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Bianchi I Spacetimes in Chiral–Quintom Theory

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Abstract: In this paper, we study anisotropic exact solutions in the homogeneous Bianchi I background geometry in a multifield theory. Specifically, we consider the Chiral–Quintom theory, which is an extension of the Chiral theory, because at least one of the scalar fields can have negative energy density. Moreover, the Quintom theory can be recovered when one of the free parameters of the theory vanishes. We find that Kasner-like and anisotropic exponential solutions exist for specific functional forms of the scalar field potential. Finally, Noether symmetry analysis is applied for the classification of the theory according to the admitted symmetries. Conservation laws are determined, while we show that the Kasner-like solution is the analytic solution for the given model.

Keywords: Bianchi spacetimes; scalar field; cosmology

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1. Introduction

Cosmic inflation describes the accelerated period in the early stage of cosmological history [1–3]. Inflation has been considered a solution to long-standing problems about the structure of the universe, such as the flatness problem and horizon problems. Indeed, the inflationary mechanism surpasses the requirement for the specific initial conditions in cosmological history [4,5]. On the other hand, recent cosmological observations indicate that, at the present time, the universe is under a second accelerated phase, known as late-time acceleration attributed to the so-called dark energy [6]. In the context of general relativity, acceleration occurs when there is a matter source that has a negative equation of state parameter and provides effective "repulsive" (anti-)gravitational force.

The introduction of scalar fields in gravitational theory gives a very simple mechanism for the description of the acceleration phases of the universe. In the minimally coupled scalar field theory, the antigravitating behavior occurs when the scalar field potential dominates [7–14]. Furthermore, for the description of the late-time acceleration of the universe, phantom scalar fields with negative energy density have been proposed [15–18]. For the phantom fields, the equation of state parameter can cross the phantom divide line and take values lower than minus one. However, in order to solve the various problems, such as the appearance of ghosts, and to describe the general cosmological history, multiscalar field models have been considered.

The Quintom model [19,20] is a well studied two-scalar field cosmological model, where one of the fields is quintessence and the second field is a phantom field. The novelty of the Quintom theory is that the effective equation of state parameter can cross the phantom divide line more than once without the appearance of ghosts. Another multiscalar field model of special interest is the Chiral model [21], which has been used to describe a multifield inflation known as hyperbolic inflation [22,23]. The analytic solution for the hyperbolic inflationary model was derived recently in [24]. The equation of state parameter in the Chiral model has as a lower bound the cosmological constant limit. However, because of quantum transitions in the early universe, it can surpass that limit, and the effective equation of state parameter crosses the phantom divide line [25].

Recently, a generalization of the Chiral model was proposed in [26], where the scalar fields can have negative energy density. This cosmological model has similarities with the Quintom theory. Indeed, the equation of state parameter can cross the phantom divide line more than once without the appearance of ghosts. In addition, it was found that this specific model reproduced the epoch for hyperbolic inflation [27]. Furthermore, the presence of the spatial curvature was investigated in [28,29]. In particular, it was found that this specific model solved the flatness problem.

Anisotropic and inhomogeneous exact solutions play an important role in the description of the very early universe before inflation, since they can describe the small anisotropic inhomogeneities in the cosmic observations [30,31]. The cosmic "no-hair" conjecture states that the future state of an accelerated universe is an isotropic universe [32]. In [33], anisotropic spacetimes were used to explain the CMB polarization and its implications for CMB anomalies. On the other hand, some anisotropic dark energy models were investigated in [34,35].

The first analytic result, which supported the cosmic "no-hair" conjecture was derived in [36]. Specifically, it was found that the presence of a positive cosmological constant in Bianchi anisotropic spacetimes provided expanding Bianchi spacetimes, which evolved to expanding de Sitter universes, see also the discussion in [37]. In the context of Chiral theory, anisotropic spacetimes were investigated in [38–40], while some other studies of scalar fields in anisotropic background spaces were presented in [41–43] and references therein.

In [38], exact anisotropic solutions in Chiral theory were determined, and it was found that there exist exact anisotropic solutions for Bianchi III or Kantowski–Sachs background geometry where the two scalar fields contribute to cosmological history. Moreover, anisotropic Kasner-like solutions, which belong to the Bianchi I family of spacetimes, were not supported by the Chiral theory [39]. However, when a gauge field coupled to the scalar field was introduced, anisotropic Bianchi I exact spacetimes were provided by the Chiral model [40]. In this work, we focus our analysis on the existence of anisotropic Bianchi I exact and analytic solutions for the Chiral–Quintom model proposed in [26].

Kasner spacetime [44] is one of the first anisotropic and homogeneous exact solutions derived in the literature and describes an empty Bianchi I universe. The Kasner metric depends on three parameters, which are constrained by two algebraic relations, so it is a one-parameter family of solutions. There are many applications of the Kasner metric, see for instance [45–47]. Moreover, Kasner spacetime describes the asymptotic behavior of the Mixmaster universe, Bianchi IX metric, when the effects of the spatial curvature are negligible. Kasner-like metrics [48–56], which are Kasner-like solutions with generalized Kasner-algebraic relations, are also of special interest. The structure of the paper is as follows.

In Section 2, we present the considered gravitational model, which is that of the Chiral–Quintom theory in a homogeneous and anisotropic Bianchi I background geometry. Exact solutions, which describe anisotropic geometries with power-law and exponential scale factors, are derived in Section 3. The existence of Kasner-like exact solutions are investigated. In Section 4, we perform a detailed analysis of the field equations by using the Noether symmetry approach. From this analysis, we can infer the existence of invariant functions and conservation laws for the field equations, which can be used to construct analytic solutions. Finally, in Section 5 we summarize our results.

2. Chiral–Quintom Theory

We assume the four-dimensional geometry with metric tensor $g_{\mu\nu}(x^\kappa)$ and the multi-scalar field gravitational model with gravitational action integral [57]

$$S=\int dx^4\sqrt{-g}\left(R-\frac{1}{2}g_{\mu\nu}H_{AB}\left(\Phi^C\right)\Phi^{A,\mu}\Phi^{B,\nu}-V\left(\Phi^C\right)\right), \tag{1}$$

where $R=R(x^\kappa)$ is the Ricciscalar of the background geometry $g_{\mu\nu}(x^\kappa)$.

The components of vector field $\Phi^A(x^\kappa)$ describe the scalar fields of the theory. In our analysis, we assume two scalar fields, namely $\phi(x^\kappa)$, $\psi(x^\kappa)$; that is, $\Phi^A = (\phi(x^\kappa), \psi(x^\kappa))^T$. Thus, $H_{AB}(\Phi^C(x^\kappa))$ is a two-dimensional symmetric tensor; that is, $H_{AB} = H_{BA}$ and describes the geometry in which the two scalar fields lie. The interaction of the scalar fields in the kinetic components is provided by the metric tensor H_{AB}. Finally, $V(\Phi^C(x^\kappa))$ is the potential function, which drives the dynamics and the cosmological evolution.

In the Chiral–Quintom theory, the gravitational action integral (2) is defined as follows [26]

$$S = \int dx^4 \sqrt{-g}\left(R - \frac{1}{2}g_{\mu\nu}\left(\varepsilon_1 \phi^{,\mu(x^\kappa)}\phi^{,\nu(x^\kappa)} + \varepsilon_2 e^{2\kappa\phi(x^\kappa)}\psi(x^\kappa)^{,\mu}\psi(x^\kappa)^{,\nu}\right) - V(\phi)\right), \tag{2}$$

where $\varepsilon_1, \varepsilon_2$ have the constraints $(\varepsilon_1)^2 = 1$ and $(\varepsilon_2)^2 = 1$. The value -1 indicates that the corresponding scalar field is phantom-like [26].

The Chiral model is recovered when ε_1 and ε_2 are positive numbers [22]. Parameter κ plays a more important role, since it is related to the curvature of the two dimensional spacetime $H_{AB}(\Phi^C(x^\kappa))$, and a nonzero value is essential in order for the hyperbolic inflation to occur [22]. Indeed, for $\kappa = 0$, the curvature of $H_{AB}(\Phi^C)$ vanishes, and the Chiral–Quintom model reduces to the Quintom theory [19]. The later model however does not reproduce the hyperbolic inflation. In this study, we consider a nonzero coupling constant parameter, κ.

For the background space, we consider that of the anisotropic and homogeneous Bianchi I spacetime

$$ds^2 = -N^2(t)dt^2 + A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2, \tag{3}$$

where $A(t)$, $B(t)$, and $C(t)$ are the three scale factors, and $N(t)$ is the lapse function.

We prefer to work on the Misner variables, where the line element reads

$$ds^2 = -N^2(t)dt^2 + e^{2\alpha}\left(e^{-2\beta_+(t)}dx^2 + e^{\beta_+(t)+\sqrt{3}\beta_-(t)}dy^2 + e^{\beta_+(t)-\sqrt{3}\beta_-(t)}dz^2\right), \tag{4}$$

where now $\alpha(t)$ is the scale factor of a three-dimensional hypersurface, and $\beta_+(t)$, $\beta_-(t)$ are the two anisotropic parameters.

When $\dot{\beta}_+(t) = 0$, $\dot{\beta}_-(t) = 0$, where $\frac{d\beta_\pm}{dt} = \dot{\beta}_\pm$, the line element (4) reduces to the spatially flat Friedmann–Lemaître–Robertson–Walker spacetime. Parameter $N(t)$ is the lapse function where, without loss of generality, we select $N(t) = 1$.

For the background geometry described by the line element (4), the corresponding field equations that follow from the variational of the action integral (2) are

$$3H^2 - \frac{3}{4}\left((\dot{\beta}_+)^2 + (\dot{\beta}_-)^2\right) = \frac{1}{2}\left(\varepsilon_1\dot{\phi}^2 + \varepsilon_2 e^{2\kappa\phi}\dot{\psi}^2\right) + V(\phi), \tag{5}$$

$$2\dot{H} + 3H^2 + \frac{3}{4}\left((\dot{\beta}_+)^2 + (\dot{\beta}_-)^2\right) = -\frac{1}{2}\left(\varepsilon_1\dot{\phi}^2 + \varepsilon_2 e^{2\kappa\phi}\dot{\psi}^2\right) + V(\phi), \tag{6}$$

$$\ddot{\beta}_+ + 3H\dot{\beta}_+ = 0, \tag{7}$$

$$\ddot{\beta}_- + 3H\dot{\beta}_- = 0, \tag{8}$$

$$\varepsilon_1(\ddot{\phi} + 3H\dot{\phi}) + V_{,\phi} = e^{2\kappa\phi}\varepsilon_2\kappa\dot{\psi}^2, \tag{9}$$

$$\ddot{\psi} + 3H\dot{\psi} = -2\kappa\dot{\psi}\dot{\phi}. \tag{10}$$

where $H = \dot{\alpha}$ is the Hubble function.

3. Anisotropic Exact Solutions

In this section, we investigate the existence of exact solutions of special interest for the field equations. Firstly, let us recover the Kasner vacuum solution by assuming $\phi(t) = 0$, $\psi(t) = 0$, and $V(\phi) = 0$.

Then, for the specific functional forms

$$H(t) = \frac{H_0}{t},\ \beta_+(t) = \beta_+^0 \ln t \ \text{and}\ \beta_-(t) = \beta_-^0 \ln t, \tag{11}$$

from Equations (5)–(8), we obtain

$$H_0 = \frac{1}{3} \ \text{and}\ \left(\beta_+^0\right)^2 + \left(\beta_-^0\right)^2 = \frac{4}{9}. \tag{12}$$

The later two algebraic expressions are the so-called Kasner relations expressed in the Misner variables.

3.1. Singular Solution

We consider now the case where the two scalar fields contribute to the field equations; that is, $\dot{\phi}(t)\dot{\psi}(t) \neq 0$, and the Bianchi spacetime is described by the singular solution

$$\alpha(t) = p \ln t,\ \beta_+(t) = \beta_+^0 \ln t, \text{and}\ \beta_-(t) = \beta_-^0 \ln t\ . \tag{13}$$

This anisotropic solution corresponds to the family of Kasner-like solutions with initial cosmological singularity when $t = 0$. Recall that we have assumed the constant lapse function $N(t) = 1$.

From the field Equations (7) and (8), we derive $p = \frac{1}{3}$; thus, the remaining field equations read

$$\left(\beta_+^0\right)^2 + \left(\beta_-^0\right)^2 - \frac{4}{9} + \frac{2}{3}t^2\left(2V + \varepsilon_1\dot{\phi}^2 + \varepsilon_2 e^{2\kappa\phi}\dot{\psi}^2\right) = 0, \tag{14}$$

$$\left(\beta_+^0\right)^2 + \left(\beta_-^0\right)^2 - \frac{4}{9} - \frac{2}{3}t^2\left(2V - \varepsilon_1\dot{\phi}^2 - \varepsilon_2 e^{2\kappa\phi}\dot{\psi}^2\right) = 0, \tag{15}$$

$$\varepsilon_1\dot{\phi} + t\left(V_{,\phi} - e^{2\kappa\phi}\varepsilon_2\kappa\dot{\psi}^2 + \varepsilon_1\ddot{\phi}\right) = 0, \tag{16}$$

and

$$\ddot{\psi} + \dot{\psi}\left(\frac{1}{t} + 2\kappa\dot{\phi}\right) = 0. \tag{17}$$

Equation (17) provides $\dot{\psi} = \psi_0 \frac{e^{-2\kappa\phi}}{t}$. Hence, by replacing this in the rest of the equations, we obtain $V(\phi) = 0$, and

$$\phi(t) = \frac{1}{\kappa}\ln(\Phi(t)), \tag{18}$$

with

$$\Phi(t) = \pm\psi_0\sqrt{\frac{6\varepsilon_2}{\xi}}\sinh\left(\Phi_1 \pm i\kappa\sqrt{\frac{\xi}{6\varepsilon_1}}\ln t\right), \tag{19}$$

where $\xi = 9\left(\left(\beta_+^0\right)^2 + \left(\beta_-^0\right)^2 - \frac{4}{9}\right)$, and Φ_1 is an integration constant.

Consequently, in order for the scalar field $\phi(t)$ to be a real field, $\varepsilon_1\varepsilon_2 < 0$, which means power-law solutions exist, i.e., Kasner-like solutions, only when one of the scalar fields is phantom-like, and the cosmological model is that of the Chiral–Quintom theory.

3.2. Exponential Solution

Now, we assume the nonsingular solution with

$$\alpha(t) = H_0 t\ ,\ \beta_+(t) = \beta_+^0 t \ \text{and}\ \beta_-(t) = \beta_-^0 t\ . \tag{20}$$

We substitute this into the field equations, and we find

$$\psi(t) = \psi_0 e^{-3H_0 t - 2\kappa\phi} \;,\; V(\phi) = 3H_0^2, \tag{21}$$

and

$$\phi = \frac{1}{\kappa}\ln(\Phi(t)), \tag{22}$$

in which

$$\Phi(t) = \pm\kappa\left(i\frac{\psi_0}{3H_0}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}e^{-3H_0 t} - \Phi_1\right), \tag{23}$$

where Φ_1 is an integration constant.

Consequently, in order for a real solution to exist, $\frac{\varepsilon_2}{\varepsilon_1} < 0$; that is, one of the scalar fields is phantom, the other is quintessence. The cosmological model is that of Chiral–Quintom theory.

4. Noether Symmetry Analysis

The application of symmetry analysis is a powerful method for the construction of conservation laws and invariant functions necessary for the analytical study of nonlinear dynamical systems [58–60]. In the case of dynamical systems, which follow from a variational principle, Noether's theorems provide a system method for the derivation of conservation laws [58].

In cosmological studies, the Noether symmetry approach has been widely applied. For a review on the subject, we refer the reader to [61]. The analysis of the cosmological field equations with the requirement for the field equations to admit conservation laws generated by Noether's theorems has been used in two ways. Indeed, new conservation laws have been constructed for the nonlinear field equations, which led to the derivation of new analytic solutions [62–65]. Moreover, Noether symmetry analysis has been applied as a classification method for the determination of the unknown functions of the given theorem. This approach has geometric characteristics because Noether symmetries are related to the geometry where the dynamical variables are defined [66,67]. We omit the presentation of the basic properties of the Noether symmetry analysis, which can be found in [61].

For the Chiral–Quintom model of our analysis, the cosmological field equations are derived by the variation of the point-like Lagrangian

$$\mathcal{L} = \mathcal{L}(\alpha, \dot{\alpha}, \beta_+, \dot{\beta}_+, \beta_-, \dot{\beta}_-, \phi, \dot{\phi}, \psi, \dot{\psi}), \tag{24}$$

where

$$\mathcal{L} = e^{3\alpha}\left(6\dot{\alpha}^2 - \frac{3}{4}\left(\left(\dot{\beta}_+\right)^2 + \left(\dot{\beta}_-\right)^2\right) - \frac{1}{2}\left(\varepsilon_1\dot{\phi}^2 + \varepsilon_2 e^{2\kappa\phi}\dot{\psi}^2\right) + V(\phi)\right). \tag{25}$$

The point-like Lagrangian (25) describes an autonomous dynamical system where, for an arbitrary potential function $V(\phi)$, it admits the Noether symmetry vector field $X_1 = \partial_t$. The corresponding conservation law is the Hamiltonian function

$$\mathcal{H} = e^{3\alpha}\left(6\dot{\alpha}^2 - \frac{3}{4}\left(\left(\dot{\beta}_+\right)^2 + \left(\dot{\beta}_-\right)^2\right) - \frac{1}{2}\left(\varepsilon_1\dot{\phi}^2 + \varepsilon_2 e^{2\kappa\phi}\dot{\psi}^2\right) - V(\phi)\right), \tag{26}$$

where from the constraint Equation (5), $\mathcal{H} = 0$. However, for specific functional forms of $V(\phi)$ additional symmetries may exist. There exist two cases, $V_A(\phi) = V_0 e^{-\lambda\phi}$ and $V_B(\phi) = 0$, where additional Noether point symmetries for the Lagrangian (25) exist. For other forms of potential functions, symmetries may exist, including generalized symmetries, hidden symmetries, and others.

We focus on the exponential potential $V_A(\phi) = V_0 e^{-\lambda\phi}$. The admitted Noether symmetries are

$$X_2 = \partial_{\beta_+} \;,\; X_3 = \partial_{\beta_-} \;,$$

$$X_4 = \beta_- \partial_{\beta_+} - \beta_+ \partial_{\beta_-} \ , \ X_5 = \partial_\psi \ ,$$

and

$$X_6 = 2t\partial_t + \frac{2}{3}\partial_\alpha + \frac{4}{\lambda}\left(\partial_\phi - \kappa\psi\partial_\psi\right) \ .$$

The corresponding Noetherian conservation laws are

$$I_2(X_2) = e^{3\alpha}\dot{\beta}_+ \ , \ I_3(X_3) = e^{3\alpha}\dot{\beta}_-, \tag{27}$$

$$I_4(X_4) = e^{3\alpha}\left(\beta_-\dot{\beta}_+ - \beta_+\dot{\beta}_-\right), \tag{28}$$

$$I_5(X_5) = e^{3\alpha}e^{2\kappa\phi}\dot{\psi}, \tag{29}$$

$$I_6(X_6) = 2t\mathcal{H} - 4e^{3\alpha}\left(\dot{\alpha} - \frac{\varepsilon_1}{\lambda}\dot{\phi} + \frac{\varepsilon_2}{\lambda}e^{2\kappa\phi}\kappa\psi\dot{\psi}\right). \tag{30}$$

By using the constraint Equation (5), conservation law $I_6(X_6)$ reads $I_6(X_6) = -4e^{3\alpha}(\dot{\alpha} - \frac{\varepsilon_1}{\lambda}\dot{\phi} + \frac{\varepsilon_2}{\lambda}e^{2\kappa\phi}\kappa\psi\dot{\psi})$.

We can easily see that the set of conservation laws are not in involution. The conservation law $I_6(X_6)$ is written as

$$I_6(X_6) = -4e^{3\alpha}\left(\dot{\alpha} - \frac{\varepsilon_1}{\lambda}\dot{\phi} + \frac{\varepsilon_2}{\lambda}\kappa I_5 \psi\right). \tag{31}$$

Thus, we can not infer the Liouville integrability property of the field equations.

A question which arises is whether we can use the invariant functions defined by the vector field X_6 in order to construct an exact solution. Indeed, the Lie invariants, which correspond to X_6, are

$$\alpha(t) = \frac{1}{3}\ln t + \alpha_0 \ , \ \phi = \frac{2}{\lambda}\ln t + \phi_0, \text{and } \psi = \psi_0 t^{-2\frac{\kappa}{\lambda}}. \tag{32}$$

By replacing these in the field equations for the α_0, ϕ_0, and ψ_0 constants, it follows that $\psi_0 = 0$, $V_0 = 0$, and the Kasner-like relation is

$$\left(\beta_+^0\right)^2 + \left(\beta_-^0\right)^2 - \frac{4}{9\lambda^2}e^{6\alpha_0}\left(\lambda^2 - 6\varepsilon_1\right) = 0, \tag{33}$$

which is nothing other than the exact solution of a minimally coupled scalar field without any contribution of the potential function or the generalized Kasner-like solution for a five-dimensional Brane.

For the case of the zero potential function $V_B(\phi) = 0$, the admitted Noether symmetries are

$$X_2 \ , \ X_3 \ , \ X_4 \ , \ X_5 \ ,$$

$$Y_6 = \partial_\phi - \kappa\psi\partial_\psi \ , \ Y_7 = \frac{2}{3}\partial_\alpha$$

and

$$Y_8 = \psi\partial_\phi - \left(\frac{\kappa}{2}\psi - \frac{\varepsilon_1}{2\kappa}e^{-2\kappa\phi}\right)\partial_\psi.$$

The corresponding conservation laws are $I_2(X_2)$, $I_3(X_3)$, $I_4(X_4)$, $I_5(X_5)$, and

$$I_6(Y_6) = e^{3\alpha}\left(\frac{\varepsilon_1}{\lambda}\dot{\phi} - \frac{\varepsilon_2}{\lambda}e^{2\kappa\phi}\kappa\psi\dot{\psi}\right) \ , \tag{34}$$

$$I_7(Y_7) = 2t\mathcal{H} - 4e^{3\alpha}\dot{\alpha} \ , \tag{35}$$

$$I_8(Y_8) = e^{3\alpha}\left(\varepsilon_1\psi\dot{\phi} - \varepsilon_2 e^{2\kappa\phi}\dot{\psi}\left(\frac{\kappa}{2}\psi - \frac{\varepsilon_1}{2\kappa}e^{-2\kappa\phi}\right)\right). \tag{36}$$

In contrast to the exponential potential function, in this case, there exist at least three conservation laws, which are in involution and independent; that is, the field equations form a Liouville integrable system.

We proceed with the derivation of the analytic solution.

Analytic Solution for $V_B(\phi)$

With the use of the conservation laws $I_2(X_2),\ I_3(X_3)$, and $I_5(X_5)$ the field equations can be written with the use of the Hamiltonian formalism as

$$\mathcal{H} = \frac{e^{-3\alpha}}{12\varepsilon_1}\left(6p_\phi^2 + \varepsilon_1\left(9\left(I_2^2 + I_3^2\right) + 6e^{-2\kappa\phi}I_5^2 - p_\alpha^2\right)\right) \equiv 0, \tag{37}$$

where

$$\dot{\alpha} = \frac{1}{6}p_\alpha e^{-3\alpha}\ ,\ \dot{\phi} = -\frac{p_\phi}{\varepsilon_1}e^{-3\alpha}\ . \tag{38}$$

Consequently, the field equations are

$$\dot{p}_\alpha = 0\ ,\ \dot{p}_\phi = e^{-3\alpha-2\kappa\phi}\varepsilon_2\kappa I_5^2. \tag{39}$$

The analytic solution is derived

$$\alpha(t) = \frac{1}{3}\ln\left(\frac{p_\alpha}{2}(t-t_0)\right). \tag{40}$$

which is the Kasner-like solution found in the previous section.

We conclude that the Kasner-like solution is the analytic solution for the given cosmological model.

5. Conclusions

In this study, we determined the exact cosmological solutions for the field equations in the Chiral–Quintom theory with anisotropic and homogeneous background geometry. The Chiral–Quintom theory belongs to the multiscalar field theories, and it is an extension of the Chiral model where now at least one of the scalar fields can have a negative energy density; that is, it has a phantom behavior. The theory extends the Quintom multifield theory, where the kinetic part of the scalar fields defines a two-dimensional manifold of non-zero constant curvature.

The cosmological field equations form a nonlinear dynamical system of ordinary differential equations, the dependent variables, the scale factors $\{\alpha(t), \beta_+(t), \beta_-(t)\}$, and the two scalar fields $\{\phi(t), \psi(t)\}$. We investigated the existence of exact anisotropic solutions, which belong to the family of Kasner-like spacetimes and to the accelerated exponential geometries, by giving the explicitly functional form of the scale factors $\{\alpha(t), \beta_+(t), \beta_-(t)\}$. These two geometries, described by the latter exact solutions, were provided by the Chiral–Quintom theory for an appropriate functional form of the scalar field potential.

Moreover, because the cosmological field equations form a Hamiltonian dynamical system, and a Lagrangian function exists, we applied the Noether symmetry approach for the investigation of conservation laws. In particular, the Noether symmetry conditions were used to constrain the potential function according to the admitted Noether symmetries for the field equations. Two potential functions were derived, the exponential potential and the zero potential, where, for these two potential functional forms, extra conservation laws related to point symmetries exist. For the zero potential, we were able to infer the integrability property of the field equations, where we proved that the Kasner-like solution was the general analytic solution for this specific cosmological model. On the other hand, for the exponential potential, we were not able to prove the integrability property for the field equations and to write the analytic solution as in the case of the spatially Friedmann–Lemaître–Robertson–Walker background geometry with matter source [24].

The existence of a Kasner-family anisotropic exact solution indicates that the field equations admit actual solutions for the anisotropic initial condition. That is, anisotropy is supported by the cosmological model, and when there is not a potential function, the spacetime retains anisotropy. However, when a constant scalar field potential appears, then the resulting spacetime has exponential scale factors leading to an inflationary universe described by the isotropic de Sitter spacetime [36].

In a future work we plan to investigate the stability properties of the Chiral–Quintom theory and extend the analysis presented in [39] for the Chiral model. Such analysis is essential in order to understand the global evolution of the field equations for other potential functions, whether the Chiral–Quintom theory solves the isotropization problem, and whether anisotropic initial conditions can lead to hyperbolic inflation.

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