Omega_{X,0}$ denotes the proportion of the X-fluid at the present time ($a = 1$). Assuming that the X-fluid is alone in the universe (or dominates the other species) and solving the Friedmann Equation (16) with the density from Equation (22), we find that the scale factor and the density of the X-fluid evolve with time as

$$a = \left[\frac{3}{2}(1+\alpha_X)\left(\frac{8\pi G\rho_0\Omega_{X,0}}{3}\right)^{1/2} t\right]^{\frac{2}{3(1+\alpha_X)}}, \qquad \rho_X = \frac{1}{6\pi G(1+\alpha_X)^2 t^2}, \tag{23}$$

if $\alpha_X > -1$, and as

$$a \propto e^{\left(\frac{8\pi G\rho_X}{3}\right)^{1/2} t}, \qquad \rho_X = \mathrm{cst}, \tag{24}$$

if $\alpha_X = -1$.

For the radiation ($\alpha_r = 1/3$):

$$P_r = \frac{1}{3}\rho_r c^2, \qquad \frac{\rho_r}{\rho_0} = \frac{\Omega_{r,0}}{a^4},$$
$$a = \left[2\left(\frac{8\pi G\rho_0\Omega_{r,0}}{3}\right)^{1/2} t\right]^{\frac{1}{2}}, \qquad \rho_r = \frac{3}{32\pi G t^2}. \tag{25}$$

For baryonic matter ($\alpha_b = 0$):

$$P_b = 0, \qquad \frac{\rho_b}{\rho_0} = \frac{\Omega_{b,0}}{a^3},$$
$$a = \left[\frac{3}{2}\left(\frac{8\pi G\rho_0\Omega_{b,0}}{3}\right)^{1/2} t\right]^{\frac{2}{3}}, \qquad \rho_b = \frac{1}{6\pi G t^2}. \tag{26}$$

For dark matter ($\alpha_{\rm dm} = 0$):

$$P_{\rm dm} = 0, \qquad \frac{\rho_{\rm dm}}{\rho_0} = \frac{\Omega_{\rm dm,0}}{a^3},$$
$$a = \left[\frac{3}{2}\left(\frac{8\pi G\rho_0\Omega_{\rm dm,0}}{3}\right)^{1/2} t\right]^{\frac{2}{3}}, \qquad \rho_{\rm dm} = \frac{1}{6\pi G t^2}. \tag{27}$$

For stiff matter ($\alpha_s = 1$):

$$P_s = \rho_s c^2, \qquad \frac{\rho_s}{\rho_0} = \frac{\Omega_{s,0}}{a^6},$$
$$a = \left[3\left(\frac{8\pi G\rho_0\Omega_{s,0}}{3}\right)^{1/2} t\right]^{\frac{1}{3}}, \qquad \rho_s = \frac{1}{24\pi G t^2}. \tag{28}$$

For the dark energy ($\alpha_{\rm de} = -1$):

$$P_{\rm de} = -\rho_{\rm de} c^2, \qquad \frac{\rho_{\rm de}}{\rho_0} = \Omega_{\rm de,0},$$
$$a \propto e^{\left(\frac{8\pi G\rho_{\rm de}}{3}\right)^{1/2} t}, \qquad \rho_{\rm de} = \text{cst}. \tag{29}$$

The equation of state of the dark energy (vacuum energy) leads to a constant energy density implying a phase of exponential expansion (de Sitter). However, we will not need the equation of state of the dark energy in our model because the acceleration of the expansion of the universe will be taken into account in the equation of state (or in the potential) of the scalar field.

2.3. Canonical Scalar Field

A canonical scalar field minimally coupled to gravity evolves according to the Klein–Gordon (KG) equation[8]

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0, \tag{30}$$

where $V(\phi)$ is the potential of the scalar field. The scalar field tends to run down the potential towards lower energies. The density and the pressure of the scalar field are given by

$$\rho_\phi c^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{31}$$

We can check that these equations imply the energy conservation Equation (see Appendix A)

$$\frac{d\rho_\phi}{dt} + 3H\left(\rho_\phi + \frac{P_\phi}{c^2}\right) = 0. \tag{32}$$

When the kinetic term dominates the potential term ($\dot{\phi}^2/2 \gg V(\phi)$), we obtain the equation of state $P_\phi \sim \rho_\phi c^2$ of stiff matter. This is the kination regime [63]. When the potential term dominates the kinetic term ($\dot{\phi}^2/2 \ll V(\phi)$), we obtain the equation of state $P_\phi \sim -\rho_\phi c^2$ of vacuum or dark energy. This is the inflation (de Sitter) regime [15]. Inversely, if the scalar field is described by the stiff equation of state $P_\phi = \rho_\phi c^2$, we find $V(\phi) = 0$ and $\phi = (c^2/12\pi G)^{1/2} \ln t + \text{cst}$ corresponding to the kination regime. If the scalar field is described by the vacuum or dark energy equation of state $P_\phi = -\rho_\phi c^2$, we find $\dot{\phi} = 0$, i.e., $\phi = \text{cst}$ and $V(\phi) = \text{cst}$ corresponding to the de Sitter regime (see Section 5.6).

3. Scalar Field with a Quadratic Equation of State

3.1. General Equations

We assume that the scalar field is described by a quadratic equation of state of the form [39–47][9]

$$P_\phi = -(\alpha+1)\frac{\rho_\phi^2}{\rho_P}c^2 + \alpha\rho_\phi c^2 - (\alpha+1)\rho_\Lambda c^2, \tag{33}$$

where $\rho_P = 5.16\times 10^{99}\,\mathrm{g\,m^{-3}}$ is the Planck density (see footnote [2]) and $\rho_\Lambda = 5.96\times 10^{-24}\,\mathrm{g\,m^{-3}}$ is the cosmological density. The linear term, characterized by a constant α, may represent radiation ($\alpha = 1/3$), dark matter ($\alpha = 0$ or $\alpha \simeq 0$), stiff matter ($\alpha = 1$), or even be a new species.[10] The equation of state parameter $w_\phi = P_\phi/(\rho_\phi c^2)$ and the squared speed of sound $c_s^2 = P'_\phi(\rho_\phi)$ are given by

$$w_\phi = -(\alpha+1)\frac{\rho_\phi}{\rho_P} + \alpha - (\alpha+1)\frac{\rho_\Lambda}{\rho_\phi}, \tag{34}$$

$$\frac{(c_s^2)_\phi}{c^2} = -2(\alpha+1)\frac{\rho_\phi}{\rho_P} + \alpha. \tag{35}$$

Solving the energy conservation Equation (32) with the equation of state (33), we find that the density of the scalar field is given in excellent approximation by (see Appendix B)

$$\rho_\phi = \frac{\rho_P}{(a/a_1)^{3(\alpha+1)}+1} + \rho_\Lambda, \tag{36}$$

where a_1 is a constant of integration. To obtain Equation (36), we have used the fact that $\rho_\Lambda/\rho_P \ll 1$. This equation is essentially exact as $\rho_\Lambda/\rho_P \sim 10^{-123}$ is extremely small. It can be rewritten as

$$\frac{\rho_\phi}{\rho_0} = \frac{\Omega_{P,0}}{(a/a_1)^{3(\alpha+1)}+1} + \Omega_{\Lambda,0}, \tag{37}$$

where $\rho_0 c^2$ is the present value of the energy density and we have introduced the notations $\Omega_{P,0} \equiv \rho_P/\rho_0$ and $\Omega_{\Lambda,0} \equiv \rho_\Lambda/\rho_0$ ($\Omega_{\Lambda,0}$ represents the present proportion of dark energy in the universe while $\Omega_{P,0}$ is just a convenient notation). After the period of inflation (see below), we can make the approximation

$$\frac{\rho_\phi}{\rho_0} \simeq \frac{\Omega_{P,0}}{(a/a_1)^{3(\alpha+1)}} + \Omega_{\Lambda,0}. \tag{38}$$

In this manner, we see that $\Omega_{P,0}a_1^{3(\alpha+1)}$ represents the present proportion $\Omega_{\alpha,0}$ of the α-fluid in the universe. Therefore, the constant a_1 is given by

$$a_1 = \left(\frac{\Omega_{\alpha,0}}{\Omega_{P,0}}\right)^{\frac{1}{3(1+\alpha)}}. \tag{39}$$

We can then rewrite Equation (37) under the equivalent form

$$\frac{\rho_\phi}{\rho_0} = \frac{\Omega_{\alpha,0}}{a^{3(\alpha+1)} + \frac{\Omega_{\alpha,0}}{\Omega_{P,0}}} + \Omega_{\Lambda,0}. \tag{40}$$

Combining Equations (33)–(36) and using again the fact that $\rho_\Lambda/\rho_P \ll 1$, we obtain in excellent approximation

$$\rho_\phi = \frac{\rho_P + \rho_\Lambda(a/a_1)^{3(\alpha+1)}}{(a/a_1)^{3(\alpha+1)}+1}, \tag{41}$$

$$P_\phi = \frac{-\rho_\Lambda c^2(a/a_1)^{6(\alpha+1)} + \alpha\rho_P c^2(a/a_1)^{3(\alpha+1)} - \rho_P c^2}{\left[(a/a_1)^{3(\alpha+1)}+1\right]^2}, \tag{42}$$

$$w_\phi = \frac{-\rho_\Lambda (a/a_1)^{6(\alpha+1)} + \alpha\rho_P (a/a_1)^{3(\alpha+1)} - \rho_P}{\left[(a/a_1)^{3(\alpha+1)} + 1\right]\left[\rho_P + \rho_\Lambda (a/a_1)^{3(\alpha+1)}\right]}, \tag{43}$$

$$\frac{(c_s^2)_\phi}{c^2} = \frac{\alpha(a/a_1)^{3(\alpha+1)} - \alpha - 2}{(a/a_1)^{3(\alpha+1)} + 1}. \tag{44}$$

We note that the denominator in Equation (43) can be replaced by $\rho_\Lambda (a/a_1)^{6(\alpha+1)} + \rho_P (a/a_1)^{3(\alpha+1)} + \rho_P$ with the same degree of approximation.

If the universe contains only the scalar field, the deceleration parameter q is given by Equation (18). Using Equations (34) and (43), we obtain

$$q = \frac{1+3\alpha}{2} - \frac{3}{2}(\alpha+1)\frac{\rho}{\rho_P} - \frac{3}{2}(\alpha+1)\frac{\rho_\Lambda}{\rho} \tag{45}$$

and

$$q = \frac{-\rho_\Lambda (a/a_1)^{6(\alpha+1)} + \frac{3\alpha+1}{2}\rho_P (a/a_1)^{3(\alpha+1)} - \rho_P}{\left[(a/a_1)^{3(\alpha+1)} + 1\right]\left[\rho_P + \rho_\Lambda (a/a_1)^{3(\alpha+1)}\right]}. \tag{46}$$

The complete analytical solution $a(t)$ of the Friedmann Equation (16) with Equation (36) is given by [42,44]

$$\begin{aligned} &\frac{1}{\sqrt{\kappa}}\ln\left[1 + 2\kappa(a/a_1)^{3(\alpha+1)} + 2\sqrt{\kappa(1 + (a/a_1)^{3(\alpha+1)} + \kappa(a/a_1)^{6(\alpha+1)})}\right] \\ &\qquad - \ln\left[\frac{2 + (a/a_1)^{3(\alpha+1)} + 2\sqrt{1 + (a/a_1)^{3(\alpha+1)} + \kappa(a/a_1)^{6(\alpha+1)}}}{(a/a_1)^{3(\alpha+1)}}\right] \\ &\qquad\qquad = 3(\alpha+1)\left(\frac{8\pi}{3}\right)^{1/2}\frac{t}{t_P} + C, \end{aligned} \tag{47}$$

with $t_P = 1/\sqrt{G\rho_P} = (\hbar G/c^5)^{1/2} = 5.39 \times 10^{-44}$ s (Planck time). This solution describes the evolution of the universe from an early de Sitter era to a late de Sitter era bridged by a decelerating algebraic expansion (α-era). For the particular values $\alpha = 0, 1/3, 1$, the density evolves with the scale factor as

$$\rho_\phi = \frac{\rho_P}{1 + (a/a_1)^3} + \rho_\Lambda \quad (\alpha = 0), \qquad \rho_\phi = \frac{\rho_P}{1 + (a/a_1)^4} + \rho_\Lambda \quad (\alpha = 1/3),$$
$$\rho_\phi = \frac{\rho_P}{1 + (a/a_1)^6} + \rho_\Lambda \quad (\alpha = 1). \tag{48}$$

To better understand the physical meaning of the different terms involved in the foregoing equations, we successively study how these equations can be simplified in the early and in the late universe. We only give the equations that will be needed in the subsequent analysis, and we refer to our previous papers [39–47] for a more thorough discussion.

3.2. Early Universe

In the early universe, where the density is high, the equation of state (33) reduces to

$$P_\phi = -(\alpha+1)\frac{\rho_\phi^2}{\rho_P}c^2 + \alpha\rho_\phi c^2. \tag{49}$$

This amounts to neglecting the contribution of dark energy ($\rho_\Lambda = 0$). The equation of state parameter and the squared speed of sound are given by

$$w_\phi = -(\alpha+1)\frac{\rho_\phi}{\rho_P} + \alpha, \tag{50}$$

$$\frac{(c_s^2)_\phi}{c^2} = -2(\alpha+1)\frac{\rho_\phi}{\rho_P} + \alpha. \tag{51}$$

Solving the energy conservation Equation (32) with the equation of state (49), we find that the density of the scalar field is

$$\frac{\rho_\phi}{\rho_P} = \frac{1}{(a/a_1)^{3(1+\alpha)}+1} \qquad \text{with} \qquad a_1 = \left(\frac{\Omega_{\alpha,0}}{\Omega_{P,0}}\right)^{\frac{1}{3(1+\alpha)}}. \tag{52}$$

It can be rewritten as

$$\frac{\rho_\phi}{\rho_0} = \frac{\Omega_{\alpha,0}}{a^{3(\alpha+1)} + \frac{\Omega_{\alpha,0}}{\Omega_{P,0}}}. \tag{53}$$

Then, we find

$$\frac{P_\phi}{\rho_P c^2} = \frac{\alpha(a/a_1)^{3(\alpha+1)} - 1}{\left[(a/a_1)^{3(\alpha+1)}+1\right]^2}, \tag{54}$$

$$w_\phi = \frac{\alpha(a/a_1)^{3(\alpha+1)} - 1}{(a/a_1)^{3(\alpha+1)}+1}, \tag{55}$$

$$\frac{(c_s^2)_\phi}{c^2} = \frac{\alpha(a/a_1)^{3(\alpha+1)} - \alpha - 2}{(a/a_1)^{3(\alpha+1)}+1}. \tag{56}$$

If the universe contains only the scalar field, the deceleration parameter is given by

$$q = \frac{1+3\alpha}{2} - \frac{3}{2}(\alpha+1)\frac{\rho}{\rho_P}, \tag{57}$$

$$q = \frac{(1+3\alpha)(a/a_1)^{3(\alpha+1)} - 2}{2\left[(a/a_1)^{3(\alpha+1)}+1\right]}. \tag{58}$$

When $a \ll a_1$, the density of the scalar field given by Equation (52) tends to a constant $\rho_\phi \simeq \rho_P$ equal to the Planck density. This leads to a phase of early inflation where the scale factor increases exponentially rapidly with time (early de Sitter era). When $a \gg a_1$, the density of the scalar field decreases algebraically as $\rho_\phi/\rho_0 = \Omega_{\alpha,0}/a^{3(\alpha+1)}$. In that case, it behaves as an α-fluid with a linear equation of state $P_\phi = \alpha\rho_\phi c^2$. The scale factor increases algebraically as $t^{2/[3(1+\alpha)]}$ (the expansion is decelerating if $\alpha > -1/3$). The equation of state (49) thus describes the smooth transition between a phase of inflation and an α-era in the early universe. The transition takes place at $a \simeq a_1$. This equation of state is studied in detail in [41,44]. The temporal evolution of the scale factor $a(t)$ is given analytically by

$$\sqrt{(a/a_1)^{3(\alpha+1)}+1} - \ln\left(\frac{1+\sqrt{(a/a_1)^{3(\alpha+1)}+1}}{(a/a_1)^{3(\alpha+1)/2}}\right) = \frac{3}{2}(\alpha+1)\left(\frac{8\pi}{3}\right)^{1/2}\frac{t}{t_P} + C. \tag{59}$$

For the particular values $\alpha = 0, 1/3, 1$, the density evolves with the scale factor as

$$\frac{\rho_\phi}{\rho_P} = \frac{1}{1+(a/a_1)^3} \quad (\alpha=0), \qquad \frac{\rho_\phi}{\rho_P} = \frac{1}{1+(a/a_1)^4} \quad (\alpha=1/3),$$
$$\frac{\rho_\phi}{\rho_P} = \frac{1}{1+(a/a_1)^6} \quad (\alpha=1). \tag{60}$$

If we follow approach (A) of the Introduction, it is relevant to take $\alpha = 1/3$ (radiation) or possibly $\alpha = 1$ (stiff fluid) in the early universe.

3.3. Late Universe

In the late universe, where the density is low, the equation of state (33) reduces to

$$P_\phi = \alpha\rho_\phi c^2 - (\alpha+1)\rho_\Lambda c^2. \tag{61}$$

This amount to neglecting quantum effects ($\rho_P \to +\infty$). The equation of state parameter and the squared speed of sound are given by

$$w_\phi = \alpha - (\alpha+1)\frac{\rho_\Lambda}{\rho_\phi}, \tag{62}$$

$$\frac{(c_s^2)_\phi}{c^2} = \alpha. \tag{63}$$

Solving the energy conservation Equation (32) with the equation of state (61), we find that the density of the scalar field is

$$\frac{\rho_\phi}{\rho_\Lambda} = \frac{1}{(a/a_2)^{3(1+\alpha)}} + 1 \qquad \text{with} \qquad a_2 = \left(\frac{\Omega_{\alpha,0}}{\Omega_{\Lambda,0}}\right)^{\frac{1}{3(1+\alpha)}}. \tag{64}$$

We note that

$$\left(\frac{a_1}{a_2}\right)^{3(\alpha+1)} = \frac{\rho_\Lambda}{\rho_P} \sim 10^{-123}. \tag{65}$$

Equation (64) can be rewritten as

$$\frac{\rho_\phi}{\rho_0} = \frac{\Omega_{\alpha,0}}{a^{3(\alpha+1)}} + \Omega_{\Lambda,0}. \tag{66}$$

Then, we find

$$\frac{P_\phi}{\rho_\Lambda c^2} = \frac{\alpha}{(a/a_2)^{3(\alpha+1)}} - 1, \tag{67}$$

$$w_\phi = \frac{\alpha(a_2/a)^{3(\alpha+1)} - 1}{(a_2/a)^{3(\alpha+1)} + 1}. \tag{68}$$

If the universe contains only the scalar field, the deceleration parameter is given by

$$q = \frac{1+3\alpha}{2} - \frac{3}{2}(\alpha+1)\frac{\rho_\Lambda}{\rho}, \tag{69}$$

$$q = \frac{(1+3\alpha)(a_2/a)^{3(\alpha+1)} - 2}{2\left[(a_2/a)^{3(\alpha+1)} + 1\right]}. \tag{70}$$

When $a \gg a_2$, the density of the scalar field given by Equation (64) tends to a constant $\rho_\phi \simeq \rho_\Lambda$ equal to the cosmological density. This leads to a phase of late accelerating expansion (or late inflation) where the scale factor increases exponentially rapidly with time (late de Sitter era). When $a \ll a_2$, the density of the scalar field decreases algebraically as $\rho_\phi/\rho_0 = \Omega_{\alpha,0}/a^{3(\alpha+1)}$. In that case, it behaves as an α-fluid with a linear equation of state $P_\phi = \alpha\rho_\phi c^2$. The scale factor increases algebraically as $t^{2/[3(1+\alpha)]}$ (the expansion is decelerating if $\alpha > -1/3$). The equation of state (61) thus describes the transition between an α-era and a phase of accelerating expansion (dark energy) in the late universe. The transition takes place at $a \simeq a_2$. This equation of state is studied in detail in [42,44]. The temporal evolution of the scale factor $a(t)$ and density $\rho_\phi(t)$ is given analytically by

$$\frac{a}{a_2} = \sinh^{\frac{2}{3(1+\alpha)}}\left[\frac{3}{2}(1+\alpha)\left(\frac{8\pi}{3}\right)^{1/2}\frac{t}{t_\Lambda}\right], \qquad \frac{\rho_\phi}{\rho_\Lambda} = \frac{1}{\tanh^2\left[\frac{3}{2}(1+\alpha)\left(\frac{8\pi}{3}\right)^{1/2}\frac{t}{t_\Lambda}\right]}, \tag{71}$$

with $t_\Lambda = 1/\sqrt{G\rho_\Lambda} = (8\pi/\Lambda)^{1/2} = 1.46 \times 10^{18}$ s (cosmological time). For the particular values $\alpha = 0, 1/3, 1$, the density evolves with the scale factor as

$$\frac{\rho_\phi}{\rho_\Lambda} = \frac{1}{(a/a_2)^3} + 1 \qquad (\alpha = 0), \qquad \frac{\rho_\phi}{\rho_\Lambda} = \frac{1}{(a/a_2)^4} + 1 \qquad (\alpha = 1/3),$$
$$\frac{\rho_\phi}{\rho_\Lambda} = \frac{1}{(a/a_2)^6} + 1 \qquad (\alpha = 1). \tag{72}$$

If we follow approach (A) of the Introduction, it is relevant to take $\alpha = 0$ (pressureless matter) in the late universe. In that case, the equation of state is constant $P_\phi = -\rho_\Lambda c^2$. This UDM model is equivalent to the ΛCDM model not only to 0-th order in perturbation theory (background) but to all orders, even in the nonlinear clustering regime [42,53,54].

3.4. Intermediate Regime

In the intermediate regime, valid after the early inflation and before the late accelerating expansion of the universe, the equation of state of the scalar field reduces to

$$P_\phi = \alpha \rho_\phi c^2. \tag{73}$$

This amounts to neglecting quantum effects ($\rho_P \to +\infty$) and dark energy ($\rho_\Lambda = 0$). The equation of state parameter and the squared speed of sound are given by

$$w_\phi = \alpha, \tag{74}$$

$$\frac{(c_s^2)_\phi}{c^2} = \alpha. \tag{75}$$

Solving the energy conservation Equation (32) with the equation of state (73), we find that the density of the scalar field is

$$\frac{\rho_\phi}{\rho_0} = \frac{\Omega_{\alpha,0}}{a^{3(\alpha+1)}}. \tag{76}$$

If the universe contains only the scalar field, the deceleration parameter is given by

$$q = \frac{1+3\alpha}{2}. \tag{77}$$

The temporal evolution of the scale factor $a(t)$ and density $\rho_\phi(t)$ is given analytically by

$$a = \left[\frac{3}{2}(1+\alpha)\left(\frac{8\pi G\rho_0\Omega_{\alpha,0}}{3}\right)^{1/2} t\right]^{\frac{2}{3(1+\alpha)}}, \qquad \rho_\phi = \frac{1}{6\pi G(1+\alpha)^2 t^2}. \tag{78}$$

The expansion is decelerating for $\alpha > -1/3$. For the particular values $\alpha = 0, 1/3, 1$, the density evolves with the scale factor as

$$\frac{\rho_\phi}{\rho_0} = \frac{\Omega_{\mathrm{dm},0}}{a^3} \qquad (\alpha = 0), \qquad \frac{\rho_\phi}{\rho_0} = \frac{\Omega_{\mathrm{r},0}}{a^4} \qquad (\alpha = 1/3),$$
$$\frac{\rho_\phi}{\rho_0} = \frac{\Omega_{\mathrm{s},0}}{a^6} \qquad (\alpha = 1). \tag{79}$$

3.5. Complete Evolution of the Universe

Regrouping the foregoing results, we see that the quadratic equation of state (33) describes in a unified manner an early inflation followed by a phase of decelerated expansion (α-era) and, finally, a late accelerating expansion. This equation of state is studied in detail in [44]. As discussed in the introduction, the equation of state has a drawback in the sense that it cannot describe both the radiation and matter eras. To solve this problem, we have proposed two approaches.

If we follow approach (A), we have to consider that α changes in the course of time (i.e., with the density of the universe). In that case, we are led to considering the equation of state (49) in the early universe with $\alpha = 1/3$ (or $\alpha = 1$) and the equation of state (61) in the late universe with $\alpha = 0$. We can then solve the Friedmann Equation (16) with Equation (53) or (66) in these two epochs and then match the results to obtain the whole evolution of the universe. This is equivalent to solving the Friedmann equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \tag{80}$$

with the density[11]

$$\frac{\rho}{\rho_0} = \frac{\Omega_{\rm r,0}}{\frac{\Omega_{\rm r,0}}{\Omega_{P,0}} + a^4} + \frac{\Omega_{\rm m,0}}{a^3} + \Omega_{\Lambda,0}. \tag{81}$$

This is the procedure adopted in [39,40,42,47]. Equations (80) and (81) determine the evolution of the scale factor $a(t)$. They describe successively the phases of inflation, radiation, pressureless matter, and dark energy. The ΛCDM model is recovered by ignoring the early inflation, i.e., by taking $\Omega_{P,0} \to +\infty$ (i.e., $\rho_P \to +\infty$ or $\hbar \to 0$) in Equation (81).

If we follow approach (B), we can consider that α is fixed in the quadratic equation of state (33) of the scalar field (the value $\alpha = 1/3$ or $\alpha = 1$ may be the most relevant). Then, we have to account for the presence of other species such as stiff matter, radiation, baryonic matter, and dark matter. Finally, we have to solve the Friedmann Equation (19) with Equation (40) for the scalar field and Equations (25)–(28) for the other species (except the one that has been incorporated in the equation of state of the scalar field). In the simplest model, ignoring stiff matter, taking $\alpha = 1/3$ in the equation of state of the scalar field and treating dark matter as an independent species with $\alpha_X = 0$, this leads again to Equations (80) and (81). Alternatively, taking $\alpha = 1$ in the equation of state of the scalar field and treating radiation and dark matter as independent species with $\alpha_X = 1/3$ and $\alpha_X = 0$ respectively, we find

$$\frac{\rho}{\rho_0} = \frac{\Omega_{\rm s,0}}{\frac{\Omega_{\rm s,0}}{\Omega_{P,0}} + a^6} + \frac{\Omega_{\rm r,0}}{a^4} + \frac{\Omega_{\rm m,0}}{a^3} + \Omega_{\Lambda,0}. \tag{82}$$

Equations (80) and (82) determine the evolution of the scale factor $a(t)$. They describe successively the phases of inflation, stiff matter, radiation, pressureless matter, and dark energy. As in footnote [11], in order to avoid a spurious divergence of the energy density at $a = 0$, the radiation and matter component terms $\Omega_{\rm r,0}/a^4$ and $\Omega_{\rm m,0}/a^3$ have to be introduced at a sufficiently late time, i.e., after the inflation era when $\rho \ll \rho_P$. The possibility that a radiation (or a stiff matter) era occurs before the inflation era, leading to a big bang singularity, is considered in Appendix E. See also Ref. [47] for more general models generalizing Equations (81) and (82).

The complete history of the universe determined by Equations (80) and (81) has been described in [39,40,42,44,47] (see in particular Figure 14 of [42]). Although the evolution of the background density is the same in approaches (A) and (B), the scalar field potential is different in these two approaches, especially in the late universe, as discussed in Sections 4 and 5 below.

4. Scalar Field in the Presence of *X*-Fluids

In this section, we derive the potential $V(\phi)$ of a real scalar field characterized by an equation of state $P_\phi(\rho_\phi)$ in the presence of other species. We provide general results using standard techniques [64,65] and apply them to the quadratic equation of state (33) of our model in the presence of X-fluids (radiation, matter, stiff matter...).

4.1. General Results

Let us consider a canonical scalar field defined by Equations (30)–(32). From Equation (31), we find

$$\dot{\phi}^2 = (w_\phi + 1)\rho_\phi c^2, \tag{83}$$

where we have introduced the equation of state parameter $w_\phi = P_\phi/\rho_\phi c^2$. Using $\dot{\phi} = (d\phi/da)Ha$ and the Friedmann Equation (19), we find that the relation between the scalar field and the scale factor is given by[12]

$$\frac{d\phi}{da} = \left(\frac{3c^2}{8\pi G}\right)^{1/2} \frac{\sqrt{1+w_\phi}}{a} \sqrt{\frac{\rho_\phi}{\sum_X \rho_X + \rho_\phi}}. \tag{84}$$

On the other hand, according to Equation (31), the potential of the scalar field is given by

$$V = \frac{1}{2}(1 - w_\phi)\rho_\phi c^2. \tag{85}$$

Therefore, we find that the potential of the canonical scalar field in the presence of other species is determined in parametric form by the equations

$$\phi(a) = \left(\frac{3c^2}{8\pi G}\right)^{1/2} \int_0^a \sqrt{1 + w_\phi(x)} \sqrt{\frac{\rho_\phi(x)}{\sum_X \rho_X(x) + \rho_\phi(x)}} \frac{dx}{x}, \tag{86}$$

$$\frac{V(a)}{\rho_0 c^2} = \frac{1}{2}\left[1 - w_\phi(a)\right] \frac{\rho_\phi(a)}{\rho_0}, \tag{87}$$

where we have taken the origin of the scalar field ($\phi = 0$) at $a = 0$.

4.2. Quadratic Equation of State

For the quadratic equation of state (33), using the results of Section 3, we have in excellent approximation (using the fact that $\rho_\Lambda/\rho_P \ll 1$):

$$\frac{\rho_\phi}{\rho_0} = \frac{\Omega_{\alpha,0}}{a^{3(\alpha+1)} + \frac{\Omega_{\alpha,0}}{\Omega_{P,0}}} + \Omega_{\Lambda,0}, \qquad \frac{\rho_\phi}{\rho_0} = \frac{\Omega_{P,0}}{1 + \frac{\Omega_{P,0}}{\Omega_{\alpha,0}} a^{3(\alpha+1)}} + \Omega_{\Lambda,0}, \tag{88}$$

$$w_\phi = \frac{-\frac{\Omega_{\Lambda,0}\Omega_{P,0}}{\Omega_{\alpha,0}^2} a^{6(\alpha+1)} + \alpha \frac{\Omega_{P,0}}{\Omega_{\alpha,0}} a^{3(\alpha+1)} - 1}{\left[1 + \frac{\Omega_{P,0}}{\Omega_{\alpha,0}} a^{3(\alpha+1)}\right]\left[1 + \frac{\Omega_{\Lambda,0}}{\Omega_{\alpha,0}} a^{3(\alpha+1)}\right]}, \tag{89}$$

$$1 + w_\phi = \frac{(\alpha+1)\frac{\Omega_{P,0}}{\Omega_{\alpha,0}} a^{3(\alpha+1)}}{\left[1 + \frac{\Omega_{P,0}}{\Omega_{\alpha,0}} a^{3(\alpha+1)}\right]\left[1 + \frac{\Omega_{\Lambda,0}}{\Omega_{\alpha,0}} a^{3(\alpha+1)}\right]}, \tag{90}$$

$$1 - w_\phi = \frac{(1-\alpha)\frac{\Omega_{P,0}}{\Omega_{\alpha,0}} a^{3(\alpha+1)} + 2\frac{\Omega_{\Lambda,0}\Omega_{P,0}}{\Omega_{\alpha,0}^2} a^{6(\alpha+1)} + 2}{\left[1 + \frac{\Omega_{P,0}}{\Omega_{\alpha,0}} a^{3(\alpha+1)}\right]\left[1 + \frac{\Omega_{\Lambda,0}}{\Omega_{\alpha,0}} a^{3(\alpha+1)}\right]}, \tag{91}$$

$$(1 - w_\phi)\frac{\rho_\phi}{\rho_0} = \frac{2\Omega_{\Lambda,0} a^{6(\alpha+1)} \left(\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}\right)^2 + (1-\alpha) a^{3(\alpha+1)} \frac{\Omega_{P,0}^2}{\Omega_{\alpha,0}} + 2\Omega_{P,0}}{\left[1 + \frac{\Omega_{P,0}}{\Omega_{\alpha,0}} a^{3(\alpha+1)}\right]^2}. \tag{92}$$

On the other hand, for the X-fluids we have

$$\frac{\rho_X}{\rho_0} = \frac{\Omega_{X,0}}{a^{3(1+\alpha_X)}}. \tag{93}$$

As a result, we find that the scalar field potential of the vacuumon unifying the early inflation and the late accelerating expansion of the universe in the presence of X-fluids is determined by the equations

$$\phi(a)=\left(\frac{3c^2}{8\pi G}\right)^{1/2}\sqrt{1+\alpha}\int_0^a \frac{\left(\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}\right)^{1/2}x^{3(\alpha+1)/2}}{\sqrt{1+\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}x^{3(\alpha+1)}}\sqrt{1+\frac{\Omega_{\Lambda,0}}{\Omega_{\alpha,0}}x^{3(\alpha+1)}}} \times\sqrt{\frac{\frac{\Omega_{\alpha,0}}{x^{3(\alpha+1)}+\frac{\Omega_{\alpha,0}}{\Omega_{P,0}}}+\Omega_{\Lambda,0}}{\sum_X \frac{\Omega_{X,0}}{x^{3(1+\alpha_X)}}+\frac{\Omega_{\alpha,0}}{x^{3(\alpha+1)}+\frac{\Omega_{\alpha,0}}{\Omega_{P,0}}}+\Omega_{\Lambda,0}}}\frac{dx}{x}, \tag{94}$$

$$\frac{V(a)}{\rho_0 c^2}=\frac{1}{2}\Omega_{P,0}\frac{2+(1-\alpha)\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}a^{3(\alpha+1)}+2\frac{\Omega_{\Lambda,0}\Omega_{P,0}}{\Omega_{\alpha,0}^2}a^{6(\alpha+1)}}{\left[\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}a^{3(\alpha+1)}+1\right]^2}. \tag{95}$$

In the early universe ($\Omega_\Lambda=0$), we obtain

$$\phi(a)=\left(\frac{3c^2}{8\pi G}\right)^{1/2}\sqrt{1+\alpha}\int_0^a \frac{\left(\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}\right)^{1/2}x^{3(\alpha+1)/2}}{\sqrt{1+\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}x^{3(\alpha+1)}}}\sqrt{\frac{\frac{\Omega_{\alpha,0}}{x^{3(\alpha+1)}+\frac{\Omega_{\alpha,0}}{\Omega_{P,0}}}}{\sum_X \frac{\Omega_{X,0}}{x^{3(1+\alpha_X)}}+\frac{\Omega_{\alpha,0}}{x^{3(\alpha+1)}+\frac{\Omega_{\alpha,0}}{\Omega_{P,0}}}}}\frac{dx}{x}, \tag{96}$$

$$\frac{V(a)}{\rho_0 c^2}=\frac{1}{2}\Omega_{P,0}\frac{2+(1-\alpha)\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}a^{3(\alpha+1)}}{\left[\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}a^{3(\alpha+1)}+1\right]^2}. \tag{97}$$

In the late universe ($\Omega_P\to+\infty$), we obtain[13]

$$\phi(a)=\left(\frac{3c^2}{8\pi G}\right)^{1/2}\sqrt{1+\alpha}\int^a \frac{1}{\sqrt{1+\frac{\Omega_{\Lambda,0}}{\Omega_{\alpha,0}}x^{3(\alpha+1)}}}\sqrt{\frac{\frac{\Omega_{\alpha,0}}{x^{3(\alpha+1)}}+\Omega_{\Lambda,0}}{\sum_X \frac{\Omega_{X,0}}{x^{3(1+\alpha_X)}}+\frac{\Omega_{\alpha,0}}{x^{3(\alpha+1)}}+\Omega_{\Lambda,0}}}\frac{dx}{x}, \tag{98}$$

$$\frac{V(a)}{\rho_0 c^2}=\frac{1}{2}(1-\alpha)\frac{\Omega_{\alpha,0}}{a^{3(\alpha+1)}}+\Omega_{\Lambda,0}. \tag{99}$$

These equations determine the potential of the vacuumon in the presence of X-fluids. Although it is not possible to obtain the scalar field potential $V(\phi)$ analytically in the general case, it can always be obtained numerically.[14] The scalar field potential $V(\phi)$ can be obtained analytically in the absence of other species (see Section 5). In the presence of X-fluids, it can be obtained analytically in particular situations of physical interest depending on the approach considered.

In approach (A), the value of α changes with the epoch during the evolution of the universe. It is equal to $\alpha=1$ (stiff matter) or $\alpha=1/3$ (radiation) in the early universe and to $\alpha=0$ (matter) in the late universe. In this approach, we do not have to take into account the presence of other species, as their effect has been incorporated in the equation of state of the scalar field. Therefore, we can consider that the scalar field is alone in the universe but that α has a different value in the early and in the late universe. The scalar field potential is determined by Equations (96) and (97) with $\Omega_{X,0}=0$ and $\alpha=1$ or $\alpha=1/3$ in the early universe (leading to Equations (119) and (120)) and by Equations (98) and (99) with $\Omega_{X,0} = 0$ and $\alpha = 0$ in the late universe (leading to Equation (137)). These cases are treated in Sections 5.2 and 5.3.

In approach (B), the value of α is fixed. We then have to consider two situations:

(i) We first consider a universe without stiff matter. In that case, it is relevant take $\alpha = 1/3$ in the quadratic equation of state (33) so that the scalar field accounts for the inflation era, the radiation era and the late acceleration of the universe. Then, we have to add baryonic matter and dark matter as independent species (they can be treated as a single species with $\alpha_X = 0$). In the early universe, we can approximate the equation of state of the scalar field by Equation (49). As matter is negligible in the early universe, we can consider that the scalar field is alone in the universe at that epoch. The scalar field potential is then determined by Equations (96) and (97) with $\Omega_{X,0} = 0$. This situation, which describes the transition between the inflation era and the radiation era, is treated in Section 5.2. It leads to the same potential (120) as in approach (A). In the late universe, we can approximate the equation of state of the scalar field by Equation (61). On other hand, we have to take into account the presence of matter as an independent species. In that case, the scalar field potential is determined by Equations (98) and (99) with $\alpha_X = 0$. Unfortunately, the integral in Equation (98) cannot be performed analytically (see Section 8). However, at very late time, the contribution of matter becomes negligible and the scalar field is alone in the universe. In that case, its potential is given by Equation (136). We note that the scalar field potential in the late universe in approach (B) is very different from its expression in approach (A) as it corresponds to Equation (132) with $\alpha = 1/3$ instead of $\alpha = 0$. Finally, in the intermediate era between the early inflation and the late accelerating expansion of the universe, we can approximate the equation of state of the scalar field by Equation (73). We also have to take into account the presence of matter as an independent species. The scalar field potential is then determined by Equations (96) and (97) with $\Omega_{P,0} \to +\infty$ and $\alpha_X = 0$ or by Equations (98) and (99) with $\Omega_{\Lambda,0} = 0$ and $\alpha_X = 0$. This situation, which describes the period where the universe contains radiation and matter, is treated in Section 7 leading to Equation (188) with $\alpha = 1/3$ and $\alpha_X = 0$.

(ii) We now consider a universe with stiff matter. In that case, it is relevant to take $\alpha = 1$ so that the scalar field accounts for the inflation era, the stiff matter era and the late acceleration of the universe. Then, we have to add radiation ($\alpha_X = 1/3$) and matter ($\alpha_X = 0$) as independent species. In the early universe, we can approximate the equation of state of the scalar field by Equation (49). As radiation and matter are negligible in the (very) early universe, we can consider that the scalar field is alone in the universe at that epoch. The scalar field potential is then determined by Equations (96) and (97) with $\Omega_{X,0} = 0$. This situation, which describes the transition between the inflation era and the stiff matter era, is treated in Section 5.2. It leads to the same potential (119) as in approach (A).[15] In the late universe, we can approximate the equation of state of the scalar field by Equation (61). On other hand, we have to take into account the presence of matter ($\alpha_X = 0$) as an independent species. In that case, the scalar field potential is determined by Equations (98) and (99) with $\alpha_X = 0$. The integral in Equation (98) can be performed analytically (see Section 8). However, the potential is independent of ϕ and given by Equation (135) as when the scalar field is alone in the universe (which is the case at very late time when the contribution of matter becomes negligible). We note that the scalar field potential in the late universe in approach (B) is very different from its expression in approach (A) as it corresponds to Equation (132) with $\alpha = 1$ instead of $\alpha = 0$. Finally, in the period between the early inflation and the late accelerating expansion, we can approximate the equation of state of the scalar field by Equation (73). We also have to take into account the presence of radiation and matter as independent species. The scalar field potential is then determined by Equations (96) and (97) with $\Omega_{P,0} \to +\infty$ and $\alpha_X = \{1/3, 0\}$ or by Equations (98) and (99) with $\Omega_{\Lambda,0} = 0$ and $\alpha_X = \{1/3, 0\}$.

This situation describes the period where the universe contains stiff matter, radiation, and matter. In this general situation involving a scalar field (stiff matter) and two external fluids (radiation and matter), the scalar field potential cannot be obtained analytically. However, if we neglect the matter contribution (at sufficiently early times) or the radiation contribution (at sufficiently late times), we have just one external fluid (radiation or matter) and the scalar field potential can be obtained analytically. This situation, which describes the period where the universe contains stiff matter and radiation or stiff matter and matter, is treated in Section 7, leading to Equation (188) with $\alpha = 1$ and $\alpha_X = 1/3$ or $\alpha_X = 0$. Note that even if the stiff matter (played by the scalar field) is subdominant in that period, it remains important because it will ultimately lead to the late accelerating expansion of the universe.

5. Scalar Field Alone

In this section, we consider the case of a scalar field alone in the universe (without X-fluids) and recover the results of our earlier papers [39–47].

5.1. Vacuumon

In the absence of X-fluids, the potential of a canonical scalar field is determined in parametric form by the equations (see Equations (86) and (87) with $\rho_X = 0$)

$$\phi(a) = \left(\frac{3c^2}{8\pi G}\right)^{1/2} \int_0^a \sqrt{1+w_\phi(x)}\,\frac{dx}{x}, \tag{100}$$

$$\frac{V(a)}{\rho_0 c^2} = \frac{1}{2}\left[1-w_\phi(a)\right]\frac{\rho_\phi(a)}{\rho_0}. \tag{101}$$

In particular, the potential of the vacuumon described by the quadratic equation of state (33) is determined by the equations (see Equations (94) and (95) with $\rho_X = 0$)

$$\phi(a) = \left(\frac{3c^2}{8\pi G}\right)^{1/2} \sqrt{1+\alpha} \int_0^a \frac{\left(\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}\right)^{1/2} x^{3(\alpha+1)/2}}{\sqrt{1+\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}x^{3(\alpha+1)}}\sqrt{1+\frac{\Omega_{\Lambda,0}}{\Omega_{\alpha,0}}x^{3(\alpha+1)}}}\,\frac{dx}{x}, \tag{102}$$

$$\frac{V(a)}{\rho_0 c^2} = \frac{1}{2}\Omega_{P,0}\frac{2+(1-\alpha)\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}a^{3(\alpha+1)} + 2\frac{\Omega_{\Lambda,0}\Omega_{P,0}}{\Omega_{\alpha,0}^2}a^{6(\alpha+1)}}{\left[\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}a^{3(\alpha+1)}+1\right]^2}. \tag{103}$$

Introducing the notations

$$x = a^{3(\alpha+1)/2}\left(\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}\right)^{1/2}, \qquad \psi = \left(\frac{8\pi G}{3c^2}\right)^{1/2}\frac{3\sqrt{\alpha+1}}{2}\phi, \tag{104}$$

and

$$\kappa = \frac{\rho_\Lambda}{\rho_P} = \frac{\Lambda\hbar G}{8\pi c^5} = 1.16\times 10^{-123}, \tag{105}$$

we can rewrite Equations (102) and (103) under the form

$$\psi = \int_0^x \frac{ds}{(\kappa s^2+1)^{1/2}(s^2+1)^{1/2}} = \mathrm{sc}^{-1}(x, 1-\kappa), \tag{106}$$

$$V = \frac{1}{2}\rho_P c^2 \frac{2+(1-\alpha)x^2+2\kappa x^4}{(x^2+1)^2}, \tag{107}$$

where sc is the Jacobian Elliptic function. This returns Equations (119) and (120) of [44]. We note that ψ is independent of α. Equations (106) and (107) define the potential $V(\psi)$ in parametric form with the parameter x going from 0 to $+\infty$. The scalar field goes from $\psi = 0$ when $x = 0$ to $\psi_{\rm max} = \int_0^{+\infty} ds/[(\kappa s^2+1)^{1/2}(s^2+1)^{1/2}] = K(1-\kappa)$ when $x \to +\infty$, where K is the complete Elliptic integral of the first kind ($K(1-\kappa) \simeq (1/2)\ln(16/\kappa)$ for $\kappa \ll 1$ giving $\psi_{\rm max} \simeq 142.84\ldots$). Eliminating x between Equations (106) and (107), we obtain [44]

$$V(\psi) = \frac{1}{2}\rho_P c^2 \frac{2+(1-\alpha)\,{\rm sc}(\psi,1-\kappa)^2 + 2\kappa\,{\rm sc}(\psi,1-\kappa)^4}{[{\rm sc}(\psi,1-\kappa)^2+1]^2} \qquad (0 \le \psi \le \psi_{\rm max}). \tag{108}$$

This is the general expression of the potential of the vacuumon.

Noting that $x = (a/a_1)^{3(\alpha+1)/2}$, where a_1 is defined by Equation (39), and using Equation (106), the relation between the scalar field and the scale factor is

$$(a/a_1)^{3(\alpha+1)/2} = {\rm sc}(\psi, 1-\kappa). \tag{109}$$

We can then express all the parameters of Section 3.1 as a function of ψ instead of a. In particular, the energy density and the pressure of the scalar field are given by

$$\begin{aligned} \rho_\phi &= \frac{\rho_P}{{\rm sc}^2(\psi,1-\kappa)+1} + \rho_\Lambda, \\ P_\phi &= \frac{-\rho_\Lambda c^2 {\rm sc}^4(\psi,1-\kappa) + \alpha\rho_P c^2 {\rm sc}^2(\psi,1-\kappa) - \rho_P c^2}{[{\rm sc}^2(\psi,1-\kappa)+1]^2}. \end{aligned} \tag{110}$$

The temporal evolution of the scale factor and of the scalar field is discussed in detail in our previous papers [39–47].

5.2. Early Vacuumon (Inflaton)

In the early universe ($\Omega_\Lambda = 0$), Equations (102) and (103) reduce to

$$\phi(a) = \left(\frac{3c^2}{8\pi G}\right)^{1/2} \sqrt{1+\alpha} \int_0^a \frac{\left(\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}\right)^{1/2} x^{3(\alpha+1)/2}}{\sqrt{1+\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}x^{3(\alpha+1)}}} \frac{dx}{x}, \tag{111}$$

$$\frac{V(a)}{\rho_0 c^2} = \frac{1}{2}\Omega_{P,0} \frac{2+(1-\alpha)\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}a^{3(\alpha+1)}}{\left(\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}a^{3(\alpha+1)}+1\right)^2}. \tag{112}$$

Introducing the notations from Equation (104), we find

$$\psi = \int_0^x \frac{ds}{(1+s^2)^{1/2}} = \sinh^{-1}(x), \tag{113}$$

$$V = \frac{1}{2}\rho_P c^2 \frac{2+(1-\alpha)x^2}{(x^2+1)^2}, \tag{114}$$

which corresponds to Equations (106) and (107) with $\kappa = 0$. This returns Equations (122) and (123) of [44]. Eliminating x between these equations we obtain [44]

$$V(\psi) = \frac{1}{2}\rho_P c^2 \frac{(1-\alpha)\cosh^2\psi + \alpha + 1}{\cosh^4\psi} \qquad (\psi \ge 0). \tag{115}$$

This is the potential of the vacuumon in the early universe (playing the role of the inflaton). It is associated to the equation of state (49) valid in the early universe (see Section 8.1 of [42]). For $\psi \to 0$, it can be expanded in Taylor series as

$$\frac{V(\psi)}{\rho_P c^2} \simeq 1 - \frac{3+\alpha}{2}\psi^2 + \frac{9+5\alpha}{6}\psi^4 + \ldots \tag{116}$$

For $\psi \to +\infty$, we obtain the exponential asymptotic behaviors

$$\frac{V(\psi)}{\rho_P c^2} \sim 2(1-\alpha)e^{-2\psi}, \qquad (\alpha \neq 1), \tag{117}$$

$$\frac{V(\psi)}{\rho_P c^2} \sim 16e^{-4\psi}, \qquad (\alpha = 1). \tag{118}$$

We note that the coefficient $\alpha = 1$ (stiff matter) plays a special role as it leads to a faster decay of the potential. In that case [45,46]

$$V(\psi) = \frac{\rho_P c^2}{\cosh^4 \psi} \qquad (\alpha = 1). \tag{119}$$

For the coefficient $\alpha = 1/3$ (radiation), we have [42]

$$V(\psi) = \frac{1}{3}\rho_P c^2 \frac{\cosh^2 \psi + 2}{\cosh^4 \psi} \qquad (\alpha = 1/3). \tag{120}$$

For $\alpha = 0$ (pressureless matter), we have [42]

$$V(\psi) = \frac{1}{2}\rho_P c^2 \frac{\cosh^2 \psi + 1}{\cosh^4 \psi} \qquad (\alpha = 0). \tag{121}$$

The potentials corresponding to $\alpha = 1/3$ and $\alpha = 1$ (relevant in the early universe) are plotted in Figure 1.

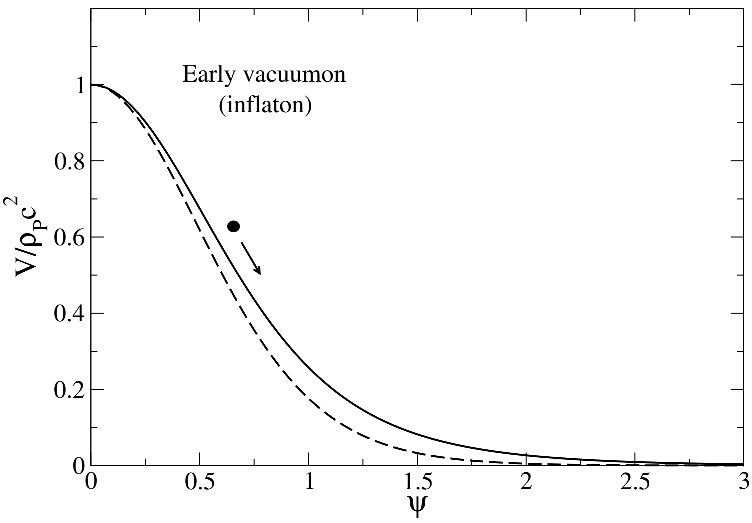

Figure 1. Potential of the scalar field in the early universe for $\alpha = 1/3$ (solid line) and $\alpha = 1$ (dashed line). The early vacuumon plays the role of the inflaton. The field tends to run down the potential.

Recalling that $x = (a/a_1)^{3(\alpha+1)/2}$ and using Equation (113), the relation between the scalar field and the scale factor is

$$(a/a_1)^{3(\alpha+1)/2} = \sinh \psi. \tag{122}$$

We can then express all the parameters of Section 3.2 as a function of ψ instead of a. In particular, the energy density and the pressure of the scalar field are given by

$$\rho_\phi = \frac{\rho_P}{\cosh^2\psi}, \qquad \frac{P_\phi}{\rho_P c^2} = \frac{\alpha \sinh^2\psi - 1}{\cosh^4\psi}. \tag{123}$$

The temporal evolution of the scalar field is discussed in detail in our previous papers [39–47]. According to Equations (59) and (122) we have

$$\cosh\psi - \ln\left(\frac{1+\cosh\psi}{\sinh\psi}\right) = \frac{3}{2}(\alpha+1)\left(\frac{8\pi}{3}\right)^{1/2}\frac{t}{t_P} + C. \tag{124}$$

Remark 1. *For $a \to 0$, we can simplify Equation (111) into*

$$\phi(a) = \left(\frac{3c^2}{8\pi G}\right)^{1/2}\sqrt{1+\alpha}\int_0^a \left(\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}\right)^{1/2} x^{3(\alpha+1)/2}\,\frac{dx}{x}, \tag{125}$$

yielding

$$\phi(a) = \frac{2}{3}\left(\frac{3c^2}{8\pi G}\right)^{1/2}\frac{1}{\sqrt{1+\alpha}}\left(\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}\right)^{1/2} a^{\frac{3}{2}(1+\alpha)}. \tag{126}$$

Substituting Equation (126) into Equation (112) we obtain the quadratic potential

$$\frac{V(\phi)}{\rho_0 c^2} = \Omega_{P,0} - \frac{3\pi G}{c^2}(1+\alpha)(3+\alpha)\Omega_{P,0}\phi^2. \tag{127}$$

This corresponds to the first term in the expansion of the potential from Equation (116).

5.3. Late Vacuumon (Quintessence)

In the late universe ($\Omega_P \to +\infty$), Equations (102) and (103) reduce to

$$\phi(a) = \left(\frac{3c^2}{8\pi G}\right)^{1/2}\sqrt{1+\alpha}\int^a \frac{1}{\sqrt{1+\frac{\Omega_{\Lambda,0}}{\Omega_{\alpha,0}}x^{3(\alpha+1)}}}\,\frac{dx}{x}, \tag{128}$$

$$\frac{V(a)}{\rho_0 c^2} = \frac{(1-\alpha)\Omega_{\alpha,0}}{2a^{3(\alpha+1)}} + \Omega_{\Lambda,0}. \tag{129}$$

These expressions are valid for sufficiently large values of a. Introducing the notations from Equations (104) and (105), we find

$$\psi = \int^x \frac{ds}{s(1+\kappa s^2)^{1/2}} = -\sinh^{-1}\left(\frac{1}{\sqrt{\kappa}x}\right) + \psi_{\max}, \tag{130}$$

$$V = \frac{1}{2}\rho_P c^2\left(\frac{1-\alpha}{x^2} + 2\kappa\right), \tag{131}$$

which correspond to Equations (106) and (107) with $x \gg 1$. Here, $\psi_{\max}$ appears as a constant of integration that can be obtained by matched asymptotics (see below). We recover Equations (125) and (126) of [44]. Eliminating x between these equations we obtain [44]

$$V(\psi) = \frac{1}{2}\rho_\Lambda c^2\left[(1-\alpha)\cosh^2(\psi_{\max}-\psi) + \alpha + 1\right] \qquad (\psi \le \psi_{\max}). \tag{132}$$

This is the potential of the vacuumon in the late universe (playing the role of the quintessence). It is associated to the equation of state (61) valid in the late universe (see Section 8.1 of [42]). For $\psi \to \psi_{\rm max}$, it can be expanded in Taylor series as

$$\frac{V(\psi)}{\rho_\Lambda c^2} \simeq 1 + \frac{1-\alpha}{2}(\psi_{\rm max} - \psi)^2 + \frac{1-\alpha}{6}(\psi_{\rm max} - \psi)^4 + \ldots \tag{133}$$

For $\psi \to -\infty$, we obtain the exponential asymptotic behavior

$$\frac{V(\psi)}{\rho_\Lambda c^2} \simeq \frac{1}{8}(1-\alpha)e^{2(\psi_{\rm max} - \psi)}, \qquad (\alpha \neq 1). \tag{134}$$

For $\alpha = 1$ (stiff matter), the scalar field potential is constant

$$V(\psi) = \rho_\Lambda c^2 \qquad (\alpha = 1). \tag{135}$$

For $\alpha = 1/3$ (radiation), we have

$$V(\psi) = \frac{1}{2}\rho_\Lambda c^2 \left[\frac{2}{3}\cosh^2(\psi_{\rm max} - \psi) + \frac{4}{3}\right] \qquad (\alpha = 1/3). \tag{136}$$

For $\alpha = 0$ (pressureless matter), we have

$$V(\psi) = \frac{1}{2}\rho_\Lambda c^2 \left[\cosh^2(\psi_{\rm max} - \psi) + 1\right] \qquad (\alpha = 0). \tag{137}$$

The potential corresponding to $\alpha = 0$ (relevant in the late universe) is plotted in Figure 2. This is the scalar field potential associated with the ΛCDM model viewed as a UDM model described by the constant equation of state $P_\phi = -\rho_\Lambda c^2$.

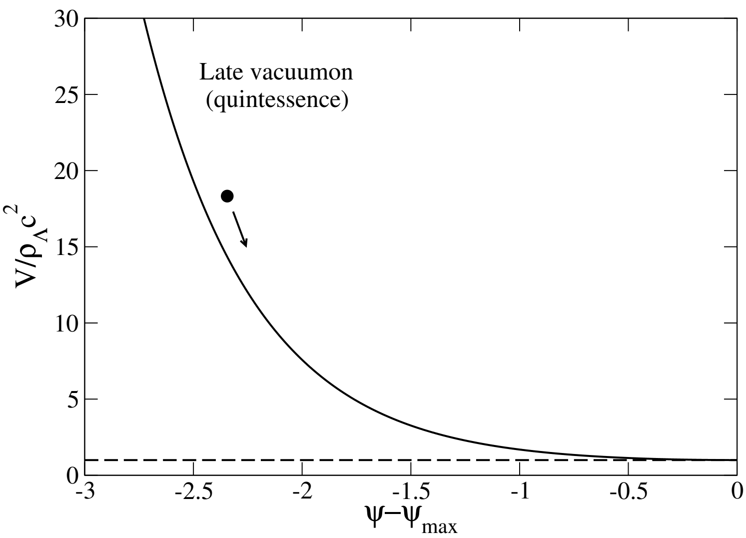

Figure 2. Potential of the scalar field in the late universe for $\alpha = 0$. The late vacuumon plays the role of quintessence. The field tends to run down the potential.

Noting that $x\sqrt{\kappa} = (a/a_2)^{3(\alpha+1)/2}$, where a_2 is defined by Equation (64), and using Equation (130), the relation between the scalar field and the scale factor is

$$(a_2/a)^{3(\alpha+1)/2} = \sinh(\psi_{\rm max} - \psi). \tag{138}$$

We can then express all the parameters of Section 3.3 as a function of ψ instead of a. In particular, the energy density and the pressure of the scalar field are given by

$$\rho_\phi = \rho_\Lambda \cosh^2(\psi_{\rm max} - \psi), \qquad \frac{P_\phi}{\rho_\Lambda c^2} = \alpha \sinh^2(\psi_{\rm max} - \psi) - 1. \tag{139}$$

The temporal evolution of the scalar field is discussed in detail in our previous papers [39–47]. According to Equations (71) and (138) we have

$$\psi_{\rm max} - \psi = \sinh^{-1}\left\{1/\sinh\left[\frac{3}{2}(1+\alpha)\left(\frac{8\pi}{3}\right)^{1/2}\frac{t}{t_\Lambda}\right]\right\}. \tag{140}$$

Remark 2. *For $a \to +\infty$, we can simplify Equation (128) into*

$$\phi(a) = \phi_{\rm max} - \left(\frac{3c^2}{8\pi G}\right)^{1/2}\sqrt{1+\alpha}\int_a^{+\infty}\frac{1}{\sqrt{\frac{\Omega_{\Lambda,0}}{\Omega_{\alpha,0}}x^{3(1+\alpha)}}}\frac{dx}{x}, \tag{141}$$

yielding

$$\phi(a) = \phi_{\rm max} - \frac{2}{3}\left(\frac{3c^2}{8\pi G}\right)^{1/2}\frac{1}{\sqrt{1+\alpha}}\left(\frac{\Omega_{\alpha,0}}{\Omega_{\Lambda,0}}\right)^{1/2}\frac{1}{a^{\frac{3}{2}(1+\alpha)}}. \tag{142}$$

Substituting Equation (142) into Equation (129) we obtain the quadratic potential

$$\frac{V(\phi)}{\rho_0 c^2} = \Omega_{\Lambda,0} + \frac{3\pi G}{c^2}(1-\alpha)(1+\alpha)\Omega_{\Lambda,0}(\phi_{\rm max} - \phi)^2. \tag{143}$$

This corresponds to the first term in the expansion of the potential from Equation (133).

5.4. Matched Asymptotics

Using matched asymptotics, we find that the potential [44]

$$\begin{aligned} V_{\rm approx}(\psi) &= \frac{1}{2}\rho_P c^2 \frac{(1-\alpha)\cosh^2\psi + \alpha + 1}{\cosh^4\psi} \\ &+ \frac{1}{2}\rho_\Lambda c^2\left[(1-\alpha)\cosh^2(\psi_{\rm max} - \psi) + \alpha + 1\right] \\ &- 2\rho_P c^2(1-\alpha)e^{-2\psi} \qquad (0 \le \psi \le \psi_{\rm max}) \end{aligned} \tag{144}$$

provides a good approximation of the exact vacuumon potential given by Equation (108). It unifies the inflaton potential in the early universe (early vacuumon) and the quintessence potential in the late universe (late vacuumon) defined by Equations (115) and (132) respectively. In addition, matched asymptotics provides the value of the constant of integration $\psi_{\rm max}$ that appears in Equation (130). Indeed, by comparing Equations (117) and (134) we obtain $\psi_{\rm max} \simeq (1/2)\ln(16/\kappa) = 142.84\ldots$ [44] in perfect agreement with the asymptotic expression of the exact value $\psi_{\rm max} = K(1-\kappa)$ for $\kappa \ll 1$ (see Section 5.1).

Remark 3. *For $\alpha = 1$, the approximate potential from Equation (144) takes the particularly simple form*

$$V_{\rm approx}(\psi) = \frac{\rho_P c^2}{\cosh^4\psi} + \rho_\Lambda c^2 \qquad (\alpha = 1). \tag{145}$$

5.5. Intermediate Regime

In the intermediate regime valid after the early inflation and before the late accelerating expansion of the universe ($\Omega_P \to +\infty$ and $\Omega_\Lambda = 0$), the scalar field is described by the linear equation of state (73). In that case, Equations (102) and (103) reduce to

$$\phi(a) = \left(\frac{3c^2}{8\pi G}\right)^{1/2} \sqrt{1+\alpha} \int^a \frac{dx}{x}, \tag{146}$$

$$\frac{V(a)}{\rho_0 c^2} = \frac{1}{2}(1-\alpha)\frac{\Omega_{\alpha,0}}{a^{3(\alpha+1)}}. \tag{147}$$

These equations can also be obtained from Equations (111) and (112) with $\Omega_P \to +\infty$ or from Equations (128) and (129) with $\Omega_\Lambda = 0$. Integrating Equation (146), we obtain

$$\phi = \left(\frac{3c^2}{8\pi G}\right)^{1/2} \sqrt{1+\alpha}\, \ln a + \phi_*, \tag{148}$$

where ϕ_* is a constant of integration. Combining Equations (147) and (148), we obtain the exponential potential

$$\frac{V(\phi)}{\rho_0 c^2} = \frac{1}{2}(1-\alpha)\Omega_{\alpha,0} e^{-3\left(\frac{8\pi G}{3c^2}\right)^{1/2}\sqrt{1+\alpha}(\phi-\phi_*)}. \tag{149}$$

This result is equivalent to Equations (117) and (134) obtained long after the primordial inflation or long before the late accelerating expansion of the universe. This determines the constant of integration $\psi_* = (1/2)\ln(4\Omega_{P,0}/\Omega_{\alpha,0})$. For $\alpha = 1$ (stiff matter), we find

$$\phi = \left(\frac{3c^2}{8\pi G}\right)^{1/2} \sqrt{2}\, \ln a + \phi_*, \qquad V(\phi) = 0 \qquad (\alpha = 1), \tag{150}$$

corresponding to the regime of kination where the potential vanishes. Comparing Equation (150) with Equations (118) and (135) we see that the limits $\alpha \to 1$ and $\phi \to \pm\infty$ do not commute. For $\alpha = -1$ (vacuum energy), we find

$$\phi = \phi_*, \qquad V(\phi) = \rho_* c^2 \qquad (\alpha = -1), \tag{151}$$

corresponding to the de Sitter regime where $\dot{\phi} = 0$.

Combining Equations (73), (76), and (148), we can express the density and the pressure as a function of the scalar field as

$$P_\phi = \alpha\rho_\phi c^2 = \alpha\Omega_{\alpha,0}\rho_0 c^2 e^{-3\sqrt{\alpha+1}\left(\frac{8\pi G}{3c^2}\right)^{1/2}(\phi-\phi_*)}. \tag{152}$$

According to Equations (78) and (148), the temporal evolution of the scalar field is

$$\phi(t) = \left(\frac{3c^2}{8\pi G}\right)^{1/2} \frac{2}{3\sqrt{1+\alpha}} \ln\left[\frac{3}{2}(1+\alpha)\left(\frac{8\pi G\rho_0\Omega_{\alpha,0}}{3}\right)^{1/2} t\right] + \phi_*. \tag{153}$$

5.6. Constant Scalar Field Potential

We have found different solutions corresponding to a constant scalar field potential (see Equations (135), (150) and (151)):

(i) For the equation of state $P_\phi = \rho_\phi c^2$ of stiff matter ($\alpha = 1$), we have (see Sections 3.4 and 5.5)

$$\rho_\phi \propto \frac{1}{a^6}, \qquad a \propto t^{1/3}, \qquad \rho_\phi = \frac{1}{24\pi G t^2}, \tag{154}$$

$$V(\phi) = 0, \qquad \phi = \left(\frac{3c^2}{4\pi G}\right)^{1/2} \ln a + \phi_*, \qquad \phi = \left(\frac{c^2}{12\pi G}\right)^{1/2} \ln t + \mathrm{cst}. \tag{155}$$

This solution describes a pure stiff matter era.

(ii) For the equation of state $P_\phi = -\rho_\phi c^2$ of vacuum or dark energy ($\alpha = -1$), we have (see Sections 3.4 and 5.5)

$$\rho_\phi = \mathrm{cst}, \qquad a \propto e^{\left(\frac{8\pi G \rho_\phi}{3}\right)^{1/2} t}, \qquad V(\phi) = \mathrm{cst}, \qquad \phi = \mathrm{cst}. \tag{156}$$

This solution describes a pure de Sitter era.

(iii) For the affine equation of state $P_\phi = \rho_\phi c^2 - 2\rho_\Lambda c^2$, we have (see Sections 3.3 and 5.3)

$$\frac{\rho_\phi}{\rho_\Lambda} = \frac{1}{(a/a_2)^6} + 1, \qquad \frac{a}{a_2} = \sinh^{1/3}\left[3\left(\frac{8\pi}{3}\right)^{1/2} \frac{t}{t_\Lambda}\right], \qquad \frac{\rho_\phi}{\rho_\Lambda} = \frac{1}{\tanh^2\left[3\left(\frac{8\pi}{3}\right)^{1/2} \frac{t}{t_\Lambda}\right]}, \tag{157}$$

$$V(\psi) = \rho_\Lambda c^2, \qquad \psi_{\max} - \psi = \sinh^{-1}\left[\left(\frac{a_2}{a}\right)^3\right], \qquad \psi_{\max} - \psi = \sinh^{-1}\left\{1/\sinh\left[3\left(\frac{8\pi}{3}\right)^{1/2} \frac{t}{t_\Lambda}\right]\right\}. \tag{158}$$

This solution describes a stiff matter era followed by a de Sitter era (late inflation).[16] Therefore, several laws of evolution may correspond to the same potential.

6. Parameters of the Scalar Field

In this section, we determine the parameters (mass, self-interaction constant, and scattering length) of the scalar field in the early and late universe by comparing the form of the potential close to its maximum and minimum with the normal form of a quartic potential given by Equation (A54).

6.1. Early Vacuumon (Inflaton)

In the early universe, the potential of the vacuumon is given by Equation (115). For $\psi \to 0$, it can be expanded up to fourth order yielding Equation (116). Comparing Equation (116) with Equation (A54), we obtain the following results (see Appendix D):

(i) The maximum value of the potential (corresponding to $\psi = 0$) is equal to the Planck energy density

$$V_{\max} = \rho_P c^2, \tag{159}$$

where $\rho_P = c^5/\hbar G^2 = 5.16 \times 10^{99}\,\mathrm{g\,m^{-3}}$ is the Planck density.

(ii) The squared mass of the scalar field is given by

$$m^2 = f_e(\alpha) \frac{8\pi}{3} M_P^2 \tag{160}$$

with

$$f_e(\alpha) = -\frac{9}{4}(\alpha+1)(\alpha+3). \tag{161}$$

We find $f_e = -27/4$ for $\alpha = 0$, $f_e = -10$ for $\alpha = 1/3$, and $f_e = -18$ for $\alpha = 1$. The mass of the early vacuumon (inflaton) is imaginary ($f_e < 0$) and its modulus $|m|$ is of the order of the Planck mass $M_P = (\hbar c/G)^{1/2} = 2.18 \times 10^{-5}$ g $= 1.22 \times 10^{19}$ GeV/c^2.

(iii) The dimensionless self-interaction constant of the scalar field is given by

$$\frac{\lambda}{8\pi} = g_e(\alpha) 3\pi \tag{162}$$

with

$$g_e(\alpha) = (\alpha+1)^2(9+5\alpha). \tag{163}$$

We find $g_e = 9$ for $\alpha = 0$, $g_e = 512/27$ for $\alpha = 1/3$, and $g_e = 56$ for $\alpha = 1$. The self-interaction constant of the early vacuumon (inflaton) is positive ($g_e > 0$), corresponding to a repulsive self-interaction of order 1.

(iv) The dimensional self-interaction constant of the scalar field is given by

$$\lambda_s = \frac{27\pi}{4} \frac{g_e(\alpha)}{|f_e(\alpha)|} \frac{G\hbar^2}{c^2}. \tag{164}$$

We find $g_e/|f_e| = 4/3$ for $\alpha = 0$, $g_e/|f_e| = 256/135$ for $\alpha = 1/3$, and $g_e/|f_e| = 28/9$ for $\alpha = 1$. The dimensional self-interaction constant of the early vacuumon (inflaton) is of order $G\hbar^2/c^2 = 5.15 \times 10^{-71}$ eV cm^3.

(v) The scattering length of the bosons associated with the scalar field is given by

$$a_s = \frac{27}{32} \frac{g_e(\alpha)}{\sqrt{|f_e(\alpha)|}} 2\sqrt{\frac{8\pi}{3}} l_P. \tag{165}$$

We find $g_e^2/|f_e| = 12$ for $\alpha = 0$, $g_e^2/|f_e| = 131072/3645$ for $\alpha = 1/3$, and $g_e^2/|f_e| = 1568/9$ for $\alpha = 1$. The scattering length a_s of the bosons associated with the early vacuumon (inflaton) is of the order of the Planck length $l_P = GM_P/c^2 = (G\hbar/c^3)^{1/2} = 1.62 \times 10^{-35}$ m which corresponds to the semi Schwarzschild radius associated with the Planck mass M_P.

In theories of extended supergravity, the mass of a scalar field is quantized according to the rule [66]

$$m^2 = n_* \frac{H^2\hbar^2}{c^4}, \tag{166}$$

where n_* is an integer and H is the Hubble factor in the de Sitter era. As $H^2 = 8\pi G\rho_P/3$ during the early inflation, we can rewrite Equation (166) as

$$m^2 = n_* \frac{8\pi G\rho_P\hbar^2}{3c^4} = n_* \frac{8\pi}{3} M_P^2. \tag{167}$$

The quantum of mass ($n_* = 1$) in the early universe is the rescaled Planck mass $M_P^* = \sqrt{8\pi/3} M_P$. Comparing Equation (167) with Equation (160), the quantization rule implies that $f_e(\alpha)$ should be an integer n_*. We see that this quantization rule is realized for the integer value $n_* = -10$ when $\alpha = 1/3$ (radiation) and for the integer value $n_* = -18$ when $\alpha = 1$ (stiff) [47,67]. By contrast, $f_e(\alpha)$ is not an integer when $\alpha = 0$ (matter). This may be connected to the fact that the index $\alpha = 0$ is not justified in the early universe as there is no matter there in the usual sense.

6.2. Late Vacuumon (Quintessence)

In the late universe, the potential of the vacuumon is given by Equation (132). For $\psi \to \psi_{\rm max}$, it can be expanded up to fourth order yielding Equation (133).[17] Comparing Equation (133) with Equation (A54), with the substitution $\phi \to \phi_{\rm max} - \phi$, we obtain the following results (see Appendix D):

(i) The minimum value of the potential (corresponding to $\psi = \psi_{\rm max}$) is equal to the cosmological energy density

$$V_{\rm min} = \rho_\Lambda c^2, \tag{168}$$

where $\rho_\Lambda = \Lambda/8\pi G = 5.96 \times 10^{-24}\,{\rm g\,m^{-3}}$ is the cosmological density.

(ii) The squared mass of the scalar field is given by

$$m^2 = f_l(\alpha)\frac{1}{3}m_\Lambda^2 \tag{169}$$

with

$$f_l(\alpha) = \frac{9}{4}(\alpha+1)(1-\alpha). \tag{170}$$

We find $f_l = 9/4$ for $\alpha = 0$, $f_l = 2$ for $\alpha = 1/3$, and $f_l = 0$ for $\alpha = 1$. The mass of the late vacuumon (quintessence) is real ($f_l > 0$) and of the order of the cosmon mass $m_\Lambda = \hbar\sqrt{\Lambda}/c^2 = 2.08 \times 10^{-33}\,{\rm eV/c^2}$, except for $\alpha = 1$ (stiff matter) where $m = 0$. Our approach provides therefore a physical interpretation of the cosmon mass as being the mass of the scalar field responsible for the dark energy in the late universe. To the best of our knowledge, this interpretation has not been given before.

(iii) The dimensionless self-interaction constant of the scalar field is given by

$$\frac{\lambda}{8\pi} = g_l(\alpha)3\pi\frac{\rho_\Lambda}{\rho_P} \tag{171}$$

with

$$g_l(\alpha) = (\alpha+1)^2(1-\alpha). \tag{172}$$

We find $g_l = 1$ for $\alpha = 0$, $g_l = 32/27$ for $\alpha = 1/3$, and $g_l = 0$ for $\alpha = 1$. The self-interaction constant of the late vacuumon (quintessence) is positive ($g_l > 0$), corresponding to a repulsive self-interaction of order $\rho_\Lambda/\rho_P \sim 10^{-123}$, except for $\alpha = 1$ (stiff matter) where $\lambda = 0$.

(iv) The dimensional self-interaction constant of the scalar field is given by

$$\lambda_s = \frac{27\pi}{4}\frac{g_l(\alpha)}{f_l(\alpha)}\frac{G\hbar^2}{c^2}. \tag{173}$$

We find $g_l/f_l = 4/9$ for $\alpha = 0$, $g_l/f_l = 16/27$ for $\alpha = 1/3$, and $g_l/f_l \to 8/9$ for $\alpha \to 1$. The dimensional self-interaction constant of the late vacuumon (quintessence) is of order $G\hbar^2/c^2 = 5.15 \times 10^{-71}\,{\rm eV\,cm^3}$, except for $\alpha = 1$ (stiff matter) where $\lambda_s = 0$ (note that the result obtained in the limit $\alpha \to 1$ is different from the result obtained for $\alpha = 1$).

(v) The scattering length of the bosons associated with the scalar field is given by

$$a_s = \frac{27}{32}\frac{g_l(\alpha)}{\sqrt{f_l(\alpha)}}\frac{2}{\sqrt{3}}r_\Lambda. \tag{174}$$

We find $g_l/\sqrt{f_l} = 2/3$ for $\alpha = 0$, $g_l/\sqrt{f_l} = 16\sqrt{2}/27$ for $\alpha = 1/3$, and $g_l/\sqrt{f_l} \to 0$ for $\alpha \to 1$. The scattering length a_s of the bosons associated with the late vacuumon (quintessence) is of the order of the cosmon radius $r_\Lambda = Gm_\Lambda/c^2 = G\hbar\sqrt{\Lambda}/c^4 = 2.75 \times 10^{-96}$ m, which corresponds to the semi Schwarzschild radius associated with the cosmon mass m_Λ, except for $\alpha = 1$ (stiff matter) where $a_s = 0$.

As $H^2 = 8\pi G\rho_\Lambda/3$ during the late inflation, we can rewrite the quantization rule (166) as

$$m^2 = n_* \frac{8\pi G\rho_\Lambda \hbar^2}{3c^4} = n_* \frac{1}{3} m_\Lambda^2. \tag{175}$$

The quantum of mass ($n_* = 1$) in the late universe is the rescaled cosmon mass $m_\Lambda^* = \frac{1}{\sqrt{3}} m_\Lambda$.[18] Comparing Equation (169) with Equation (175), the quantization rule implies that $f_I(\alpha)$ should be an integer n_*. We see that this quantization rule is realized for the integer value $n_* = 2$ when $\alpha = 1/3$ (radiation) and for the integer value $n_* = 0$ when $\alpha = 1$ (stiff matter) [47,67]. By contrast, $f_I(\alpha)$ is not an integer when $\alpha = 0$ (matter). We recall that our model of late universe with $\alpha = 0$ is equivalent to the ΛCDM model interpreted as a UDM model. Therefore, the quintessence field associated with the standard ΛCDM model ($\alpha = 0$) does not satisfy the quantization rule (166) [47,67]. This suggests (provided that this quantization rule *must* hold, which is not firmly established) that the relevant value of α to take is $\alpha = 1/3$ or $\alpha = 1$, and that dark matter should be treated as an independent species, as in scenario (B).

Remark 4. *In this paper, we have described the late universe by a polytrope of index $n = -1$ [42]. For $\alpha = 0$ and $n = -1$, our model is equivalent to the ΛCDM model. If we consider a more general polytropic equation of state with $\alpha = 0$ and an arbitrary index $n < 0$ (see Equation (A45) and Ref. [42]), we find that the mass of the scalar field in the late universe is given by (see Appendix D and [47,67])*

$$m^2 = -\frac{1+2n}{n^2} \frac{6\pi G\rho_\Lambda \hbar^2}{c^4}. \tag{176}$$

We note that this mass presents a maximum as a function of n for $n = -1$. Interestingly, this maximum selects the ΛCDM model among all the polytropic models of the form of Equation (A45) with $\alpha = 0$ and $n < 0$ [47,67].

7. Scalar Field in the Presence of One Fluid in the Intermediate Regime

7.1. General Results

In this section, we consider the complete model of Section 4 in the intermediate regime between the two de Sitter eras. In this regime, the scalar field has a linear equation of state $P_\phi = \alpha\rho_\phi c^2$. We use approach (B) in which the value of α is fixed. As we have seen, relevant values of α are $\alpha = 1$ (stiff matter) or $\alpha = 1/3$ (radiation). On the other hand, we have to take into account the presence of X-fluids which also have a linear equation of state $P_X = \alpha_X \rho_X c^2$. If the scalar field evolves in the presence of radiation, one has $\alpha_r = 1/3$. If it evolves in the presence of matter, one has $\alpha_m = 0$. For the sake of generality, we consider arbitrary values of α and α_X but we assume that $\alpha \geq -1$ and $\alpha_X \geq -1$ so that the universe has not a phantom behavior.

The energy density of the scalar field and of the X-fluids evolve with the scale factor as

$$\frac{\rho_\phi}{\rho_0} = \frac{\Omega_{\alpha,0}}{a^{3(1+\alpha)}}, \qquad \frac{\rho_X}{\rho_0} = \frac{\Omega_{X,0}}{a^{3(1+\alpha_X)}}. \tag{177}$$

The total energy density is therefore

$$\frac{\rho}{\rho_0} = \sum_X \frac{\Omega_{X,0}}{a^{3(1+\alpha_X)}} + \frac{\Omega_{\alpha,0}}{a^{3(1+\alpha)}}. \tag{178}$$

The time evolution $a(t)$ of the scale factor is determined by the Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\left(\sum_X \rho_X + \rho_\phi\right), \tag{179}$$

where the right hand side is given by Equation (178).[19] On the other hand, the scalar field potential is determined by the equations[20]

$$\phi(a) = \left(\frac{3c^2}{8\pi G}\right)^{1/2}\sqrt{1+\alpha}\int_0^a \sqrt{\frac{\frac{\Omega_{\alpha,0}}{x^{3(\alpha+1)}}}{\sum_X \frac{\Omega_{X,0}}{x^{3(1+\alpha_X)}} + \frac{\Omega_{\alpha,0}}{x^{3(\alpha+1)}}}}\,\frac{dx}{x}, \tag{180}$$

$$\frac{V(a)}{\rho_0 c^2} = \frac{1}{2}(1-\alpha)\Omega_{\alpha,0}\frac{1}{a^{3(\alpha+1)}}. \tag{181}$$

In the following, in order to have analytical results, we assume that there is just one X-fluid in addition to the scalar field. It can represent for example radiation ($\alpha_X = 1/3$) or matter ($\alpha_X = 0$). The case of two X-fluids (radiation and matter) in addition to the scalar field could be treated numerically. We obtain the form of the potential in the limiting case where one perfect fluid dominates over the other. In our model, the foregoing equations are valid only in the intermediate regime between the two de Sitter eras, where the equation of state of the scalar field can be approximated by a linear law, so they have a restricted domain of validity. However, in our mathematical analysis below, we shall study Equations (177)–(181) for all times because they can be relevant in other contexts, for example in the case where the fluid describes dark matter and the scalar field describes dark energy (see Section 7.3).

7.2. Hyperbolic Potential

We first assume $\alpha < \alpha_X$. In that case, the X-fluid dominates at early times and the scalar field dominates at late times. Equation (180) can be rewritten as

$$\phi(a) = \left(\frac{3c^2}{8\pi G}\right)^{1/2}\sqrt{1+\alpha}\int_0^a \frac{dx}{x\sqrt{1+\frac{\Omega_{X,0}}{\Omega_{\alpha,0}}\frac{1}{x^{3(\alpha_X-\alpha)}}}}. \tag{182}$$

Making the change of variables

$$X = \frac{\Omega_{X,0}}{\Omega_{\alpha,0}}\frac{1}{x^{3(\alpha_X-\alpha)}}, \tag{183}$$

we obtain

$$\phi(a) = \left(\frac{3c^2}{8\pi G}\right)^{1/2}\sqrt{1+\alpha}\frac{1}{3(\alpha_X-\alpha)}\int_{\frac{\Omega_{X,0}}{\Omega_{\alpha,0}}\frac{1}{a^{3(\alpha_X-\alpha)}}}^{+\infty}\frac{dX}{X\sqrt{1+X}}. \tag{184}$$

Using the identity

$$\int \frac{1}{\sqrt{x+1}}\frac{dx}{x} = \ln\left(\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right) = -2\sinh^{-1}\left(\frac{1}{\sqrt{x}}\right), \tag{185}$$

we find

$$\phi(a) = \left(\frac{3c^2}{8\pi G}\right)^{1/2}\sqrt{1+\alpha}\frac{2}{3(\alpha_X-\alpha)}\sinh^{-1}\left[\sqrt{\frac{\Omega_{\alpha,0}}{\Omega_{X,0}}}a^{\frac{3}{2}(\alpha_X-\alpha)}\right]. \tag{186}$$

Inversely,

$$a = \left(\frac{\Omega_{X,0}}{\Omega_{\alpha,0}}\right)^{\frac{1}{3(\alpha_X-\alpha)}} \sinh^{\frac{2}{3(\alpha_X-\alpha)}}\left[\left(\frac{8\pi G}{3c^2}\right)^{1/2}\frac{1}{\sqrt{1+\alpha}}\frac{3(\alpha_X-\alpha)}{2}\phi\right]. \tag{187}$$

Combining Equation (187) with Equation (181), we obtain the hyperbolic potential

$$\begin{aligned} V(\phi) &= \frac{1}{2}\rho_0 c^2 \Omega_{\alpha,0}(1-\alpha)\left(\frac{\Omega_{\alpha,0}}{\Omega_{X,0}}\right)^{\frac{\alpha+1}{\alpha_X-\alpha}} \\ &\times \sinh^{-\frac{2(\alpha+1)}{\alpha_X-\alpha}}\left[\left(\frac{8\pi G}{3c^2}\right)^{1/2}\frac{1}{\sqrt{1+\alpha}}\frac{3(\alpha_X-\alpha)}{2}\phi\right]. \end{aligned} \tag{188}$$

We now assume $\alpha > \alpha_X$. In that case, the scalar field dominates at early times and the X-fluid dominates at late times. We note that the integral from Equation (182) diverges at $x = 0$. We can easily circumvent this problem by redefining the additive constant in the scalar field so that

$$\phi(a) = -\left(\frac{3c^2}{8\pi G}\right)^{1/2}\sqrt{1+\alpha}\int_a^{+\infty}\frac{dx}{x\sqrt{1+\frac{\Omega_{X,0}}{\Omega_{\alpha,0}}\frac{1}{x^{3(\alpha_X-\alpha)}}}}. \tag{189}$$

With this convention, we can check that the previous Equations (186)–(188) remain valid.

Finally, when $\alpha = \alpha_X$, Equation (180) reduces to

$$\phi(a) = \left(\frac{3c^2}{8\pi G}\right)^{1/2}\sqrt{1+\alpha}\frac{1}{\sqrt{1+\frac{\Omega_{X,0}}{\Omega_{\alpha,0}}}}\int_1^a\frac{dx}{x}, \tag{190}$$

where we have chosen the origin of the scalar field at $a = 1$ in order to avoid spurious divergences. This yields

$$\phi(a) = \left(\frac{3c^2}{8\pi G}\right)^{1/2}\sqrt{1+\alpha}\left(\frac{\Omega_{\alpha,0}}{\Omega_{X,0}+\Omega_{\alpha,0}}\right)^{1/2}\ln a. \tag{191}$$

Inversely,

$$a = e^{\left(\frac{8\pi G}{3c^2}\right)^{1/2}\frac{1}{\sqrt{1+\alpha}}\left(\frac{\Omega_{X,0}+\Omega_{\alpha,0}}{\Omega_{\alpha,0}}\right)^{1/2}\phi}. \tag{192}$$

Combining Equation (192) with Equation (181) we obtain the exponential potential

$$\frac{V(\phi)}{\rho_0 c^2} = \frac{1}{2}(1-\alpha)\Omega_{\alpha,0}e^{-3\left(\frac{8\pi G}{3c^2}\right)^{1/2}\sqrt{1+\alpha}\left(\frac{\Omega_{X,0}+\Omega_{\alpha,0}}{\Omega_{\alpha,0}}\right)^{1/2}\phi}. \tag{193}$$

For a scalar field alone ($\Omega_{X,0} = 0$) we recover Equation (149).

Below, we consider particular limits of the hyperbolic potential (188).

7.2.1. Power-Law Potential

When the terms in brackets in Equations (186)–(188) go to zero, we find

$$\phi(a) = \left(\frac{3c^2}{8\pi G}\right)^{1/2}\sqrt{1+\alpha}\frac{2}{3(\alpha_X-\alpha)}\sqrt{\frac{\Omega_{\alpha,0}}{\Omega_{X,0}}}a^{\frac{3}{2}(\alpha_X-\alpha)}, \tag{194}$$

$$a=\left(\frac{\Omega_{X,0}}{\Omega_{\alpha,0}}\right)^{\frac{1}{3(\alpha_X-\alpha)}}\left[\left(\frac{8\pi G}{3c^2}\right)^{1/2}\frac{1}{\sqrt{1+\alpha}}\frac{3(\alpha_X-\alpha)}{2}\phi\right]^{\frac{2}{3(\alpha_X-\alpha)}},\tag{195}$$

$$V(\phi)=\frac{1}{2}\rho_0 c^2\Omega_{\alpha,0}(1-\alpha)\left(\frac{\Omega_{\alpha,0}}{\Omega_{X,0}}\right)^{\frac{\alpha+1}{\alpha_X-\alpha}}\left[\left(\frac{8\pi G}{3c^2}\right)^{1/2}\frac{1}{\sqrt{1+\alpha}}\frac{3(\alpha_X-\alpha)}{2}\phi\right]^{-\frac{2(\alpha+1)}{\alpha_X-\alpha}}.\tag{196}$$

This corresponds to the situation where the X-fluid dominates the scalar field. In that case, we obtain a power-law potential. When $\alpha<\alpha_X$, this approximation is valid at sufficiently early times ($R\rightarrow 0$, $\phi\rightarrow 0^+$ and $V\rightarrow +\infty$). When $\alpha>\alpha_X$, it is valid at sufficiently late times ($R\rightarrow +\infty$, $\phi\rightarrow 0^-$ and $V\rightarrow 0$). Pure (inverse) power-law potentials were introduced by Peebles and Ratra [30,31]. They were used in relation to tracker solutions [22,23,33].

7.2.2. Exponential Potential

When the terms in brackets in Equations (186)–(188) go to $+\infty$, we find

$$\phi(a)=\left(\frac{3c^2}{8\pi G}\right)^{1/2}\sqrt{1+\alpha}\frac{2}{3(\alpha_X-\alpha)}\ln\left[2\sqrt{\frac{\Omega_{\alpha,0}}{\Omega_{X,0}}}a^{\frac{3}{2}(\alpha_X-\alpha)}\right],\tag{197}$$

$$a=\left(\frac{\Omega_{X,0}}{\Omega_{\alpha,0}}\right)^{\frac{1}{3(\alpha_X-\alpha)}}\frac{1}{2^{\frac{2}{3(\alpha_X-\alpha)}}}e^{\left(\frac{8\pi G}{3c^2}\right)^{1/2}\frac{1}{\sqrt{1+\alpha}}\phi},\tag{198}$$

$$V(\phi)=\frac{1}{2}\rho_0 c^2\Omega_{\alpha,0}(1-\alpha)\left(\frac{\Omega_{\alpha,0}}{\Omega_{X,0}}\right)^{\frac{\alpha+1}{\alpha_X-\alpha}}2^{\frac{2(\alpha+1)}{\alpha_X-\alpha}}e^{-3\left(\frac{8\pi G}{3c^2}\right)^{1/2}\sqrt{1+\alpha}\,\phi}.\tag{199}$$

This corresponds to the situation where the scalar field dominates the X-fluid. In that case, we obtain an exponential potential as when the scalar field with a linear equation of state is alone in the universe (see Section 5.5).[21] When $\alpha<\alpha_X$, this approximation is valid at sufficiently late times ($R\rightarrow +\infty$, $\phi\rightarrow +\infty$ and $V\rightarrow 0$). When $\alpha>\alpha_X$, it is valid at sufficiently early times ($R\rightarrow 0$, $\phi\rightarrow -\infty$ and $V\rightarrow +\infty$). Pure exponential potentials were introduced by Halliwell [69] (see also [31,70–75]). Exponential potentials arise very naturally in all models of unification with gravity such as Kaluza–Klein theories, supergravity theory or string theory. Most theories undergoing dimensional reduction to an effective four-dimensional theory yield exponential potentials.

7.3. Comparison with Other Works

The hyperbolic potential (188) has been obtained by several authors [18,76–81] using different methods. In these models, the fluid characterized by an equation of state $P_X=\alpha_X\rho_X c^2$ with $0\le\alpha_X\le 1$ describes radiation ($\alpha_X=1/3$) or dark matter ($\alpha_X=0$) and the scalar field characterized by an equation of state $P_\phi=\alpha\rho_\phi c^2$ with $-1<\alpha\le -1/3$ (e.g., $\alpha=-0.65$ [79] or $\alpha=-2/3$ [80]) describes DE. As $\alpha<\alpha_X$ these models describe a radiation or dark matter era followed by a dark energy era. During the radiation or dark matter era, the universe is decelerating and the scale factor increases algebraically as $t^{1/2}$ or $t^{2/3}$ (see Equation (23)). During the dark energy era, the universe is accelerating and the scale factor increases algebraically as $t^{2/[3(1+\alpha)]}$ (power-law inflation). In the radiation and dark matter eras, the scalar field is subdominant and the potential $V(\phi)$ has an inverse power-law behavior (tracking). In the dark energy era, the scalar field dominates the other species (X-fluids) and the potential $V(\phi)$ has an exponential behavior as when a scalar field with a linear equation of state is alone in the universe. We thus achieve a tracker solution that can drive the universe into its current inflationary state. The behavior of the potential as an inverse power-law potential at early times avoids the fine-tuning problem and the cosmic coincidence problem [22]. Its exponential behavior at late time drives the universe into a power-law inflationary stage in agreement with the observations. In this model, the scalar field describes only dark energy, and we have to add radiation and (dark) matter as additional species.

In the present work, the hyperbolic potential (188) is obtained in a particular limit of a more general model based on a scalar field with a quadratic equation of state in the presence of X-fluids. It is valid in the intermediate regime between the early and late inflation. The scalar field models dark radiation ($\alpha = 1/3$) or stiff matter ($\alpha = 1$) and the fluid models radiation ($\alpha_X = 1/3$) or matter ($\alpha_X = 0$). As $\alpha \geq \alpha_X$, we are always in the situation where the scalar field dominates at early times and the X-fluid dominates at late times (note that the scalar field dominates again in the late universe due to the constant term $-(\alpha+1)\rho_\Lambda c^2$ in its equation of state which has been ignored in the intermediate regime). Therefore, in this intermediate period, the scalar field potential is initially exponential and then becomes algebraic.

Remark 5. *The* Λ*CDM model corresponds to* $\alpha_X = 0$ *and* $\alpha = -1$*. In that case,* $\phi = \mathrm{cst}$ *and* $V(\phi) = \mathrm{cst}$*. The* Λ*CDM model can also be obtained by taking* $\alpha_X = -1$ *and* $\alpha = 0$*. In that case, the scalar field has a potential*

$$V(\phi) = \frac{1}{2}\rho_0 c^2 \Omega_{X,0} \sinh^2\left[\left(\frac{8\pi G}{3c^2}\right)^{1/2} \frac{3}{2}\phi\right]. \tag{200}$$

8. Scalar Field in the Presence of One Fluid in the Late Universe

In this section, we consider the complete model of Section 4 in the late universe (we consider just one X-fluid in addition to the scalar field). In that case, the scalar field has an affine equation of state $P_\phi = \alpha\rho_\phi c^2 - (\alpha+1)\rho_\Lambda c^2$. The scalar field potential is determined by Equations (98) and (99) which can be rewritten as

$$\phi(a) = \left(\frac{3c^2}{8\pi G}\right)^{1/2} \sqrt{1+\alpha} \int^a \frac{1}{\sqrt{\frac{\Omega_{X,0}}{\Omega_{\alpha,0}} x^{3(\alpha-\alpha_X)} + 1 + \frac{\Omega_{\Lambda,0}}{\Omega_{\alpha,0}} x^{3(1+\alpha)}}} \frac{dx}{x}, \tag{201}$$

$$\frac{V(a)}{\rho_0 c^2} = \frac{(1-\alpha)\Omega_{\alpha,0}}{2a^{3(\alpha+1)}} + \Omega_{\Lambda,0}. \tag{202}$$

The scalar field is described by the index $\alpha = 1/3$ (radiation) or $\alpha = 1$ (stiff) and the X-fluid is described by the index $\alpha_X = 0$ (matter). Unfortunately, the integral cannot be calculated analytically in general except in the degenerate case $\alpha = \alpha_X$ (see below) which is not relevant in our situation. However, in the very late universe, we can ignore the term in $\Omega_{X,0}$ in Equation (201). In that case, we are led back to the equations (128) and (129) valid for a scalar field alone in the universe leading to the potential from Equation (132). Note that for $\alpha = 1$ (stiff matter), the potential from Equation (202) is constant:

$$V(\phi) = \rho_\Lambda c^2 \qquad (\alpha = 1), \tag{203}$$

whether or not an X-fluid is present. In addition, for $\alpha = 1$ and $\alpha_X = 0$ the integral in Equation (201) can be computed analytically, giving

$$\phi(a) = \frac{\sqrt{2}}{3}\left(\frac{3c^2}{8\pi G}\right)^{1/2} \ln\left[\frac{a^3}{2 + \frac{\Omega_{X,0}}{\Omega_{\alpha,0}} a^3 + 2\sqrt{1 + \frac{\Omega_{X,0}}{\Omega_{\alpha,0}} a^3 + \frac{\Omega_{\Lambda,0}}{\Omega_{\alpha,0}} a^6}}\right] + \mathrm{cst}. \tag{204}$$

For $a \gg 1$, it reduces to

$$\phi(a) = -\frac{2\sqrt{2}}{3}\left(\frac{3c^2}{8\pi G}\right)^{1/2}\frac{\Omega_{\alpha,0}}{\Omega_{X,0}}\sqrt{\frac{\Omega_{X,0}}{\Omega_{\alpha,0}}\frac{1}{a^3}+\frac{\Omega_{\Lambda,0}}{\Omega_{\alpha,0}}}+\mathrm{cst}. \tag{205}$$

This expression provides the correction to Equation (128) due to the presence of the X-fluid.

Remark 6. *It is interesting at a general level to consider the case $\alpha = \alpha_X$ in Equation (201) where the integral can be calculated analytically. This corresponds to a situation where a scalar field described by the affine equation of state $P_\phi = \alpha\rho_\phi c^2 - (\alpha+1)\rho_\Lambda c^2$ evolves in the presence of an X-fluid described by the linear equation of state $P_X = \alpha_X\rho_X c^2$ with $\alpha_X = \alpha$.*[22] *In that case, Equation (201) reduces to*

$$\phi(a) = -\left(\frac{3c^2}{8\pi G}\right)^{1/2}\sqrt{1+\alpha}\frac{1}{\sqrt{\frac{\Omega_{X,0}}{\Omega_{\alpha,0}}+1}}\int_a^{+\infty}\frac{1}{\sqrt{1+\frac{\Omega_{\Lambda,0}}{\Omega_{X,0}+\Omega_{\alpha,0}}x^{3(1+\alpha)}}}\frac{dx}{x}, \tag{206}$$

where we have chosen the origin of the scalar field at $a \to +\infty$ in order to avoid spurious divergences at $x = 0$.[23] *Making the change of variables*

$$X = \frac{\Omega_{\Lambda,0}}{\Omega_{X,0}+\Omega_{\alpha,0}}x^{3(\alpha+1)}, \tag{207}$$

we obtain

$$\phi(a) = -\left(\frac{3c^2}{8\pi G}\right)^{1/2}\frac{1}{\sqrt{\frac{\Omega_{X,0}}{\Omega_{\alpha,0}}+1}}\frac{1}{3\sqrt{\alpha+1}}\int_{\frac{\Omega_{\Lambda,0}}{\Omega_{X,0}+\Omega_{\alpha,0}}a^{3(\alpha+1)}}^{+\infty}\frac{1}{\sqrt{1+X}}\frac{dX}{X}. \tag{208}$$

Using the identity from Equation (185), we find

$$\phi(a) = -\left(\frac{3c^2}{8\pi G}\right)^{1/2}\frac{1}{\sqrt{\frac{\Omega_{X,0}}{\Omega_{\alpha,0}}+1}}\frac{2}{3\sqrt{\alpha+1}}\mathrm{argsinh}\left(\sqrt{\frac{\Omega_{X,0}+\Omega_{\alpha,0}}{\Omega_{\Lambda,0}}}\frac{1}{a^{3(\alpha+1)/2}}\right). \tag{209}$$

Inversely,

$$a = \left(\frac{\Omega_{X,0}+\Omega_{\alpha,0}}{\Omega_{\Lambda,0}}\right)^{\frac{1}{3(\alpha+1)}}\frac{1}{\sinh^{\frac{2}{3(\alpha+1)}}\left[-\frac{3}{2}\left(\frac{8\pi G}{3c^2}\right)^{1/2}\sqrt{1+\alpha}\sqrt{\frac{\Omega_{X,0}}{\Omega_{\alpha,0}}+1}\,\phi\right]}. \tag{210}$$

Finally, combining Equation (210) with Equation (202), we obtain the hyperbolic potential

$$\frac{V(\phi)}{\rho_0 c^2} = \frac{1}{2}(1-\alpha)\frac{\Omega_{\alpha,0}\Omega_{\Lambda,0}}{\Omega_{X,0}+\Omega_{\alpha,0}}\sinh^2\left[-\frac{3}{2}\left(\frac{8\pi G}{3c^2}\right)^{1/2}\sqrt{1+\alpha}\sqrt{\frac{\Omega_{X,0}}{\Omega_{\alpha,0}}+1}\,\phi\right]+\Omega_{\Lambda,0}. \tag{211}$$

For a scalar field alone ($\Omega_{X,0} = 0$), we recover Equation (132).

9. Spectrum of Fluctuations in the Primordial Universe

In this work and in previous ones [39–47], we have developed a model of early inflation based on the quadratic equation of state (49). This model is able to describe the evolution of the homogeneous background and to account for a primordial phase of exponential expansion (de Sitter era) followed by a stiff matter era ($\alpha = 1$) or a radiation era ($\alpha = 1/3$). It also provides a graceful exit to the de Sitter era. However, explaining the evolution of the homogeneous background is not sufficient. A relevant model of inflation must also reproduce the observed spectrum of fluctuations in the primordial universe. In this section, we apply the Hamilton–Jacobi formalism of inflation [60] to a scalar field described by the equation of state (49) and we determine the Hubble hierarchy parameters and the spectral indices. We show that the obtained results are in severe disagreement with the observations. These negative results suggest that the vacuumon potential $V(\phi)$ is just an effective classical potential that cannot be used to compute the fluctuations in the primordial universe. A fully quantum mechanics approach may be required.

9.1. Hamilton–Jacobi Formalism

Let us first briefly review the Hamilton–Jacobi formalism of inflation following Ref. [60]. This formalism is very general. In particular, it does not rely on the slow roll approximation. This is important because, as we shall see, our model of inflation is never in the slow roll regime.[24] In this section, we assume that the scalar field is alone in the primordial universe. The basic equations of the problem are the KG equation

$$\ddot{\phi}+3H\dot{\phi}+\frac{dV}{d\phi}=0 \tag{212}$$

and the Friedmann equation

$$H^2=\frac{8\pi G}{3}\rho_\phi, \tag{213}$$

where

$$\rho_\phi c^2=\frac{1}{2}\dot{\phi}^2+V(\phi) \tag{214}$$

is the energy density of the scalar field. In the following, we assume that the function $H(\phi)$ is known. It is called the generating function. Specifying $H(\phi)$ is equivalent to specifying the equation of state $P_\phi(\rho_\phi)$ or the potential $V(\phi)$ of the scalar field. Taking the time derivative of Equations (213) and (214), combining the results with Equation (212), and using $\dot{H}=H'(\phi)\dot{\phi}$, we find

$$\dot{\phi}=-\frac{c^2}{4\pi G}H', \tag{215}$$

where $'$ denotes derivative with respect to ϕ. Using $aH=\dot{a}=a'\dot{\phi}$ and Equation (215), we obtain

$$aH=-\frac{c^2}{4\pi G}H'a'. \tag{216}$$

Integrating this equation, we find that

$$a(\phi)=\exp\left\{-\frac{4\pi G}{c^2}\int\frac{H}{H'}\,d\phi\right\}. \tag{217}$$

This equation determines the scale factor as a function of ϕ. On the other hand, combining Equations (213)–(215), we find

$$V(\phi)=\frac{3c^2}{8\pi G}\left[H^2-\frac{c^2}{12\pi G}(H')^2\right]. \tag{218}$$

This equation determines the scalar field potential. In the slow-roll approximation where $\dot{\phi}^2\ll V(\phi)$ and $|\ddot{\phi}|\ll V'(\phi)$, the scalar field potential is given by

$$V_{\rm SR}(\phi)=\frac{3c^2}{8\pi G}H^2. \tag{219}$$

The first Hubble hierarchy parameter is defined by

$$\epsilon_H=-\frac{d\ln H}{d\ln a}=\frac{c^2}{4\pi G}\left(\frac{H'}{H}\right)^2, \tag{220}$$

where we have used Equation (216) to obtain the second equality. Using $\ddot{a} = (d/dt)(Ha) = a\dot{H} + H\dot{a}$, implying $\ddot{a}/a = H^2 + \dot{H} = H^2(1 + \dot{H}/H^2) = H^2(1 + \dot{H}a/H\dot{a})$, we find that

$$\frac{\ddot{a}}{a} = H^2(1 - \epsilon_H). \tag{221}$$

Therefore, ϵ_H is related to the deceleration parameter from Equation (14) by

$$q = \epsilon_H - 1. \tag{222}$$

The parameter ϵ_H gives information about the acceleration of the universe. During inflation ($\ddot{a} > 0$), we have $\epsilon_H < 1$. Inflation ends when $\ddot{a} = 0$ yielding $\epsilon_H(t_{\rm end}) = 1$. The number of e-folding before inflation ends is defined by

$$N(t) = \int_t^{t_{\rm end}} H\,dt = \int_t^{t_{\rm end}} \frac{\dot{a}}{a}\,dt = \int_t^{t_{\rm end}} \frac{da}{a} = \ln\left[\frac{a(t_{\rm end})}{a(t)}\right]. \tag{223}$$

It can be written in terms of the scalar field as

$$N(\phi) = \int_t^{t_{\rm end}} H\,dt = \frac{4\pi G}{c^2}\int_{\phi_{\rm end}}^{\phi} \frac{H}{H'}\,d\phi = \int_{\phi_{\rm end}}^{\phi} \frac{1}{\epsilon_H}\frac{H'}{H}\,d\phi, \tag{224}$$

where we have used Equation (215) to obtain the second equality and Equation (220) to obtain the last equality. Here, $\phi_{\rm end}$ is the value of the scalar field at the end of inflation. It is determined by the condition $\epsilon_H(\phi_{\rm end}) = 1$. The pressure of the scalar field is given by

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{225}$$

Summing Equations (214) and (225), we find

$$P_\phi + \rho_\phi c^2 = \dot{\phi}^2. \tag{226}$$

Substituting Equations (213) and (215) into Equation (226), we obtain

$$P_\phi = \frac{3c^2}{8\pi G}H^2\left[\frac{c^2}{6\pi G}\left(\frac{H'}{H}\right)^2 - 1\right]. \tag{227}$$

This equation determines the pressure as a function of ϕ. Finally, the energy density $\rho_\phi(\phi)$ of the scalar field is given by

$$\rho_\phi = \frac{3H^2}{8\pi G}. \tag{228}$$

Eliminating ϕ between Equations (227) and (228), we obtain the equation of state $P_\phi(\rho_\phi)$.

9.2. Application to Our Model of Inflation

We now apply the Hamilton–Jacobi formalism to our model of early inflation. The generating function of our model is[25]

$$H(\phi) = \left(\frac{8\pi G\rho_P}{3}\right)^{1/2}\frac{1}{\cosh\psi} \quad \text{with} \quad \psi = \left(\frac{8\pi G}{3c^2}\right)^{1/2}\frac{3\sqrt{\alpha+1}}{2}\phi. \tag{229}$$

Using Equations (227) and (228), we obtain the pressure

$$P_\phi = \frac{\rho_P c^2}{\cosh^2\psi}\left(\alpha - \frac{\alpha+1}{\cosh^2\psi}\right) \tag{230}$$

and the energy density

$$\rho_\phi = \frac{\rho_P}{\cosh^2\psi}. \tag{231}$$

This returns the results from Equation (123). Eliminating ψ between these equations, we obtain the equation of state

$$P_\phi = -(\alpha+1)\frac{\rho_\phi^2}{\rho_P}c^2 + \alpha\rho_\phi c^2. \tag{232}$$

This returns the result from Equation (49). Using Equation (218), we obtain the scalar field potential

$$V(\phi) = \frac{1}{2}\rho_P c^2 \frac{(1-\alpha)\cosh^2\psi + \alpha + 1}{\cosh^4\psi}. \tag{233}$$

This returns the result from Equation (115). Using Equation (217), we obtain the scale factor

$$a(\phi) \propto (\sinh\psi)^{\frac{2}{3(\alpha+1)}}. \tag{234}$$

This returns the result from Equation (122). Using Equations (223) and (234), we find that the number of e-folding before inflation ends is given by

$$N(\phi) = \frac{2}{3(\alpha+1)}\ln\left[\frac{\sinh(\psi_{\rm end})}{\sinh\psi}\right]. \tag{235}$$

In our model, we find $N_i \sim \ln(a_1/l_P) \sim 67$ [44]. Using Equations (220) and (222), we obtain the first Hubble hierarchy parameter and the deceleration parameter through the relation

$$\epsilon_H = q + 1 = \frac{3}{2}(\alpha+1)\tanh^2\psi. \tag{236}$$

The condition $\epsilon_H = 1$ (i.e., $q = 0$) gives

$$\tanh^2(\psi_{\rm end}) = \frac{2}{3(\alpha+1)}. \tag{237}$$

This equation determines the value $\psi_{\rm end}$ of the scalar field at the end of the inflation. Therefore, inflation takes place between $\psi = 0$ and $\psi_{\rm end}$. The second and third Hubble hierarchy parameters are defined by

$$\eta_H = -\frac{d\ln H'}{d\ln a} = \frac{c^2}{4\pi G}\frac{H''}{H} \tag{238}$$

and

$$\xi_H^2 = \left(\frac{c^2}{4\pi G}\right)^2 \frac{H'''H'}{H^2}, \tag{239}$$

where we have used Equation (216) to obtain the second equality in Equation (238). For the generating function from Equation (229), we find

$$\eta_H = \frac{3}{2}(\alpha+1)\frac{\cosh^2\psi - 2}{\cosh^2\psi} \tag{240}$$

and

$$\xi_H^2 = \frac{9}{4}(\alpha+1)^2\frac{\cosh^2\psi - 1}{\cosh^4\psi}(\cosh^2\psi - 6). \tag{241}$$

We note that $\eta_H \to -\frac{3}{2}(\alpha+1) \neq 0$ when $\psi \to 0$, so we are never in the slow roll regime unless $\alpha \simeq -1$ (which is a very peculiar situation [41]). The scalar spectral index n_s is given by [60]

$$n_s - 1 = 2\eta_H - 4\epsilon_H. \tag{242}$$

Using Equations (236) and (238), we obtain

$$n_s - 1 = -3(\alpha+1). \tag{243}$$

In our model, the scalar spectral index n_s turns out to be independent of ϕ. The running scalar spectral index $n_{\rm run}$ is given by [60]

$$n_{\rm run} = 10\epsilon_H\eta_H - 8\epsilon_H^2 - 2\xi_H^2. \tag{244}$$

Using Equations (236), (238), and (241), we find

$$n_{\rm run} = 0. \tag{245}$$

In our model, the running scalar spectral index vanishes whatever the values of ϕ and α. The gravitational wave spectral index n_T is given by [60]

$$n_T = -2\epsilon_H \tag{246}$$

and the tensor-to-scalar amplitude ratio is given by [60]

$$r = 4\epsilon_H. \tag{247}$$

In our model, using Equation (236), we find that

$$r = -2n_T = 6(\alpha+1)\tanh^2\psi. \tag{248}$$

Comparison with Planck data: The values of the parameters of inflation obtained by the Planck collaboration [82] are $n_s = 0.9603 \pm 0.0073$, $n_{\rm run} = -0.0134 \pm 0.0090$, $r < 0.11$ and $N = 60 - 70$. A value of n_s close to 1 means that we have nearly scale-dependent density perturbations. For $\alpha = 1$ (stiff matter), $\alpha = 1/3$ (radiation) or $\alpha = 0$ (matter), the value of n_s predicted by our model is very far from unity. This implies that we are not in the slow roll regime where ϵ_H and η_H are small. For $\alpha = 1$, we find $n_s = -5$; for $\alpha = 1/3$, we find $n_s = -3$; and for $\alpha = 0$, we find $n_s = -2$. It is only for $\alpha \simeq -1$ that n_s becomes close to unity. However, in that case, we cannot describe the graceful exit to inflation (i.e., the smooth transition from inflation to a stiff matter or a radiation era). Therefore, for relevant values of α, our inflationary model cannot account for the fluctuations in the primordial universe. This is probably because the vacuumon potential $V(\phi)$ (see Equation (233)) is just an *effective* classical field that cannot be used to compute the spectrum of fluctuations in the primordial universe. It may be therefore necessary to develop a fully quantum model of inflation.

10. Conclusions

We have developed the model of universe introduced in our previous papers [39–47]. This model assumes that the universe is filled with an exotic fluid described by a quadratic equation of state. The presence of this fluid accounts for a phase of early inflation, followed by a decelerated expansion, and finally a late accelerating expansion. Therefore, our quadratic equation of state unifies inflation, relativistic or nonrelativistic matter, and dark energy in the spirit of a generalized Chaplygin gas. Stiff matter, radiation, and (dark or baryonic) matter may be incorporated in this exotic fluid at different periods of its evolution or treated as independent species. We have given the scalar field representation of this exotic fluid and determined its potential in the absence or presence of other species. For a pure scalar field the potential can be expressed in terms of the Jacobian Elliptic function. We have shown that the mass of the early vacuumon (inflaton) is equal to the Planck mass M_P while the mass of the late vacuumon (quintessence) is equal to the cosmon mass m_Λ. Our model is able to describe the complete history of the universe from an early de Sitter era with density ρ_P to a late de Sitter era with density ρ_Λ, bridged by a phase of decelerated expansion. In our model, the periods of early and late inflation are described by two polytropic equations of state with index $n_e = +1$ and $n_l = -1$, respectively. This makes our model of universe very symmetric, the Planck density ρ_P in the early universe playing a role similar to the cosmological density ρ_Λ in the late universe. They represent fundamental upper and lower density bounds differing by 123 orders of magnitude. Our model is also fully consistent with the ΛCDM model at late times and completes it by incorporating in a natural manner a phase of early inflation that avoids the primordial (big bang) singularity. Therefore, in our model, the universe exists eternally in the past and in the future and does not present singularities. This has been called the aioniotic universe [42]. We have also made the connection between our model [39–47] based on a quadratic equation of state and the RVM [57–59] based on a quartic dependence of the cosmological constant on the Hubble parameter. In this connection, our exotic fluid can be viewed as a mixture of a barotropic fluid (representing stiff matter, radiation, or dark matter) and vacuum energy. Furthermore, our scalar field [42,44] corresponds to what Basilakos et al. [58] have called later the vacuumon. Despite its interest, our model cannot account for the spectrum of fluctuations in the early universe. This suggests that the vacuumon potential is just an effective classical potential that cannot be directly used to compute the fluctuations in the early universe. A fully quantum field theory may be required to achieve that goal.

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Appendix A. Energy Conservation Equation for a Real Scalar Field

Starting from the KG Equation (30), we can derive the energy conservation Equation (32) for a real scalar field as follows. Taking the time derivative of the energy density from Equation (31), we obtain

$$\frac{d\rho_\phi}{dt} = \frac{1}{c^2}\left[\ddot{\phi} + V'(\phi)\right]\dot{\phi}. \tag{A1}$$

Combining Equation (A1) with Equation (30), we find that

$$\frac{d\rho_\phi}{dt} = -\frac{3H}{c^2}\dot{\phi}^2. \tag{A2}$$

Summing the density and the pressure from Equation (31), we find

$$\dot{\phi}^2 = \rho_\phi c^2 + P_\phi. \tag{A3}$$

Substituting Equation (A3) into Equation (A2) we obtain the energy conservation Equation (32). Inversely, from Equations (31) and (32) we can derive the KG Equation (30).

Appendix B. Expression of the Energy Density of the Scalar Field Described by our Quadratic Equation of State

Let us consider a scalar field described by the quadratic equation of state (33). From this equation, we find

$$\frac{P_\phi}{c^2}+\rho_\phi=(\alpha+1)\left(-\frac{\rho_\phi^2}{\rho_P}+\rho_\phi-\rho_\Lambda\right). \tag{A4}$$

The two roots of the term in brackets are

$$\rho_\phi^{(\pm)}=\frac{\rho_P}{2}\left(1\pm\sqrt{1-4\frac{\rho_\Lambda}{\rho_P}}\right). \tag{A5}$$

As $\kappa=\rho_\Lambda/\rho_P\sim 10^{-123}\ll 1$, we can make the approximations

$$\rho_\phi^{(+)}\simeq\rho_P \quad \text{and} \quad \rho_\phi^{(-)}\simeq\rho_\Lambda. \tag{A6}$$

These approximations are essentially exact as κ is extremely small. Therefore, Equation (A4) can be written in excellent approximation as

$$\frac{P_\phi}{c^2}+\rho_\phi=(\alpha+1)\frac{1}{\rho_P}(\rho_P-\rho_\phi)(\rho_\phi-\rho_\Lambda). \tag{A7}$$

Substituting this expression into the energy conservation Equation (32), we obtain

$$-\int\frac{\rho_P\,d\rho_\phi}{(\rho_P-\rho_\phi)(\rho_\phi-\rho_\Lambda)}=3(\alpha+1)\ln\left(\frac{a}{a_1}\right), \tag{A8}$$

where a_1 is a constant of integration. This equation can be rewritten as

$$-\int\left(\frac{1}{\rho_P-\rho_\phi}+\frac{1}{\rho_\phi-\rho_\Lambda}\right)d\rho_\phi=3(\alpha+1)\ln\left(\frac{a}{a_1}\right), \tag{A9}$$

where we have used $\rho_P\gg\rho_\Lambda$. We then find

$$\frac{\rho_P-\rho_\phi}{\rho_\phi-\rho_\Lambda}=\left(\frac{a}{a_1}\right)^{3(\alpha+1)}, \tag{A10}$$

giving

$$\rho=\frac{\rho_P+\rho_\Lambda\left(\frac{a}{a_1}\right)^{3(\alpha+1)}}{1+\left(\frac{a}{a_1}\right)^{3(\alpha+1)}}. \tag{A11}$$

Using $\rho_P\gg\rho_\Lambda$ again, this relation can be rewritten as

$$\rho=\rho_\Lambda+\frac{\rho_P}{1+\left(\frac{a}{a_1}\right)^{3(\alpha+1)}}. \tag{A12}$$

The constant of integration a_1 can be obtained by applying Equation (A12) at $a=1$ where $\rho=\rho_0$. This yields

$$a_1=\left(\frac{\rho_0-\rho_\Lambda}{\rho_P}\right)^{\frac{1}{3(\alpha+1)}}, \tag{A13}$$

where we have used $\rho_P \gg \rho_\Lambda$. We note that $\rho_0 - \rho_\Lambda = \rho_{\alpha,0}$ can be interpreted as the present density of the α-fluid. Equation (A12) can finally be written as

$$\rho = \rho_\Lambda + \frac{\rho_P}{1 + \frac{\rho_P}{\rho_0 - \rho_\Lambda} a^{3(\alpha+1)}}. \tag{A14}$$

Remark A1. *Actually, it is possible to integrate Equation (32) with Equation (A4) without making any approximation as detailed in Appendix C of [44]. However, in view of the extremely small value of $\kappa = \rho_\Lambda/\rho_P \sim 10^{-123} \ll 1$, this exact expression is not particularly useful for our purposes.*

Appendix C. Connection between the RVM and our Quadratic Equation of State

In this appendix, we discuss the connection between the RVM [57–59] and our model [39–47]. The RVM assumes that the cosmological constant $\Lambda(H)$ depends on the Hubble parameter and that this dependence is specified by a quartic function of H (actually, a quadratic function of H^2). This relationship is motivated by particle physics and the renormalization group approach. On the other hand, our phenomenological model assumes that the universe is filled with an exotic fluid described by the quadratic equation of state (33). We show that the two models are consistent and complementary to each other.

Appendix C.1. Approach Based on a Quartic $\Lambda(H)$

Let us first recall the main lines of the RVM [57–59]. For an N-species system, the conservation law reads

$$\sum_N \left[\dot\rho_N + 3H\left(\rho_N + \frac{P_N}{c^2}\right)\right] = 0, \tag{A15}$$

where $\rho_N c^2$ and P_N denote the energy density and the pressure of each species, respectively. This equation expresses the conservation of the total energy (summed over all species present in the universe). If we consider a system composed of just one X-fluid + vacuum, we find

$$\dot\rho_X + \dot\rho_\Lambda + 3H\left(\rho_X + \rho_\Lambda + \frac{P_X}{c^2} + \frac{P_\Lambda}{c^2}\right) = 0. \tag{A16}$$

In the general case, the cosmological constant $\Lambda(t)$ and, correspondingly, the vacuum energy density

$$\rho_\Lambda(t) = \frac{\Lambda(t)}{8\pi G} \tag{A17}$$

may depend on time.[26] The equation of state of vacuum is given by

$$P_\Lambda = -\rho_\Lambda c^2. \tag{A18}$$

Therefore, Equation (A16) reduces to

$$\dot\rho_X + 3H\left(\rho_X + \frac{P_X}{c^2}\right) = -\dot\rho_\Lambda. \tag{A19}$$

We also recall the Friedmann equation

$$\frac{3H^2}{8\pi G} = \rho_X + \rho_\Lambda, \tag{A20}$$

where the right hand side includes the energy density of the X-fluid + the energy density of vacuum. We note that, according to Equation (A19), the X-fluid and the vacuum energy are coupled to each other and are not treated as independent species. Introducing the total density and the total pressure[27],

$$\rho = \rho_X + \rho_\Lambda, \qquad P = P_X + P_\Lambda, \tag{A21}$$

we can rewrite Equations (A19) and (A20) as

$$\dot{\rho} + 3H\left(\rho_X + \frac{P_X}{c^2}\right) = 0 \tag{A22}$$

and

$$\frac{3H^2}{8\pi G} = \rho. \tag{A23}$$

We now assume that the X-fluid has a linear equation of state

$$P_X = w_X \rho_X c^2, \tag{A24}$$

where w_X is constant. Combining Equations (A22)–(A24), we obtain

$$\dot{H} + 4\pi G(1 + w_X)\rho_X = 0. \tag{A25}$$

Using Equation (A21), this equation can be rewritten as

$$\dot{H} + 4\pi G(1 + w_X)(\rho - \rho_\Lambda) = 0. \tag{A26}$$

Finally, using Equation (A23), we find

$$\dot{H} + \frac{3}{2}(1 + w_X)H^2 = 4\pi G(1 + w_X)\rho_\Lambda. \tag{A27}$$

As $H = \dot{a}/a$, we can also write this equation as

$$aHH' + \frac{3}{2}(1 + w_X)H^2 = 4\pi G(1 + w_X)\rho_\Lambda, \tag{A28}$$

where $H' = dH/da$. The foregoing equations are valid whether or not Λ (and therefore ρ_Λ) is constant. Therefore, these equations are general. Now, based on results of particle physics and the renormalization group approach, the authors of [57–59] argue that the cosmological constant is a function $\Lambda = \Lambda(H)$ of the Hubble parameter given by

$$\Lambda(H) = 3c_0 + 3\nu H^2 + 3\alpha_s \frac{H^4}{H_I^2}, \tag{A29}$$

where the coefficients c_0, ν, α_s, and H_I can be determined in principle by particle physics. Substituting the vacuum energy density $\rho_\Lambda(H)$ from Equations (A17) and (A29) into Equation (A27) or (A28), we obtain a differential equation for H (see below). The solution of this equation gives $H(t)$ or $H(a)$. We can then obtain $\rho(a)$ from Equation (A23) and $\rho_\Lambda(a)$ from Equations (A17) and (A29). Finally, we can obtain $\rho_X(a)$ from $\rho_X(a) = \rho(a) - \rho_\Lambda(a)$ and $a(t)$ from $\dot{a}/a = H(a)$. Specifically, for the quartic $\Lambda(H)$ relation from Equation (A29), the differential equations (A27) and (A28) for H read

$$\dot{H} + \frac{3}{2}(1 + w_X)H^2 = \frac{3}{2}(1 + w_X)\left(c_0 + \nu H^2 + \alpha_s \frac{H^4}{H_I^2}\right) \tag{A30}$$

and

$$aHH' = \frac{3}{2}(1+w_X)c_0 + \frac{3}{2}(1+w_X)(\nu-1)H^2 + \frac{3}{2}(1+w_X)\alpha_s\frac{H^4}{H_I^2}. \quad \text{(A31)}$$

Equation (A31) can be solved by making the same approximations as in Appendix B[28], yielding

$$H^2 = \frac{c_0}{1-\nu} + \frac{H_0^2 - \frac{c_0}{1-\nu}}{\frac{H_0^2 - \frac{c_0}{1-\nu}}{(1-\nu)\frac{H_I^2}{\alpha_s}} + a^{3(1+w_X)(1-\nu)}}. \quad \text{(A32)}$$

Appendix C.2. Approach Based on a Quadratic $P(\rho)$

Let us now recall the main lines of our approach [39–47]. We assume that the universe is filled with an exotic fluid characterized by a barotropic equation of state $P(\rho)$. This fluid may correspond to a scalar field[29], but this is not compulsory. The energy conservation equation is given by

$$\dot{\rho} + 3H\left(\rho + \frac{P}{c^2}\right) = 0. \quad \text{(A33)}$$

As $H = \dot{a}/a$, we can rewrite this equation as

$$a\rho' + 3\left(\rho + \frac{P}{c^2}\right) = 0, \quad \text{(A34)}$$

where $\rho' = d\rho/da$. We also recall the Friedmann equation

$$\frac{3H^2}{8\pi G} = \rho. \quad \text{(A35)}$$

Based on heuristic considerations [39–47], we assume that the pressure $P(\rho)$ is a quadratic function of the energy density given by

$$P = -(\alpha+1)\frac{\rho^2}{\rho_P}c^2 + \alpha\rho c^2 - (\alpha+1)\rho_\Lambda c^2. \quad \text{(A36)}$$

Substituting $P(\rho)$ from Equation (A36) into Equation (A34), we obtain a differential equation for ρ (see below). The solution of this differential equation gives $\rho(a)$. We can then obtain $a(t)$ by solving the Friedmann Equation (A35). Specifically, for the quadratic equation of state $P(\rho)$ from Equation (A36), the differential Equation (A34) for ρ reads

$$a\rho' + 3(\alpha+1)\left(-\frac{\rho^2}{\rho_P} + \rho - \rho_\Lambda\right) = 0, \quad \text{(A37)}$$

which is solved in Appendix B to give

$$\rho = \rho_\Lambda + \frac{\rho_0 - \rho_\Lambda}{\frac{\rho_0 - \rho_\Lambda}{\rho_P} + a^{3(\alpha+1)}}. \quad \text{(A38)}$$

This solution was first obtained in Refs. [39,40,42,44,47]. The complete analytical solution $a(t)$ of the Friedmann Equation (A35) with Equation (A38) is given in [42,44]. This solution (see Equation (47)) describes a universe going from an early de Sitter era to a late de Sitter era passing by an intermediate phase of decelerated expansion.

Appendix C.3. Connection between the Two Models

The connection between our model and the RVM can be obtained by identifying the density and the pressure of the exotic fluid in our model with the total density and pressure (X-fluid + vacuum) in the RVM. This identification guarantees that the scalar field is the same in the two approaches (see footnote [27]). We can now proceed as follows. Using Equations (A16), (A20), and (A21), we obtain

$$\dot{\rho}+3H\left(\rho+\frac{P}{c^2}\right)=0 \qquad \text{and} \qquad \frac{3H^2}{8\pi G}=\rho, \tag{A39}$$

which are equivalent to Equations (A33) and (A35). On the other hand, according to Equations (A18), (A21), and (A24) we have

$$\begin{aligned} P &= w_X\rho_X c^2-\rho_\Lambda c^2 \\ &= w_X(\rho-\rho_\Lambda)c^2-\rho_\Lambda c^2 \\ &= w_X\rho c^2-(w_X+1)\rho_\Lambda c^2. \end{aligned} \tag{A40}$$

Substituting $\rho_\Lambda(H)$ from Equations (A17) and (A29) into Equation (A40), we obtain

$$P=w_X\rho c^2-(w_X+1)\frac{3c^2}{8\pi G}\left(c_0+\nu H^2+\alpha_s\frac{H^4}{H_I^2}\right). \tag{A41}$$

Finally using Equation (A23), we find

$$P=-c_0(w_X+1)\frac{3c^2}{8\pi G}+[w_X-(w_X+1)\nu]\rho c^2-(w_X+1)\frac{c^2}{8\pi G}\frac{3\alpha_s}{H_I^2}\left(\frac{8\pi G}{3}\right)^2\rho^2. \tag{A42}$$

Therefore, the RVM yields a quadratic equation of state which is equivalent to the quadratic equation of state of our model (see Equation (A36)) up to a change of notations

$$\alpha+1=(w_X+1)(1-\nu), \qquad \rho_\Lambda=\frac{3c_0}{8\pi G(1-\nu)}, \qquad \rho_P=\frac{3H_I^2(1-\nu)}{8\pi G\alpha_s}. \tag{A43}$$

Using this correspondence, we find that Equation (A32) is equivalent to Equation (A38) as it should be. Some comments are in order:

1. We face again the "problem" mentioned in the introduction in the sense that the RVM can accommodate only one X-fluid at a time. In the early universe, this fluid corresponds to the radiation ($w_X=1/3$) and in the late universe this fluid corresponds to the matter ($w_X=0$). Therefore, we can use Equation (A42) in the early universe with $w_X=1/3$ and in the late universe with $w_X=0$ but we cannot use it to describe the whole evolution of the universe including the successive periods of inflation, radiation, matter, and dark energy. In other words, one has to adapt w_X to the period under consideration. This is similar to approach (A) in our model.
2. The RVM determines the evolution of the X-fluid density $\rho_X(t)$ and of the vacuum energy density $\rho_\Lambda(t)$ while our model, based on the equation of state (A36), determines only the evolution of the *total* density $\rho(t)=\rho_X(t)+\rho_\Lambda(t)$, not the evolution of $\rho_X(t)$ and $\rho_\Lambda(t)$ *individually* (they do not appear explicitly in our model). Therefore, the RVM implies Equation (A36) (under the form of Equation (A42)) but this is not reciprocal. In this sense, the RVM contains more information than our model. However, it is not quite clear if, during the early inflation for example, we can really disentangle the radiation from the vacuum energy as in the RVM or if there is just one fluid described by the equation of state (A36), as in our model, which successively behaves as vacuum energy then as radiation.

3. Comparing Equations (A36) and (A42), we see that the coefficient of the linear term is given by

$$\alpha = w_X - (w_X + 1)\nu. \tag{A44}$$

In the matter era, we have $w_X = 0$ implying $\alpha = -\nu$. Therefore, the RVM suggests that $\alpha \neq 0$ in the matter era even though $w_X = 0$. A nonvanishing value of α could be accounted for in our model by assuming that dark matter has an effective temperature so that $P = \alpha\rho c^2$ with $\alpha = k_B T_{\rm eff}/mc^2$. Taking $v_c \sim (k_B T/m)^{1/2} \sim 100\,{\rm km/s}$ from the rotation curves of the galaxies, we find $\alpha \sim 10^{-7}$.[30] This estimate is consistent with the RVM provided that $\nu < 0$ and $|\nu| \sim 10^{-7}$. It is argued in [57,58] that ν can be positive or negative and that $|\nu| \ll 1$, probably in the range $|\nu| = 10^{-6} - 10^{-3}$, and with a typical value $|\nu| = O(10^{-3})$.

4. Except for the slight differences mentioned above, the RVM [57–59] and our model [39–47] are consistent and complementary to each other and both represent an interesting modelling of the evolution of the universe.

Appendix D. Parameters of the Scalar Field in the General Case

In this appendix, we determine the parameters of the scalar field in a more general situation than the one exposed in Section 6.

Appendix D.1. Generalized Polytropic Equation of State

We consider a scalar field described by a generalized polytropic equation of state of the form [41–43]

$$P_\phi = \alpha\rho_\phi c^2 - (\alpha+1)\rho_\phi c^2 \left(\frac{\rho_\phi}{\rho_*}\right)^{1/n}. \tag{A45}$$

This is the sum of a linear equation of state $P_\phi = \alpha\rho_\phi c^2$ and a polytropic equation of state $P_\phi = K\rho_\phi^\gamma$ with a polytropic index $\gamma = 1 + 1/n$ and a negative polytropic constant $K = -(\alpha+1)c^2/{\rho_*}^{1/n} < 0$. We assume $-1 < \alpha \leq 1$ to simplify the discussion (see [41–43] for more general results). If $w = P_\phi/(\rho_\phi c^2) \geq -1$ (non-phantom universe), the energy conservation equation (32) with the equation of state (A45) can be integrated into

$$\rho_\phi = \frac{\rho_*}{\left[1 + (a/a_*)^{3(1+\alpha)/n}\right]^n}, \tag{A46}$$

where a_* is a constant of integration.

When $n > 0$, Equations (A45) and (A46) describe the transition between a phase of early inflation and a phase of algebraic expansion. For $a \ll a_*$ (de Sitter era), we obtain $\rho_\phi \simeq \rho_*$. We can thus identify ρ_* with the Planck density ρ_P (see footnote [2]). For $a \gg a_*$ (α-era), we obtain $\rho_\phi \simeq \rho_*/(a/a_*)^{3(1+\alpha)}$ and $P \sim \alpha\rho_\phi c^2$. Therefore, the scale factor a_* marks the transition between an early de Sitter era and an α-era. For $n = 1$, we recover the results of Section 3.2. The general case is treated in [41].

When $n < 0$, Equations (A45) and (A46) describe the transition between a phase of algebraic expansion and a phase of late inflation. For $a \ll a_*$ (α-era), we obtain $\rho_\phi \simeq \rho_*/(a/a_*)^{3(1+\alpha)}$ and $P \sim \alpha\rho_\phi c^2$. For $a \gg a_*$ (de Sitter era), we obtain $\rho_\phi \simeq \rho_*$. We can thus identify ρ_* with the cosmological density ρ_Λ. Therefore, the scale factor a_* marks the transition between an α-era and a late de Sitter era. For $n = -1$, we recover the results of Section 3.3. The general case is treated in [42].

Appendix D.2. Scalar Field Potential

The scalar field potential corresponding to the equation of state (A45) is [42]

$$V(\psi) = \frac{1}{2}\rho_* c^2 \frac{(1-\alpha)\cosh^2\psi + \alpha + 1}{\cosh^{2(n+1)}\psi}, \tag{A47}$$

where we have defined

$$\psi = \left(\frac{8\pi G}{3c^2}\right)^{1/2} \frac{3\sqrt{\alpha+1}}{2n}(\phi + \mathrm{cst}). \tag{A48}$$

For $\psi \to 0$, it can be expanded into

$$\frac{V(\psi)}{\rho_* c^2} \simeq 1 - \frac{1+\alpha+2n}{2}\psi^2 + \frac{2+2\alpha+4n+3\alpha n+3n^2}{6}\psi^4 + \ldots \tag{A49}$$

For $\psi \to \pm\infty$, we obtain the asymptotic behaviors

$$\frac{V(\psi)}{\rho_* c^2} \sim 2^{2n-1}(1-\alpha)e^{-2n|\psi|} \qquad (\alpha \neq 1), \tag{A50}$$

$$\frac{V(\psi)}{\rho_* c^2} \sim 2^{2(n+1)}e^{-2(n+1)|\psi|} \qquad (\alpha = 1). \tag{A51}$$

We note that the coefficient $\alpha = 1$ of stiff matter plays a special role as it leads to a faster decay of the potential. In that case, Equation (A47) becomes

$$V(\psi) = \frac{\rho_* c^2}{\cosh^{2(n+1)}\psi} \qquad (\alpha = 1). \tag{A52}$$

The relation between the scalar field and the scale factor is [42]

$$(a/a_*)^{3(\alpha+1)/2n} = \sinh\psi. \tag{A53}$$

This relation allows us to express ρ_ϕ, P_ϕ, etc. as a function of ψ instead of a.

Appendix D.3. Normal form of the Potential

The approximate expression (A49) of the scalar field potential for $\psi \to 0$ can be compared with the normal form of a quartic potential

$$V = V_0 + \frac{m^2c^4}{2\hbar^2}\phi^2 + \frac{\lambda c^3}{4\hbar}\phi^4, \tag{A54}$$

where V_0 is the value of the potential at $\phi = 0$, m is the mass of the scalar field, and λ is the dimensionless self-interaction constant. When the scalar field describes the wave function of a Bose–Einstein condensate (BEC), we have the relation[31]

$$\frac{\lambda}{8\pi} = \frac{2a_s}{3\lambda_C} = \frac{2a_s|m|c}{3\hbar}, \tag{A55}$$

where a_s is the scattering length of the bosons and $\lambda_C = \hbar/(|m|c)$ is their Compton wavelength. One can also introduce the dimensional self-interaction constant

$$\lambda_s = \frac{4\pi a_s \hbar^2}{|m|} = \frac{3\lambda\hbar^3}{|m|^2 c}. \tag{A56}$$

Comparing Equation (A49) with Equation (A54), and recalling Equation (A48), we obtain the following results:

(i) The value of the potential at $\phi = 0$ is given by

$$V_0 = \rho_* c^2. \tag{A57}$$

(ii) The squared mass of the scalar field is given by

$$m^2 = f(\alpha, n) m_*^2, \tag{A58}$$

where

$$f(\alpha, n) = -\frac{9(\alpha+1)}{4n^2}(1+\alpha+2n) \tag{A59}$$

and

$$m_* = \left(\frac{8\pi G\rho_* \hbar^2}{3c^4}\right)^{1/2}. \tag{A60}$$

(iii) The dimensionless self-interaction constant of the scalar field is given by

$$\frac{\lambda}{8\pi} = g(\alpha, n)\frac{\lambda_*}{8\pi}, \tag{A61}$$

where

$$g(\alpha, n) = \frac{(\alpha+1)^2}{n^4}(2+2\alpha+4n+3\alpha n+3n^2) \tag{A62}$$

and

$$\frac{\lambda_*}{8\pi} = 3\pi\frac{\rho_*}{\rho_P}. \tag{A63}$$

(iv) The dimensional self-interaction constant of the scalar field is given by

$$\lambda_s = \frac{27\pi}{4}\frac{g(\alpha, n)}{|f(\alpha, n)|}\frac{G\hbar^2}{c^2}, \tag{A64}$$

where

$$\frac{G\hbar^2}{c^2} = 5.15 \times 10^{-71}\,\mathrm{eV\,cm^3}. \tag{A65}$$

(v) The scattering length of the bosons is given by

$$a_s = \frac{27}{32}\frac{g(\alpha, n)}{\sqrt{|f(\alpha, n)|}} r_*, \tag{A66}$$

where

$$r_* = \frac{2Gm_*}{c^2} \tag{A67}$$

is the effective Schwarzschild (or gravitational) radius of a particle of mass m_*.

Remark A2. *The potential (A47) depends on a single parameter—the density ρ_*—which determines m_*, λ_*, and r_*. These quantities are related by*

$$\frac{\lambda_*}{8\pi} = \frac{9}{8}\left(\frac{m_*}{M_P}\right)^2 = \frac{9}{32}\left(\frac{r_*}{l_P}\right)^2 = 3\pi\frac{\rho_*}{\rho_P}. \tag{A68}$$

Appendix D.4. The Early Universe

In the early universe, the characteristic density ρ_* is the Planck density

$$\rho_* = \rho_P = \frac{c^5}{\hbar G^2} = 5.16 \times 10^{99}\,\mathrm{g\,m^{-3}}. \tag{A69}$$

We then obtain

$$m_* = \left(\frac{8\pi G\rho_P\hbar^2}{3c^4}\right)^{1/2} = \left(\frac{8\pi}{3}\right)^{1/2} M_P, \tag{A70}$$

where

$$M_P = \left(\frac{\hbar c}{G}\right)^{1/2} = 2.18 \times 10^{-5}\,\mathrm{g} \tag{A71}$$

is the Planck mass. We also obtain

$$r_* = \left(\frac{8\pi}{3}\right)^{1/2}\frac{2GM_P}{c^2} = 2\left(\frac{8\pi}{3}\right)^{1/2} l_P, \tag{A72}$$

where

$$l_P = \frac{GM_P}{c^2} = \left(\frac{G\hbar}{c^3}\right)^{1/2} = 1.62 \times 10^{-35}\,\mathrm{m} \tag{A73}$$

is the Planck length (the semi Schwarzschild radius of a particle of mass M_P). Finally,

$$\frac{\lambda_*}{8\pi} = 3\pi. \tag{A74}$$

Appendix D.5. The Late Universe

In the late universe, the characteristic density ρ_* is the cosmological density

$$\rho_* = \rho_\Lambda = \frac{\Lambda}{8\pi G} = 5.96 \times 10^{-24}\,\mathrm{g\,m^{-3}}. \tag{A75}$$

We then obtain

$$m_* = \left(\frac{8\pi G\rho_\Lambda\hbar^2}{3c^4}\right)^{1/2} = \frac{m_\Lambda}{\sqrt{3}}, \tag{A76}$$

where

$$m_\Lambda = \frac{\hbar\sqrt{\Lambda}}{c^2} = 2.08 \times 10^{-33}\,\mathrm{eV/c^2} \tag{A77}$$

is the cosmon mass.[32] We also obtain

$$r_* = \frac{2r_\Lambda}{\sqrt{3}}, \tag{A78}$$

where

$$r_\Lambda = \frac{Gm_\Lambda}{c^2} = \frac{G\hbar\sqrt{\Lambda}}{c^4} = 2.75 \times 10^{-96}\,\mathrm{m} \tag{A79}$$

is the cosmon radius (the semi Schwarzschild radius of a particle of mass m_Λ). Finally,

$$\frac{\lambda_*}{8\pi}=3\pi\frac{\rho_\Lambda}{\rho_P}=1.09\times 10^{-122}. \quad \text{(A80)}$$

Remark A3. *We note that*

$$\frac{m_\Lambda}{M_P}=\frac{r_\Lambda}{l_P}=\sqrt{\frac{8}{3}}\left(\frac{\lambda_*}{8\pi}\right)^{1/2}=\sqrt{8\pi}\left(\frac{\rho_\Lambda}{\rho_P}\right)^{1/2}=\left(\frac{G\hbar\Lambda}{c^5}\right)^{1/2}=1.70\times 10^{-61}. \quad \text{(A81)}$$

Appendix E. Scalar Field in the Presence of One Fluid in the Early Universe

In the main text, we have assumed in approach (B) that the early universe is dominated by the scalar field and that the X-fluids appear later in the evolution of the universe. In that case, there is first a phase of inflation followed by an α-era both due to the scalar field. As the scalar field is alone in the early universe, its potential is given by Equation (115). The α-era may correspond to a stiff matter era ($\alpha=1$) or to a radiation era ($\alpha=1/3$). If the α-era corresponds to a stiff matter era ($\alpha=1$), we have to introduce an X-fluid with $\alpha_X=1/3$ to represent the subsequent phase of radiation while this is not necessary if the α-era corresponds to a radiation era ($\alpha=1/3$).[33] Then, there is a matter era due to another X-fluid with $\alpha_X=0$, and finally a late inflation due to the constant pressure of the scalar field.

In this appendix, we consider the possibility that an X-fluid coexists with the scalar field since the beginning of the universe. In that case, the X-fluid dominates the scalar field for $a\to 0$ and leads to a big bang singularity where $\rho_X\propto a^{-3(1+\alpha_X)}$. This X-era is followed by an inflation era due to the scalar field when $\rho_X\ll\rho_P$. Then, the evolution of the universe is dominated by the X-fluid or by the α-fluid (different possibilities can arise depending on the values of the parameters).[34] In any case, we exit this period through a radiation era due to the scalar field (if $\alpha=1/3$) or to the X-fluid (if $\alpha_X=1/3$). Then, there is a matter era due to another X-fluid with $\alpha_X=0$, and finally a late inflation era due to the constant negative pressure of the scalar field.

Let us determine the potential of the scalar field in the presence of the X-fluid in the early universe. In that case, the scalar field has a quadratic equation of state $P_\phi=-(\alpha+1)(\rho_\phi^2/\rho_P)c^2+\alpha\rho_\phi c^2$. The scalar field potential is determined by Equations (96) and (97). For small a they reduce to

$$\phi(a)=\left(\frac{3c^2}{8\pi G}\right)^{1/2}\sqrt{1+\alpha}\int_0^a\left(\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}\right)^{1/2}x^{3(\alpha+1)/2}\sqrt{\frac{\Omega_{P,0}}{\frac{\Omega_{X,0}}{x^{3(1+\alpha_X)}}+\Omega_{P,0}}}\,\frac{dx}{x}, \quad \text{(A82)}$$

$$\frac{V(a)}{\rho_0c^2}=\Omega_{P,0}\left[1-\frac{1}{2}(\alpha+3)\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}a^{3(\alpha+1)}\right]. \quad \text{(A83)}$$

The scalar field is described by the index $\alpha=1/3$ (radiation) or $\alpha=1$ (stiff matter) and the X-fluid is described by the index $\alpha_X=1$ (stiff matter) or $\alpha_X=1/3$ (radiation), respectively. Unfortunately, neither (96) nor Equation (A82) can be calculated analytically. However, when the X-fluid dominates, Equation (A82) can be simplified further into

$$\phi(a)=\left(\frac{3c^2}{8\pi G}\right)^{1/2}\sqrt{1+\alpha}\frac{\Omega_{P,0}}{\sqrt{\Omega_{\alpha,0}\Omega_{X,0}}}\int_0^a x^{\frac{3}{2}(\alpha+\alpha_X+2)-1}\,dx, \quad \text{(A84)}$$

giving

$$\phi(a)=\left(\frac{3c^2}{8\pi G}\right)^{1/2}\frac{2}{3}\frac{\sqrt{1+\alpha}}{\alpha+\alpha_X+2}\frac{\Omega_{P,0}}{\sqrt{\Omega_{\alpha,0}\Omega_{X,0}}}a^{\frac{3}{2}(\alpha+\alpha_X+2)}. \quad \text{(A85)}$$

Substituting Equation (A85) into Equation (A83), we find that the scalar field potential is given in this regime by

$$V(\phi) = \rho_P c^2 - \frac{1}{2}\rho_P c^2(\alpha+3)\frac{\Omega_{P,0}}{\Omega_{\alpha,0}}\left[\left(\frac{8\pi G}{3c^2}\right)^{1/2}\frac{3}{2}\frac{\alpha+\alpha_X+2}{\sqrt{1+\alpha}}\frac{\sqrt{\Omega_{\alpha,0}\Omega_{X,0}}}{\Omega_{P,0}}\phi\right]^{\frac{2(\alpha+1)}{\alpha+\alpha_X+2}}. \quad \text{(A86)}$$

Notes

1 It is expected, but not observationally established, that the periods of acceleration are exponential, corresponding to a de Sitter stage.

2 The Planck density corresponds to a mass scale $\sim 10^{19}$ GeV/c^2. Actually, the scale of primordial inflation could be three orders of magnitude smaller, corresponding to the grand unification theory (GUT) scale $\sim 10^{16}$ GeV/c^2. In the following, for convenience, we shall identify the scale of primordial inflation to the Planck scale. If another scale turns out to be more relevant for our problem, we just have to replace the Planck density ρ_P by the corresponding density in the equations.

3 This can be viewed as an "initial conditions problem" or as a "fine tuning problem". Indeed, as dark matter and dark energy evolve at different rates with the universe expansion, conditions in the early universe must be set very carefully in order for them to be comparable to the ones existing today.

4 We stress that there is no initial singularity in our model [45,46] as the stiff matter era starts after the inflation era (de Sitter) during which the density is constant (finite).

5 Note that Basilakos et al. [58] did not derive the complete energy density evolution nor the complete scalar field potential obtained in Equations (86), (106), and (121) of [44], but only their asymptotic expressions in the early and late universe.

6 Apparently, their formula (40) in [59] contains a mistake. The potential associated with the inflation + stiff matter era should read like Equation (F.42) of [46].

7 This is because the other species enter into the Friedmann equation as additional components of the energy density and therefore alter the evolution of the scale factor and of the Hubble constant with respect to the free scalar field.

8 In this paper, we assume that the scalar field is real. The cosmological evolution of a complex scalar field is considered in [52,61,62]. On the other hand, in order to simplify the equations, we make the change of notation $\phi \to c\phi$.

9 We will see in Sections 4 and 5 how to relate the potential $V(\phi)$ of the scalar field to its equation of state $P_\phi(\rho)$.

10 As discussed in approach (A) of the Introduction, α may change with the density of the universe. Therefore, its value may depend on the epoch under consideration.

11 To avoid a spurious divergence of the energy density at $a = 0$, the matter component term $\Omega_{\mathrm{m},0}/a^3$ has to be introduced at a sufficiently late time, i.e., after the inflation era when $\rho \ll \rho_P$.

12 We assume a non-phantom universe $w_\phi > -1$. We also assume that the scalar field ϕ increases with the scale factor a so that $d\phi/da \geq 0$.

13 We have left the lower limit of integration undetermined as the expression of the integrand is only valid for sufficiently large values of a. The lower limit of integration has to be obtained by matching the solutions in the early and late universe.

14 For example, we can take $\alpha = 1$ (stiff matter) in the equation of state of the scalar field and add radiation and matter as additional species.

15 The case where a radiation era occurs before the inflation era, leading to a big-bang singularity, is considered in Appendix E.

16 This is a particular case of the general solution given in [42].

17 For $\alpha = 1$ (stiff matter), the scalar field potential is constant $V = \rho_\Lambda c^2$ (see Equation (135)).

18 A similar quantization rule was introduced by Wesson [68]. By using the dimensional reduction from higher dimensional relativity and by assuming that the Compton wavelength of a particle cannot take any value, he proposed that the mass is quantized according to the rule $m = (n_*\hbar/c^2)\sqrt{\Lambda/3} = n_* m_\Lambda^*$, where n_* is an integer (this differs from Equation (175) in that it involves n_* instead of $\sqrt{n_*}$). Hence, m_Λ^* is the minimum mass corresponding to the ground state $n_* = 1$. In our model, the mass of the scalar field associated with the ΛCDM model ($\alpha = 0$) is $(3/2)m_\Lambda^*$.

19 Some analytical solutions of the Friedmann equation involving two or more fluids with a linear equation of state (e.g., stiff matter, radiation, matter, or dark energy) are given in [45].

20 These equations can be obtained from Equations (86) and (87) by using the fact that both the scalar field and the X-fluids have a linear equation of state. They can also be recovered from Equations (94) and (95) by taking $\Omega_{\Lambda,0} = 0$ and $\Omega_{P,0} \to +\infty$.

21 The two potentials (149) and (199) are equivalent provided that the terms depending on X in Equation (199) are included in the constant of integration ϕ_* appearing in Equation (149).

22 In the framework of our model, this corresponds to the situation where the scalar field describes dark radiation ($\alpha = 1/3$) and the X-fluid describes normal radiation ($\alpha_X = 1/3$). This also corresponds to the situation where the scalar field describes dark matter ($\alpha = 0$) and the X-fluid describes baryonic matter ($\alpha_X = 0$).

23 This assumes $\rho_\Lambda \neq 0$. The case $\rho_\Lambda = 0$ leads to Equations (190)–(193).

24 This is basically why we do not find a good agreement with the observations.

25 The function $H(\phi)$ can be obtained from Equations (104), (123), and (213). However, in line with the Hamilton–Jacobi formalism, we shall proceed here the other way round and take Equation (229) as a starting point from which all the results of our model can be derived.

26 In this section, the time-dependent vacuum energy density $\rho_\Lambda(t)$ should not be confused with the constant cosmological density $\rho_\Lambda = 5.96 \times 10^{-24}\,\mathrm{g\,m^{-3}}$ appearing in Equation (33).

27 In the scalar field representation of the RVM [58,59], the scalar field is associated with the *total* density and pressure. This implies that the X-fluid is part of the scalar field. In other words, the vacuumon is not associated to the vacuum alone, but to the vacuum + X-fluid. We also recall that the X-fluid changes with the epoch considered (early or late universe).

28 We show in Appendix C.3 that the calculations in the RVM are equivalent to those of Appendix B.

29 In that case, ρ and P represent the energy density and the pressure of the scalar field denoted ρ_ϕ and P_ϕ in the main text.

30 This value is consistent with the condition $\alpha \leq 10^{-7}$ necessary to avoid the presence of oscillations in the matter power spectrum (see Appendix C in [52]).

31 The factor $2/3$ arises because we consider a real scalar field (see [83] for more details). In principle, Equation (A55) makes sense only if m is a positive real number. However, in order to treat all possible situations, we formally extend this formula to the case where m is imaginary ($m^2 < 0$) by taking its modulus $|m|$.

32 This mass scale is often interpreted as the smallest mass of the elementary particles predicted by string theory [84] or as the upper bound on the mass of the graviton [85]. The mass m_Λ also represents the quantum of mass in theories of extended supergravity [66]. The mass scale m_Λ is simply obtained by equating the Compton wavelength of the particle $\lambda_C = \hbar/mc$ with the Hubble radius $R_\Lambda = c/H_0$ (the typical size of the visible universe) giving $m_\Lambda = \hbar H_0/c^2 \sim \hbar\sqrt{\Lambda}/c^2$ (as $H_0^2 \sim G\rho_\Lambda \sim \Lambda$). The mass m_Λ corresponds to Wesson's [68] minimum mass interpreted as a quantum of dark energy (Wesson's maximum mass $M_\Lambda = (4/3)\pi\rho_0 R_\Lambda^3 = c^3/2GH_0 = 9.20 \times 10^{55}$ g is of the order of the mass of the universe). The mass scales M_Λ and m_Λ also appear in Refs. [51,86] and represent the mass of the visible universe and the minimum mass of the bosons. Böhmer and Harko [87] proposed to call the elementary particle of dark energy having the mass m_Λ the "cosmon". Cosmons were originally introduced by Peccei et al. [88] to name scalar fields that could dynamically adjust the cosmological constant to zero (see also [89–91]). The name cosmon was also used in a different context [92] to designate a very light scalar particle (dilaton) of mass $\sim 10^{-3}\,\mathrm{eV/c^2}$ which could mediate new macroscopic forces in the submillimeter range.

33 We could, however, introduce an X-fluid with $\alpha_X = 1/3$ to distinguish the radiation due to the scalar field (dark radiation) from the ordinary radiation due to photons or other relativistic particles.

34 For example, if $\alpha_X = 1$ and $\alpha = 1/3$ we generically have a stiff matter era (X) followed by an inflation era (SF) and a radiation era (SF). By contrast, if $\alpha_X = 1/3$ and $\alpha = 1$ we generically have a radiation era (X) followed by an inflation era (SF), a stiff matter era (SF) and a radiation era (X).

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