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Vaidya and Generalized Vaidya Solutions by Gravitational Decoupling

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Abstract: In this paper, we apply the gravitational decoupling method for dynamical systems in order to obtain a new type of solution that can describe a hairy dynamical black hole. We consider three cases of decoupling. The first one is the simplest and most well known when the mass function is the function only of space coordinate r. The second case is a Vaidya spacetime case when the mass function depends on time v. Finally, the third case represents the generalization of these two cases: the mass function is the function of both r and v. We also calculate the apparent horizon and singularity locations for all three cases.

Keywords: gravitational decoupling; Vaidya spacetimes; hairy black hole

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1. Introduction

Black holes are one of the most fascinating objects in our Universe. Currently, we can make direct observations of them via detection of their gravitational waves [1,2] or black hole shadow [3,4].

The famous no-hair theorem states that a black hole might have only three charges: the mass M, angular momentum J, and electric charges Q [5]. However, it can be shown that black holes can have other charges and there is so-called soft hair [6]. Among other possibilities for evading the no-hair theorem is to use the gravitational decoupling method [7–9].

It is well known that obtaining the analytical solution of the Einstein equations is a difficult task in most cases. We know that we can obtain an analytical solution of the spherically symmetric spacetime in the case of the perfect fluid as the gravitational source. However, if we consider the more realistic case when the perfect fluid is coupled to another matter, it is nearly impossible to obtain the analytical solution. In papers [7–9], it was shown using the Minimal Geometric Deformation (MGD) [10,11] method that we can decouple the gravitational sources, for example, one can write the energy-momentum tensor T_{ik} as:

$$T_{ik} = \tilde{T}_{ik} + \alpha\Theta_{ik}\,. \tag{1}$$

where $\tilde{T}_{ik}$ is the energy-momentum tensor of the perfect fluid and α is the coupling constant to the energy-momentum tensor Θ_{ik}. It is possible to solve Einstein's field equations for a gravitational source whose energy-momentum tensor is expressed as (1) by solving Einstein's field equations for each component $\tilde{T}_{ik}$ and Θ_{ik} separately. Then, by a straightforward superposition of the two solutions, we obtain the complete solution corresponding to the source T_{ik}. Since Einstein's field equations are non-linear, the MGD decoupling represents a novel and useful method in the search for and analysis of solutions, especially when we face scenarios beyond trivial cases, such as the interior of stellar systems with

gravitational sources more realistic than the ideal perfect fluid, or even when we consider alternative theories, which usually introduce new features that are difficult to deal with.

Moreover, there is only the gravitational interaction between two sources, i.e.,

$$T^{ik}_{;k} = 0 \rightarrow \tilde{T}^{ik}_{;k} = \alpha \Theta^{ik}_{;k} = 0\,. \tag{2}$$

This fact allows us to think about Θ_{ik} as dark matter. By applying the gravitational decoupling method, one can obtain well-known black hole solutions with hair [12,13]. However, this method is applied only to static or stationary cases. That is, we obtain only an eternal hairy black hole solution. If one wants to understand the process of these hairy black hole formations, then one should consider the gravitational collapse of the matter cloud. The problem is that in the general case, the (MGD) method is not applicable due to its violation of condition (2). The gravitational decoupling of a dynamical system is still a problem. One of the first successful decouplings of the dynamical system was performed in Ref. [14].

In this paper, we offer a model of the gravitational decoupling of dynamical systems, which can be used to investigate the question of gravitational collapse to a hairy black hole. By using the hairy Schwarzschild black hole solution obtained in Ref. [12], we introduce the Eddington–Finkelstein coordinates in order to consider the non-zero right hand side of the Einstein equations. In this case, the mass M is not a constant; however, it is the mass function of time v and the radial coordinate r. As a result, we obtained the Vaidya and generalized Vaidya solutions. In the Vaidya case, the energy-momentum tensor $\tilde{T}_{ik}$ represents the null dust. In the generalized Vaidya case, the $\tilde{T}_{ik}$ represents the mixture of two matter fields—type I and type II [15,16]

The Vaidya spacetime is the so-called radiating Schwarzschild solution [17] and is one of the first examples of a cosmic censorship conjecture violation [18]. The Vaidya spacetime is widely used in many astrophysical applications with strong gravitational fields. In general relativity, this spacetime assumed added importance with the completion of the junction conditions at the surface of the star by Santos [19]. The pressure at the surface is non-zero, and the star dissipates energy in the form of heat flux. This made it possible to study dissipation and physical features associated with gravitational collapse, as shown by Herrera et al. [20–22] . Some recent studies of the temperature properties inside the radiating star include Reddy et al. [23], Thirukkanesh et al. [24], and Thirukkanesh and Govender [25]. The metric in Ref. [26] may be extended to include both null dust and null string fluids leading to the generalized Vaidya spacetime. The properties of the generalized Vaidya metric have been studied by Hussain [27], Wang and Wu [16], and Glass and Krisch [28,29]. Maharaj et al. [30,31] modeled a radiating star with a generalized Vaidya atmosphere in general relativity. A detailed study of continual gravitational collapse of these spacetimes in the context of the cosmic censorship conjecture was performed in Refs. [32–37]. In the geometrical context, gravitational collapse has been considered in Lovelock gravity theory [38], black holes in dynamical cosmology backgrounds [39], and in electromagnetic fluids [40]. The influences of dust, radiation, quintessence, and the cosmological constant are included in these studies. The conformal symmetries and embedding properties of the generalized Vaidya metric were studied in Refs. [41,42]. Other properties of this spacetimes can be found in Refs. [43,44].

The paper is organized as follows: In Section 2 we introduce the Eddington–Finkelstein coordinates for the hairy Schwarzschild metric and consider the solution of the Einstein equation with mass as the function of the radial coordinate r only. In Section 3, we obtain the hairy Vaidya solution by solving the Einstein equations with mass depending on the time v only and calculate the apparent horizon for this metric. In Section 4, the general case $M = M(v, r)$ is considered in order to obtain the generalized hairy Vaidya spacetime, and the apparent horizon, in this case, is also calculated. Section 5 is the conclusion.

2. The Perfect Fluid Case

The hairy Schwarzschild spacetime obtained by gravitational decoupling [12], which satisfies all energy conditions [15,45], has the following form:

$$ds^2 = -e^{\mu}dt^2 + e^{-\lambda}dr^2 + r^2d\theta^2 + r^2\sin\theta^2 d\varphi^2\,, \tag{3}$$

where the metric coefficients are:

$$e^{\mu} = e^{-\lambda} = 1 - \frac{2M}{r} + \alpha\exp\left(\frac{-r}{(\sigma - \alpha l/2)}\right). \tag{4}$$

Here, α is the coupling constant, l is a new charge (hair) of a black hole [12], σ is the parameter related to the Misner–Sharp mass, and M is a mass of a black hole, which is given by:

$$M = \mathbf{M} + \frac{\alpha l}{2}\,. \tag{5}$$

$\mathbf{M}$ is the usual Schwarzschild mass. The kinematic properties of the solution (3) has been intensively studied in Ref. [46]. Moreover, the authors showed that parameters α and l can mimic the Kerr spacetime and gave the numerical values for the supermassive black holes at Ark 564 and NGC 1365. The influence of a primary hair on the thermodynamics of a black hole (3) has been investigated in Ref. [47]. As we have pointed out in the introduction, gravitational decoupling allows us to consider the Einstein equations for each source separately; however, the Schwarzschild solution is the vacuum solution. That is, to obtain (3) one should put $\tilde{T}_{ik} = 0$ and consider how an additional source Θ_{ik} changes the vacuum Schwarzschild metric. So, the metric (3) is the solution of the following Einstein equation:

$$G_{ik} = -\alpha\Theta_{ik} \tag{6}$$

where the energy-momentum tensor Θ_{ik} represents anisotropic fluid. This energy-momentum tensor satisfies the strong and dominant energy condition for $r \geq 2M$ [12]. It has the following form:

$$\begin{aligned} p_t = \Theta_2^2 &= \frac{(\alpha l + r - 2\sigma)\alpha\exp(\frac{2r}{\alpha l - 2\sigma})}{4r\pi(\alpha l - 2\sigma)^2}\,, \\ P_r = -\rho = \Theta_1^1 &= -\frac{\alpha\exp(\frac{2r}{\alpha l - 2\sigma})(\alpha l + 2r - 2\sigma)}{8(\alpha l - 2\sigma)r^2\pi}. \end{aligned} \tag{7}$$

Here, ρ is the energy density of an additional matter source and P_r and P_t are the radial and tangential pressure, respectively.

To obtain the line element (3) in Eddington–Finkelstein coordinates one should perform the following coordinate transformation [47]

$$dt = dv + \frac{rdr}{\left(-\alpha e^{-\frac{2r}{-\alpha l + 2M}}r + 2M - r\right)} \tag{8}$$

Then, one obtains the hairy Schwarzschild spacetime in Eddington–Finkelstein coordinates:

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{\alpha}{\exp\left(\frac{r}{-\alpha l + 2\sigma}\right)^2}\right)dv^2 + 2dvdr + r^2d\theta^2 + r^2\sin\theta^2 d\varphi^2 \tag{9}$$

We know that the energy-momentum tensor of the generalized Vaidya spacetime represents the mixture of two matter fields—type I (the null dust) and type II (the null string) [16]. We can obtain type I if we assume that the mass function M depends upon the time v, and we can acquire type II if the mass function depends on the radial coordinate

r; furthermore, we can obtain their combination if the mass function is the function of both v and r. We begin our consideration by assuming that M is the function of the r coordinate only ($M = M(r)$). With this assumption, the Einstein tensor components G_{ik} for the metric (9) are given by:

$$G_0^0 = \frac{\alpha(\alpha l + 2r - 2\sigma)\exp\left(\frac{2r}{\alpha l - 2\sigma}\right) - 2M'(r)(\alpha l - 2\sigma)}{r^2(\alpha l - 2\sigma)} \tag{10}$$

$$G_1^1 = \frac{\alpha(\alpha l + 2r - 2\sigma)\exp\left(\frac{2r}{\alpha l - 2\sigma}\right) - 2M'(r)(\alpha l - 2\sigma)}{r^2(\alpha l - 2\sigma)} \tag{11}$$

$$G_2^2 = \frac{2\alpha(\alpha l + 2r - 2\sigma)\exp\left(\frac{2r}{\alpha l - 2\sigma}\right) - M''(r)(\alpha l - 2\sigma)^2}{r(\alpha l - 2\sigma)^2} \tag{12}$$

$$G_3^3 = \frac{2\alpha(\alpha l + 2r - 2\sigma)\exp\left(\frac{2r}{\alpha l - 2\sigma}\right) - M''(r)(\alpha l - 2\sigma)^2}{r(\alpha l - 2\sigma)^2}, \tag{13}$$

with the energy-momentum tensor:

$$T_{ik} = \tilde{T}_{ik} + \alpha\Theta_{ik} \tag{14}$$

where Θ_{ik} is the energy-momentum tensor (7) and $\tilde{T}_{ik}$ is the energy-momentum tensor of the metric (9) with $\alpha = 0$. We write it in the following form:

$$\tilde{T}_{ik} = (\hat{\rho} + \hat{p})(l_i n_k + n_i l_k) + \hat{p}\tilde{g}_{ik}\,. \tag{15}$$

where $\tilde{g}_{ik}$ is the metric tensor (9) with $\alpha = 0$. $\hat{\rho}$ and $\hat{p}$ are the energy density and the pressure of the matter $\tilde{T}_{ik}$. l_i and n_i are two null vectors, which are given by:

$$\begin{aligned} n_i &= \frac{1}{2}\left(1 - \frac{2M(r)}{r}\right)\delta_i^0 - \delta_i^1\,, \\ l_i &= \delta_i^0\,, \\ l_i l^i &= n_i n^i = 0\,, \\ n_i l^i &= -1\,. \end{aligned} \tag{16}$$

First of all, let us find the Einstein equation in the case $\alpha = 0$:

$$\begin{aligned} \hat{\rho} &= -\frac{2M'(r)}{r^2}\,, \\ \hat{p} &= \frac{M''(r)}{r}\,. \end{aligned} \tag{17}$$

The Einstein tensor components, in this case, are given by:

$$\begin{aligned} \tilde{G}_0^0 &= -\frac{2M'(r)}{r^2}\,, \\ \tilde{G}_1^1 &= -\frac{2M'(r)}{r^2}\,, \\ \tilde{G}_2^2 &= -\frac{M''(r)}{r}\,, \\ \tilde{G}_3^3 &= -\frac{M''(r)}{r}\,. \end{aligned} \tag{18}$$

Comparing (10), (11), (12), (13) and (18), one can easily decouple the initial Einstein tensor into $\tilde{G}_{ik}$, ($\alpha = 0$), and $\hat{G}_{ik}$, which correspond to the metric of minimal geometric deformation. So, one has:

$$\begin{aligned}
\hat{G}_0^0 = \rho = -P_r &= \frac{\alpha(\alpha l + 2r - 2\sigma)\exp\left(\frac{2r}{\alpha l - 2\sigma}\right)}{r^2(\alpha l - 2\sigma)}, \\
\hat{G}_1^1 &= \frac{\alpha(\alpha l + 2r - 2\sigma)\exp\left(\frac{2r}{\alpha l - 2\sigma}\right)}{r^2(\alpha l - 2\sigma)}, \\
\hat{G}_2^2 &= \frac{2\alpha(\alpha l + 2r - 2\sigma)\exp\left(\frac{2r}{\alpha l - 2\sigma}\right)}{r(\alpha l - 2\sigma)^2}, \\
\hat{G}_3^3 &= \frac{2\alpha(\alpha l + 2r - 2\sigma)\exp\left(\frac{2r}{\alpha l - 2\sigma}\right)}{r(\alpha l - 2\sigma)^2}.
\end{aligned} \tag{19}$$

The energy-momentum tensor must satisfy the conservation equation, which is automatically satisfied through the Einstein equation:

$$T^{ik}_{;k} = \tilde{T}^{ik}_{;k} + \alpha\Theta^{ik}_{;k} = 0\,. \tag{20}$$

Hence, for two sources, one has either energy exchange between two matter fields:

$$\tilde{T}^{ik}_{;k} = -\alpha\Theta^{ik}_{;k} \neq 0\,, \tag{21}$$

or purely the gravitation interaction of two sources:

$$\tilde{T}^{ik}_{;k} = \alpha\Theta^{ik}_{;k} = 0\,. \tag{22}$$

The last condition means that Θ_{ik} corresponds to the dark matter due to only gravitational interaction. In our case, from the condition $\tilde{T}^{ik}_{;k} = 0$, it follows that $\Theta^{ik}_{;k} = 0$, i.e., there is no energy exchange between two sources. Let us introduce the generalized density $\tilde{\rho}$ and pressure $\tilde{P}$ for the metric (9):

$$\tilde{\rho} = \frac{-\alpha(\alpha l + r - 2\sigma)\exp\left(\frac{2r}{\alpha l - 2\sigma}\right) + 2M''(r)(\alpha l - 2\sigma)^2}{8r^2\pi(\alpha l - 2\sigma)^2} \tag{23}$$

$$\tilde{P} = \frac{2\alpha(\alpha l + r - 2\sigma)\exp\left(\frac{2r}{\alpha l - 2\sigma}\right) - M''(r)(\alpha l - 2\sigma)^2}{8r\pi(\alpha l - 2\sigma)^2} \tag{24}$$

It is worth noticing that this decoupling was introduced in Ref. [7]. We have transformed it to Eddington–Finkelstein coordinates (9) because it is a effective tool to obtain Vaidya and generalized Vaidya solutions by gravitational decoupling. We also notice that a new gravitational source Θ_{ik} changes the location of the apparent horizon. To prove it, let us consider the expansion Θ_l of outgoing null geodesic congruence:

$$e^{\gamma}\Theta_l = \frac{2}{r}\left(1 - \frac{2M(r)}{r} + \frac{\alpha}{\exp\left(\frac{r}{-\alpha l + 2\sigma}\right)^2}\right). \tag{25}$$

So, to obtain the apparent horizon, one should solve the following equation:

$$1 - \frac{2M(r)}{r} + \frac{\alpha}{\exp\left(\frac{r}{-\alpha l + 2\sigma}\right)^2} = 0\,. \tag{26}$$

One should note that this solution is static. It means that the apparent horizon coincides with the event horizon. In a dynamical case, it is not true, and the location of the event horizon is the big question. The only thing that we know is that in a dynamical case, the radius of the apparent horizon r_{ah} is bigger than the event horizon location $r_{eh}(r_{ah} \geq r_{eh})$. The horizon of this metric is a canonical one [48] if the following condition is held:

$$\frac{dB}{dr}|_{r=r_h} < 1\,. \tag{27}$$

where r_h is the solution of (26) and

$$B(r) \equiv 2M(r) - r\frac{\alpha}{\exp\left(\frac{r}{-\alpha l+2\sigma}\right)^2}\,. \tag{28}$$

To understand the structure of a singularity of a new solution, one should count the Kretschmann scalar $K = R_{iklm}R^{iklm}$. Here, we do not investigate the question of the global structure of this singularity. The main question, which we are interested in now, is that a new solution does not generate new singularities except for $r = 0$. In the next section, the singularity location will be at $r = 0$ only due to the fact that the mass function M depends only on the time v. However, when M is the function of r, the structure of the point $r = 0$ is not so clear. For example, when one considers the generalized Vaidya solution (without a hair), $r = 0$ is not always a singular point [49]. The Kretschmann scalar for metric (3) with $M = M(r)$ is given by:

$$\begin{aligned} K = \frac{1}{(\alpha l-2\sigma)^4 r^6}\Big(&-16r\big(r^4M''(r) + 2r^2(\alpha l - r - 2\sigma)M'(r) + \\ &\Big(2r^2 + (-2\alpha l + 4\sigma)r(\alpha l - 2\sigma)^2\Big)M(r)\Big)\alpha(\alpha l - 2\sigma)^2\exp\left(\tfrac{2r}{\alpha l-2\sigma}\right) + \\ &4r^2\Big(2r^2 + (\alpha l - 2\sigma)^2\Big)^2\alpha^2\exp\left(\tfrac{4r}{\alpha l-2\sigma}\right) + \\ &4(\alpha l - 2\sigma)^4\big(r^4M''(r)^2 + \big(-4r^3M'(r) + 4r^2M(r)\big)M''(r) + \\ &8r^2M'(r)^2 - 16rMM'(r) + 12M(r)^2\big)\Big) \end{aligned} \tag{29}$$

3. Vaidya Solution by Gravitational Decoupling

The Vaidya spacetime by gravitational decoupling is obtained by the assumption that the mass in (9) is the function of the time v:

$$ds^2 = -\left(1 - \frac{2M(v)}{r} + \frac{\alpha}{\exp\left(\frac{r}{-\alpha l+2\sigma}\right)^2}\right)dv^2 + 2dvdr + r^2d\theta^2 + r^2\sin\theta^2 d\varphi^2 \tag{30}$$

The Einstein tensor components for this metric are given by:

$$\begin{aligned} G_0^0 &= \frac{\alpha(\alpha l + 2r - 2\sigma)\exp\left(\frac{2r}{\alpha l-2\sigma}\right)}{r^2(\alpha l - 2\sigma)}\,, \\ G_0^1 &= \frac{2\dot{M}(v)}{r^2}\,, \\ G_1^1 &= \frac{\alpha(\alpha l + 2r - 2\sigma)\exp\left(\frac{2r}{\alpha l-2\sigma}\right)}{r^2(\alpha l - 2\sigma)}\,, \\ G_2^2 &= \frac{2\alpha(\alpha l + 2r - 2\sigma)\exp\left(\frac{2r}{\alpha l-2\sigma}\right)}{r(\alpha l - 2\sigma)^2}\,, \\ G_3^3 &= \frac{2\alpha(\alpha l + 2r - 2\sigma)\exp\left(\frac{2r}{\alpha l-2\sigma}\right)}{r(\alpha l - 2\sigma)^2}\,. \end{aligned} \tag{31}$$

Here, the decoupling is quite simple. First of all, one can see that the Einstein tensor $\hat{G}^i_k$ is the same as in the previous section (19) and the only non-vanishing component of the Einstein tensor $\tilde{G}^i_k$, which corresponds with case $\alpha = 0$, is

$$\tilde{G}^1_0 = \frac{2\dot{M}(v)}{r^2} = -\mu\,. \tag{32}$$

Here, μ is the energy density. The energy-momentum tensor $\tilde{T}_{ik}$ represents null dust:

$$\begin{aligned} \tilde{T}_{ik} &= \mu L_i L_k\,, \\ L_i &= \delta^0_i\,. \end{aligned} \tag{33}$$

Such as in the previous case, we have only gravitational interaction between two matter sources:

$$\tilde{T}^{ik}_{;k} = \Theta^{ik}_{;k} = 0\,. \tag{34}$$

Now, we calculate the expansion Θ_l in order to obtain the apparent horizon equation:

$$e^{\gamma}\Theta_l = \frac{2}{r}\left(1 - \frac{2M(v)}{r} + \frac{\alpha}{\exp\left(\frac{r}{-\alpha l + 2\sigma}\right)^2}\right). \tag{35}$$

As in the previous section, the apparent horizon equation is:

$$1 - \frac{2M(v)}{r} + \frac{\alpha}{\exp\left(\frac{r}{-\alpha l + 2\sigma}\right)^2} = 0\,. \tag{36}$$

The singularity location in the metric (30) is at $r = 0$, which can be seen from the Kretschmann scalar:

$$\begin{aligned} K = \frac{1}{(\alpha l - 2\sigma)^4 r^6}\Big(&- 16\left(\alpha^2 l^2 - 2l(r + 2\sigma)\alpha + 2r^2 + 4r\sigma + 4\sigma^2\right)\times \\ &r(\alpha l - 2\sigma)^2 \alpha M(v) \exp\left(\frac{2r}{\alpha l - 2\sigma}\right) + \\ 4r^2\alpha^2\left(\alpha^2 l^2 - 4\alpha l\sigma + 2r^2 + 4\sigma^2\right)^2 &\exp\left(\frac{4r}{\alpha l - 2\sigma}\right) + 48M(v)^2(\alpha l - 2\sigma)^4\Big) \end{aligned} \tag{37}$$

4. Generalized Vaidya Spacetime by Gravitational Decoupling

Finally, if we consider the mass in (9) as the function of both time v and the space coordinate r, we obtain the generalized Vaidya spacetime by gravitational decoupling:

$$ds^2 = -\left(1 - \frac{2M(v,r)}{r} + \frac{\alpha}{\exp\left(\frac{r}{-\alpha l + 2\sigma}\right)^2}\right)dv^2 + 2dvdr + r^2 d\theta^2 + r^2 \sin\theta^2 d\varphi^2\,. \tag{38}$$

This metric represents the Einstein equation solution of three sources: the null dust, the null perfect fluid, and new field Θ_{ik}. The Einstein tensor is given by:

$$
\begin{aligned}
G_0^0 &= \frac{\alpha(\alpha l + 2r - 2\sigma)\exp\left(\frac{2r}{\alpha l - 2\sigma}\right) - 2M'(v,r)(\alpha l - 2\sigma)}{r^2(\alpha l - 2\sigma)}\,, \\
G_0^1 &= \frac{2\dot{M}(v,r)}{r^2}\,, \\
G_1^1 &= \frac{\alpha(\alpha l + 2r - 2\sigma)\exp\left(\frac{2r}{\alpha l - 2\sigma}\right) - 2M'(v,r)(\alpha l - 2\sigma)}{r^2(\alpha l - 2\sigma)}\,, \\
G_2^2 &= \frac{2\alpha(\alpha l + 2r - 2\sigma)\exp\left(\frac{2r}{\alpha l - 2\sigma}\right) - M''(v,r)(\alpha l - 2\sigma)^2}{r(\alpha l - 2\sigma)^2}\,, \\
G_3^3 &= \frac{2\alpha(\alpha l + 2r - 2\sigma)\exp\left(\frac{2r}{\alpha l - 2\sigma}\right) - M''(v,r)(\alpha l - 2\sigma)^2}{r(\alpha l - 2\sigma)^2}\,.
\end{aligned}
\tag{39}
$$

We can decouple this tensor into two: one corresponds to the Θ_{ik} matter field and, exactly as in (19), the other Einstein tensor corresponds to the energy-momentum tensor $\tilde{T}_{ik}$, which is a mixture of the two energy-momentum tensors of type-I and type-II matter fields:

$$
\tilde{T}_{ik} = \tilde{T}_{ik}^{nulldust} + \tilde{T}_{ik}^{nullstring}\,. \tag{40}
$$

Here, $\tilde{T}_{ik}^{nulldust}$ is from (33) and $\tilde{T}_{ik}^{nullstring}$ is from (15). The Einstein tensor corresponding to the case $\alpha = 0$ is given by:

$$
\begin{aligned}
G_0^0 &= -\frac{2M'(v,r)}{r^2}\,, \\
G_0^1 &= \frac{2\dot{M}(v,r)}{r^2}\,, \\
G_1^1 &= -\frac{2M'(v,r)}{r^2}\,, \\
G_2^2 &= -\frac{M''(v,r)}{r}\,, \\
G_3^3 &= -\frac{M''(v,r)}{r}\,.
\end{aligned}
\tag{41}
$$

To obtain the mass function, one should impose the equation of the state $P = \xi\rho$. Then, the mass function is given by:

$$
\begin{aligned}
M(v,r) &= C(v) + D(v)r^{1-2\xi}\,, \\
\xi &\neq \frac{1}{2}\,, \xi \in [-1\,,1]\,.
\end{aligned}
\tag{42}
$$

where $C(v)$ and $D(v)$ are arbitrary functions of time v. The energy conditions for the tensor Θ_{ik} were obtained in Ref. [12] and are the same in this case; however, weak, strong, and dominant energy conditions in all three cases demand:

$$
\mu \geq 0\,, \hat{\rho} \geq 0\,, \hat{\rho} \geq \hat{P}\,, \hat{P} \geq 0\,. \tag{43}
$$

The interaction between $\tilde{T}_{ik}$ and Θ_{ik} is purely gravitational, i.e.,:

$$
\tilde{T}^{ik}_{;k} = \Theta^{ik}_{;k} = 0\,. \tag{44}
$$

Calculating the expansion Θ_l for null outgoing geodesic congruence

$$e^{\gamma}\Theta_l = \frac{2}{r}\left(1 - \frac{2M(v,r)}{r} + \frac{\alpha}{\exp\left(\frac{r}{-\alpha l+2\sigma}\right)^2}\right). \tag{45}$$

one can easily see that $g_{00} = 0$ is again the apparent horizon equation:

$$1 - \frac{2M(v,r)}{r} + \frac{\alpha}{\exp\left(\frac{r}{-\alpha l+2\sigma}\right)^2} = 0\,. \tag{46}$$

The Kretschmann scalar shows that for all three cases the singularity location at $r = 0$ [1]:

$$\begin{aligned} K = \tfrac{1}{(\alpha l-2\sigma)^4 r^6}\Big(&-16r\big(r^4M''(v,r) + 2r^2(\alpha l - r - 2\sigma)M'(v,r) + \\ &\Big(2r^2 + (-2\alpha l + 4\sigma)r + (\alpha l - 2\sigma)^2\Big)M(v,r)\big)\alpha(\alpha l - 2\sigma)^2\exp^{\frac{2r}{\alpha l-2\sigma}} + \\ &4r^2\Big(2r^2 + (\alpha l - 2\sigma)^2\Big)^2\alpha^2\exp^{\frac{4r}{\alpha l-2\sigma}} + \\ &4(\alpha l - 2\sigma)^4\big(r^4M''(v,r)^2 + \\ &\big(-4r^3M'(v,r) + 4r^2M(v,r)\big)M''(v,r) + \\ &8r^2M'(v,r)^2 - 16rM(v,r)M'(v,r) + 12M(v,r)^2\big)\Big) \end{aligned} \tag{47}$$

5. Conclusions

In this work, using the gravitational decoupling method, we obtained new dynamical solutions—Vaidya and generalized Vaidya spacetimes. Despite the fact that the g_{00} component of Vaidya spacetimes depends on time, we can easily decouple two (Vaidya spacetime) or three (generalized one) gravitational sources. Moreover, we preserve the conservation laws for the energy-momentum tensor. It means that there is no energy exchange between these matter fields, and they interact only by gravitation. This fact allows us to consider Θ_{ik} as a dark matter source. The results of this paper will allow us to consider the gravitational collapse problem and how the new matter field might affect the gravitational collapse process. In this paper, we briefly considered the structure of the obtained spacetimes, i.e., we calculated only the apparent horizon and singularity location and proved that the apparent horizon equation is always $g_{00} = 0$ and the singularity is located at $r = 0$. The Vaidya metric describes a dynamical spacetime instead of a static spacetime as the Schwarzschild or Reissner–Nordstrom metrics do. In the real world, astronomical bodies gain mass when they absorb radiation, and they lose mass when they emit radiation, which means that the space time around them is time dependent. As we pointed out, the Vaidya spacetime can be used as the simplest model of gravitational collapse. New solutions by the gravitational decoupling method allow us to investigate the question of how an additional matter field will affect the gravitational collapse process. When we consider the gravitational collapse of Vaidya spacetimes, one might expect the naked singularity to form. New solutions can tell us how Θ_{ik} will influence the result of the gravitational collapse. Vaidya spacetimes are currently widely used and the important question of the global structure of new solutions is the direction of future research. We have already explained that Θ_{ik} can be thought of as the energy-momentum tensor of a dark matter. So, the obtained solution can tell us how the well-known properties of the Vaidya spacetimes change when an additional matter field is present. These properties should also be studied in the future.

We consider the additional matter source Θ_{ik} to be static in this paper. However, it is interesting if one can decouple the Einstein equations, which can be achieved if the parameter σ connected to the Misner–Sharp mass is also time-depended.

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Note

[1] Here, we only show that these solutions do not generate new singularities except for the singular point at $r = 0$. However, as we pointed out earlier, $r = 0$ might be a regular point in the case $M = M(r)$.

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