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# Sign Switching Dark Energy from a Running Barrow Entropy

**Sofia Di Gennaro [1] and Yen Chin Ong [1,2,*]**

[1] Center for Gravitation and Cosmology, College of Physical Science and Technology, Yangzhou University, 180 Siwangting Road, Yangzhou 225002, China
[2] Shanghai Frontier Science Center for Gravitational Wave Detection, School of Aeronautics and Astronautics, Shanghai Jiao Tong University, Shanghai 200240, China
* Correspondence: ycong@yzu.edu.cn

**Abstract:** Barrow proposed that the area law of the entropy associated with a horizon might receive a "fractal correction" due to quantum gravitational effects—in place of $S \propto A$, we have instead $S \propto A^{1+\delta/2}$, where $0 \leqslant \delta \leqslant 1$ measures the deviation from the standard area law ($\delta = 0$). Based on black hole thermodynamics, we argue that the Barrow entropy should run (i.e., energy scale dependent), which is reasonable given that quantum gravitational corrections are expected to be important only in the high-energy regime. When applied to the Friedmann equation, we demonstrate the possibility that such a running Barrow entropy index could give rise to a dynamical effective dark energy, which is asymptotically positive and vanishing, but negative at the Big Bang. Such a sign switching dark energy could help to alleviate the Hubble tension. Other cosmological implications are discussed.

**Keywords:** Barrow entropy; running Barrow index; modified Friedmann equation; varying gravitational constant; cosmological constant

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**Citation:** Di Gennaro, S.; Ong, Y.C. Sign Switching Dark Energy from a Running Barrow Entropy. *Universe* **2022**, *8*, 541. https://doi.org/10.3390/universe8100541

Academic Editors: Emmanuel N. Saridakis and Panayiotis Stavrinos

Received: 4 October 2022
Accepted: 14 October 2022
Published: 19 October 2022



## 1. Introduction: Barrow Entropy and Quantum Gravity

One of the most striking properties of gravity is that black holes—and by extension, apparent horizons—have entropy that scales with the surface area [1,2]. Specifically, the Bekenstein–Hawking entropy is

$$S = \frac{A}{4}\frac{k_B c^3}{G\hbar}, \tag{1}$$

where $k_B$, $c$, and $G$ are the Boltzmann constant, the speed of light, and Newton's gravitational constant, respectively.

In different approaches to quantum gravity, this expression receives different forms of correction (see Section 2). Recently, Barrow [3] proposed that, due to quantum gravity corrections, black holes might exist with extremely wrinkled surfaces, such that the event horizon is like, e.g., a Koch fractal surface. The entropy expression becomes

$$S = \left(\frac{A}{A_0}\right)^{1+\frac{\delta}{2}}, \tag{2}$$

where $0 \leqslant \delta \leqslant 1$ is the parameter that governs how "fractalized" the surface has become, and we shall refer to it as the "Barrow entropy index" (BEI). It is not quite clear whether we should have $S = \frac{k_B}{4}\left(\frac{A}{\ell_p^2}\right)^{1+\delta/2}$ or $S = k_B\left(\frac{A}{4\ell_p^2}\right)^{1+\delta/2}$. Barrow worked with $S \propto A/4 \approx A$ from the beginning (before the correction), and thus, this issue was not discussed. However, the constant factor $1/4$, whether it is raised to the power of $1+\delta/2$ or not, does not affect the results much, except possibly when dealing with microscopic black holes, since the area would be very large anyway, so we use the form of Equation (2), in line with most literature, for easy comparison. However, for definiteness, we set $A_0 = 4\ell_p^2$, where $\ell_p$ is the Planck length $\ell_p = \sqrt{G\hbar/c^3}$. Hereinafter, we employ the Planck units, so that $\hbar = G = c = k_B = 1$.

Cosmological and black hole shadow constraints (assuming a fixed $\delta$) have put an upper bound on $\delta$, which is typically $\delta \lesssim O(10^{-3})$ or $O(10^{-4})$ [4–9].

In this work, we argue that $\delta$ has to be a function of the energy scale and study the implications of Barrow entropy for cosmology (by the expansion of the universe, $\delta$ is thus a function of cosmic time). Such a possibility was also raised in [7,10,11]. We expect that, as quantum gravitational effects become more pronounced near the Big Bang, $\delta \to 1$. As the universe expands and cools, $\delta \to 0$ monotonically. Incidentally, in the context of an asymptotically flat Schwarzschild black hole, this would mean that, as the black hole evaporates and its temperature increases, $\delta$ would approach unity (the effect of a running BEI on Hawking evaporation will be studied elsewhere).

The running Barrow entropy index gives rise to an additional term in the modified Friedmann equation, which we propose to be identified with an effective dynamical dark energy, which is asymptotically zero (hence, small at late time). Surprisingly, the effective dark energy has a negative sign right about the Big Bang, i.e., the Universe started out as an anti-de Sitter-like spacetime. Interestingly, this may help to relax the Hubble tension—the mismatch between the locally measured expansion rate of the universe and the inferred rate from early universe via the cosmic microwave background (CMB) [12,13].

In addition, following a recent work of Sheykhi [14], which considered the fixed $\delta$ scenario, we shall identify the correction on the right-hand side (r.h.s.) of the Friedmann equation as an effective modification on the gravitational constant, instead of the matter field. This has the advantage that the resulting fixed $\delta$ cosmology satisfies the generalized second law (GSL), whereas if the matter sector is modified instead of the geometry, then the GSL fails [15]. The viability of the GSL in the context of the interacting Barrow holographic dark energy model was also investigated in [16].

## 2. Barrow Entropy Index Is Energy Scale Dependent

In a recent work [17], Chen et al. studied black holes in asymptotically safe gravity, in which the gravitational constant also runs [18] (see [19,20] for cosmological studies). They showed that the area law receives a well-known logarithmic correction in the quantum gravity literature:

$$S = \frac{A}{4} + C\ln(A); \tag{3}$$

see [21] for a review. This at least suggests that the reverse implication—a modification of the Bekenstein–Hawking entropy leading to a varying effective gravitational constant (as in the BEI case)—is perhaps not so surprising.

Here, we remark that, although the Barrow entropy is supposed to quantify a quantum gravitational correction, it is *not* the—arguably more understood—quantum gravity effect that gives rise to Equation (3). (In fact, the effects of having both corrections together were considered in [22].) The sign of the constant $C$ in front of the logarithmic term in Equation (3) is usually negative [21], but positive in some works, such as [17,23]. In [24], Gour argued that a positive constant might be required if the area fluctuation is taken into account, whereas for a fixed area, the corrections due to the number of microstates (that describe the black hole) would give a negative constant. We have not much to add to this discussion. Instead, we would like to point out that the Barrow entropy, Equation (2), if expanded out as a series assuming small $\delta$ (when the quantum gravity correction is small), yields, up to the first order in $\delta$,

$$S = \frac{A}{4} + \frac{A}{8}\ln\left(\frac{A}{4}\right)\delta. \tag{4}$$

Note the presence of the area as a coefficient of the logarithmic correction. Thus, there is a risk that the "correction" term can be of the same order as the leading term. In fact, for any fixed $\delta \ll 1$, the expression will still deviate significantly from the Bekenstein–Hawking area law for a sufficiently *large* black hole, precisely when we expect quantum gravity corrections to be *small*. This provides a strong argument for a running BEI—$\delta$ should at least scale inversely proportional to $A\ln(A)$ to keep the subleading term small at $O(1)$. Such a running BEI would guarantee that quantum gravity correction is small when the black hole is large (i.e., when the energy scale is small, since the Hawking temperature is inversely proportional to the mass). Another option is to consider a $\delta$ that runs like $1/A$, then the Barrow entropy would just be the same as the standard logarithmic correction (at least up to the first order in $\delta$), but in this work, we take the former view, with the hope of uncovering new physics.

## 3. Dark Energy From a Running Barrow Entropy Index

Sheykhi derived the correction to the Friedmann equation due to the Barrow entropy (with a fixed BEI) in [14]:

$$\left(H^2+\frac{k}{a^2}\right)^{1-\frac{\delta}{2}} = \frac{8\pi}{3}\rho\left[\frac{2-\delta}{2+\delta}\frac{A_0^{1+\frac{\delta}{2}}}{4(4\pi)^{\frac{\delta}{2}}}\right] =: \frac{8\pi\rho}{3}G_{\text{eff}}. \tag{5}$$

In other words, the matter field feels an effective gravitational constant[1] $G_{\text{eff}}$ instead of $G$. This is one of the main results of [14].

This was obtained by applying the first law of thermodynamics to the energy flux through the apparent horizon of the Friedmann–Lemaître–Robertson–Walker (FLRW) universe, instead of the black hole horizon [23]. The expression for the horizon is

$$r_{\text{AH}} = \left(H^2+\frac{k}{a^2}\right)^{-\frac{1}{2}}. \tag{6}$$

In the following, we assumed a spatially flat universe ($k=0$).

Allowing the Barrow entropy index to run, the change in entropy would be:

$$\begin{aligned} dS &= d\left(\frac{4\pi r^2}{A_0}\right)^{1+\frac{\delta}{2}} \\ &= \left(\frac{4\pi r^2}{A_0}\right)^{1+\frac{\delta}{2}}\left[\frac{(2+\delta)\dot{r}}{r}+\ln\left(\frac{4\pi r^2}{A_0}\right)\frac{\dot{\delta}}{2}\right]dt. \end{aligned} \tag{7}$$

Then, using the first law of thermodynamics and the continuity equation, we can show, following [14], that

$$\begin{aligned} dE &= -\frac{1}{2\pi r}\left(\frac{4\pi r^2}{A_0}\right)^{1+\frac{\delta}{2}}\left[\frac{(2+\delta)\dot{r}}{r}+\ln\left(\frac{4\pi r^2}{A_0}\right)\frac{\dot{\delta}}{2}\right]dt \\ &= -4\pi r^3 H(\rho+p)dt, \end{aligned} \tag{8}$$

and, consequently,

$$\frac{(4\pi)^{\frac{\delta}{2}}}{2\pi A_0^{1+\frac{\delta}{2}}}\left[(2+\delta)r^{\delta-3}+\ln\left(\frac{4\pi r^2}{A_0}\right)\frac{r^{\delta-2}}{2}\frac{d\delta}{dr}\right]dr = -\frac{\dot{\rho}}{3}dt. \tag{9}$$

We note that $r_{\text{AH}} = 1/H$. In a sensible cosmology, $H$ decreases with time, so $r_{\text{AH}}$ is an increasing function of time. In other words, a larger $r$ corresponds to a later time. From the black hole thermodynamics argument as per Section 2, we want $\delta$ to at least decrease as $\delta \sim \text{const.}/r^2\ln(r^2)$ at large $r$. At early time, it should approach unity. For definiteness, we can take a function that is always smaller than the asymptotic behavior $\text{const.}/r^2\ln(r^2)$, so a natural choice is $\delta(r) = e^{-r}$.

The differential form of the modified Friedmann equation is equivalent to

$$(2-\delta)r^{\delta-3}dr + \frac{2-\delta}{2+\delta}\ln\left(\frac{4\pi r^2}{A_0}\right)\frac{r^{\delta-2}}{2}\frac{d\delta}{dr}dr$$
$$= -\frac{8\pi}{3}\underbrace{\left[\frac{2-\delta}{2+\delta}\frac{A_0^{1+\frac{\delta}{2}}}{4(4\pi)^{\frac{\delta}{2}}}\right]}_{=:G_{\text{eff}}} d\rho, \tag{10}$$

where $G_{\text{eff}}$ is the effective gravitational constant, the same as defined in the fixed BEI case [14], as in Equation (5). We immediately notice that $G_{\text{eff}}$ is monotonically decreasing as $\delta \to 0$, with $G_{\text{eff}}(\delta = 1) = 1/(3\sqrt{\pi}) \approx 0.189$ near the Big Bang, which is about 20%, the percent value of unity (in Planck units). We defer the possible cosmological implications of this varying $G_{\text{eff}}$ to the Discussion Section.

For now, we note that, since $\delta$ is a function of $H$, the Friedmann equation in the case of a running BEI cannot be easily solved by direct integration. We can, however, still analyze the late time and early time behavior separately. At late time, we chose some $r^*$ sufficiently large such that $1/r^2 \ln(r^2) \ll 1$ is as small as we wish. We have $\delta \sim 0$ (more precisely, we may say that $\delta$ is small enough that it is slowly varying compared to $r$, so we can take some small value of fixed $\delta$ when integrating with respect to $r$). Then, with $A_0 = 4$ in Planck units, the first term yields (with $R > r^*$)

$$2\int_{\tilde{r}}^{R} r^{-3}dr = 2\left[-\frac{1}{2r^2}\right]_{r^*}^{R} = \frac{1}{r^{*2}} - \frac{1}{R^2}. \tag{11}$$

The r.h.s. of Equation (10) integrates to give the matter density

$$-\frac{8\pi}{3}[\rho]_{\rho^*}^{\rho(R)}. \tag{12}$$

Thus, in the absence of the second term in Equation (10), this will just give the usual Friedmann equation (as we took $R \to \infty$; note $\rho(R) \to 0$ in this limit, because matter density decreases as the universe expands):

$$H^2 = \frac{8\pi\rho}{3} \tag{13}$$

evaluated at the time that corresponds to $r^*$.

Therefore, we can interpret the second term in Equation (10) as the differential form of the effective dark energy (so that Equation (10) yields the form[2] $H^2 - \Lambda/3 = 8\pi\rho/3$):

$$\frac{d\Lambda_{\text{eff}}}{3} = \frac{2-\delta}{2+\delta}\ln\left(\frac{4\pi r^2}{A_0}\right)\frac{r^{\delta-2}}{2}\frac{d\delta}{dr}dr. \tag{14}$$

If we use the ansatz $1/r^2 \ln(r^2)$ for $\delta$, then

$$\frac{d\Lambda_{\text{eff}}}{3} = -\frac{2-\delta}{2+\delta}\ln\left(\frac{4\pi r^2}{A_0}\right)$$
$$\times \frac{r^{\delta-2}}{2}\left[\frac{2}{r^3\ln(r^2)} + \frac{2}{r^3(\ln(r^2))^2}\right]dr. \tag{15}$$

At late time, $\delta \sim 0$, we have an upper bound:

$$-\frac{\Lambda_{\text{eff}}}{3}\bigg|_{\Lambda(r^*)}^{\Lambda(R)} \sim \frac{1}{2}\int_{r^*}^{R} \frac{\ln(\pi r^2)}{r^2}\left[\frac{2}{r^3\ln(r^2)} + \frac{2}{r^3(\ln(r^2))^2}\right]dr \tag{16}$$

$$< \frac{1}{2}\int_{r^*}^{R} \frac{\ln(\pi r^2)}{r^2}\frac{4}{r^3\ln(r^2)}dr \tag{17}$$

$$= 2\int_{r^*}^{R} \frac{1}{r^5}\frac{1}{\ln(r^2)}[\ln\pi + \ln(r^2)]dr \tag{18}$$

$$< 2\int_{r^*}^{R}\left(\frac{1}{r^5} + \frac{1}{r^5}\right)dr = 4\int_{r^*}^{R}\frac{1}{r^5}dr \tag{19}$$

$$= \left[-\frac{1}{r^4}\right]_{r^*}^{R} = \frac{1}{r^{*4}} - \frac{1}{R^4}. \tag{20}$$

That is,

$$\frac{\Lambda_{\text{eff}}(r^*)}{3} < \frac{1}{r^{*4}} \tag{21}$$

as $R \to \infty$. Since $r^*$ was chosen to be very large, $\Lambda_{\text{eff}}$ must be very small, and furthermore, it becomes smaller at later time. In fact, the integral Equation (16) yields

$$\Lambda_{\text{eff}}(r^*) \sim \frac{3}{4}\left[\frac{1}{r^{*4}}\frac{\ln(\pi r^{*2})}{\ln(r^{*2})} - 2(1-\ln(\pi))\text{Ei}(-4\ln(r^*))\right],$$

where Ei is the exponential integral function. One can check numerically that it is positive and decreases towards 0 very rapidly. Such an asymptotically vanishing effective cosmological constant scenario was previously discussed in [26–31]. Next, we show that, although the universe is late time de Sitter (dS)-like, it is anti-de Sitter (AdS)-like in the very early universe. Such a behavior had been observed, for example, in non-commutative quantum field theory [32].

In the vicinity of the Big Bang, the effective dark energy term depends on the profile of $\delta$ (how $\delta$ approaches $1/r^2\ln(r^2)$). For example, if $\delta = 1$ for some time after the Big Bang, then $d\delta/dr = 0$, and there is no effective dark energy. However, if we chose as we did previously $\delta = e^{-r}$, then the effective dark energy at the Big Bang is (with $\delta \sim 1$ and $0 < \varepsilon \ll 1$)

$$\frac{\Lambda_{\text{eff}}}{3}\bigg|_{\Lambda_{\text{eff}}(\varepsilon)}^{\Lambda_{\text{eff}}(2\varepsilon)} \sim -\frac{1}{3}\int_{\varepsilon}^{2\varepsilon}\ln(\pi r^2)\frac{e^{-r}}{2r}dr. \tag{22}$$

The integration interval is kept small near the Big Bang to ensure that $\delta = e^{-r}$ is nearly constant, so that we can approximate as follows:

$$\Lambda_{\text{eff}}\big|_{\Lambda_{\text{eff}}(\varepsilon)}^{\Lambda_{\text{eff}}(2\varepsilon)} \sim -\int_{\varepsilon}^{2\varepsilon}\ln(\pi r^2)\frac{e^{-r}}{2r}dr \sim -\varepsilon\ln(\pi\varepsilon^2)\frac{e^{-\varepsilon}}{2\varepsilon}. \tag{23}$$

That is,

$$\frac{\Lambda_{\text{eff}}(2\varepsilon) - \Lambda_{\text{eff}}(\varepsilon)}{\varepsilon} \sim -\ln(\pi\varepsilon^2)\frac{e^{-\varepsilon}}{2\varepsilon}. \tag{24}$$

In the limit $\varepsilon \to 0$, this yields $d\Lambda_{\text{eff}}/d\varepsilon \to \infty$. It follows that[3] $\Lambda_{\text{eff}} \to -\infty$ at the Big Bang with a log divergence.[4] It is worth commenting that [5] gives a strong constraint for the smallness of $\delta$ (assuming $\delta$ to be fixed) during the Big Bang nucleosynthesis (BBN) epoch. Thus, we know that $\delta$ cannot be slowly varying in the early universe if $\delta$ starts out near unity at the Big Bang. In fact, this is partly the reason why we chose $\delta = e^{-r}$ in our toy model, as it decays rather quickly at the beginning. Here, however, we used the mathematical fact that any function is nearly constant if the interval of its domain is sufficiently small.

An infinitely large negative cosmological "constant" is not good (the universe might immediately re-collapse), though we should also note that this is a logarithmic divergence

$\sim O(\ln \varepsilon)$, which is still "smaller" in magnitude than the $1/\varepsilon$-divergence in the modified Hubble term[5] (from the first term in Equation (10)), which is proportional to

$$\int_{\varepsilon}^{2\varepsilon} r^{-2} dr \sim \varepsilon \frac{1}{\varepsilon^2} = \frac{1}{\varepsilon}. \tag{25}$$

Of course, we could in principle choose other functions to obtain a finite nonzero value. Regardless, the integral:

$$\Lambda_{\text{eff}}(\varepsilon) \sim -\int_{\varepsilon}^{2\varepsilon} \ln\left(\pi r^2\right) \frac{1}{2r} \frac{d\delta}{dr} dr. \tag{26}$$

is always negative because $d\delta/dr < 0$ *and* $\ln(\pi r^2) < 0$ for small $r$, as long as $\delta$ is a strictly decreasing function. Therefore, $\Lambda_{\text{eff}}$ started out negative and crossed zero at some point in time (at $r_H = \sqrt{1/\pi} \approx 0.564$), and at late time, it is small and positive. By continuity, this suggests that $\Lambda_{\text{eff}}$ reached a maximum value at some point in the past, before decreasing towards its current small value $10^{-122}$. Another possibility is that $\delta$ contains a factor that cancels the $\ln(\pi r^2)$ term; then, in principle, the transition for AdS to dS can happen at an even later time, depending on the zero of the function. Similar scenarios have been proposed in the literature [33–40], with [35,36], that the transition from the AdS phase to the dS phase occurred around the recombination epoch, while [37] argued that the transition occurred as late as at redshift $z \approx 2.32$, which triggered the late time acceleration. It is also worth mentioning that [35,41] proposed that the AdS–dS transition could help to alleviate the Hubble tension. In fact, [41] argued that the $S_8$ tension [42,43] (equivalently, the $\sigma_8$ tension) is also alleviated, among other improvements; see also [44]. In [45], the authors concluded that solving both the $H_0$ and $\sigma_8$ tensions would require that $G_{\text{eff}} < G$ at some redshift $z$, if the equation of state satisfies $w(z) \leqslant -1$ (otherwise, $w(z)$ must cross the phantom divide). Thus, it is worth mentioning that a running BEI also causes a varying gravitational constant, and furthermore, from Equation (5), our $G_{\text{eff}}$, in the Planck units, always satisfies

$$\frac{1}{3\sqrt{\pi}} < G_{\text{eff}} = \frac{2-\delta}{2+\delta} \frac{1}{\pi^{\frac{\delta}{2}}} < 1 = G. \tag{27}$$

for all values of admissible $\delta \in [0,1]$. However, there are also arguments that state that this is unlikely to alleviate the Hubble tension by itself [46]; see also the subtleties discussed in the next section.

## 4. Discussion

In this work, we argued that if the Barrow entropy does encode some form of quantum gravitational correction, then in order to be consistent with the black hole thermodynamics of large black holes (for which quantum gravity correction should be small), the BEI must vary according to the energy scale. In terms of the horizon area, it should decay at least as fast as $1/(A \ln A)$.

When applied to the cosmological context, we showed that the modified Friedmann equation has many surprising properties: the universe started out with an effective negative cosmological constant, but at late time, it became positive and small, decaying towards zero asymptotically. As already explored in the literature, such a scenario could help to alleviate the Hubble tension and the $S_8$ tension. See, however, in ref. [47,48].

Despite the fact that the BEI is, thus far, only a phenomenological parametrization of quantum gravity effects, it is interesting to see that it could potentially explain some cosmological mysteries. In fact, although the explicit form of $\delta$ can only be determined once we have a better understanding of how to derive the Barrow entropy from a theory of quantum gravity, the features we obtained are quite generic as long as $\delta$ is assumed to be monotonically decreasing with the energy scale. The shortcoming is that we still do not know the exact behavior of $\delta$, especially at early time. Another caveat is that in deriving Equation (8), we used the first law as $dE = -TdS = \dot{\rho} V dt$, in which the volume change was neglected, which amounts to the assumption that $\dot{r} \ll Hr$. This holds at late time with exponential late time expansion, but may not hold near the Big Bang; this will depend on the explicit solution of the scale factor, which in turn depends on the profile of $\delta$.

Lastly, we comment on the varying effective gravitational constant. Whenever there is a varying $G$, one might worry that the continuity equation (the one employed in Equation (8) is the standard one $\dot{\rho} = -3H(\rho + p)$ that follows from $\nabla_\mu T^{\mu\nu} = 0$) may be modified since $\nabla_\mu(GT^{\mu\nu})$ may no longer imply $\nabla_\mu(T^{\mu\nu}) = 0$. However, the "bare" gravitational constant $G$ that appears at the level of the action is not the same as the effective $G_{\text{eff}}$ that appears in the modified Friedmann Equation (10). Furthermore, the Barrow entropy itself contains $A_0 = 4G$, which is the "bare" $G$. The fact that $G_{\text{eff}}$ is smaller in the early universe could suggest that we may be able to ameliorate the arrow of time problem [49–58] following the same line of thought as Greene et al. [59], as well as Sloan and Ellis [60]. (Note that to ameliorate the $H_0$ tension, one usually requires that $G_{\text{eff}} < G$ at late time. However, in our case, this condition always holds, even in the early universe.) However, this would require a very careful examination of the structure formation process by studying the perturbation equation. Typically, the effective gravitational constant that governs structure formation may not be the same as the $G_{\text{eff}}$ that appears in the background Friedmann equation, which may also differ from the "bare" $G$. A concrete example was provided in [61]. Structure formation in the case of a fixed BEI was recently studied in [10]. In fact, a varying gravitational constant would cause many issues, and early time cosmology would need to be re-examined in close detail [62] (for example, the CMB angular power spectrum would be modified [46,62]).

In conclusion, the effects of a running BEI requires a deeper investigation. There is also a need to better understand how Barrow entropy can possibly arise from a theory of quantum gravity. Does the fractalized geometry only hold for horizons or any spacetime hypersurface in general? What about the spacetime manifold itself? Barrow entropy is not the first proposal that involves fractal geometry in gravity. It has previously been proposed that the spacetime dimension becomes fractalized and decreases towards the Planck scale [63–68], which is a different modification (a two-sphere in such a fractalized geometry has a *lower* dimension [63], not higher, as per Barrow's proposal), but perhaps, one could obtain some ideas about how to derive Barrow entropy from these different approaches. Connections with fractional quantum mechanics [69] and other notions of entropies in relation to holographic dark energy [11,70–72] should also be further investigated.

**Author Contributions:** Conceptualization, Y.C.O.; validation, Y.C.O.; formal analysis, S.D.G. and Y.C.O.; investigation, S.D.G.; writing–original draft preparation, Y.C.O.; writing—review and editing, S.D.G.; supervision, Y.C.O.; funding acquisition, Y.C.O. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research was funded by the National Natural Science Foundation of China grant number 11922508.

**Acknowledgments:** Y.C.O. thanks Brett McInnes for useful discussions.

**Conflicts of Interest:** The authors declare no conflict of interest.

## Notes

1 We remark that [25], on the contrary, incorporated the extra term in a new energy density $\rho_{\text{DE}}$, which is attributed to dark energy, instead of modifying the gravitational constant. We can check that the two are equivalent by taking Equation (2.12) of [25] and substituting their expressions for $\Lambda$ and $\beta$. The difference between the two expressions is given by the fact that [25] considered only the case $k = 0$ and kept the integration constant $C$, which in [14] was set to 0 or considered as part of the total energy density $\rho$. In our work, we use the form of [14]. As we shall see below, when allowing the BEI to run, we would have an extra term that can be interpreted as a dynamical dark energy, different from the identification in [25] for the fixed index case.

2 $\frac{1}{3}\int_{\Lambda_{\text{eff}}(r^*)}^{\Lambda_{\text{eff}}(R)} d\Lambda_{\text{eff}} = \frac{1}{3}[\Lambda_{\text{eff}}(R) - \Lambda_{\text{eff}}(r^*)]$. This means that the observed value of the dark energy would be $\Lambda_{\text{eff}}(r^*) - \Lambda_{\text{eff}}(\infty)$ in the limit $R \to \infty$. Without loss of generality, we can take $\Lambda_{\text{eff}}(\infty) \equiv 0$, so that $\Lambda_{\text{eff}}(r^*)$ is the observed value of the effective cosmological constant.

[3] As per Footnote 4, the observed value of the effective cosmological constant over the interval $(\varepsilon, 2\varepsilon)$ is $\Lambda_{\text{eff}}(\varepsilon) - \Lambda_{\text{eff}}(2\varepsilon) < 0$. Since they are both comparable in magnitude (negative and with the same divergence properties), we can just refer to $\Lambda(\varepsilon)$ as the effective cosmological constant at the Big Bang.

[4] Indeed, a function $f(\varepsilon) = \ln(\pi\varepsilon^2)e^{-\varepsilon} < 0$ for small $\varepsilon$, satisfies $f(2\varepsilon) - f(\varepsilon) \to \infty$ and $f'(\varepsilon) \to \infty$. Another example to illustrate the same phenomenon, albeit with a different divergence, is to consider $f(x) = -1/x$. This function satisfies $f(2\varepsilon) - f(\varepsilon) \to \infty$ and $f'(\varepsilon) = 1/\varepsilon^2 \to \infty$. Colloquially, $f$ climbs out from $-\infty$ at the origin with an infinite positive slope.

[5] Since $H \to \infty$, there is still a Big Bang singularity in this model.

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