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Review

Covariant Derivative of Fermions and All That

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Abstract: We present detailed pedagogical derivation of covariant derivative of fermions and some related expressions, including commutator of covariant derivatives and energy-momentum tensor of a free Dirac field. On top of that, local conformal transformations for a Dirac fermion in curved spacetime are considered and we obtain the expression for the energy-momentum tensor on the cosmological background.

Keywords: covariant derivative; Dirac fermions; curved space-time; conformal symmetry

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1. Introduction

The formulation of classical fields on an arbitrary curved background can be seen as an important element of general relativity. On the other hand, this is one of the first steps in constructing the quantum field theory on a curved background (see, e.g., [1–7]). The main difficulty is related to that, in general, it is not possible to start from the usual consideration based on the representations of the Lorentz group, because the general Riemann space (one can call it metric background) has much less symmetries compared to the Minkowski space. One can expect to achieve a systematic construction of the fields on de Sitter or anti-de Sitter spaces (see, e.g., [8] and references therein) but this is not what we actually need in both classical and semiclassical gravity, since the metric backgrounds of our interest are much more general than these two examples.

The simplest possible solution for formulating classical fields on an arbitrary metric background consists of the covariant generalization of the flat space-time expressions. This means, one has to start from the action of the field in the locally flat reference frame and construct the covariant expression which reduce to the flat-space action in this special frame. Such a procedure is sometimes called covariantization, it is rather simple for scalars and vectors, and essentially more involved for spinor fields. In what follows, we consider the main steps in formulating such a covariant generalization of the flat spacetime action for a Dirac fermion. The action, in the flat case, is

$$S_f \;=\; i\int d^4x\; \bar{\psi}\big(\gamma^a\partial_a - im\big)\psi\,, \tag{1}$$

where $a = 0,1,2,3$ are Minkowski-space indices. Along with this main problem, we shall present necessary details of the tetrad formalism, calculate dynamical energy-momentum tensor of spinor field and discuss the conformal properties of this field. The last step will be derivation of the energy-momentum tensor of spinor field on the cosmological background, in the case when spinor depends only on time and not on the space coordinates. All these calculations and considerations are pretty well-known and we do not pretend at all to say a new word in this field. The purpose of the work is mainly pedagogical, namely we intend to present the detailed derivations which can be useful for the one who wants to learn the subject.

The last note is that the original version of this manuscript was supposed to be part of the book [7]. However, in this book another (albeit equivalent) scheme of constructing fermion fields in curved space was chosen, based on the application of the group theory methods. The approach used in the mentioned book utilizes the fact that the spinor connection, which will be defined below, is a connection for a local Lorentz gauge group. This group is present also in curved background and one can use it to derive both spinor connection and the commutator of covariant derivatives. The interested reader may find it useful to make a comparison between the two approaches.

The paper is organized as follows. Section 2 discusses the tetrad (also called vierbein) formalism, including the detailed consideration of what means the covariant derivative of the tetrad. In Section 3 we describe the construction of the covariant derivative of Dirac fermion. Section 4 is deriving the commutator of covariant derivatives and its connection to the Riemann tensor. Section 5 describes local conformal transformations in the action of spinors in curved spacetime. In this section, we consider the n-dimensional spacetime, different from the rest of the paper dealing with $n = 4$. In Section 6 we present a detailed derivation of the action of Dirac fermion for a weak gravitational field, obtain the energy-momentum tensor $T_{\mu\nu}$ for the Dirac field and show the connection of its trace with the conformal symmetry. Section 7 shows the calculation of $T_{\mu\nu}$ on the cosmological background and explains the problems of taking the massless limit in the naive way. Finally, in Section 8 we draw our conclusions. The notations include the signature $\eta_{\alpha\beta} = \mathrm{diag}\,(+\,-\,-\,-)$, the definition of the Riemann tensor

$$R^{\lambda}{}_{\tau\alpha\beta} = \partial_\alpha \Gamma^{\lambda}{}_{\tau\beta} - \partial_\beta \Gamma^{\lambda}{}_{\tau\alpha} + \Gamma^{\lambda}{}_{\gamma\alpha}\Gamma^{\gamma}{}_{\tau\beta} - \Gamma^{\lambda}{}_{\gamma\beta}\Gamma^{\gamma}{}_{\tau\alpha}, \tag{2}$$

Ricci tensor $R^{\alpha}{}_{\mu\alpha\nu} = R_{\mu\nu}$, and its trace $R = R_{\mu\nu}g^{\mu\nu}$, i.e., the Ricci scalar. Finally, our notations for symmetrization can be easily understood from the following two examples:

$$A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji}) \qquad \text{and} \qquad B_{(i|k|j)} = \frac{1}{2}(B_{ikj} + B_{jki}). \tag{3}$$

2. Tetrad Formalism and Covariant Derivatives

It is well-known that general relativity is a theory of the metric field. At the same time, in many cases, it proves useful to use other variables for describing the gravitational field. In particular, for defining the covariant derivative of a fermion, one need an object called tetrad. The German name *vierbein* is also frequently used. The names indicate that the object is four-dimensional, but the formalism described below can be easily generalized to the space of any dimension and also does not depend on the signature of the metric. Anyway, for the sake of definiteness we will refer to the four-dimensional space-time with the $M_{1,3}$ signature.

Let us start by defining the tetrad formalism on the $M_{1,3}$ Riemann space. Locally, at point P, one can introduce the flat metric η_{ab}. This means that the vector basis includes four orthonormal vectors $\mathbf{e}_a$, such that $\mathbf{e}_a \cdot \mathbf{e}_b = \eta_{ab}$. Here $\mathbf{e}_a$ are 4-dimensional vectors in the tangent space to the manifold of our interest at the point P. Furthermore, X^a are local coordinates on $M_{1,3}$, which can be also seen as coordinates in the tangent space in a close vicinity of the point P. With respect to the general coordinates x^μ, we can write $\mathbf{e}_a = e_a^\mu \mathbf{e}_\mu$, where $\mathbf{e}_\mu$ is a corresponding local basis and e_a^μ are transition coefficients from one basis to another. Hence

$$\begin{aligned}
e_a^\mu &= \frac{\partial x^\mu}{\partial X^a}, \qquad e_\nu^a = \frac{\partial X^a}{\partial x^\nu}, \quad \text{and therefore} \\
e_a^\mu e^{a\nu} &= e_a^\mu e_b^\nu \eta^{ab} = g^{\mu\nu}, \\
e_\mu^a e_{a\nu} &= g_{\mu\nu}, \quad e_\mu^a e_b^\mu = \delta_b^a, \quad e_\mu^a e_a^\alpha = \delta_\mu^\alpha.
\end{aligned} \tag{4}$$

We assume that the Greek indices $\mu, \nu, \ldots$ are raised and lowered by the covariant metrics $g_{\mu\nu}$ and $g^{\mu\nu}$, while the indices $a, b, c \ldots$ are raised and lowered by η_{ab} and η^{ab}.

The transition from one type or indices to another is done using tetrad, since it is just a change of basis. Taking this into account, in some cases, we admit objects with mixed indices, e.g.,

$$T^{a}_{\cdot\mu} \;=\; T^{\nu}_{\cdot\mu}\, e^{a}_{\nu} \;=\; T^{a}_{\cdot b}\, e^{b}_{\mu}\,. \tag{5}$$

Formula (4) show that the descriptions in terms of the metric and in terms of the tetrad are equivalent. The same concerns also the invariant volume of integration. It is easy to show that

$$\det e^{a}_{\mu} = \sqrt{|g|}\,, \qquad g = \det g_{\mu\nu}\,, \tag{6}$$

because

$$g = \det\left(e^{a}_{\mu}e^{b}_{\nu}\eta_{ab}\right) \;=\; \left(\det e^{a}_{\mu}\right)^{2}\cdot\,\det\eta_{ab}\,. \tag{7}$$

Indeed, the local frame coordinates X^{a} are not unique, since even for the locally Minkowski metric one is allowed to make transformations $\mathbf{e}'_{a} = \Lambda^{b}_{a'}e_{b}$, such that[1]

$$\mathbf{e}'_{a}\cdot\mathbf{e}'_{c} \;=\; \Lambda^{b}_{a'}\Lambda^{d}_{c'}\,\mathbf{e}_{b}\cdot\mathbf{e}_{d} \;=\; \eta_{bd}\,\Lambda^{b}_{a'}\Lambda^{d}_{c'} \;=\; \eta_{ac}\,. \tag{8}$$

In the flat space-time case, such frame rotations with constant $\Lambda^{b}_{a'}$ form Lorentz group. The situation changes in curved space-time manifolds, because then $\Lambda^{a}_{b'}$ depends on the point P. For example, we can consider an infinitesimal transformation

$$\Lambda^{a}_{b'} \;=\; \delta^{a}_{b'} + \Omega^{a}_{b'}(x)\,, \tag{9}$$

which preserves the form of the tangent-space metric. Then (8) gives

$$(\delta^{a}_{b'} + \Omega^{a}_{b'})(\delta^{d}_{c'} + \Omega^{d}_{c'})\,\eta_{ad} \;=\; \eta_{bc}\,. \tag{10}$$

It is an easy exercise to show that in this case, $\Omega_{ab}(x) \;=\; -\,\Omega_{ba}(x)$.

Let us consider the covariant derivative in tetrad formalism and introduce the notion of spin connection. In the metric formalism, the covariant derivative of a vector V^{μ} is

$$\nabla_{\nu}V^{\mu} \;=\; \partial_{\nu}V^{\mu} + \Gamma^{\mu}_{\lambda\nu}V^{\lambda}\,, \tag{11}$$

where the affine connection is given by the standard Christoffel symbol expression,

$$\Gamma^{\lambda}_{\mu\nu} \;=\; \frac{1}{2}\,g^{\lambda\tau}\left(\partial_{\mu}g_{\tau\nu} + \partial_{\nu}g_{\tau\mu} - \partial_{\tau}g_{\mu\nu}\right). \tag{12}$$

Our purpose is to construct a version of covariant derivative for the objects such as tetrad, which have local Lorentz indices. Obviously, this is a non-trivial task, because tetrad is not a tensor and therefore its covariant derivative is not a conventional object. Hence our constructions will necessary involve certain amount of *ad hoc* assumptions. In what follows, we shall see that there is a scheme of consistent consideration of the problem.

It is easy to understand that (different from the flat space-time) in the Lorentz frame X^{a} the covariant derivative of the same vector cannot be just equal to $\partial_{a}V^{b}$, because otherwise we come to the contradiction. The details will become clear below, let us just say that qualitatively the reason is that any shift from the point P means the change from one tangent space to another and hence deriving the corresponding difference between two values of the vector requires an additional definition. Let us suppose that the desired covariant derivative is a linear operator and satisfies Leibniz rule. The general expression satisfying these conditions is

$$\nabla_{a}V^{b} \;=\; \partial_{a}V^{b} + \tilde{\omega}^{a}_{\cdot\,ca}V^{c}\,, \tag{13}$$

where $\tilde{\omega}^{a}_{\cdot\,ca}$ are some unknown coefficients. Later on we will discuss their relation to the spin connection. In the flat space-time $\tilde{\omega}^{a}_{\cdot\,ca} = 0$.

We request that the vector components satisfy $V^\mu = e^\mu_a V^a$ and the tensor components satisfy $\nabla_\nu V^\mu = e^a_\nu e^\mu_b \nabla_a V^b$. Then, according to (11) and (13), we have

$$\begin{aligned} \nabla_\lambda V^\mu &= \partial_\lambda V^\mu + \Gamma^\mu_{\tau\lambda} V^b = e^a_\lambda e^\mu_b \nabla_a V^b = e^a_\lambda e^\mu_b \left(\partial_a V^b + \tilde{\omega}^{b}_{\cdot\,ca} V^c\right) \\ &= \delta^\mu_\tau \partial_\lambda V^\tau + V^\tau e^a_\lambda e^\mu_b \partial_a e^b_\tau + e^\mu_b e^c_b e^c_\tau V^\tau \tilde{\omega}^{b}_{\cdot\,c\lambda} \\ &= \partial_\lambda V^\mu + V^\tau (e^\mu_b \partial_\lambda e^b_\tau + e^\mu_b e^c_\tau \tilde{\omega}^{b}_{\cdot\,c\lambda})\,. \end{aligned} \tag{14}$$

Therefore, we arrive at the equation for $\tilde{\omega}^{b}_{\cdot\,c\lambda}$,

$$\Gamma^\mu_{\tau\lambda} = e^\mu_b\, \partial_\lambda e^b_\tau + e^\mu_d e^c_\tau\, \tilde{\omega}^{d}_{\cdot\,c\lambda}\,. \tag{15}$$

Multiplying the last equation by $e^a_\mu e^{\tau b}$, we arrive at the following solution:

$$\tilde{\omega}^{ab}_{\cdot\cdot\,\lambda} = e^a_\mu e^{\tau b} \Gamma^\mu_{\tau\lambda} - e^{\tau b} \partial_\lambda e^a_\tau\,. \tag{16}$$

One of the important features of the last expression is antisymmetry in (a, b), that means $\tilde{\omega}^{ab}_{\cdot\cdot\,\mu} = -\tilde{\omega}^{ba}_{\cdot\cdot\,\mu}$. In order to see this, consider the sum

$$\begin{aligned} \tilde{\omega}^{ab}_{\cdot\cdot\,\mu} + \tilde{\omega}^{ba}_{\cdot\cdot\,\mu} &= e^a_\nu e^{\lambda b} \Gamma^\nu_{\lambda\mu} + e^b_\nu e^{\lambda a} \Gamma^\nu_{\lambda\mu} - e^{\lambda b} \partial_\mu e^a_\lambda - e^{\lambda a} \partial_\mu e^b_\lambda \\ &= \frac{1}{2}\left(\partial_\lambda g_{\nu\mu} + \partial_\mu g_{\nu\lambda} - \partial_\nu g_{\mu\lambda}\right) \cdot \left(e^{a\nu} e^{\lambda b} + e^{a\lambda} e^{\nu b}\right) - e^{b\lambda} \partial_\mu e^a_\lambda - e^{a\lambda} \partial_\mu e^b_\lambda \\ &= e^{a\nu} e^{\lambda b} e^c_\nu \partial_\mu e_{c\lambda} + e^{a\nu} e^{\lambda b} e_{c\lambda} \partial_\mu e^c_\nu - e^{\lambda b} \partial_\mu e^a_\lambda - e^{a\lambda} \partial_\mu e^b_\lambda = 0\,. \end{aligned}$$

Another observation follows from Equations (11) and (13). We can rewrite (11) in the form

$$\nabla_\nu (V^a e^\mu_a) = \partial_\mu (V^a e^\mu_a) + \Gamma^\mu_{\lambda\nu}\, V^a e^\lambda_a \tag{17}$$

and construct the "covariant derivative" of the tetrad $\nabla_\nu e^\mu_a$ in a way consistent with the Leibnitz rule and with both (17) and (7). Then

$$\begin{aligned} \nabla_\nu V^\mu &= \partial_\nu V^\mu + \Gamma^\mu_{\lambda\nu} V^\lambda = \nabla_\nu (V^b e^\mu_b) = e^a_\nu \nabla_a (V^b e^\mu_b) \\ &= e^a_\nu e^\mu_b \nabla_a V^b + e^a_\nu V^b \nabla_a e^\mu_b = e^a_\nu e^\mu_b (\partial_a V^b + \tilde{\omega}^{b}_{\cdot\,ca} V^c) + e^a_\nu V^b \nabla_a e^\mu_b\,. \end{aligned} \tag{18}$$

At the same time, we have

$$\partial_\nu V^\mu = e^a_\nu \partial_a (V^b e^\mu_b) = e^a_\nu e^\mu_b \partial_a V^b + V^b e^a_\nu \partial_a e^\mu_b\,. \tag{19}$$

By combining (18) and (19) one can easily obtain the relation

$$V^\lambda e^c_\lambda\, \tilde{\omega}^{b}_{\cdot ca} e^a_\nu e^\mu_b + e^a_\nu e^b V^\lambda \nabla_a e^\mu_b = V^\lambda e^b_\lambda e^a_\nu \partial_a e^\mu_b + V^\lambda \Gamma^\mu_{\lambda\nu}\,. \tag{20}$$

Since this relation should be true for all V^λ, we arrive at

$$e^a_\nu e^b_\lambda \nabla_a e^\mu_b = e^a_\nu e^b_\lambda\, \partial_a e^\mu_b + \Gamma^\mu_{\lambda\nu} - e^a_\nu e^c_\lambda e^\mu_b\, \tilde{\omega}^{b}_{\cdot\,ca}\,. \tag{21}$$

Multiplying this equation by $e^\nu_d e^\lambda_e$ one can get (changing some indices)

$$\nabla_a e^\mu_b = \partial_a e^\mu_b + \Gamma^\mu_{\lambda\nu} e^\lambda_b e^\nu_a - \tilde{\omega}^{cb}_{\cdot\cdot\,\nu} e^\nu_a e^\mu_c\,. \tag{22}$$

Since this is a tensor quantity, we can multiply it by e^a_τ and obtain the final result

$$\nabla_\tau e^\mu_b = e^a_\tau \partial_a e^\mu_b + \Gamma^\mu_{\lambda\tau} e^\lambda_b - \tilde{\omega}^{c}_{\cdot\,b\tau}\, e^\mu_c\,. \tag{23}$$

One can show that, also[2],

$$\nabla_\tau e_{\mu a} \;=\; e^b_\tau \partial_b e_{\mu a} - \Gamma^\lambda_{\mu\tau} e^\lambda_a - \tilde{\omega}_{a\cdot\tau}^{\;\;b}\, e_{\mu b}\,. \tag{24}$$

Finally, by direct replacement of (16) one can easily check that both covariant derivatives vanish, $\nabla_\tau e^\mu_b = 0$ and $\nabla_\tau e_{\mu a} = 0$. This fact is nothing else but a direct consequence of the metricity property of covariant derivative, because

$$\nabla_\tau g_{\mu\nu} \;=\; \nabla_\tau \big(e^a_\mu e^b_\nu \eta_{ab} \big) \;=\; 2\eta_{ab}\, e^a_\mu \cdot \nabla_\tau e^b_\nu \;=\; 0\,. \tag{25}$$

3. Covariant Derivative of Dirac Fermion

Let us construct a covariant generalization of the action of a spinor field (1). The desirable expression has the form

$$S_f \;=\; i\int d^4x \sqrt{-g}\,\bar{\psi}\big(\gamma^\mu \nabla_\mu - im\big)\psi\,. \tag{26}$$

Since the part related to the integration volume element is relatively simple, what has to be done is to generalize the gamma-matrices and to construct the covariant derivative.

Let us use tetrad and define the curved-space gamma-matrices as $\gamma^\mu = e^\mu_a \gamma^a$. Now the indices of the the new gamma-matrices are lowered and raised by means of the covariant metrics $g_{\mu\nu}$ and $g^{\mu\nu}$. It is easy to see that the new gamma-matrices satisfy the curved-space version of Clifford algebra,

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu \;=\; g_{\mu\nu}\,. \tag{27}$$

The next observation is that since $\gamma^\mu = e^\mu_a \gamma^a$ and γ^a are constant matrices (see Appendix A for the details), the vanishing covariant derivative $\nabla_\alpha e^\mu_b = 0$ means $\nabla_\alpha \gamma^\mu = 0$.

The construction of covariant derivative is a little bit more involved. Let us define

$$\nabla_\mu \psi \;=\; \partial_\mu \psi + \frac{i}{2}\,\omega_{\mu\cdot\cdot}^{\;\,ab}\,\sigma_{ab}\,\psi\,, \quad \text{where} \quad \sigma_{ab} = \frac{i}{2}\big(\gamma_a\gamma_b - \gamma_b\gamma_a\big)\,. \tag{28}$$

One can regard (28) as a hypothesis which can be proved or disproved. $\omega_{\mu\cdot\cdot}^{\;\,ab}$ are antisymmetric coefficients of the *spinor connection*, which have to be found from the requirement of covariance. Taking a conjugate of (28), we arrive at

$$\nabla_\mu \bar{\psi} \;=\; \partial_\mu \bar{\psi} - \frac{i}{2}\,\omega_{\mu\cdot\cdot}^{\;\,ab}\,\bar{\psi}\,\sigma_{ab}\,. \tag{29}$$

Using (28) and (29), it is easy to verify an identity

$$\nabla_\alpha\big(\bar{\psi}\psi\big) = \partial_\alpha\big(\bar{\psi}\psi\big), \tag{30}$$

which is a natural result for a scalar combination $\bar{\psi}\psi$.

In order to obtain the equation for spinor connection, consider the relation for a composite vector

$$\nabla_\mu(\bar{\psi}\gamma^\alpha\psi) \;=\; \partial_\mu(\bar{\psi}\gamma^\alpha\psi) + \Gamma^\alpha_{\nu\mu}\bar{\psi}\gamma^\nu\psi\,. \tag{31}$$

A simple calculation using Leibniz rule yields

$$\begin{aligned}
&\nabla_\mu\bar{\psi}\cdot\gamma^\alpha\psi \,+\, \bar{\psi}(\nabla_\mu\gamma^\alpha)\psi + \bar{\psi}\gamma^\alpha\nabla_\mu\psi \;=\; \partial_\mu\bar{\psi}\cdot\gamma^\alpha\psi + \bar{\psi}\partial_\mu\gamma^\alpha\psi + \bar{\psi}\gamma^\alpha\partial_\mu\psi \\
&+\Gamma^\alpha_{\nu\mu}\bar{\psi}\gamma^\nu\psi \,-\, \frac{i}{2}\omega_{\mu\cdot\cdot}^{\;\,ab}\,\bar{\psi}\,\sigma_{ab}\gamma^\alpha\psi + \frac{i}{2}\omega_{\mu\cdot\cdot}^{\;\,ab}\bar{\psi}\gamma^\alpha\sigma_{ab}\psi \;=\; \bar{\psi}\gamma^a(\partial_\mu e^\alpha_a)\psi + \bar{\psi}\Gamma^\alpha_{\nu\mu}\gamma^\nu\psi\,.
\end{aligned}$$

As this equality should be valid for any field ψ, we arrive at

$$-\frac{1}{4}\,\omega_{\mu\,\cdot\cdot}^{\;\;ab}\,e^{\alpha}_{c}\,\big[\gamma^{c}(\gamma_a\gamma_b-\gamma_b\gamma_a)-(\gamma_a\gamma_b-\gamma_b\gamma_a)\gamma^{c}\big]\;=\;\gamma^{c}\big(e^{\nu}_{c}\Gamma^{\alpha}_{\mu\nu}+\partial_\mu e^{\alpha}_{c}\big)\,. \tag{32}$$

By means of the well-known relations

$$\begin{aligned}\gamma^{c}\gamma_a\gamma_b &= 2\delta^{c}_{a}\gamma_b-\gamma_a\gamma^{c}\gamma_b\,,\\ 2\delta^{c}_{a}\gamma_b-2\delta^{c}_{b}\gamma_a+\gamma_a\gamma_b\gamma^{c}-\gamma^{c}\gamma_b\gamma_a &= -2\delta^{c}_{b}\gamma_a+2\gamma_b\delta^{c}_{a}-\gamma_b\gamma_a\gamma^{c}\end{aligned}$$

we get

$$\omega_{\mu\,\cdot\cdot}^{\;\;ab}\,\big(e^{\alpha}_{b}\gamma_a-e^{\alpha}_{a}\gamma_b\big)\;=\;\gamma^{c}\big(e^{\tau}_{c}\Gamma^{\alpha}_{\tau\mu}+\partial_\mu e^{\alpha}_{c}\big)\,. \tag{33}$$

The last equation can be easily solved owing to the antisymmetry of $\omega_{\mu\,\cdot\cdot}^{\;\;ab}$, the result is

$$\begin{aligned}\omega_{\mu\,\cdot\cdot}^{\;\;ab} &= -\frac{1}{2}\,\tilde{\omega}^{ab}_{\cdot\cdot\;\mu}=\frac{1}{2}\,\big(e^{b}_{\tau}e^{\lambda a}\Gamma^{\tau}_{\lambda\mu}-e^{\lambda a}\partial_\mu e^{b}_{\lambda}\big)\\ &= \frac{1}{4}\,\big(e^{b}_{\tau}e^{\lambda a}-e^{a}_{\tau}e^{\lambda b}\big)\Gamma^{\tau}_{\lambda\mu}+\frac{1}{4}\big(e^{\lambda b}\partial_\mu e^{a}_{\lambda}-e^{\lambda a}\partial_\mu e^{b}_{\lambda}\big)\,.\end{aligned} \tag{34}$$

Here we established the relation with Equation (16) and presented both compact and explicitly antisymmetric forms of the spinor connection. Finally, the expression for the covariant derivative is (28) with the spinor connection (34). At this point, we can say that the construction of the action (26) is complete. In the next sections, we will derive and discuss some additional relevant formulas and features.

4. Commutator of Covariant Derivatives

The next necessary step is to derive the commutator of two covariant derivatives acting on a Dirac spinor. Consider

$$\begin{aligned}\nabla_\nu\nabla_\mu\Psi &= \nabla_\nu\big(\partial_\mu\Psi+\frac{i}{2}\,\omega_{\mu\,\cdot\cdot}^{\;\;ab}\sigma_{ab}\Psi\big)\\ &= \partial_\nu(\nabla_\mu\Psi)+\frac{i}{2}\,\omega_{\nu\,\cdot\cdot}^{\;\;ab}\,\sigma_{ab}\,\nabla_\mu\Psi-\Gamma^{\lambda}_{\mu\nu}\nabla_\lambda\Psi\\ &= \partial_\nu\partial_\mu\Psi+\frac{i}{2}\,\partial_\nu\omega_{\mu\,\cdot\cdot}^{\;\;ab}\,\sigma_{ab}\Psi+\frac{i}{2}\,\omega_{\mu\,\cdot\cdot}^{\;\;ab}\,\sigma_{ab}\,\partial_\nu\Psi+\frac{i}{2}\,\omega^{ab}_{\nu}\,\sigma_{ab}\,\partial_\mu\Psi\\ &- \frac{1}{4}\,\omega_{\nu\,\cdot\cdot}^{\;\;ab}\sigma_{ab}\,\omega_{\mu\,\cdot\cdot}^{\;\;cd}\sigma_{cd}\Psi-\Gamma^{\lambda}_{\mu\nu}\partial_\lambda\Psi-\frac{i}{2}\,\Gamma^{\lambda}_{\mu\nu}\,\omega_{\lambda\,\cdot\cdot}^{\;\;ab}\,\sigma_{ab}\Psi\,.\end{aligned} \tag{35}$$

By using this expression and the relation

$$\gamma_a\gamma_b\gamma_c\gamma_d\;=\;2\eta_{bc}\gamma_a\gamma_d-2\eta_{ac}\gamma_b\gamma_d+2\gamma_c\gamma_a\eta_{bd}-2\gamma_c\gamma_b\eta_{ad}+\gamma_c\gamma_d\gamma_a\gamma_b, \tag{36}$$

it is not difficult to obtain the commutator

$$[\nabla_\mu,\nabla_\nu]\psi\;=\;\nabla_\mu\nabla_\nu\psi-\nabla_\mu\nabla_\nu\psi=-\frac{1}{4}\,R_{\mu\nu\,\cdot\cdot}^{\;\;\;ab}\,\gamma_a\gamma_b\,\psi\,, \tag{37}$$

where

$$R_{\mu\nu\,\cdot\cdot}^{\;\;\;ab}\;=\;\partial_\mu\omega_{\nu\,\cdot\cdot}^{\;\;ab}-\partial_\nu\omega_{\mu\,\cdot\cdot}^{\;\;ab}+\omega_{\mu\,\cdot\cdot}^{\;\;ac}\,\omega_{\nu c\,\cdot}^{\;\;\;b}-\omega_{\nu\,\cdot\cdot}^{\;\;ac}\,\omega_{\mu c\,\cdot}^{\;\;\;b} \tag{38}$$

is a new notation, which becomes clear if we prove that there is a direct relation with the Riemann tensor,

$$R_{\mu\nu ab}\;=\;R_{\mu\nu\rho\sigma}\,e^{\rho}_{a}\,e^{\sigma}_{b}\,. \tag{39}$$

The proof consists of a direct substitution of Equation (16) with the definition (34), in the expression (38) and some algebra which we leave as an exercise to the interested reader.

Using (37) and (39), one can write a useful covariant expression for the commutator,

$$[\nabla_\mu, \nabla_\nu]\psi \;=\; -\frac{1}{4}\, R_{\mu\nu\rho\sigma}\, \gamma^\rho \gamma^\sigma\, \psi\,. \tag{40}$$

One of the applications of the previous expression is the possibility of a “doubling” of the covariant Dirac Equation (26). Taking the product

$$\big(\gamma^\mu \nabla_\mu - im\big)\,\big(\gamma^\nu \nabla_\nu + im\big) \;=\; \gamma^\mu \nabla_\mu\, \gamma^\nu \nabla_\nu + m^2\,, \tag{41}$$

one can obtain

$$\begin{aligned} \gamma^\mu \nabla_\mu\, \gamma^\nu \nabla_\nu &= \frac{1}{2}\, \gamma^\mu \gamma^\nu \big(\nabla_\mu \nabla_\nu + \nabla_\mu \nabla_\nu\big) + \frac{1}{2}\, \gamma^\mu \gamma^\nu \big(\nabla_\mu \nabla_\nu - \nabla_\mu \nabla_\nu\big) \\ &= \frac{1}{2}\, \big(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu\big) \nabla_\mu \nabla_\nu + \frac{1}{2}\, \big(\nabla_\mu \nabla_\nu - \nabla_\mu \nabla_\nu\big) \\ &= g^{\mu\nu}\, \nabla_\mu \nabla_\nu + \frac{1}{8}\, R_{\mu\nu\rho\sigma}\, \gamma^\rho \gamma^\sigma \gamma^\mu \gamma^\nu \;=\; \Box - \frac{1}{4}\, R\,. \end{aligned} \tag{42}$$

In the last step we used an identity

$$R_{\mu\nu\rho\sigma}\, \gamma^\rho \gamma^\sigma \gamma^\mu \gamma^\nu \;=\; -2R\,, \tag{43}$$

which can be easily proved by using covariant version of relation (36) and the algebraic properties of the Riemann tensor. We leave the verification of this identity as one more exercise for the interested reader.

The two relations, i.e., (40) and the Lichnerowicz formula (42) play important roles in differential geometry and quantum field theory in curved space.

5. Local Conformal Transformation

It is important to consider the conformal transformation of the metric and fermion field. For the sake of generality, we consider the theory in n spacetime (or Euclidean space, as there is no difference at this level). The action is a direct generalization of (26),

$$S_f \;=\; i \int d^n x \sqrt{-g}\, \bar{\psi} \big(\gamma^\mu \nabla_\mu - im\big) \psi\,. \tag{44}$$

The definition of the gamma matrices and covariant derivatives do not change and the unique difference comes from the algebra of the gamma matrices. For the *global* conformal transformation

$$g_{\mu\nu} \to g_{\mu\nu}\, e^{2\lambda}\,, \qquad \psi \to \psi\, e^{d_\psi \lambda}\,, \qquad \bar{\psi} \to \bar{\psi}\, e^{d_\psi \lambda}\,, \qquad \lambda = const, \tag{45}$$

we easily find that the conformal weight of the spinor field that provides the invariance of the *massless* part of the action (44), is $d_\psi = \frac{1-n}{2}$. In what follows, we assume this value. For obvious reasons, the conformal transformation for the conjugated fermion $\bar{\psi}$ has the same conformal weight as in the case of ψ.

Using the global symmetry as a hint, consider the *local* conformal transformation

$$g_{\mu\nu} \;=\; \bar{g}_{\mu\nu}\, e^{2\sigma}\,, \qquad \psi \;=\; \psi_*\, e^{d_\psi \sigma}\,, \qquad \bar{\psi} \;=\; \bar{\psi}_*\, e^{d_\psi \sigma}\,. \tag{46}$$

In what follows we will omit space-time arguments, but always assume that $\sigma = \sigma(x)$. Furthermore, all metric-dependent quantities with bars are constructed using the fiducial metric $\bar{g}_{\mu\nu}$. Also, we use compact notation for the partial derivative, e.g., $\sigma_{,\lambda} = \partial_\lambda \sigma$. Let us note that many formulas related to local conformal transformations of curvature tensors, their contractions, etc, can be found in [10].

The transformation of the elements of the action (44) provides

$$g^{\mu\nu} = \bar{g}^{\mu\nu}\, e^{-2\sigma}\,, \quad \sqrt{-\bar{g}} = \sqrt{-g}\, e^{n\sigma}\,, \quad e^{\mu}_{a} = \bar{e}^{\mu}_{a}\, e^{-\sigma}\,, \quad e^{b}_{\mu} = \bar{e}^{b}_{\mu}\, e^{\sigma} \tag{47}$$

and, consequently,

$$\gamma^{\mu} = \bar{\gamma}^{\mu}\, e^{-\sigma}\,, \quad \gamma_{\mu} = \bar{\gamma}_{\mu}\, e^{\sigma}\,. \tag{48}$$

One can use this set of formulas to check the relation $d_{\psi} = \frac{1-n}{2}$. For the Christoffel symbols and spinor connection, Equations (12) and (34) give

$$\begin{aligned} \Gamma^{\lambda}_{\alpha\beta} &= \bar{\Gamma}^{\lambda}_{\alpha\beta} + \delta^{\lambda}_{\alpha}\sigma_{,\beta} + \sigma_{,\alpha}\delta^{\lambda}_{\beta} - \sigma_{,\tau}\bar{g}^{\lambda\tau}\bar{g}_{\alpha\beta}\,, \\ \omega_{\mu\,\cdot\cdot}^{\;\;ab} &= \bar{\omega}_{\mu\,\cdot\cdot}^{\;\;ab} + \big(\bar{e}^{a}_{\mu}\,\bar{e}^{\lambda b} - \bar{e}^{b}_{\mu}\,\bar{e}^{\lambda a}\big)\,\sigma_{,\lambda}\,. \end{aligned} \tag{49}$$

For the contraction, we get

$$\gamma^{\mu}\nabla_{\mu}\psi = \bar{\gamma}^{\mu}e^{-\sigma}\Big\{\partial_{\mu}\psi - \frac{i}{4}\,\bar{\omega}_{\mu\,\cdot\cdot}^{\;\;ab}\,\sigma_{ab}\psi - \frac{i}{4}\,\big(\bar{e}^{a}_{\mu}\bar{e}^{\lambda b} - \bar{e}^{b}_{\mu}\bar{e}^{\lambda a}\big)\sigma_{,\lambda}\cdot\frac{i}{2}\,(\gamma_{a}\gamma_{b} - \gamma_{b}\gamma_{a})\psi\Big\}$$

$$= \bar{\gamma}^{\mu}\bar{e}^{-\sigma}\Big\{\bar{\nabla}_{\mu}\psi + \frac{1}{4}(\bar{\gamma}_{\mu}\bar{\gamma}^{\lambda} - \bar{\gamma}^{\lambda}\bar{\gamma}_{\mu})\,\sigma_{,\lambda}\psi\Big\} \tag{50}$$

$$= e^{-\sigma}\Big\{\bar{\gamma}^{\mu}\bar{\nabla}_{\mu}\psi + \frac{n-1}{2}\,\bar{\gamma}^{\lambda}\,\sigma_{,\lambda}\psi\Big\} = e^{-\frac{n-3}{2}\sigma}\,\bar{\gamma}^{\mu}\bar{\nabla}_{\mu}\psi_{*}\,. \tag{51}$$

Substituting (51), (46), and (47) in (44), we arrive at the transformation law

$$i\int d^4x\sqrt{-g}\,\bar{\psi}\big(\gamma^{\mu}\nabla_{\mu} - im\big)\psi = i\int d^4x\sqrt{-\bar{g}}\,\bar{\psi}_{*}\big(\bar{\gamma}^{\mu}\bar{\nabla}_{\mu} - im\,e^{\sigma}\big)\psi_{*}\,. \tag{52}$$

Thus, the massless action is invariant under the local conformal transformation.

One can restore local conformal symmetry in the general action (26), by replacing the mass by a scalar field φ, i.e., forming the Yukawa interaction term. The detailed consideration of this issue is beyond the scope of the present contribution and one has to consult, e.g., [7].

6. Energy-Momentum Tensor for the Dirac Field

Consider the derivation of the *dynamical*[3] $T_{\mu\nu}$ for the Dirac field. The definition of this tensor is

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\,\frac{\delta S_f}{\delta g_{\mu\nu}}\,. \tag{53}$$

There are several nontrivial details in using this definition. First of all, action (26) is constructed from the tetrad and not from the metric. On top of that, this action is not Hermitian. In order to fix the last issue, one can reformulate the action in the equivalent Hermitian form,

$$S_f = \frac{i}{2}\int d^4x\sqrt{-g}\,\big(\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi - \nabla_{\mu}\bar{\psi}\,\gamma^{\mu}\psi + 2im\big). \tag{54}$$

The difference with the original expression (26) is the integrals of total derivative term. This integral is irrelevant for the variational derivatives, but the action (54) gives a consistent result in a more straightforward way and we shall use this form of the action.

Concerning the first problem, we shall need the variation of the tetrad corresponding to the variation of the metric

$$g_{\alpha\beta} \to g'_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}. \tag{55}$$

The solution of this problem is [11,12] (see also Chapter 9 of [4])

$$
\begin{aligned}
e'^{\,a}_{\mu} &= e^a_\mu - \frac{1}{2}\,e^a_\nu h^\nu{}_\mu + \frac{3}{8}\,e^a_\nu h^\nu{}_\lambda h^\lambda{}_\mu + \dots \\
e'^{\,\alpha}_{b} &= e^\alpha_b + \frac{1}{2}\,e^\beta_b h^\alpha_\beta - \frac{1}{8}\,e^\beta_b h^\lambda_\beta h^\alpha_\lambda + \dots
\end{aligned}
\tag{56}
$$

The first variations of other relevant quantities has the form

$$
\begin{aligned}
\delta\sqrt{-g} &= \frac{1}{2}\sqrt{-g}h\,, \quad \delta g^{\mu\nu} = -h^{\mu\nu}\,, \quad \delta e^c_\mu = \frac{1}{2}\,h^\nu_\mu e^c_\nu\,, \quad \delta e^\rho_b = -\frac{1}{2}\,h^\rho_\lambda e^\lambda_b\,, \\
\delta\Gamma^\lambda_{\alpha\beta} &= \frac{1}{2}\left(\nabla_\alpha h^\lambda_\beta + \nabla_\beta h^\lambda_\alpha - \nabla^\lambda h_{\alpha\beta}\right), \quad \delta\gamma^\mu = -\frac{1}{2}\,h^\mu_\nu\,\gamma^\nu\,.
\end{aligned}
\tag{57}
$$

Here all indices are raised and lowered with the background metrics $g^{\mu\nu}$ and $g_{\mu\nu}$ and $h = g^{\mu\nu}h_{\mu\nu}$. Furthermore, direct calculation using (34) and (57) yields

$$
\delta\omega_{\mu\,\cdot\cdot}^{\;\;ab} = -\,\delta\omega_{\mu\,\cdot\cdot}^{\;\;ba} = \frac{1}{2}\,\delta(e^b_\tau e^{\lambda a}\Gamma^\tau_{\lambda\mu} - e^{\lambda a}\partial_\mu e^b_\lambda) = \frac{1}{2}\left(e^{a\tau}e^{b\lambda} - e^{b\tau}e^{a\lambda}\right)\nabla_\lambda h_{\mu\tau}\,. \tag{58}
$$

One more useful relation is

$$
\bar\psi\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma^\mu\psi = \bar\psi\gamma^\mu\partial_\mu\psi - \partial_\mu\bar\psi\gamma^\mu\psi - \frac{i}{4}\,\omega_{\mu\,\cdot\cdot}^{\;\;ab}\,\bar\psi\left(\gamma^\mu\sigma_{ab} + \sigma_{ab}\gamma^\mu\right)\psi\,. \tag{59}
$$

Consider the term depending on the variation of $\omega_{\mu\,\cdot\cdot}^{\;\;ab}$, defined by (58),

$$
\begin{aligned}
\delta_\omega S_f &= -\frac{i}{4}\int d^4x\sqrt{-g}\,\bar\psi\left(\gamma^\mu\sigma_{ab} + \sigma_{ab}\gamma^\mu\right)\psi\,\delta\omega_{\mu\,\cdot\cdot}^{\;\;ab} \\
&= -\frac{i}{4}\int d^4x\sqrt{-g}\,\frac{1}{2}\,\left(e^{a\tau}e^{b\lambda} - e^{b\tau}e^{a\lambda}\right)\left(\nabla_\lambda h_{\mu\tau}\right)e^{\mu c}\,\bar\psi(\gamma_c\sigma_{ab} + \sigma_{ab}\gamma_c)\psi\,.
\end{aligned}
\tag{60}
$$

After integration by parts, we get

$$
\delta_\omega S_f = \frac{i}{8}\int d^4x\sqrt{-g}\,h_{\mu\tau}e^{\mu c}\left(e^{\tau a}e^{\lambda b} - e^{\tau b}e^{\lambda a}\right)\nabla_\lambda\left[\bar\psi(\gamma_c\sigma_{ab} + \sigma_{ab}\gamma_c)\psi\right]. \tag{61}
$$

It is important that the term in the brackets [...] is antisymmetric in ab. Hence one can simplify the last expression, using the expression for σ_{ab},

$$
\delta_\omega S_f = -\frac{1}{8}\int d^4x\sqrt{-g}\,h_{\mu\nu}\,e^{\mu c}e^{\nu a}e^{\lambda b}\nabla_\lambda\left[\bar\psi(\gamma_c\gamma_a\gamma_b - \gamma_c\gamma_b\gamma_a + \gamma_a\gamma_b\gamma_c - \gamma_b\gamma_a\gamma_c)\psi\right].
$$

Replacing $\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}$, we arrive at

$$
\delta_\omega S_f = -\frac{1}{8}\int d^4x\sqrt{-g}\,h_{\mu\nu}\nabla_\lambda\left[\bar\psi(g^{\lambda\mu}\gamma^\nu - g^{\lambda\nu}\gamma^\mu)\psi\right] = 0\,. \tag{62}
$$

Thus, the $\delta\omega_{\mu\,\cdot\cdot}^{\;\;ab}$ is irrelevant as its contribution vanish. Hence we arrive at

$$
\begin{aligned}
\delta S_f &= \frac{i}{2}\int d^4x\sqrt{-g}\Big\{\frac{1}{2}\left(hg^{\mu\nu} - h^{\mu\nu}\right)\left(\bar\psi\gamma_\nu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma_\nu\psi\right) - ihm\bar\psi\psi\Big\} \\
&= \int d^4x\sqrt{-g}\,h_{\alpha\beta}\,\Big\{\frac{i}{4}g^{\alpha\beta}\left(\bar\psi\gamma^\lambda\nabla_\lambda\psi - \nabla_\lambda\bar\psi\gamma^\lambda\psi\right) - \frac{1}{2}\,g^{\alpha\beta}m\bar\psi\psi \\
&\quad - \frac{i}{4}\left(\bar\psi\gamma^\alpha\nabla^\beta\psi - \nabla^\alpha\bar\psi\gamma^\beta\psi\right)\Big\}\,.
\end{aligned}
\tag{63}
$$

The expression (63) is interesting by itself, as it describes an interaction of massive Dirac fermion with a weak gravitational field. In particular, it can be used for elaborating the nonrelativistic approximation to the fermions, or for deriving the equations of motion for the spinning charged classical particle in a weak relativistic field [13,14] (also the forthcoming work [15] with more references).

On the other hand, using the definition (53), the energy-momentum tensor of the fermion has the form

$$T_{\mu\nu} \,=\, \frac{i}{2}\,[\bar{\psi}\gamma_{(\mu}\nabla_{\nu)}\psi - \nabla_{(\mu}\bar{\psi}\gamma_{\nu)}\psi] - \frac{i}{2}\,g_{\mu\nu}\,[\bar{\psi}\gamma^\lambda\nabla_\lambda\psi - \nabla_\lambda\bar{\psi}\gamma^\lambda\psi] \,+\, m\bar{\psi}\psi\,g_{\mu\nu}. \tag{64}$$

Let us take a trace of the tensor (64). We get

$$T^\mu_\mu \;=\; T_{\mu\nu}g^{\mu\nu} \,=\, -\frac{3i}{2}\,(\bar{\psi}\gamma^\lambda\nabla_\lambda\psi - \nabla_\lambda\bar{\psi}\gamma^\lambda\psi) \,+\, 4m\bar{\psi}\psi\,. \tag{65}$$

Using the equations of motion

$$\gamma^\mu\nabla_\mu\psi = -im\psi \quad \text{and} \quad \nabla_\mu\bar{\psi}\gamma^\mu \,=\, im\bar{\psi}, \tag{66}$$

the on shell trace is

$$T^\mu_\mu\Big|_{on\ shell} \;=\; m\bar{\psi}\psi\,. \tag{67}$$

For a massless theory this expression vanishes, in accordance with our previous finding that the massless fermion theory is invariant under local conformal transformation.

There is an alternative way of deriving the trace (65). Let us remember that the definition (53) is valid for any kind of a matter field S_m. Take a variational derivative of the action with respect to the conformal factor σ and using (46), we get

$$\frac{\delta' S_m}{\delta\sigma} \,=\, \frac{\delta S_m}{\delta g_{\mu\nu}}\,\frac{\delta g_{\mu\nu}}{\delta\sigma} \,=\, \frac{\delta S_m}{\delta g_{\mu\nu}}\,2\,\bar{g}_{\mu\nu}\,e^{2\sigma} \,=\, \frac{\delta S_m}{\delta g_{\mu\nu}}\,2\,g_{\mu\nu} \,=\, -\sqrt{-g}\,T^{\mu\nu}\,g_{\mu\nu}. \tag{68}$$

Here the prime over variation in the first expression means that we take into account only the dependence of the metric and not of the matter fields. Formula (68) is valid, in particular, for the Dirac fermions in curved space, and can be used for getting T^μ_μ.

Adding variations with respect to the matter fields, we get

$$\frac{1}{\sqrt{-g}}\,\frac{\delta S_m}{\delta\sigma} \,=\, \frac{1}{\sqrt{-g}}\Big(2g_{\mu\nu}\,\frac{\delta S_m}{\delta g_{\mu\nu}} \,+\, \sum_k d_k\,\Phi_k\frac{\delta S_m}{\delta\Phi_k}\Big), \tag{69}$$

where d_k is the conformal weight of the field Φ_k (e.g., $d_k = -1$ for scalars and $d_k = -3/2$ for spinors in $n = 4$). Obviously, (69) is equivalent to (68) on shell, when $\delta S_m/\delta\Phi_k = 0$. Thus, the expression for the trace (67) can be obtained, even for the massive nonconformal case, by the following sequence of steps: *(i)* Rewriting the action (26) using the parametrization (46); *(ii)* Taking the variational derivative with respect to σ; *(iii)* Replacing $\sigma \to 0$ and $\bar{g}_{\mu\nu} \to g_{\mu\nu}$ and (off shell) the same for the spinor field.

The considerations presented above and its result (69) are valid for all matter fields. In the conformal case, the *r.h.s.* is zero, which establishes the relation between local conformal symmetry (i.e., independence on σ) and the vanishing on the mass shell expression for the trace of the energy-momentum tensor.

7. Derivation of $T_{\mu\nu}$ on the Cosmological Background

In order to have an illustration for the results presented above, let us calculate the energy-momentum tensor for a free spinor on a cosmological background. It is assumed that not only metric, but also the fermion field depends only on time and not on the space coordinates. For the sake of simplicity we choose a conformally-flat metric

$$g_{\mu\nu} = a^2(\eta)\bar{g}_{\mu\nu} \;=\; e^{2\sigma(\eta)}\bar{g}_{\mu\nu}\,, \tag{70}$$

with $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. We also use the conformal time variable η, related to physical time t as $d\eta = a(t)dt$. The derivative with respect to η will be denoted by prime. It is well-known that the non-zero components of affine connection are

$$\Gamma^0_{00} = \frac{a'}{a}\,, \qquad \Gamma^0_{ik} = \frac{a'}{a}\,\delta_{ik}\,, \qquad \Gamma^i_{0k} = \frac{a'}{a}\,\delta^i_k\,. \tag{71}$$

Here and in what follows we shall use the notations $i, j, k = 1, 2, 3$ and $a, b, c = 0, 1, 2, 3$, while both sets of Latin indices correspond to the flat fiducial metric. The tetrad components can be easily derived and the curved-space gamma-matrices are given by

$$\gamma^0 = \frac{1}{a}\,\Gamma^0\,, \qquad \gamma_0 = a\,\Gamma^0\,, \qquad \gamma^k = \frac{1}{a}\,\Gamma^k\,, \qquad \gamma_k = a\,\Gamma_k\,, \tag{72}$$

where the useful notations $\Gamma^a = \gamma^a$ were introduced for the flat-space gamma's.

In order to derive the components of $T_{\mu\nu}$ one has to take care about the spinor connection first. Direct calculation gives (other components are equal to zero)

$$\begin{aligned}
\omega_{0\,\cdot\cdot}^{\;\;k0} &= \Gamma^k_{00} - e^{-\sigma}\eta^{\lambda 0}\partial_0\big(e^{\sigma}\delta^k_\lambda\big) = \Gamma^k_{00} = 0\,, \\
\omega_{0\,\cdot\cdot}^{\;\;ki} &= \Gamma^\tau_{\lambda 0}e^k_\tau e^{\lambda i} - e^{\lambda i}\partial_0 e^k_\lambda \;=\; -\frac{a'}{a}\eta^{ki} \;=\; \frac{a'}{a}\,\delta^{ki}\,, \\
\omega_{j\,\cdot\cdot}^{\;\;ik} &= \Gamma^\tau_{\lambda j}e^i_\tau e^{\lambda k} - e^{k\lambda}\partial_j e^i_\lambda = 0\,, \\
\omega_{k\,\cdot\cdot}^{\;\;oi} &= \Gamma^\tau_{\lambda k}e^o_\tau e^{\lambda i} - e^{\lambda i}\partial_k e^0_\lambda \;=\; \Gamma^0_{jk}\eta^{ij} = \frac{a'}{a}\eta_{jk}\eta^{ij} = -\frac{a}{a}\delta^i_k\,.
\end{aligned} \tag{73}$$

Thus, we arrive at

$$\nabla_0\psi \;=\; \psi' - \frac{i}{4}\omega_0^{kj}\sigma_{kj}\psi = \psi' + \frac{i}{4}\,\frac{a'}{a}\,\eta^{kj}\sigma_{kj} = \psi'\,.$$

Similarly, $\nabla_0\bar{\psi} = \bar{\psi}'$. Furthermore, it is easy to obtain

$$\begin{aligned}
\nabla_i\psi &= \partial_i\psi - \frac{i}{2}\omega_{i\,\cdot\cdot}^{\;ok}\sigma_{ok}\psi = \frac{i}{2}\,\frac{a'}{a}\,\frac{i}{2}\,2\Gamma_0\Gamma_k\psi\delta \;=\; -\frac{a'}{2a}\Gamma_o\Gamma_i\,, \\
\nabla_i\bar{\psi} &= \partial_i\bar{\psi} + \frac{i}{2}\,\omega_{i\,\cdot\cdot}^{\;0k}\bar{\psi}\,\sigma_{ok} \;=\; \frac{1}{2}\,\delta^k_i\,\frac{a'}{a}\,\bar{\psi}\Gamma_0\Gamma_k \;=\; +\frac{a'}{2a}\,\bar{\psi}\Gamma_o\Gamma_i\,.
\end{aligned}$$

Now we are in a position to calculate the components of $T_{\alpha\beta}$. According to Equation (64),

$$\begin{aligned}
T_{00} \;&=\; -\frac{i}{2}\,g_{00}\big(\bar{\psi}\gamma^0\nabla_0\psi + \bar{\psi}\gamma^k\nabla_k\psi - \nabla_0\bar{\psi}\gamma^0\psi - \nabla_k\bar{\psi}\gamma^k\psi\big) + g_{00}\bar{\psi}\psi m \\
&+\; \frac{i}{2}\,\big(\bar{\psi}\gamma_0\nabla_0\psi - \nabla_0\bar{\psi}\gamma_0\psi\big) \;=\; a^2 m\bar{\psi}\psi\,.
\end{aligned} \tag{74}$$

In a similar way, taking into account $g_{0k} = 0$, we get

$$\begin{aligned}
T_{0k} \;&=\; \frac{i}{4}\,\big(\bar{\psi}\gamma_0\nabla_k\psi + \bar{\psi}\gamma_k\nabla_0\psi - \nabla_0\bar{\psi}\gamma_k\psi - \nabla_k\bar{\psi}\gamma_0\psi\big) \\
&=\; \frac{ia}{4}\,\big(\bar{\psi}\Gamma_k\psi' - \bar{\psi}'\Gamma_k\psi\big)\,.
\end{aligned} \tag{75}$$

Because of the homogeneity and isotropy of space, in the cosmological setting this must be zero. Let us show that this is the case.

On the mass shell, we have

$$\gamma^0\nabla_0\psi + \gamma^\mu\nabla_k\psi + im\psi \;=\; \frac{1}{a}\,\Gamma^0\psi' - \frac{a'}{2a^2}\,\Gamma^k\Gamma_0\Gamma_k\psi + im\psi = 0\,. \tag{76}$$

Since $\Gamma^k\Gamma_0\Gamma_k = -3\Gamma_0$, we get

$$\frac{1}{a}\Gamma^0\big(\psi' + \frac{3}{2}\frac{a'}{a}\psi\big) + im\psi \,=\, 0\,. \tag{77}$$

Similarly, starting from $\nabla_\mu\bar{\psi}\gamma^\mu - im\psi = 0$, one can arrive at

$$\frac{1}{a}\,\Big(\bar{\psi}' + \frac{3a'}{2a}\bar{\psi}\Big)\Gamma_0 \,-\, im\bar{\psi} = 0\,. \tag{78}$$

Then, (75) becomes the following on shell expression:

$$T_{0k}\Big|_{on\ shell} \;=\; -\frac{ia}{4}\,ma\big(\bar{\psi}\Gamma_k\Gamma_0\psi + \bar{\psi}\Gamma_0\Gamma_k\psi\big) \,=\, 0\,, \tag{79}$$

exactly as required to guarantee the vanishing non-diagonal element (75).

The last part is to derive the space components,

$$\begin{aligned} T_{ki} &= -ia^2\eta_{ik}\big(\bar{\psi}\gamma^0\nabla_0\psi + \bar{\psi}\gamma^j\nabla_j\psi - \nabla_0\bar{\psi}\gamma^0\psi - \nabla_j\bar{\psi}\gamma^j\psi + 2im\bar{\psi}\psi\big) + \frac{i}{2}\big[\bar{\psi}\gamma_{(k}\nabla_{i)}\psi\big] \\ &= -\frac{ia}{2}\eta_{ik}\big(\bar{\psi}\Gamma^0\psi' - \bar{\psi}'\Gamma^0\psi - \frac{a'}{2a}\bar{\psi}\Gamma^j\Gamma_0\Gamma_j\psi - \frac{a'}{2a}\bar{\psi}\Gamma_0\Gamma_j\Gamma^j\psi\big) \\ &+\; a^2\eta_{ik}\,m\bar{\psi}\psi - \frac{ia'}{8}\,\big[\bar{\psi}\Gamma_{(k}\Gamma_{|0|}\Gamma_{i)}\psi + \bar{\psi}\Gamma_0\Gamma_{(k}\Gamma_{i)}\psi\big] \\ &= -\frac{ia}{2}\,\eta_{ik}\,\big(\bar{\psi}\Gamma_0\psi' - \bar{\psi}'\Gamma_0\psi\big) + ma^2\,\eta_{ik}\bar{\psi}\psi - \frac{ia'}{a}\,\bar{\psi}\big(\Gamma_0\Gamma_{(k}\Gamma_{i)} - \Gamma_0\Gamma_{(k}\Gamma_{i)}\big)\psi \\ &= \frac{ia}{2}\,\eta_{ik}\,\big(\bar{\psi}'\Gamma_0\psi - \bar{\psi}\Gamma_0\psi'\big) \,+\, ma^2\eta_{ik}\bar{\psi}\psi\,. \end{aligned} \tag{80}$$

On shell we get, according to (77) and (78),

$$T_{ik}\Big|_{on\ shell} \;=\; \eta_{ik}\,\Big[ma^2\bar{\psi}\psi + \frac{a^2}{2}\,m\,\big(\bar{\psi}\Gamma_0^2\psi + \bar{\psi}\Gamma_0^2\psi\big)\Big] \,=\, 0\,. \tag{81}$$

Here we used the $\Gamma_0^2 = 1$ property of the gamma-matrix.

For a massive case, we meet $T_{11} = T_{22} = T_{33} = 0$, while $\rho = T_{00} \neq 0$. This is a dust-type equation of state for the matter.

The physical interpretation of the results for diagonal component (74) and (81) is the following. First of all, for a massless case, $T_{\mu\nu}\big|_{on\ shell} = 0$ for the FRW metric. This is a natural output, since we permitted the fermion field to depend only on conformal time and *not* on the space coordinates. Therefore, we exclude the field configurations such as plane (and other, of course) waves. In the massive case, there is still a dust-like configuration and in the massless version no solutions are possible. Of course, if we permit the space-dependence, the equation of state will be the one of radiation, $T^\alpha_\mu = \,\mathrm{diag}\,\big(\rho_r\,,\,-\frac{1}{3}\delta^j_i\rho_r\big)$.

8. Conclusions

We presented in details the derivation of spinor connection, covariant derivative of Dirac fermion, and energy-momentum tensor for the fermion. Also, the conformal properties of fermions in curved space were discussed. Let us note that the same constructions can be applied to more general backgrounds, for instance for the non-Riemannian space which has not only metric, but also metric-independent torsion. The result can be found, e.g., in [4] or in the review papers [16,17]. In a more general case, when the metricity condition is not satisfied, the construction of covariant derivative of the fermions meets difficulties and the output of this procedure looks unclear, at the moment. In principle, the scheme described here should work is this and other cases, when the geometry is enriched by additional metric-independent fields.

The last observation is that the importance of the covariant formulation of spinors in curved space goes beyond the quantum field theory. Regardless spin is certainly a property of quantum theory, the fermions in curved space are important for the models of spinning particles and the last can be used as basic models for the macroscopic gravitationally interacting compact magnetic bodies. This is an important issue in view of existing experimental efforts (see, e.g., [18,19] and references therein) towards the precise measurements of the accelerations of such particles. At the moment, the existing theoretical literature on this issue is restricted to elementary spinning particles [20–23] (see also the planned continuation of the present work in [15]), but it would be certainly interesting to extend the existing understanding to the macroscopic bodies.

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Appendix A. Gamma Matrices

The three Pauli matrices are defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A1}$$

In the standard representation, the gamma (Dirac) matrices are defined as

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}, \tag{A2}$$

satisfying the Clifford algebra

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab} I, \tag{A3}$$

where I is the four-dimensional unit matrix.

It is important that neither Pauli matrices nor Dirac matrices do not form a vector with respect to the Lorentz transformation and are just constant matrices which do not change under Lorentz transformation. This important feature is compatible with (A3) and with the Lorentz covariance of the Dirac equation [7,24]. It also holds in other representations of γ^a. It is interesting that all this is true only in the Cartesian coordinates in the space section. To write the gamma matrices, e.g., in spherical coordinates one has to use the tetrads, with the corresponding change in (A3).

Notes

1 We use notations of the tensor textbook [9]. In particular, prime over the index or over the vector is the same thing, e.g., $V'_\alpha = V_{\alpha'}$.

2 This can be done by either repeating the steps leading to (23) or by directly using (23). The interested reader can compare these two ways to make this calculation.

3 An alternative definition of the *canonical* energy-momentum tensor is based on the Noether' s theorem. After this canonical energy-momentum is making symmetric we arrive at its Belinfante improvement. This symmetric version and the dynamical definition described below, give the same results in all particular flat-space cases, regardless the general proof of their equivalence is not knows, to the best of the author's knowledge. In curves space the dynamical definition is preferable for obvious reasons.

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