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Improved Soliton Solutions of Generalized Fifth Order Time-Fractional KdV Models: Laplace Transform with Homotopy Perturbation Algorithm

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Abstract: The main purpose of this research is to propose a new methodology to observe a class of time-fractional generalized fifth-order Korteweg–de Vries equations. Laplace transform along with a homotopy perturbation algorithm is utilized for the solution and analysis purpose in the current study. This extended technique provides improved and convergent series solutions through symbolic computation. The proposed methodology is applied to time-fractional Sawada–Kotera, Ito, Lax's, and Kaup–Kupershmidt models, which are induced from a generalized fifth-order KdV equation. For validity purposes, obtained and existing results at integral orders are compared. Convergence analysis was also performed by computing solutions and errors at different values in a fractional domain. Dynamic behavior of the fractional parameter is also studied graphically. Simulations affirm the dominance of the proposed algorithm in terms of accuracy and fewer computations as compared to other available schemes for fractional KdVs. Hence, the projected algorithm can be utilized for more advanced fractional models in physics and engineering.

Keywords: fractional partial differential equations; Korteweg–de Vries equations; time-fractional Sawada–Kotera equation; time-fractional Ito equation; time-fractional Lax's equation; time-fractional Kaup–Kupershmidt equation; Laplace transform; homotopy perturbation

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Citation: Qayyum, M.; Ahmad, E.; Riaz, M.B.; Awrejcewicz, J. Improved Soliton Solutions of Generalized Fifth Order Time-Fractional KdV Models: Laplace Transform with Homotopy Perturbation Algorithm. *Universe* **2022**, *8*, 563. https://doi.org/10.3390/universe8110563

Academic Editor: Yakup Yildirim

Received: 28 September 2022
Accepted: 25 October 2022
Published: 27 October 2022

1. Introduction

Modeling and analysis of physical phenomena is an essential step in scientific work. There are many important mathematical models for capturing different situations in various fields. Among many, Korteweg–de Vries models are essential due to their ability of capturing physical situations, such as the motion of long waves in shallow water with weak non-linearities. Initially, this model was derived as 1D small amplitude and long surface gravity waves. Flood analysis and ocean flow analysis are a few uses of shallow water equations. Scott Russell in 1834 [1], Boussinesq and Rayleigh in 1870 [2–4], and Korteweg and De Vries in 1895 [5] played a major role in the discovery of the KdV model. These models have many uses in collision-free hydromagnetic waves, stratified internal waves, shock wave formation [6], fluid and quantum mechanics [7], and in biology [8].

A generalized fifth-order time-fractional KdV equation is usually in the following form of non-linear partial differential equation(PDE):

$$\frac{\partial^{\beta}\mathbf{J}}{\partial\mathfrak{t}^{\beta}}+a\mathbf{J}^{2}\frac{\partial\mathbf{J}}{\partial\mathfrak{r}}+b\frac{\partial\mathbf{J}}{\partial\mathfrak{r}}\frac{\partial^{2}\mathbf{J}}{\partial\mathfrak{r}^{2}}+c\mathbf{J}\frac{\partial^{3}\mathbf{J}}{\partial\mathfrak{r}^{3}}+\frac{\partial^{5}\mathbf{J}}{\partial\mathfrak{r}^{5}}=0, \tag{1}$$

where $\mathbf{J}$ is the wave function with $\mathfrak{r}$ and $\mathfrak{t}$ as the space and time variables, respectively. In addition, β represents the fractional order parameter. The model in Equation (1) has various physical applications in acoustic magnetic propagation in plasma, incompressible and inviscid fluids, gravitational field, etc. It typically consists of one linear dispersive term and three non-linear terms. The linear dispersive term has a momentous role in the balancing of non-linearity and dispersion effects of soliton behavior [9]; a, b, and c in Equation (1) are real constant parameters. Different values of these parameters give different versions of time-fractional KdV equations. For instance, $a = 45$, $b = 15$, and $c = 15$ give a time-fractional, Sawada–Kotera equation [10]. Fixing a, b, and c as 2, 6, and 3, respectively, leads to a time-fractional Ito equation [11]. In addition, when $a = 30$, $b = 20$, and $c = 10$, it gives a time-fractional Lax's KdV equation [12,13]. Similarly, $a = 45$, $b = -15$, and $c = -15$ leads to a time-fractional Kaup–Kupershmidt equation [14]. Many researchers tried to find exact solutions of these equations in [15–18].

Since fractional calculus deals with the calculation of non-integer order derivatives and integrals, it appeals to a wide range of audiences while dealing with fractional ordinary and partial differential equations. Different numerical and analytical methods, including the homotopy analysis method [19], the residual power series method [20], the variation iteration method [21], the Adomian decomposition method [22], the Bernstein polynomials method [23], homotopy perturbation method [24], etc., are used in the literature while dealing with fractional problems. Many modifications of existing techniques are also applied to fractional KdV equations [25–27].

The homotopy perturbation method (HPM), introduced by He [28], provides a convenient technique to find analytical solutions of linear and non-linear differential equations in both fractional and integer form. To reduce errors and increase its reliability, many modifications of HPM [29–32] have been adapted in the literature. The Laplace homotopy perturbation method (LHPM), which combines HPM with Laplace transform, is an efficient and convenient modification of HPM for solving fractional and integral differential equations [33]. Results obtained from LHPM are highly accurate and straightforward without imposing any restrictions on the concerned model. In this paper, we have extended LHPM to generalized fifth-order highly non-linear time-fractional KdV models. In the rest of the manuscript, Sections 2 and 3 consist of preliminaries and the general concept of LHPM for fractional KdV equations. Convergence and error estimation are presented in Section 4. Applications and solutions of different KdV models are explained in Section 5. Discussion and conclusion are presented in Sections 6 and 7, respectively.

2. Preliminaries

Definition 1 ([34])**.** *The Laplace transform $\mathscr{L}$ of the Riemann–Liouville time-fractional integral $\mathscr{I}_t^{\beta}$ on a function $\mathcal{J}(\mathfrak{v},\mathfrak{t})$ is described as:*

$$\mathscr{L}[\mathscr{I}_t^{\beta}\mathcal{J}(\mathfrak{v},\mathfrak{t})] = s^{-\beta}\mathscr{L}[\mathcal{J}(\mathfrak{v},\mathfrak{t})], \qquad f-1<\beta\leq f. \tag{2}$$

Definition 2 ([35])**.** *The Laplace transform $\mathscr{L}$ of Caputo's time-fractional derivative $\mathscr{D}_t^{\beta}$ on a function $\mathcal{J}(\mathfrak{v},\mathfrak{t})$ is described as:*

$$\mathscr{L}[\mathscr{D}_t^{\beta}\mathcal{J}(\mathfrak{v},\mathfrak{t})] = s^{\beta}\mathscr{L}[\mathcal{J}(\mathfrak{v},\mathfrak{t})] - \sum_{j=0}^{f-1} s^{\beta-j-1}\mathcal{J}^{(f)}(\mathfrak{v},0), \qquad f-1<\beta\leq f. \tag{3}$$

where $\mathcal{J}^{(f)}(\mathfrak{v},0)$ represents the initial conditions.

Lemma 1 ([36])**.** *Let $\mathcal{J}$ be a function, then for a positive constant c, the stability result states*

$$\parallel \mathcal{J}(u) - \mathcal{J}(v) \parallel \leq c \parallel u - v \parallel, \qquad u, v \,\epsilon\, R^n. \tag{4}$$

3. Fundamental Concept of the Laplace Homotopy Perturbation Method for Fifth-Order Time-Fractional KdV Models

Let us consider a general non-linear, fifth-order, time-fractional KdV equation as:

$$\frac{\partial^{\beta}\mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{t}^{\beta}} + a\mathbf{J}^{2}(\mathfrak{x},\mathfrak{t})\frac{\partial \mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}} + b\frac{\partial \mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}}\frac{\partial^{2}\mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^{2}} + c\mathbf{J}(\mathfrak{x},\mathfrak{t})\frac{\partial^{3}\mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^{3}} + \frac{\partial^{5}\mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^{5}} = 0, \quad \mathfrak{x}\,\epsilon\,\Omega,\ \mathfrak{t} > 0,\ f-1 < \beta \leq f, \tag{5}$$

with initial conditions

$$\mathbf{J}^{(f)}(\mathfrak{x},0) = \mathcal{I}^{f}, \qquad f = 0,1,2,\ldots, \tag{6}$$

where $\mathbf{J}(\mathfrak{x},\mathfrak{t})$ is an unknown function with $\frac{\partial^{\beta}}{\partial \mathfrak{t}^{\beta}}$ as its fractional derivative; a, b, and c are constant parameters that give distant versions of time fractional KdV equations for different values.

The first step of the Laplace transform algorithm is applied on both sides of (5), which gives

$$\mathscr{L}\{\frac{\partial^{\beta}\mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{t}^{\beta}} + a\mathbf{J}^{2}(\mathfrak{x},\mathfrak{t})\frac{\partial \mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}} + b\frac{\partial \mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}}\frac{\partial^{2}\mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^{2}} + c\mathbf{J}(\mathfrak{x},\mathfrak{t})\frac{\partial^{3}\mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^{3}} + \frac{\partial^{5}\mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^{5}}\} = 0, \tag{7}$$

Using (3) gives

$$\mathscr{L}[\mathbf{J}(\mathfrak{x},\mathfrak{t})] - \left(\frac{1}{s^{\beta}}\right)\sum_{j=0}^{f-1} s^{\beta-j-1}\mathbf{J}^{(f)}(\mathfrak{x},0) + \left(\frac{1}{s^{\beta}}\right)\mathscr{L}\left\{a\mathbf{J}^{2}(\mathfrak{x},\mathfrak{t})\frac{\partial \mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}} + b\frac{\partial \mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}}\frac{\partial^{2}\mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^{2}} + c\mathbf{J}(\mathfrak{x},\mathfrak{t})\frac{\partial^{3}\mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^{3}} + \frac{\partial^{5}\mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^{5}}\right\} = 0, \tag{8}$$

Now, we can construct a homotopy

$$\mathcal{H} = (1-p)(\mathscr{L}\{\mathbf{J}(\mathfrak{x},\mathfrak{t};p)\} - \mathbf{J}_{0}(\mathfrak{x},\mathfrak{t})) + p\Bigg(\mathscr{L}\{\mathbf{J}(\mathfrak{x},\mathfrak{t};p)\} - \left(\frac{1}{s^{\beta}}\right)\sum_{j=0}^{f-1} s^{\beta-j-1}\mathbf{J}^{(f)}(\mathfrak{x},0) + \left(\frac{1}{s^{\beta}}\right)\mathscr{L}\left\{a\mathbf{J}^{2}(\mathfrak{x},\mathfrak{t})\frac{\partial \mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}} + b\frac{\partial \mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}}\frac{\partial^{2}\mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^{2}} + c\mathbf{J}(\mathfrak{x},\mathfrak{t})\frac{\partial^{3}\mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^{3}} + \frac{\partial^{5}\mathbf{J}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^{5}}\right\}\Bigg), \tag{9}$$

where $\mathbf{J}_{0}(\mathfrak{x},\mathfrak{t})$ is the initial guess, and it satisfies the given conditions.

Expanding $\mathbf{J}(\mathfrak{x},\mathfrak{t})$ in Taylor series with regard to p gives:

$$\mathbf{J}(\mathfrak{x},\mathfrak{t};p) = \sum_{m=1}^{\infty} p^{m}\mathbf{J}_{m}, \tag{10}$$

Substituting (10) into (9) and then comparing the coefficients of p, we have distinct order problems.

The problem at the first order is:

$$\mathscr{L}\{\mathbf{J}_{1}(\mathfrak{x},\mathfrak{t})\} + \mathbf{J}_{0}(\mathfrak{x},\mathfrak{t}) - \left(\frac{1}{s^{\beta}}\right)\sum_{j=0}^{f-1} s^{\beta-j-1}\mathbf{J}^{(f)}(\mathfrak{x},0) + \left(\frac{1}{s^{\beta}}\right)\mathscr{L}\left\{a\mathbf{J}_{0}^{2}(\mathfrak{x},\mathfrak{t})\frac{\partial \mathbf{J}_{0}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}} + b\frac{\partial \mathbf{J}_{0}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}}\frac{\partial^{2}\mathbf{J}_{0}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^{2}} + c\mathbf{J}_{0}(\mathfrak{x},\mathfrak{t})\frac{\partial^{3}\mathbf{J}_{0}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^{3}} + \frac{\partial^{5}\mathbf{J}_{0}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^{5}}\right\} = 0, \tag{11}$$

Employing inverse Laplace transform gives

$$\begin{aligned}\mathbf{J}_1(\mathfrak{x},\mathfrak{t}) = \mathscr{L}^{-1}\Bigg\{ -\mathbf{J}_0(\mathfrak{x},\mathfrak{t}) + \left(\frac{1}{s^{\beta}}\right)\sum_{j=0}^{f-1} s^{\beta-j-1}\mathbf{J}^{(f)}(\mathfrak{x},0)\Bigg\} - \mathscr{L}^{-1}\Bigg\{\left(\frac{1}{s^{\beta}}\right)\mathscr{L}\Bigg\{a\mathbf{J}_0^2(\mathfrak{x},\mathfrak{t}) \\ \frac{\partial \mathbf{J}_0(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}} + b\frac{\partial \mathbf{J}_0(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}}\frac{\partial^2 \mathbf{J}_0(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^2} + c\mathbf{J}_0(\mathfrak{x},\mathfrak{t})\frac{\partial^3 \mathbf{J}_0(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^3} + \frac{\partial^5 \mathbf{J}_0(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^5}\Bigg\}\Bigg\},\end{aligned} \tag{12}$$

The the problem at the kth order is:

$$\begin{aligned}\mathscr{L}\{\mathbf{J}_k(\mathfrak{x},\mathfrak{t})\} + \left(\frac{1}{s^{\beta}}\right)\mathscr{L}\Bigg\{a\mathbf{J}_{k-1}^2(\mathfrak{x},\mathfrak{t})\frac{\partial \mathbf{J}_{k-1}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}} + b\frac{\partial \mathbf{J}_{k-1}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}}\frac{\partial^2 \mathbf{J}_{k-1}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^2} \\ + c\mathbf{J}_{k-1}(\mathfrak{x},\mathfrak{t})\frac{\partial^3 \mathbf{J}_{k-1}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^3} + \frac{\partial^5 \mathbf{J}_{k-1}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^5}\Bigg\} = 0,\end{aligned} \tag{13}$$

By taking Inverse Laplace transform, we have

$$\begin{aligned}\mathbf{J}_k(\mathfrak{x},\mathfrak{t}) = \mathscr{L}^{-1}\Bigg\{ -\left(\frac{1}{s^{\beta}}\right)\mathscr{L}\Bigg\{a\mathbf{J}_{k-1}^2(\mathfrak{x},\mathfrak{t})\frac{\partial \mathbf{J}_{k-1}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}} + b\frac{\partial \mathbf{J}_{k-1}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}}\frac{\partial^2 \mathbf{J}_{k-1}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^2} \\ + c\mathbf{J}_{k-1}(\mathfrak{x},\mathfrak{t})\frac{\partial^3 \mathbf{J}_{k-1}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^3} + \frac{\partial^5 \mathbf{J}_{k-1}(\mathfrak{x},\mathfrak{t})}{\partial \mathfrak{x}^5}\Bigg\}\Bigg\},\end{aligned} \tag{14}$$

The approximate solution of the general fifth-order KdV equation is

$$\tilde{\mathbf{J}} = \mathbf{J}_0(\mathfrak{x},\mathfrak{t}) + \mathbf{J}_1(\mathfrak{x},\mathfrak{t}) + \mathbf{J}_2(\mathfrak{x},\mathfrak{t}) + \mathbf{J}_3(\mathfrak{x},\mathfrak{t}) + \dots, \tag{15}$$

Residual error is observed by substituting (15) in a given time fractional KdV Equation (5) as:

$$\mathfrak{Res} = \frac{\partial^{\beta}\tilde{\mathbf{J}}}{\partial \mathfrak{t}^{\beta}} + a\tilde{\mathbf{J}}^2\frac{\partial \tilde{\mathbf{J}}}{\partial \mathfrak{x}} + b\frac{\partial \tilde{\mathbf{J}}}{\partial \mathfrak{x}}\frac{\partial^2 \tilde{\mathbf{J}}}{\partial \mathfrak{x}^2} + c\tilde{\mathbf{J}}(\mathfrak{x},\mathfrak{t})\frac{\partial^3 \tilde{\mathbf{J}}}{\partial \mathfrak{x}^3} + \frac{\partial^5 \tilde{\mathbf{J}}}{\partial \mathfrak{x}^5}. \tag{16}$$

4. Convergence and Error Estimation of LHPM for Fractional KdV Equation

4.1. Convergence

Theorem 1. *Consider a Banach space ($\mathbb{B}[0,T]$,$\|\ .\ \|$) and suppose $\mathbf{J}_n(\mathfrak{x},\mathfrak{t})$ and $\mathbf{J}(\mathfrak{x},\mathfrak{t})$ are defined in it. Then, for a constant ζ where $0 < \zeta < 1$, the series solution in (15) converges to the solution of fractional KdV (5).*

Proof. Let $\{a_n\}$ be the sequence of partial sums of (15). We have to prove that $a_n(\mathfrak{x},\mathfrak{t})$ is a Cauchy sequence in ($\mathbb{B}$[0,T],$\|\ .\ \|$). Consider

$$\begin{aligned}\| a_{n+1}(\mathfrak{x},\mathfrak{t}) - a_n(\mathfrak{x},\mathfrak{t}) \| &= \| \mathbf{J}_{n+1}(\mathfrak{x},\mathfrak{t}) \| \\ &\le \zeta \| \mathbf{J}_n(\mathfrak{x},\mathfrak{t}) \| \\ &\le \zeta^2 \| \mathbf{J}_{n-1}(\mathfrak{x},\mathfrak{t}) \| \\ &\le \dots \le \zeta^{n+1} \| \mathbf{J}_0(\mathfrak{x},\mathfrak{t}) \|,\end{aligned} \tag{17}$$

Now, for partial sums a_n and a_m where $n,m \in \mathbb{N}$ and $n \ge m$, using triangle inequality, we obtain

$$\begin{aligned}\| a_n - a_m \| &= \| (a_n(\mathfrak{x},\mathfrak{t}) - a_{n-1}(\mathfrak{x},\mathfrak{t})) + (a_{n-1}(\mathfrak{x},\mathfrak{t}) - a_{n-2}(\mathfrak{x},\mathfrak{t})) \\ &\quad + \dots + (a_{m+1}(\mathfrak{x},\mathfrak{t}) - a_m(\mathfrak{x},\mathfrak{t})) \| \\ &\le \| a_n(\mathfrak{x},\mathfrak{t}) - a_{n-1}(\mathfrak{x},\mathfrak{t}) \| + \| a_{n-1}(\mathfrak{x},\mathfrak{t}) - a_{n-2}(\mathfrak{x},\mathfrak{t}) \| \\ &\quad + \dots + \| a_{m+1}(\mathfrak{x},\mathfrak{t}) - a_m(\mathfrak{x},\mathfrak{t}) \|,\end{aligned} \tag{18}$$

Using (17), we have

$$\begin{aligned}\| a_n - a_m \| &\leq \zeta^n \| \mathbf{J}_0(\mathfrak{x},\mathfrak{t}) \| + \zeta^{n-1} \| \mathbf{J}_0(\mathfrak{x},\mathfrak{t}) \| + \ldots + \zeta^{m+1} \| \mathbf{J}_0(\mathfrak{x},\mathfrak{t}) \| \\ &\leq (\zeta^n + \zeta^{n-1} + \ldots + \zeta^{m+1}) \| \mathbf{J}_0(\mathfrak{x},\mathfrak{t}) \| \\ &\leq \zeta^{m+1}(\zeta^{n-m-1} + \zeta^{n-m-2} + \ldots + \zeta + 1) \| \mathbf{J}_0(\mathfrak{x},\mathfrak{t}) \| \\ &\leq \zeta^{m+1}\left(\frac{1-\zeta^{n-m}}{1-\zeta}\right) \| \mathbf{J}_0(\mathfrak{x},\mathfrak{t}) \|,\end{aligned} \tag{19}$$

Since $0 < \zeta < 1$, therefore, $1 - \zeta^{n-m} < 1$. Thus, we have

$$\| a_n - a_m \| \leq \frac{\zeta^{m+1}}{1-\zeta} max|\mathbf{J}_0(\mathfrak{x},\mathfrak{t})|, \qquad \forall\ \mathfrak{t}\epsilon[0,T], \tag{20}$$

Since $\mathbf{J}_0$ is bounded, so

$$\lim_{n,m\to\infty} \| a_n(\mathfrak{x},\mathfrak{t}) - a_m(\mathfrak{x},\mathfrak{t}) \| = 0. \tag{21}$$

Hence, we have proved that $a_n(\mathfrak{x},\mathfrak{t})$ is a Cauchy sequence in Banach space ($\mathbb{B}$[0,T],$\|$. $\|$). Thus, the series solution in (15) converges to the solution of (5). □

4.2. Error Estimation

Theorem 2. *Consider the time fractional KdV Equation (5), then the maximum absolute truncation error of its solution (15) is*

$$\left|\mathbf{J}(\mathfrak{x},\mathfrak{t}) - \sum_{j=0}^{m} \mathbf{J}_j(\mathfrak{x},\mathfrak{t})\right| \leq \frac{\zeta^{m+1}}{1-\zeta} \| \mathbf{J}_0(\mathfrak{x},\mathfrak{t}) \| . \tag{22}$$

Proof. From (19) we have,

$$\| \mathbf{J}(\mathfrak{x},\mathfrak{t}) - a_m \| \leq \zeta^{m+1}\left(\frac{1-\zeta^{n-m}}{1-\zeta}\right) \| \mathbf{J}_0(\mathfrak{x},\mathfrak{t}) \|, \tag{23}$$

Since $0 < \zeta < 1$, therefore $1 - \zeta^{n-m} < 1$. Thus we have

$$\left|\mathbf{J}(\mathfrak{x},\mathfrak{t}) - \sum_{j=0}^{m} \mathbf{J}_j(\mathfrak{x},\mathfrak{t})\right| \leq \frac{\zeta^{m+1}}{1-\zeta} \| \mathbf{J}_0(\mathfrak{x},\mathfrak{t}) \| . \tag{24}$$

□

5. Solutions of Time-Fractional KdV Models Using Laplace Homotopy Perturbation Method

Example 1. *Consider the time fractional, Sawada–Kotera equation*

$$\frac{\partial^\beta \mathbf{J}}{\partial \mathfrak{t}^\beta} = -45\mathbf{J}^2\frac{\partial \mathbf{J}}{\partial \mathfrak{x}} - 15\frac{\partial \mathbf{J}}{\partial \mathfrak{x}}\frac{\partial^2 \mathbf{J}}{\partial \mathfrak{x}^2} - 15\mathbf{J}\frac{\partial^3 \mathbf{J}}{\partial \mathfrak{x}^3} - \frac{\partial^5 \mathbf{J}}{\partial \mathfrak{x}^5}, \qquad 0 < \beta \leq 1,\ \mathfrak{t} > 0, \tag{25}$$

associated with initial condition

$$\mathbf{J}(\mathfrak{x},0) = 2k^2\mathrm{sech}^2(k\mathfrak{x}), \tag{26}$$

where $k \neq 0$. The exact solution of (25) is:

$$\mathbf{J}(\mathfrak{x},\mathfrak{t}) = 2k^2\mathrm{sech}^2(k(\mathfrak{x} - 16k^4\mathfrak{t})). \tag{27}$$

Solution: Taking Laplace transform of (25) and then by using (3), we have

$$s^{\beta}\mathscr{L}[\mathbf{J}(\mathfrak{x},\mathfrak{t})]-s^{\beta-1}2k^{2}\mathrm{sech}^{2}(k\mathfrak{x})-\mathscr{L}\left\{-45\mathbf{J}^{2}\frac{\partial\mathbf{J}}{\partial\mathfrak{x}}-15\frac{\partial\mathbf{J}}{\partial\mathfrak{x}}\frac{\partial^{2}\mathbf{J}}{\partial\mathfrak{x}^{2}}-15\mathbf{J}\frac{\partial^{3}\mathbf{J}}{\partial\mathfrak{x}^{3}}-\frac{\partial^{5}\mathbf{J}}{\partial\mathfrak{x}^{5}}\right\}=0, \tag{28}$$

Using (9), homotopy can be constructed as follows:

$$\begin{aligned}\mathcal{H}=(1-p)(\mathscr{L}\{\mathbf{J}(\mathfrak{x},\mathfrak{t})\}-\mathbf{J}_{0}(\mathfrak{x},\mathfrak{t}))+p\bigg(\mathscr{L}\{\mathbf{J}(\mathfrak{x},\mathfrak{t})\}-\left(\frac{1}{s}\right)2k^{2}\mathrm{sech}^{2}(k\mathfrak{x})-\\ \left(\frac{1}{s^{\beta}}\right)\mathscr{L}\left\{-45\mathbf{J}^{2}\frac{\partial\mathbf{J}}{\partial\mathfrak{x}}-15\frac{\partial\mathbf{J}}{\partial\mathfrak{x}}\frac{\partial^{2}\mathbf{J}}{\partial\mathfrak{x}^{2}}-15\mathbf{J}\frac{\partial^{3}\mathbf{J}}{\partial\mathfrak{x}^{3}}-\frac{\partial^{5}\mathbf{J}}{\partial\mathfrak{x}^{5}}\right\}\bigg),\end{aligned} \tag{29}$$

where $\mathbf{J}_0(\mathfrak{x},\mathfrak{t})$ is the initial guess which satisfies the initial condition (26). For the current problem, the following initial guess is taken

$$\mathbf{J}_{0}(\mathfrak{x},\mathfrak{t})=2k^{2}\mathrm{sech}^{2}(k\mathfrak{x}), \tag{30}$$

Expanding $\mathbf{J}(\mathfrak{x},\mathfrak{t})$ in Taylor's series with regard to p, and then comparing the coefficients of identical powers of p, we have.

First-order problem:

$$\begin{aligned}\mathscr{L}\{\mathbf{J}_{1}(\mathfrak{x},\mathfrak{t})\}+\mathbf{J}_{0}(\mathfrak{x},\mathfrak{t})-\left(\frac{1}{s}\right)2k^{2}\mathrm{sech}^{2}(k\mathfrak{x})-\left(\frac{1}{s^{\beta}}\right)\mathscr{L}\bigg\{-45\mathbf{J}_{0}^{2}\frac{\partial\mathbf{J}_{0}}{\partial\mathfrak{x}}-15\frac{\partial\mathbf{J}_{0}}{\partial\mathfrak{x}}\frac{\partial^{2}\mathbf{J}_{0}}{\partial\mathfrak{x}^{2}}\\ -15\mathbf{J}_{0}\frac{\partial^{3}\mathbf{J}_{0}}{\partial\mathfrak{x}^{3}}-\frac{\partial^{5}\mathbf{J}_{0}}{\partial\mathfrak{x}^{5}}\bigg\}=0,\\ \mathbf{J}_{1}(\mathfrak{x},0)=0,\end{aligned} \tag{31}$$

Taking inverse Laplace transform leads to a solution at the first order:

$$\mathbf{J}_{1}(\mathfrak{x},\mathfrak{t})=\mathfrak{t}k^{7}(64\mathrm{sech}^{6}(\mathfrak{x}k)\tanh(\mathfrak{x}k)+128\mathrm{sech}^{4}(\mathfrak{x}k)\tanh^{3}(\mathfrak{x}k)+64\mathrm{sech}^{2}(\mathfrak{x}k)\tanh^{5}(\mathfrak{x}k)), \tag{32}$$

Second-order problem:

$$\begin{aligned}\mathscr{L}\{\mathbf{J}_{2}(\mathfrak{x},\mathfrak{t})\}-\left(\frac{1}{s^{\beta}}\right)\mathscr{L}\bigg\{-45\mathbf{J}_{1}^{2}\frac{\partial\mathbf{J}_{1}}{\partial\mathfrak{x}}-15\frac{\partial\mathbf{J}_{1}}{\partial\mathfrak{x}}\frac{\partial^{2}\mathbf{J}_{1}}{\partial\mathfrak{x}^{2}}-15\mathbf{J}_{1}\frac{\partial^{3}\mathbf{J}_{1}}{\partial\mathfrak{x}^{3}}-\frac{\partial^{5}\mathbf{J}_{1}}{\partial\mathfrak{x}^{5}}\bigg\}=0,\\ \mathbf{J}_{2}(\mathfrak{x},0)=0,\end{aligned} \tag{33}$$

Second-order solution:

$$\mathbf{J}_{2}(\mathfrak{x},\mathfrak{t})=-512\mathfrak{t}^{2}k^{12}\mathrm{sech}^{2}(\mathfrak{x},\mathfrak{t})(\mathrm{sech}^{2}(\mathfrak{x},\mathfrak{t})-2\tanh^{2}(\mathfrak{x},\mathfrak{t}))(\mathrm{sech}^{2}(\mathfrak{x},\mathfrak{t})+\tanh^{2}(\mathfrak{x},\mathfrak{t}))^{4}, \tag{34}$$

Third-order problem:

$$\begin{aligned}\mathscr{L}\{\mathbf{J}_{3}(\mathfrak{x},\mathfrak{t})\}-\left(\frac{1}{s^{\beta}}\right)\mathscr{L}\bigg\{-45\mathbf{J}_{2}^{2}\frac{\partial\mathbf{J}_{2}}{\partial\mathfrak{x}}-15\frac{\partial\mathbf{J}_{2}}{\partial\mathfrak{x}}\frac{\partial^{2}\mathbf{J}_{2}}{\partial\mathfrak{x}^{2}}-15\mathbf{J}_{2}\frac{\partial^{3}\mathbf{J}_{2}}{\partial\mathfrak{x}^{3}}-\frac{\partial^{5}\mathbf{J}_{2}}{\partial\mathfrak{x}^{5}}\bigg\}=0,\\ \mathbf{J}_{3}(\mathfrak{x},0)=0,\end{aligned} \tag{35}$$

Third order-solution:

$$\begin{aligned}\mathbf{J}_{3}(\mathfrak{x},\mathfrak{t})=-\frac{32768}{3}\mathfrak{t}^{3}k^{17}\mathrm{sech}^{2}(\mathfrak{x},\mathfrak{t})\tanh(\mathfrak{x},\mathfrak{t})(2\mathrm{sech}^{2}(\mathfrak{x},\mathfrak{t})-\tanh^{2}(\mathfrak{x},\mathfrak{t}))(\mathrm{sech}^{2}(\mathfrak{x},\mathfrak{t})\\ +\tanh^{2}(\mathfrak{x},\mathfrak{t}))^{6},\end{aligned} \tag{36}$$

Fourth-order problem:

$$\mathscr{L}\{\mathbf{J}_4(\mathfrak{x},\mathfrak{t})\} - \left(\frac{1}{s^\beta}\right)\mathscr{L}\left\{-45\mathbf{J}_3^2\frac{\partial \mathbf{J}_3}{\partial \mathfrak{x}} - 15\frac{\partial \mathbf{J}_3}{\partial \mathfrak{x}}\frac{\partial^2 \mathbf{J}_3}{\partial \mathfrak{x}^2} - 15\mathbf{J}_3\frac{\partial^3 \mathbf{J}_3}{\partial \mathfrak{x}^3} - \frac{\partial^5 \mathbf{J}_3}{\partial \mathfrak{x}^5}\right\} = 0, \quad \mathbf{J}_4(\mathfrak{x},0) = 0, \tag{37}$$

Fourth-order solution:

$$\mathbf{J}_4(\mathfrak{x},\mathfrak{t}) = \frac{131072}{3}\mathfrak{t}^4 k^{22}\mathrm{sech}^2(\mathfrak{x},\mathfrak{t})(\mathrm{sech}^2(\mathfrak{x},\mathfrak{t}) + \tanh^2(\mathfrak{x},\mathfrak{t}))^8(2\mathrm{sech}^4(\mathfrak{x},\mathfrak{t}) - 11\mathrm{sech}^2(\mathfrak{x},\mathfrak{t})\tanh^2(\mathfrak{x},\mathfrak{t}) + 2\tanh^4(\mathfrak{x},\mathfrak{t})), \tag{38}$$

The rest of values for $\mathbf{J}_i(\mathfrak{x},\mathfrak{t})$ with i ≥ 5 can be computed in a similar way. The approximate solution of (25) can be captured by

$$\tilde{\mathbf{J}} = \sum_{i=0}^{4}\mathbf{J}_i(\mathfrak{x},\mathfrak{t}) \tag{39}$$

Residual error is obtained by substituting (39) in (25) as follows:

$$\mathfrak{Res}\mathbf{J} = \frac{\partial^\beta \tilde{\mathbf{J}}}{\partial \mathfrak{t}^\beta} + 45\tilde{\mathbf{J}}^2\frac{\partial \tilde{\mathbf{J}}}{\partial \mathfrak{x}} + 15\frac{\partial \tilde{\mathbf{J}}}{\partial \mathfrak{x}}\frac{\partial^2 \tilde{\mathbf{J}}}{\partial \mathfrak{x}^2} + 15\tilde{\mathbf{J}}\frac{\partial^3 \tilde{\mathbf{J}}}{\partial \mathfrak{x}^3} + \frac{\partial^5 \tilde{\mathbf{J}}}{\partial \mathfrak{x}^5}. \tag{40}$$

Example 2. *Consider the following time-fractional Ito model*

$$\frac{\partial^\beta \mathbf{J}}{\partial \mathfrak{t}^\beta} = -2\mathbf{J}^2\frac{\partial \mathbf{J}}{\partial \mathfrak{x}} - 6\frac{\partial \mathbf{J}}{\partial \mathfrak{x}}\frac{\partial^2 \mathbf{J}}{\partial \mathfrak{x}^2} - 3\mathbf{J}\frac{\partial^3 \mathbf{J}}{\partial \mathfrak{x}^3} - \frac{\partial^5 \mathbf{J}}{\partial \mathfrak{x}^5}, \qquad 0 < \beta \leq 1,\ \mathfrak{t} > 0, \tag{41}$$

along with the initial condition

$$\mathbf{J}(\mathfrak{x},0) = 20k^2 - 30k^2\tanh^2(k\mathfrak{x}), \tag{42}$$

where $k \neq 0$. The exact solution of (41) is

$$\mathbf{J}(\mathfrak{x},\mathfrak{t}) = 20k^2 - 30k^2\tanh^2(k(\mathfrak{x} - 96k^4\mathfrak{t})). \tag{43}$$

Solution: Applying the Laplace transform on (41) and using (3) give the following:

$$s^\beta\mathscr{L}[\mathbf{J}(\mathfrak{x},\mathfrak{t})] - s^{\beta-1}20k^2 - 30k^2\tanh^2(k\mathfrak{x}) - \mathscr{L}\left\{-2\mathbf{J}^2\frac{\partial \mathbf{J}}{\partial \mathfrak{x}} - 6\frac{\partial \mathbf{J}}{\partial \mathfrak{x}}\frac{\partial^2 \mathbf{J}}{\partial \mathfrak{x}^2} - 3\mathbf{J}\frac{\partial^3 \mathbf{J}}{\partial \mathfrak{x}^3} - \frac{\partial^5 \mathbf{J}}{\partial \mathfrak{x}^5}\right\} = 0, \tag{44}$$

Utilizing (9) homotopy for the given problem is

$$\mathcal{H} = (1-p)(\mathscr{L}\{\mathbf{J}(\mathfrak{x},\mathfrak{t})\} - \mathbf{J}_0(\mathfrak{x},\mathfrak{t})) + p\Bigg(\mathscr{L}\{\mathbf{J}(\mathfrak{x},\mathfrak{t})\} - \left(\frac{1}{s}\right)20k^2 - 30k^2\tanh^2(k\mathfrak{x}) - \left(\frac{1}{s^\beta}\right)\mathscr{L}\left\{-2\mathbf{J}^2\frac{\partial \mathbf{J}}{\partial \mathfrak{x}} - 6\frac{\partial \mathbf{J}}{\partial \mathfrak{x}}\frac{\partial^2 \mathbf{J}}{\partial \mathfrak{x}^2} - 3\mathbf{J}\frac{\partial^3 \mathbf{J}}{\partial \mathfrak{x}^3} - \frac{\partial^5 \mathbf{J}}{\partial \mathfrak{x}^5}\right\}\Bigg), \tag{45}$$

where

$$\mathbf{J}_0(\mathfrak{x},\mathfrak{t}) = 20k^2 - 30k^2\tanh^2(k\mathfrak{x}), \tag{46}$$

By replacing (10) in (41) and then comparing coefficients of like power of p, leads to the following:

First-order problem:

$$\mathscr{L}\{\mathbf{J}_1(\mathfrak{x},\mathfrak{t})\}+\mathbf{J}_0(\mathfrak{x},\mathfrak{t})-\left(\frac{1}{s}\right)20k^2-30k^2\tanh^2(k\mathfrak{x})-\left(\frac{1}{s^\beta}\right)\mathscr{L}\left\{-2\mathbf{J}_0^2\frac{\partial\mathbf{J}_0}{\partial\mathfrak{x}}-6\frac{\partial\mathbf{J}_0}{\partial\mathfrak{x}}\frac{\partial^2\mathbf{J}_0}{\partial\mathfrak{x}^2}-3\mathbf{J}_0\frac{\partial^3\mathbf{J}_0}{\partial\mathfrak{x}^3}-\frac{\partial^5\mathbf{J}_0}{\partial\mathfrak{x}^5}\right\}=0,\quad \mathbf{J}_1(\mathfrak{x},0)=0, \tag{47}$$

Second-order problem:

$$\mathscr{L}\{\mathbf{J}_2(\mathfrak{x},\mathfrak{t})\}-\left(\frac{1}{s^\beta}\right)\mathscr{L}\left\{-2\mathbf{J}_1^2\frac{\partial\mathbf{J}_1}{\partial\mathfrak{x}}-6\frac{\partial\mathbf{J}_1}{\partial\mathfrak{x}}\frac{\partial^2\mathbf{J}_1}{\partial\mathfrak{x}^2}-3\mathbf{J}_1\frac{\partial^3\mathbf{J}_1}{\partial\mathfrak{x}^3}-\frac{\partial^5\mathbf{J}_1}{\partial\mathfrak{x}^5}\right\}=0,\quad \mathbf{J}_2(\mathfrak{x},0)=0, \tag{48}$$

Third-order problem:

$$\mathscr{L}\{\mathbf{J}_3(\mathfrak{x},\mathfrak{t})\}-\left(\frac{1}{s^\beta}\right)\mathscr{L}\left\{-2\mathbf{J}_2^2\frac{\partial\mathbf{J}_2}{\partial\mathfrak{x}}-6\frac{\partial\mathbf{J}_2}{\partial\mathfrak{x}}\frac{\partial^2\mathbf{J}_2}{\partial\mathfrak{x}^2}-3\mathbf{J}_2\frac{\partial^3\mathbf{J}_2}{\partial\mathfrak{x}^3}-\frac{\partial^5\mathbf{J}_2}{\partial\mathfrak{x}^5}\right\}=0,\quad \mathbf{J}_3(\mathfrak{x},0)=0, \tag{49}$$

Fourth-order problem:

$$\mathscr{L}\{\mathbf{J}_4(\mathfrak{x},\mathfrak{t})\}-\left(\frac{1}{s^\beta}\right)\mathscr{L}\left\{-2\mathbf{J}_3^2\frac{\partial\mathbf{J}_3}{\partial\mathfrak{x}}-6\frac{\partial\mathbf{J}_3}{\partial\mathfrak{x}}\frac{\partial^2\mathbf{J}_3}{\partial\mathfrak{x}^2}-3\mathbf{J}_3\frac{\partial^3\mathbf{J}_3}{\partial\mathfrak{x}^3}-\frac{\partial^5\mathbf{J}_3}{\partial\mathfrak{x}^5}\right\}=0,\quad \mathbf{J}_4(\mathfrak{x},0)=0, \tag{50}$$

Continuing this way, higher order problems and solutions can be formulated. The approximate solution of (41) is

$$\tilde{\mathbf{J}}=\mathbf{J}_0(\mathfrak{x},\mathfrak{t})+\mathbf{J}_1(\mathfrak{x},\mathfrak{t})+\mathbf{J}_2(\mathfrak{x},\mathfrak{t})+\mathbf{J}_3(\mathfrak{x},\mathfrak{t})+\mathbf{J}_4(\mathfrak{x},\mathfrak{t})+\dots \tag{51}$$

Residual error is derived by substituting the obtained approximate solution in (41)

$$\mathfrak{Res}\mathbf{J}=\frac{\partial^\beta\tilde{\mathbf{J}}}{\partial\mathfrak{t}^\beta}+2\tilde{\mathbf{J}}^2\frac{\partial\tilde{\mathbf{J}}}{\partial\mathfrak{x}}+6\frac{\partial\tilde{\mathbf{J}}}{\partial\mathfrak{x}}\frac{\partial^2\tilde{\mathbf{J}}}{\partial\mathfrak{x}^2}+3\tilde{\mathbf{J}}\frac{\partial^3\tilde{\mathbf{J}}}{\partial\mathfrak{x}^3}+\frac{\partial^5\tilde{\mathbf{J}}}{\partial\mathfrak{x}^5}. \tag{52}$$

Example 3. *Consider the following time-fractional Lax's KdV model*

$$\frac{\partial^\beta\mathbf{J}}{\partial\mathfrak{t}^\beta}=-30\mathbf{J}^2\frac{\partial\mathbf{J}}{\partial\mathfrak{x}}-20\frac{\partial\mathbf{J}}{\partial\mathfrak{x}}\frac{\partial^2\mathbf{J}}{\partial\mathfrak{x}^2}-10\mathbf{J}\frac{\partial^3\mathbf{J}}{\partial\mathfrak{x}^3}-\frac{\partial^5\mathbf{J}}{\partial\mathfrak{x}^5},\qquad 0<\beta\le 1,\ \mathfrak{t}>0, \tag{53}$$

with initial condition

$$\mathbf{J}(\mathfrak{x},0)=2k^2(2-3\tanh^2(k\mathfrak{x})), \tag{54}$$

where $k\neq 0$ is an arbitrary constant. The exact solution of (53) is

$$\mathbf{J}(\mathfrak{x},\mathfrak{t})=2k^2(2-3\tanh^2(k(\mathfrak{x}-56k^4\mathfrak{t}))). \tag{55}$$

Solution:
Following the similar steps mapped out in Section 3, we have
First-order problem:

$$\mathscr{L}\{\mathbf{J}_1(\mathfrak{x},\mathfrak{t})\} + \mathbf{J}_0(\mathfrak{x},\mathfrak{t}) - \left(\frac{1}{s}\right)2k^2(2-3\tanh^2(k\mathfrak{x})) - \left(\frac{1}{s^\beta}\right)\mathscr{L}\bigg\{-30\mathbf{J}_0^2\frac{\partial \mathbf{J}_0}{\partial \mathfrak{x}} - 20\frac{\partial \mathbf{J}_0}{\partial \mathfrak{x}}\frac{\partial^2 \mathbf{J}_0}{\partial \mathfrak{x}^2} - 10\mathbf{J}_0\frac{\partial^3 \mathbf{J}_0}{\partial \mathfrak{x}^3} - \frac{\partial^5 \mathbf{J}_0}{\partial \mathfrak{x}^5}\bigg\} = 0, \quad \mathbf{J}_1(\mathfrak{x},0) = 0, \tag{56}$$

Second-order problem:

$$\mathscr{L}\{\mathbf{J}_2(\mathfrak{x},\mathfrak{t})\} - \left(\frac{1}{s^\beta}\right)\mathscr{L}\bigg\{-30\mathbf{J}_1^2\frac{\partial \mathbf{J}_1}{\partial \mathfrak{x}} - 20\frac{\partial \mathbf{J}_1}{\partial \mathfrak{x}}\frac{\partial^2 \mathbf{J}_1}{\partial \mathfrak{x}^2} - 10\mathbf{J}_1\frac{\partial^3 \mathbf{J}_1}{\partial \mathfrak{x}^3} - \frac{\partial^5 \mathbf{J}_1}{\partial \mathfrak{x}^5}\bigg\} = 0, \quad \mathbf{J}_2(\mathfrak{x},0) = 0, \tag{57}$$

Third-order problem:

$$\mathscr{L}\{\mathbf{J}_3(\mathfrak{x},\mathfrak{t})\} - \left(\frac{1}{s^\beta}\right)\mathscr{L}\bigg\{-30\mathbf{J}_2^2\frac{\partial \mathbf{J}_2}{\partial \mathfrak{x}} - 20\frac{\partial \mathbf{J}_2}{\partial \mathfrak{x}}\frac{\partial^2 \mathbf{J}_2}{\partial \mathfrak{x}^2} - 10\mathbf{J}_2\frac{\partial^3 \mathbf{J}_2}{\partial \mathfrak{x}^3} - \frac{\partial^5 \mathbf{J}_2}{\partial \mathfrak{x}^5}\bigg\} = 0, \quad \mathbf{J}_3(\mathfrak{x},0) = 0, \tag{58}$$

Fourth-order problem:

$$\mathscr{L}\{\mathbf{J}_4(\mathfrak{x},\mathfrak{t})\} - \left(\frac{1}{s^\beta}\right)\mathscr{L}\bigg\{-30\mathbf{J}_3^2\frac{\partial \mathbf{J}_3}{\partial \mathfrak{x}} - 20\frac{\partial \mathbf{J}_3}{\partial \mathfrak{x}}\frac{\partial^2 \mathbf{J}_3}{\partial \mathfrak{x}^2} - 10\mathbf{J}_3\frac{\partial^3 \mathbf{J}_3}{\partial \mathfrak{x}^3} - \frac{\partial^5 \mathbf{J}_3}{\partial \mathfrak{x}^5}\bigg\} = 0, \quad \mathbf{J}_4(\mathfrak{x},0) = 0, \tag{59}$$

The inverse Laplace transform leads to the approximate solution $\tilde{\mathbf{J}}(\mathfrak{x},\mathfrak{t})$. The residual error of the current problem is

$$\mathfrak{Res}\mathbf{J} = \frac{\partial^\beta \tilde{\mathbf{J}}}{\partial \mathfrak{t}^\beta} + 30\tilde{\mathbf{J}}^2\frac{\partial \tilde{\mathbf{J}}}{\partial \mathfrak{x}} + 20\frac{\partial \tilde{\mathbf{J}}}{\partial \mathfrak{x}}\frac{\partial^2 \tilde{\mathbf{J}}}{\partial \mathfrak{x}^2} + 10\tilde{\mathbf{J}}\frac{\partial^3 \tilde{\mathbf{J}}}{\partial \mathfrak{x}^3} + \frac{\partial^5 \tilde{\mathbf{J}}}{\partial \mathfrak{x}^5}. \tag{60}$$

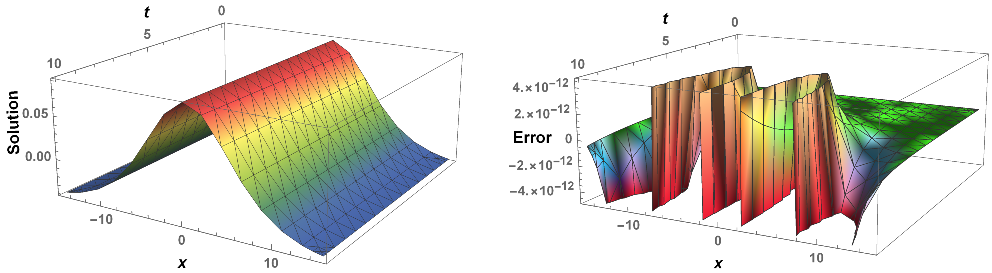
Results related to Example 3 are in Tables 1 and 2 and Figures 1–3.

Table 1. Comparison of LHPM errors with FCRPSA and mADM errors in Example 3 when $\beta = 1$ and $k = 0.01$.

t	$\mathfrak{x}$	Exact Solution	LHPM Solution	LHPM Error	FCRPSA Error [14]	mADM Error [13]
0.8	2	0.00039976	0.00039976	5.42×10^{-20}	1.34×10^{-17}	2.30×10^{-13}
	4	0.00039904	0.00039904	0	1.30×10^{-17}	4.60×10^{-13}
	6	0.00039784	0.00039784	0	1.27×10^{-17}	6.19×10^{-13}
	8	0.00039617	0.00039617	0	1.21×10^{-17}	9.21×10^{-13}
	10	0.00039404	0.00039404	0	1.14×10^{-17}	1.15×10^{-12}
5	2	0.00039976	0.00039976	5.42×10^{-20}	5.17×10^{-16}	1.42×10^{-12}
	4	0.00039904	0.00039904	5.42×10^{-20}	5.08×10^{-16}	2.84×10^{-12}
	6	0.00039784	0.00039784	5.42×10^{-20}	4.94×10^{-16}	4.26×10^{-12}
	8	0.00039617	0.00039617	5.42×10^{-20}	4.73×10^{-16}	5.68×10^{-12}
	10	0.00039404	0.00039404	0	4.49×10^{-16}	7.10×10^{-12}

Table 2. Error analysis of LHPM in fractional domain for time-fractional Lax's KdV model (Example 3) when $k = 0.14$ and $\mathfrak{x} = 4$.

$\mathfrak{t}$	$\beta = 0.2$	$\beta = 0.4$	$\beta = 0.6$	$\beta = 0.8$
0.1	3.15×10^{-8}	1.35×10^{-9}	3.43×10^{-11}	4.87×10^{-13}
0.3	9.45×10^{-8}	1.22×10^{-8}	9.27×10^{-10}	3.94×10^{-11}
0.5	1.57×10^{-7}	3.39×10^{-8}	4.29×10^{-9}	3.04×10^{-10}
0.7	2.20×10^{-7}	6.65×10^{-8}	1.17×10^{-8}	1.16×10^{-9}
0.9	2.83×10^{-7}	1.10×10^{-7}	2.50×10^{-8}	3.19×10^{-9}



(**a**) (**b**)

Figure 1. 3D graphical illustration of LHPM solution (**a**) and error (**b**) in Example 3 when $\beta = 1$ and $k = 0.15$.

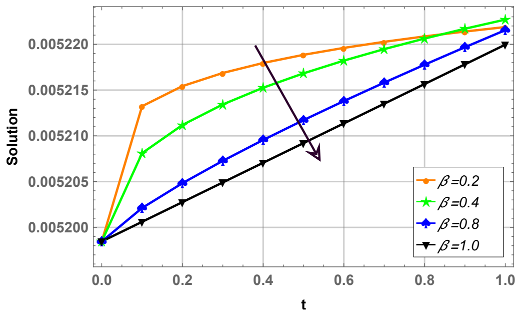


Figure 2. Effect of fractional parameter β on time-fractional Lax's KdV equation (Example 3) when $k = 0.1$ and $\mathfrak{x} = 10$.

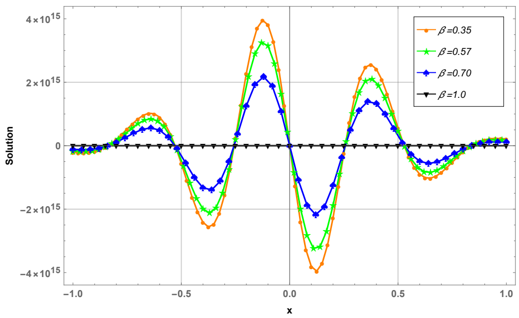


Figure 3. 2D surface formation at different values of β in Example 3 when $k = 1$, $\mathfrak{t} = 3$.

Example 4. *Consider the following time-fractional Kaup–Kupershmidt model*

$$\frac{\partial^{\beta}\mathbf{J}}{\partial\mathfrak{t}^{\beta}} = -45\mathbf{J}^2\frac{\partial\mathbf{J}}{\partial\mathfrak{x}} + 15\rho\frac{\partial\mathbf{J}}{\partial\mathfrak{x}}\frac{\partial^2\mathbf{J}}{\partial\mathfrak{x}^2} + 15\mathbf{J}\frac{\partial^3\mathbf{J}}{\partial\mathfrak{x}^3} - \frac{\partial^5\mathbf{J}}{\partial\mathfrak{x}^5}, \qquad 0 < \beta \leq 1,\ \mathfrak{t} > 0, \quad (61)$$

associated with initial condition

$$\mathbf{J}(\mathfrak{x},0)=\frac{1}{4}w^2\lambda^2\mathrm{sech}^2(\frac{w\lambda\mathfrak{x}}{2})+\frac{w^2\lambda^2}{12}, \tag{62}$$

where λ and $w\neq 0$ are arbitrary constants. The exact solution of (61) is:

$$\mathbf{J}(\mathfrak{x},\mathfrak{t})=\frac{1}{4}w^2\lambda^2\mathrm{sech}^2(\frac{-w^5\lambda\mathfrak{t}^\beta(-8\lambda^2\mu+16\mu^2+\lambda^4)}{32\beta}+\frac{w\lambda\mathfrak{x}}{2})+\frac{w^2\lambda^2}{12}, \tag{63}$$

Solution:
Using basic steps given in Section 3, we obtain the following
First-order problem:

$$\begin{aligned}\mathscr{L}\{\mathbf{J}_1(\mathfrak{x},\mathfrak{t})\}+\mathbf{J}_0(\mathfrak{x},\mathfrak{t})-\left(\frac{1}{s}\right)\frac{1}{4}w^2\lambda^2\mathrm{sech}^2(\frac{w\lambda\mathfrak{x}}{2})+\frac{w^2\lambda^2}{12}-\left(\frac{1}{s^\beta}\right)\mathscr{L}\Big\{-45\mathbf{J}_0^2\frac{\partial\mathbf{J}_0}{\partial\mathfrak{x}}&\\ +15\rho\frac{\partial\mathbf{J}_0}{\partial\mathfrak{x}}\frac{\partial^2\mathbf{J}_0}{\partial\mathfrak{x}^2}+15\mathbf{J}_0\frac{\partial^3\mathbf{J}_0}{\partial\mathfrak{x}^3}-\frac{\partial^5\mathbf{J}_0}{\partial\mathfrak{x}^5}\Big\}&=0,\\ \mathbf{J}_1(\mathfrak{x},0)&=0,\end{aligned} \tag{64}$$

Second-order problem:

$$\begin{aligned}\mathscr{L}\{\mathbf{J}_2(\mathfrak{x},\mathfrak{t})\}-\left(\frac{1}{s^\beta}\right)\mathscr{L}\Big\{-45\mathbf{J}_1^2\frac{\partial\mathbf{J}_1}{\partial\mathfrak{x}}+15\rho\frac{\partial\mathbf{J}_1}{\partial\mathfrak{x}}\frac{\partial^2\mathbf{J}_1}{\partial\mathfrak{x}^2}+15\mathbf{J}_1\frac{\partial^3\mathbf{J}_1}{\partial\mathfrak{x}^3}-\frac{\partial^5\mathbf{J}_1}{\partial\mathfrak{x}^5}\Big\}&=0,\\ \mathbf{J}_2(\mathfrak{x},0)&=0,\end{aligned} \tag{65}$$

Third-order problem:

$$\begin{aligned}\mathscr{L}\{\mathbf{J}_3(\mathfrak{x},\mathfrak{t})\}-\left(\frac{1}{s^\beta}\right)\mathscr{L}\Big\{-45\mathbf{J}_2^2\frac{\partial\mathbf{J}_2}{\partial\mathfrak{x}}+15\rho\frac{\partial\mathbf{J}_2}{\partial\mathfrak{x}}\frac{\partial^2\mathbf{J}_2}{\partial\mathfrak{x}^2}+15\mathbf{J}_2\frac{\partial^3\mathbf{J}_2}{\partial\mathfrak{x}^3}-\frac{\partial^5\mathbf{J}_2}{\partial\mathfrak{x}^5}\Big\}&=0,\\ \mathbf{J}_3(\mathfrak{x},0)&=0,\end{aligned} \tag{66}$$

Fourth-order problem:

$$\begin{aligned}\mathscr{L}\{\mathbf{J}_4(\mathfrak{x},\mathfrak{t})\}-\left(\frac{1}{s^\beta}\right)\mathscr{L}\Big\{-45\mathbf{J}_3^2\frac{\partial\mathbf{J}_3}{\partial\mathfrak{x}}+15\rho\frac{\partial\mathbf{J}_3}{\partial\mathfrak{x}}\frac{\partial^2\mathbf{J}_3}{\partial\mathfrak{x}^2}+15\mathbf{J}_3\frac{\partial^3\mathbf{J}_3}{\partial\mathfrak{x}^3}-\frac{\partial^5\mathbf{J}_3}{\partial\mathfrak{x}^5}\Big\}&=0,\\ \mathbf{J}_4(\mathfrak{x},0)&=0,\end{aligned} \tag{67}$$

The inverse Laplace transform leads to the approximate solution $\tilde{\mathbf{J}}(\mathfrak{x},\mathfrak{t})$. The residual error of the concerned problem is

$$\mathfrak{ResJ}=\frac{\partial^\beta\tilde{\mathbf{J}}}{\partial\mathfrak{t}^\beta}+45\tilde{\mathbf{J}}^2\frac{\partial\tilde{\mathbf{J}}}{\partial\mathfrak{x}}-15\rho\frac{\partial\tilde{\mathbf{J}}}{\partial\mathfrak{x}}\frac{\partial^2\tilde{\mathbf{J}}}{\partial\mathfrak{x}^2}-15\tilde{\mathbf{J}}\frac{\partial^3\tilde{\mathbf{J}}}{\partial\mathfrak{x}^3}+\frac{\partial^5\tilde{\mathbf{J}}}{\partial\mathfrak{x}^5}. \tag{68}$$

6. Discussion

In this manuscript, the Laplace transform with homotopy perturbation method is proposed for the solution of generalized fifth-order time-fractional KdV models. The proposed method is tested against different time-fractional KdV models, including Sawada–Kotera, Ito, Lax, and Kaup–Kupershmidt KdV equations. These equations belong to the prominent form of fractional KdV family hierarchy. The main focus of this paper is to propose a new method for the solution and analysis of KdV equations in a fractional environment. Graphical and numerical comparisons of the obtained results were also made to provide strong evidence for the usage of the proposed technique. Obtained results are compared with existing ones from the literature.

The proposed method is firstly applied to the Sawada–Kotera model in Example 1, and the results are shown in Tables 3 and 4. Table 3 presents a comparison of LHPM with exact and fractional conformable residual power series algorithm (FCRPSA) results. These results showed the efficiency of LHPM over FCRPSA. In Table 4, error analysis has been performed in the fractional domain by finding residual errors at different values of β. The results are consistent through out the fractional domain. It is also observed that increasing the value of the fractional parameter decreases the error. Figure 4 illustrates the three-dimensional plot of approximate solutions and errors at β = 1. The 3D solution curve (part a) depicts that at $\mathfrak{x}$ and t = 0, the crest of the wave is highest, and as the distance and time increased, the altitude of the wave started to decrease. Figure 5 represents the effect of fractional parameter β on the water waves at $\mathfrak{x}$ = 10. It is observed that for the time range between 0 and 1, increasing β decreases the water wave profile. Elevation and motion of the surface of water waves in two-dimensional form at a fixed time for various β can be seen in Figure 6. Near $\mathfrak{x}$ = 0, the altitude of the wave was highest, but it started to smoothen out as the distance increased.

Table 3. Comparison of LHPM and FCRPSA errors in Example 1 when β = 1 and k = 0.01.

t	$\mathfrak{x}$	Exact Solution	LHPM Solution	LHPM Error	FCRPSA Error [14]
0.1	2	0.00019992	0.00019992	1.38×10^{-20}	3.46×10^{-18}
	4	0.00019968	0.00019968	3.13×10^{-21}	3.46×10^{-18}
	6	0.00019928	0.00019928	2.57×10^{-20}	1.04×10^{-17}
	8	0.00019872	0.00019872	1.38×10^{-20}	1.38×10^{-17}
	10	0.00019801	0.00019801	4.48×10^{-20}	1.73×10^{-17}
0.5	2	0.00019992	0.00019992	6.59×10^{-20}	2.70×10^{-16}
	4	0.00019968	0.00019968	3.86×10^{-20}	8.22×10^{-16}
	6	0.00019928	0.00019928	2.05×10^{-20}	1.35×10^{-15}
	8	0.00019872	0.00019872	1.18×10^{-20}	1.88×10^{-15}
	10	0.00019801	0.00019801	6.15×10^{-20}	2.39×10^{-15}
0.9	2	0.00019992	0.00019992	3.71×10^{-20}	1.59×10^{-15}
	4	0.00019968	0.00019968	2.63×10^{-20}	4.75×10^{-15}
	6	0.00019928	0.00019928	1.54×10^{-20}	7.90×10^{-15}
	8	0.00019872	0.00019872	4.39×10^{-20}	1.09×10^{-14}
	10	0.00019801	0.00019801	5.10×10^{-20}	1.40×10^{-14}

Table 4. Error analysis of LHPM in fractional domain for time-fractional Sawada–Kotera model (Example 1) when k = 0.15 and $\mathfrak{x} = 1$.

t	$\beta = 0.2$	$\beta = 0.4$	$\beta = 0.6$	$\beta = 0.8$
0.1	9.50×10^{-9}	4.09×10^{-10}	1.03×10^{-11}	1.47×10^{-13}
0.3	2.85×10^{-8}	3.68×10^{-9}	2.79×10^{-10}	1.19×10^{-11}
0.5	4.75×10^{-8}	1.02×10^{-8}	1.29×10^{-9}	9.19×10^{-11}
0.7	6.65×10^{-8}	2.00×10^{-8}	3.55×10^{-9}	3.53×10^{-10}
0.9	8.55×10^{-8}	3.31×10^{-8}	7.55×10^{-9}	9.64×10^{-10}

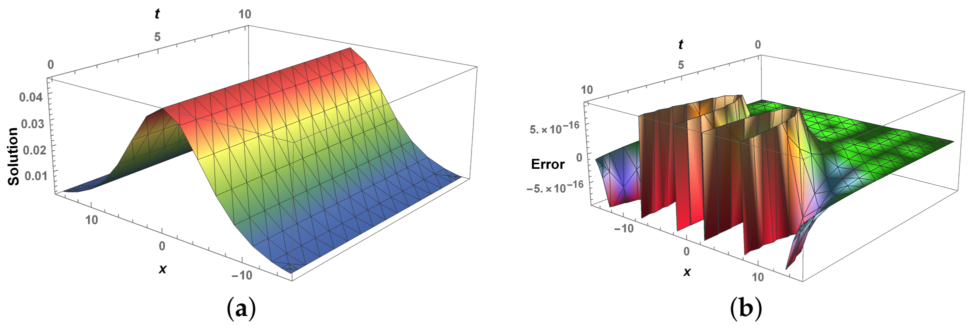


Figure 4. 3D graphical illustration of LHPM solution (**a**) and error (**b**) in Example 1 when $\beta = 1$ and $k = 0.15$.

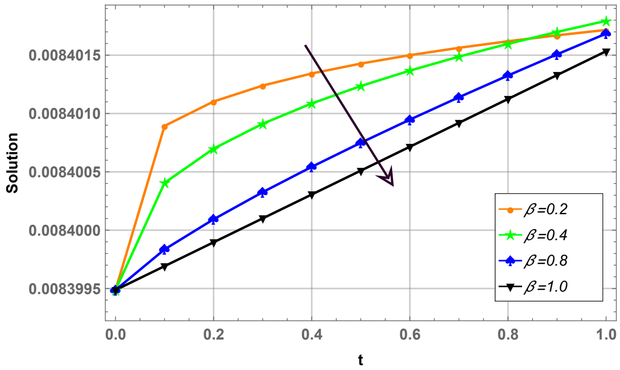


Figure 5. Effect of fractional parameter β on time-fractional Sawada–Kotera equation (Example 1) when $k = 0.1$ and $\mathfrak{x} = 10$.

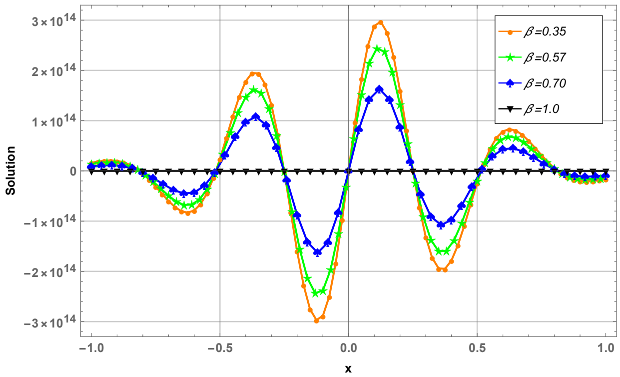
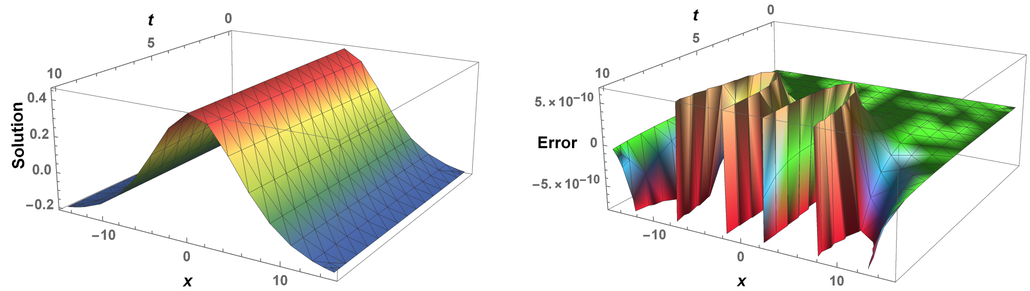


Figure 6. 2D surface formation at different values of β in Example 1 when $k = 1$, $\mathfrak{t} = 3$.

For the time-fractional Ito equation in Example 2, Table 5 is the comparison of LHPM with the modified Adomian decomposition method (mADM) and FCRPSA error at $\beta = 1$ and $k = 0.01$, whereas Table 6 displays the residual errors at different values of β. In both tables, it can be seen that LHPM is a reliable and powerful technique. Figure 7 is the LHPM solution and error graphs in 3D form, which displays the decrease in wave altitude with the increase of distance and time. The effect of different values of fractional parameter on the water waves surface at distance $\mathfrak{x} = 10$ can be seen in Figure 8, which shows that increasing β decreases the water wave level for the interval $0 < \mathfrak{t} < 1$. A 2D plot to evaluate the motion of the wave surface is displayed in Figure 9 for $k = 1$, $\mathfrak{t} = 3$, and $\beta = 0.2, 0.4, 0.8$, and 1.0. It can be seen that nearby $\mathfrak{x} = 0$, the wave is at its peak but starts declining as the distance becomes larger.



(**a**) (**b**)

Figure 7. 3D graphical illustration of LHPM solution (**a**) and error (**b**) in Example 2 when $\beta = 1$ and $k = 0.15$.

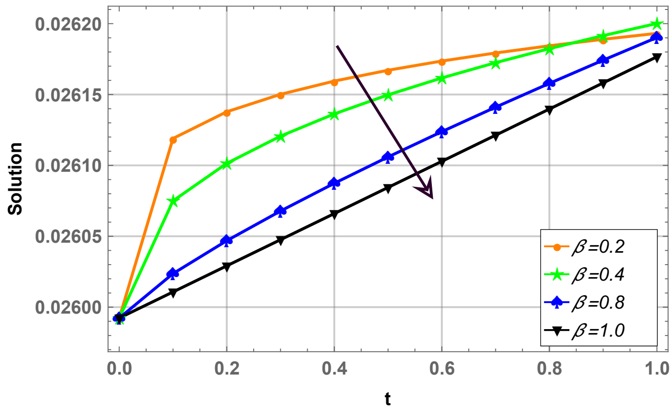


Figure 8. Effect of fractional parameter β on a time-fractional Ito equation (Example 2) when $k = 0.1$ and $\mathfrak{x} = 10$.

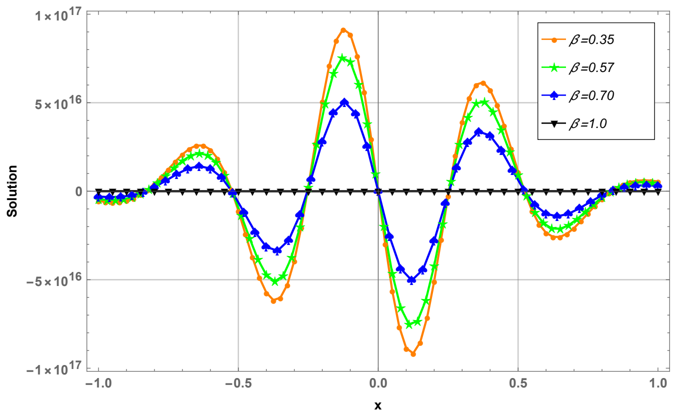


Figure 9. 2D surface formation at different values of β in Example 2 when $k = 1, \mathrm{t} = 3$.

Table 5. Comparison of LHPM errors with FCRPSA and mADM in Example 2 when β = 1 and k = 0.01.

t	$\mathfrak{x}$	Exact Solution	LHPM Solution	LHPM Error	FCRPSA Error [14]	mADM Error [13]
0.2	2	0.0019988	0.0019988	2.11×10^{-22}	2.77×10^{-17}	1.41×10^{-16}
	4	0.0019952	0.0019952	8.47×10^{-22}	3.88×10^{-16}	5.63×10^{-16}
	6	0.0019892	0.0019892	5.08×10^{-21}	4.13×10^{-16}	1.26×10^{-15}
	8	0.0019808	0.0019808	3.38×10^{-21}	1.04×10^{-15}	2.25×10^{-15}
	10	0.0019702	0.0019702	0	2.31×10^{-15}	3.52×10^{-15}
0.6	2	0.0019988	0.0019988	0	0	1.41×10^{-16}
	4	0.0019952	0.0019952	8.47×10^{-22}	3.60×10^{-16}	5.64×10^{-16}
	6	0.0019892	0.0019892	5.08×10^{-21}	4.05×10^{-16}	1.27×10^{-15}
	8	0.0019808	0.0019808	3.38×10^{-21}	9.90×10^{-16}	2.25×10^{-15}
	10	0.0019702	0.0019702	3.38×10^{-21}	2.20×10^{-15}	3.52×10^{-15}
1.0	2	0.0019988	0.0019988	0	2.77×10^{-17}	1.41×10^{-16}
	4	0.0019952	0.0019952	1.69×10^{-21}	3.33×10^{-16}	5.66×10^{-16}
	6	0.0019892	0.0019892	1.69×10^{-21}	3.94×10^{-16}	1.27×10^{-15}
	8	0.0019808	0.0019808	3.38×10^{-21}	8.99×10^{-16}	2.26×10^{-15}
	10	0.0019702	0.0019702	3.38×10^{-21}	2.00×10^{-15}	3.53×10^{-15}

Table 6. Error analysis of LHPM in a fractional domain for a time-fractional Ito model (Example 2) when k = 0.14 and $\mathfrak{x} = 1$.

t	$\beta = 0.2$	$\beta = 0.4$	$\beta = 0.6$	$\beta = 0.8$
0.1	2.16×10^{-7}	9.34×10^{-9}	2.36×10^{-10}	3.35×10^{-12}
0.3	6.50×10^{-7}	8.41×10^{-8}	6.38×10^{-9}	2.71×10^{-10}
0.5	1.08×10^{-6}	2.33×10^{-7}	2.95×10^{-8}	2.09×10^{-9}
0.7	1.51×10^{-6}	4.58×10^{-7}	8.11×10^{-8}	8.05×10^{-9}
0.9	1.95×10^{-6}	7.57×10^{-7}	1.72×10^{-7}	2.20×10^{-8}

In Example 3, a time-fractional Lax's KdV equation is solved numerically and analytically by LHPM. A comparison of FCRPSA, mADM, and LHPM absolute errors at β = 1 in Table 1 shows that LHPM is more productive than mADM and FCRPSA. Residual errors for various β values are also shown in Table 2. A 3D LHPM solution plot and an error plot (Figure 1) are also displayed. From Figure 2 at k = 0.1 and $\mathfrak{x}$ = 10, it is observed that increasing the value of the fractional parameter for the fifth-order Lax's KdV model decreases the waves level. The effect of various β values on surface waves for fixed t = 3 is shown in 2D form in Figure 3. Analysis reveals that the behaviour of escalation of a wave is similar to that of Example 2. Tables 7–9 provide a comparison of LHPM errors with the optical optimal homotopy asymptotic method (OHAM) errors of the Kaup–Kupershmidt equation (Example 4) for β = 0.5, 0.75, and 1, respectively. Observation showed that LHPM is more reliable than OHAM. A 3D solution graph and error graph of the Kaup–Kupershmidt equation can be seen in Figure 10. For different β values, surface waves level in a 2D form for t = 3 is displayed in Figure 11 which depicts a sinusoidal cycle between $\mathfrak{x}$ = −0.3 and 0.3. Moreover, wave level is declining at a greater value of $\mathfrak{x}$.

Table 7. Comparison of LHPM and OHAM errors in the Kaup–Kupershmidt model (Example 4) at different t when $\beta = 0.5$, $\lambda = 0.1$, $w = 1$, $\rho = 2.5$ and $\mu = 0$.

$\mathfrak{x}$	$t = 0.3$		$t = 0.5$		$t = 0.8$	
	OHAM [37]	**LHPM**	**OHAM** [37]	**LHPM**	**OHAM** [37]	**LHPM**
0.2	7.23×10^{-6}	4.22×10^{-9}	7.21×10^{-6}	5.41×10^{-9}	7.18×10^{-6}	6.78×10^{-9}
0.4	5.79×10^{-5}	8.55×10^{-9}	5.78×10^{-5}	1.10×10^{-8}	5.78×10^{-5}	1.38×10^{-8}
0.6	1.53×10^{-4}	1.28×10^{-8}	1.53×10^{-4}	1.65×10^{-8}	1.53×10^{-4}	2.08×10^{-8}
0.8	2.86×10^{-4}	1.71×10^{-8}	2.86×10^{-4}	2.20×10^{-8}	2.85×10^{-4}	2.78×10^{-8}

Table 8. Comparison of LHPM and OHAM errors in the Kaup–Kupershmidt equation (Example 4) at different t when $\beta = 0.75$, $\lambda = 0.1$, $w = 1$, $\rho = 2.5$ and $\mu = 0$.

$\mathfrak{x}$	$t = 0.3$		$t = 0.5$		$t = 0.8$	
	OHAM [37]	**LHPM**	**OHAM** [37]	**LHPM**	**OHAM** [37]	**LHPM**
0.2	7.22×10^{-6}	3.05×10^{-9}	7.17×10^{-6}	4.44×10^{-9}	7.12×10^{-6}	6.26×10^{-9}
0.4	5.79×10^{-5}	6.14×10^{-9}	5.78×10^{-5}	8.98×10^{-9}	5.76×10^{-5}	1.27×10^{-8}
0.6	1.53×10^{-4}	9.21×10^{-9}	1.53×10^{-4}	1.34×10^{-8}	1.52×10^{-4}	1.91×10^{-8}
0.8	2.86×10^{-4}	1.22×10^{-8}	2.85×10^{-4}	1.79×10^{-8}	2.85×10^{-4}	2.55×10^{-8}

Table 9. Comparison of LHPM and OHAM errors in the Kaup–Kupershmidt model (Example 4) at different t when $\beta = 1$, $\lambda = 0.1$, $w = 1$, $\rho = 2.5$ and $\mu = 0$.

$\mathfrak{x}$	$t = 0.3$		$t = 0.5$		$t = 0.8$	
	OHAM [37]	**LHPM**	**OHAM** [37]	**LHPM**	**OHAM** [37]	**LHPM**
0.2	7.21×10^{-6}	2.09×10^{-9}	7.14×10^{-6}	3.46×10^{-9}	7.03×10^{-6}	5.05×10^{-9}
0.4	5.79×10^{-5}	4.19×10^{-9}	5.77×10^{-5}	6.97×10^{-9}	5.74×10^{-5}	1.11×10^{-8}
0.6	1.53×10^{-4}	6.28×10^{-9}	1.52×10^{-4}	1.04×10^{-8}	1.52×10^{-4}	1.66×10^{-8}
0.8	2.86×10^{-4}	8.36×10^{-9}	2.85×10^{-4}	1.39×10^{-8}	2.85×10^{-4}	2.22×10^{-8}

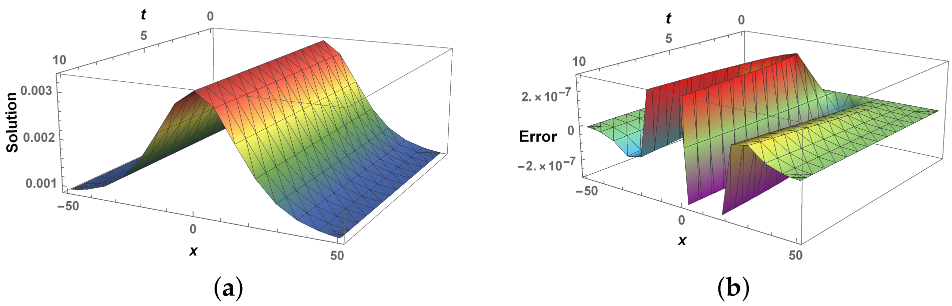


Figure 10. 3D graphical illustration of LHPM solution (**a**) and error (**b**) in Example 4 when $\beta = 1$, $\lambda = 0.1$, $w = 1$, $\rho = 2.5$, and $\mu = 0$.

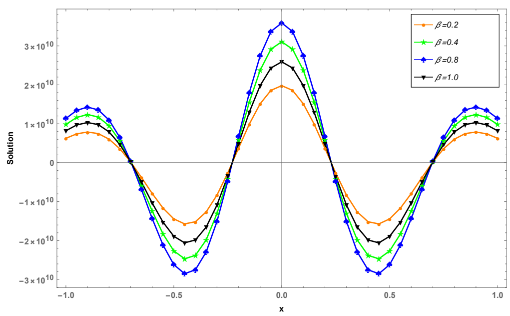


Figure 11. 2D surface formation at different values of β in Example 4 when $\lambda = 1$, $w = 1$, $\rho = 2.5$, $\mu = 1$ and $t = 3$.

7. Conclusions

This paper is focused on the analysis of generalized fifth-order time-fractional KdV models through the Laplace transform along with homotopy perturbation. For checking the validity and efficiency of the proposed method, it is applied to the Sawada–Kotera, Ito, Lax, and Kaup–Kupershmidt KdV models in fractional sense, and residual errors are computed for different values of fractional parameter in the fractional domain. LHPM solutions are obtained without the imposition of any restrictions on the structure of models. The obtained approximate solutions and errors are illustrated in three-dimensional plots for reader convenience. Plots against different values of fractional parameter on water surface level are also displayed to provide better understanding of the models. A comparison of proposed and existing techniques affirm the efficiency and accuracy of LHPM over other methods. Hence, LHPM is helpful in managing complex non-linear, fractional, higher order KdV equations with improved accuracy.

Author Contributions: Conceptualization, M.Q. and M.B.R.; methodology, E.A.; software, J.A.; validation, J.A., M.Q. and M.B.R.; formal analysis, J.A.; investigation, E.A.; resources, M.Q.; data curation, M.Q.; writing—original draft preparation, E.A.; writing—review and editing, J.A., M.Q. and M.B.R.; visualization, M.Q.; supervision, M.B.R., M.Q.; project administration, J.A.; funding acquisition, J.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Russell, J.S. Report on Waves. In Proceedings of the 14th Meeting of the British Association for the Advancement of Science, York, UK, September 1844.
2. Boussinesq, J. Theorie de l'intumescence liquide appelee onde solitaire ou de translation se propageant dans un canal rectangulaire. *CR Acad. Sci. Paris* **1871**, *72*, 755–759.
3. Boussinesq, J.V. Theorie generale des mouvements qui sont propages dans un canal rectangulaire horizontal. *CR Acad. Sci. Paris* **1871**, *73*, 256–260.
4. Boussinesq, J. Theorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. *J. Math. Pures Appl.* **1872**, *338*, 55–108.
5. Korteweg, D.J.; de Vries, G. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Lond. Edinb. Dublin Philos. Mag. J. Sci.* **1895**, *39*, 422–443. [CrossRef]
6. Al-Smadi, M.; Arqub, O.A.; Gaith, M. Numerical simulation of telegraph and Cattaneo fractional-type models using adaptive reproducing kernel framework. *Math. Methods Appl. Sci.* **2020**, *40*, 8472–8489. [CrossRef]
7. Al-Smadi, M.; Arqub, O.A.; Momani, S. Numerical computations of coupled fractional resonant Schrödinger equations arising in quantum mechanics under conformable fractional derivative sense. *Phys. Scr.* **2020**, *95*, 075218. [CrossRef]
8. Kumar, S.; Kumar, A.; Samet, B.; Dutta, H. A study on fractional host-parasitoid population dynamical model to describe insect species. *Numer. Methods Partial. Differ. Equ.* **2020**, *37*, 1673–1692. [CrossRef]
9. Ito, M. An Extension of Nonlinear Evolution Equations of the K-dV (mK-dV) Type to Higher Orders. *J. Phys. Soc. Jpn.* **1980**, *49*, 771–778. [CrossRef]
10. Iyiola, O.S. A numerical study of ito equation and Sawada–Kotera equation both of time-fractional type. *Adv. Math. Sci. J.* **2013**, *2*, 71–79.
11. Wang, J.; Xu, T.Z.; Wang, G.W. Numerical algorithm for time-fractional Sawada–Kotera equation and Ito equation with Bernstein polynomials. *Appl. Math. Comput.* **2018**, *338*, 1–11. [CrossRef]
12. Lee, C. Some remarks on the fifth-order KdV equations. *J. Math. Anal. Appl.* **2015**, *425*, 281–294. [CrossRef]
13. Bakodah, H.O. Modified Adomain Decomposition Method for the Generalized Fifth Order KdV Equations. *Am. J. Comput. Math.* **2013**, *3*, 53–58. [CrossRef]
14. Arqub, O.A.; Al-Smadi, M.; Almusawa, H.; Baleanu, D.; Hayat, T.; Alhodaly, M.; Osman, M. A novel analytical algorithm for generalized fifth-order time-fractional nonlinear evolution equations with conformable time derivative arising in shallow water waves. *Alex. Eng. J.* **2022**, *61*, 5753–5769. [CrossRef]
15. Guo, Y.; Li, D.; Wang, J. The new exact solutions of the Fifth-Order Sawada–Kotera equation using three wave method. *Appl. Math. Lett.* **2019**, *94*, 232–237. [CrossRef]
16. Liu, T. Exact Solutions to Time-Fractional Fifth Order KdV Equation by Trial Equation Method Based on Symmetry. *Symmetry* **2019**, *11*, 742. [CrossRef]

17. Park, C.; Nuruddeen, R.I.; Ali, K.K.; Muhammad, L.; Osman, M.S.; Baleanu, D. Novel hyperbolic and exponential ansatz methods to the fractional fifth-order Korteweg–de Vries equations. *Adv. Differ. Equ.* **2020**, *2020*, 1–12. [CrossRef]
18. Singh, J.; Gupta, A.; Baleanu, D. On the analysis of an analytical approach for fractional Caudrey-Dodd-Gibbon equations. *Alex. Eng. J.* **2022**, *61*, 5073–5082. [CrossRef]
19. Saratha, S.R.; Bagyalakshmi, M.; Krishnan, G.S.S. Fractional generalised homotopy analysis method for solving nonlinear fractional differential equations. *Comput. Appl. Math.* **2020**, *39*, 1–32. [CrossRef]
20. Ismail, G.M.; Abdl-Rahim, H.R.; Ahmad, H.; Chu, Y.M. Fractional residual power series method for the analytical and approximate studies of fractional physical phenomena. *Open Phys.* **2020**, *18*, 799–805. [CrossRef]
21. Ziane, D.; Cherif, M.H. Variational iteration transform method for fractional differential equations. *J. Interdiscip. Math.* **2018**, *21*, 185–199. [CrossRef]
22. Guo, P. The Adomian Decomposition Method for a Type of Fractional Differential Equations. *J. Appl. Math. Phys.* **2019**, *7*, 2459–2466. [CrossRef]
23. Song, H.; Yi, M.; Huang, J.; Pan, Y. Bernstein polynomials method for a class of generalized variable order fractional differential equations. *IAENG Int. J. Appl. Math.* **2016**, *46*, 437–444.
24. Nadeem, M.; He, J.H. The homotopy perturbation method for fractional differential equations: Part 2, two-scale transform. *Int. J. Numer. Methods Heat Fluid Flow* **2021**, *32*, 559–567. [CrossRef]
25. Veeresha, P.; Prakasha, D.G.; Singh, J. Solution for fractional forced KdV equation using fractional natural decomposition method. *AIMS Math.* **2020**, *5*, 798–810. [CrossRef]
26. Xing, Z.; Wen, L.; Wang, W. An efficient difference scheme for time-fractional KdV equation. *Comput. Appl. Math.* **2021**, *40*. [CrossRef]
27. Alderremy, A.A.; Aly, S.; Fayyaz, R.; Khan, A.; Shah, R.; Wyal, N. The Analysis of Fractional-Order Nonlinear Systems of Third Order KdV and Burgers Equations via a Novel Transform. *Complexity* **2022**, *2022*, 4935809. [CrossRef]
28. He, J.H. Homotopy perturbation technique. *Comput. Methods Appl. Mech. Eng.* **1999**, *178*, 257–262. [CrossRef]
29. He, J.H.; El-Dib, Y.O. The enhanced homotopy perturbation method for axial vibration of strings. *Facta Univ. Ser. Mech. Eng.* **2021**, *19*, 735. [CrossRef]
30. Anjum, N.; He, J.H.; Ain, Q.T.; Tian, D. He's modified homotopy perturbation method for doubly-clamped electrically actuated microbeams-based microelectromechanical system. *Facta Univ. Ser. Mech. Eng.* **2021**, *19*, 601. [CrossRef]
31. Thabet, H.; Kendre, S. Modified least squares homotopy perturbation method for solving fractional partial differential equations. *Malaya J. Mat.* **2018**, *6*, 420–427. [CrossRef]
32. Habib, S.; Batool, A.; Islam, A.; Nadeem, M.; Gepreel, K.A.; He, J.H. Study of nonlinear Hirota-Satsuma coupled kdv and coupled mkdv system with time fractional derivative. *Fractals* **2021**, *29*, 2150108. [CrossRef]
33. Johnston, S.J.; Jafari, H.; Moshokoa, S.P.; Ariyan, V.M.; Baleanu, D. Laplace homotopy perturbation method for Burgers equation with space- and time-fractional order. *Open Phys.* **2016**, *14*, 247–252. [CrossRef]
34. Yin, X.B.; Kumar, S.; Kumar, D. A modified homotopy analysis method for solution of fractional wave equations. *Adv. Mech. Eng.* **2015**, *7*, 168781401562033. [CrossRef]
35. Bohner, M.; Tun, O.; Tun, C. Qualitative analysis of caputo fractional integro-differential equations with constant delays. *Comput. Appl. Math.* **2021**, *40*, 1–17. [CrossRef]
36. Tun, C.; Tun, O. On the stability, integrability and boundedness analyses of systems of integro-differential equations with time-delay retardation. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Serie A Mat.* **2021**, *115*. [CrossRef]
37. Gupta, A.K.; Ray, S.S. The comparison of two reliable methods for accurate solution of time-fractional Kaup–Kupershmidt equation arising in capillary gravity waves. *Math. Methods Appl. Sci.* **2015**, *39*, 583–592. [CrossRef]