

Article

Metric-Affine Version of Myrzakulov $F(R, T, Q, \mathcal{T})$ Gravity and Cosmological Applications

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Abstract: We derive the full set of field equations for the metric-affine version of the Myrzakulov gravity model and also extend this family of theories to a broader one. More specifically, we consider theories whose gravitational Lagrangian is given by $F(R, T, Q, \mathcal{T}, \mathcal{D})$ where T , Q are the torsion and non-metricity scalars, $\mathcal{T}$ is the trace of the energy-momentum tensor and $\mathcal{D}$ the divergence of the dilation current. We then consider the linear case of the aforementioned theory and, assuming a cosmological setup, we obtain the modified Friedmann equations. In addition, focusing on the vanishing non-metricity sector and considering matter coupled to torsion, we obtain the complete set of equations describing the cosmological behavior of this model along with solutions.

Keywords: cosmology; torsion



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1. Introduction

Even though general relativity (GR) is undeniably one of the most beautiful and successful theories of physics, recent observational data have challenged its status [1]. Probably the most important observations that cannot be explained within the realm of GR are the early time as well as the late time accelerated expansion of our universe. This contradiction between theory and observations has led to the development of a fairly large number of alternative theories to GR which collectively go by the name of modified gravity [2]. The search for a successful alternative has been proven to be both fruitful as well as constructive in regard to our understanding of gravity.

Among this plethora of modified gravities, let us mention the metric $f(R)$ theories, the metric-affine (Palatini) $f(R)$ gravity [3–5], the teleparallel $f(T)$ gravities [6,7], the symmetric teleparallel $f(Q)$ [8,9], scalar–tensor theories [10,11], etc., and also certain extensions of them (see discussion in Section IV). Of course, the kind of modifications one chooses to adopt is a matter of personal taste. From our point of view, interesting and well-motivated alternatives are those which extend the underlying geometry of spacetime by allowing a connection that is more general than the usual Levi-Civita one. In generic settings, when no a priori restriction is imposed on the connection and the latter is regarded as another fundamental field on top of the metric, the space will be non-Riemannian [12] and possess both torsion and non-metricity. These last geometric quantities can then be computed once the affine connection is found. The theories formulated on this non-Riemannian manifold are known as metric-affine theories of gravity [13,14].

In recent years, there has been an ever-increasing interest in the metric-affine approach [5,15–29] and especially in its cosmological applications [30–41]. This interest is possibly due to the fact that the additional effects (compared to GR) that come into play in this framework have a direct geometrical interpretation. That is, the modifications are solely due to spacetime torsion and non-metricity. Furthermore, these geometric notions are excited by matter that has intrinsic structure [32,42–45]. This inner structure-generalized

geometry interrelation adds another positive characteristic to the MAG scheme. This is the framework we consider in this study.

The paper is organized as follows. Firstly, we fix conventions and briefly review some of the basic elements of non-Riemannian geometry and the physics of metric-affine gravity. We then consider an extended version of the $F(R, T, Q, \mathcal{T}, \mathcal{D})$ theory [46]. To be more specific, working in a metric-affine setup, we consider the class of theories with gravitational Lagrangians of the form $F(R, T, Q, \mathcal{T}, \mathcal{D})$, where $\mathcal{D}$ is the divergence of the dilation current, the new add-on we are establishing here. Then, we obtain the field equations for this family of theories by varying with respect to the metric and the independent affine connection. Considering a linear function F we then present a cosmological application for this model and, finally, switching off non-metricity and considering a scalar field coupled to torsion, we obtain the modified Friedmann equations and also provide solutions for this simple case.

2. Conventions/Notation

Let us now briefly go over the basic geometric as well as physical setup we are going to use and also fix notation. We consider a 4-dim non-Riemannian manifold endowed with a metric and an affine connection $(\mathcal{M}, g, \nabla)$. Our definition for the covariant derivative, for example, of a vector, will be

$$\nabla_\alpha u^\lambda = \partial_\alpha u^\lambda + \Gamma^\lambda_{\beta\alpha} u^\beta \quad (1)$$

We also define the (Cartan) torsion tensor by

$$S_{\mu\nu}{}^\lambda := \Gamma^\lambda_{[\mu\nu]} \quad (2)$$

and the non-metricity tensor as

$$Q_{\alpha\mu\nu} := -\nabla_\alpha g_{\mu\nu} \quad (3)$$

Contracting these with the metric tensor, we obtain the associated torsion and non-metricity vectors

$$S_\mu := S_{\mu\nu}{}^\nu \quad (4)$$

$$Q_\mu := Q_{\mu\alpha\beta} g^{\alpha\beta}, \quad q_\mu := Q_{\alpha\beta\mu} g^{\alpha\beta}, \quad (5)$$

respectively. In addition, since we are in four dimensions, we can also form the torsion pseudo-vector according to

$$t^\mu := \varepsilon^{\mu\alpha\beta\gamma} S_{\alpha\beta\gamma} \quad (6)$$

Given the above definitions for torsion and non-metricity, one can easily show (see, for instance, [14]) the affine connection decomposition ¹

$$\Gamma^\lambda_{\mu\nu} = N^\lambda_{\mu\nu} + \tilde{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} (Q_{\mu\nu\alpha} + Q_{\nu\alpha\mu} - Q_{\alpha\mu\nu}) - g^{\alpha\lambda} (S_{\alpha\mu\nu} + S_{\alpha\nu\mu} - S_{\mu\nu\alpha}) + \tilde{\Gamma}^\lambda_{\mu\nu} \quad (7)$$

where $N^\lambda_{\mu\nu}$ is known as the distortion tensor. Continuing, we define the curvature tensor as usual

$$R^\mu{}_{\nu\alpha\beta} := 2\partial_{[\alpha} \Gamma^\mu_{\nu|\beta]} + 2\Gamma^\mu_{\rho[\alpha} \Gamma^\rho_{\nu|\beta]} \quad (8)$$

and by a double contraction of the latter, we get the Ricci scalar

$$R := R^\mu{}_{\nu\mu\nu} g^{\nu\beta} \quad (9)$$

Then, by using decomposition (7), we obtain the post-Riemannian expansion for the Ricci scalar [14]

$$R = \tilde{R} + T + Q + 2Q_{\alpha\mu\nu} S^{\alpha\mu\nu} + 2S_\mu (q^\mu - Q^\mu) + \tilde{\nabla}_\mu (q^\mu - Q^\mu - 4S^\mu) \quad (10)$$

where $\tilde{R}$ is the Riemannian Ricci tensor (i.e., computed with respect to the Levi-Civita connection) and we have also defined the torsion and non-metricity scalars as ²

$$T := S_{\mu\nu\alpha}S^{\mu\nu\alpha} - 2S_{\mu\nu\alpha}S^{\alpha\mu\nu} - 4S_\mu S^\mu \quad (11)$$

and

$$Q := \frac{1}{4}Q_{\alpha\mu\nu}Q^{\alpha\mu\nu} - \frac{1}{2}Q_{\alpha\mu\nu}Q^{\mu\nu\alpha} - \frac{1}{4}Q_\mu Q^\mu + \frac{1}{2}Q_\mu q^\mu, \quad (12)$$

respectively. Note that with the introduction of the superpotentials ³

$$\Omega^{\alpha\mu\nu} := \frac{1}{4}Q^{\alpha\mu\nu} - \frac{1}{2}Q^{\mu\nu\alpha} - \frac{1}{4}g^{\mu\nu}Q^\alpha + \frac{1}{2}g^{\alpha\mu}Q^\nu \quad (13)$$

$$\Sigma^{\alpha\mu\nu} := S^{\alpha\mu\nu} - 2S^{\mu\nu\alpha} - 4g^{\mu\nu}S^\alpha \quad (14)$$

these can be expressed more compactly as

$$T = S_{\alpha\mu\nu}\Sigma^{\alpha\mu\nu} \quad (15)$$

$$Q = Q_{\alpha\mu\nu}\Omega^{\alpha\mu\nu} \quad (16)$$

Equation (8) is of key importance in teleparallel formulations. For instance, by imposing vanishing curvature (which also implies $R = 0$) and metric compatibility ($Q_{\alpha\mu\nu} = 0$), one obtains from (7)

$$\tilde{R} = -T + 4\tilde{\nabla}_\mu S^\mu \quad (17)$$

which is the basis of the metric teleparallel formulation. In a similar manner, the symmetric teleparallel (vanishing curvature and torsion) and also the generalized teleparallelism (only vanishing curvature) are obtained [48].

Let us now turn our attention to the matter content. In metric-affine gravity, apart from the energy-momentum tensor, which we define as usual,

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}}, \quad (18)$$

one also has to vary the matter part with respect to the affine connection. This new object, which is defined by

$$\Delta_\lambda{}^{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta \Gamma^\lambda{}_{\mu\nu}}, \quad (19)$$

is called hypermomentum [42] and encodes the microscopic characteristics of matter such as spin, dilation and shear. In the same way that the energy-momentum tensor sources spacetime curvature by means of the metric field equations, the hypermomentum is the source of spacetime torsion and non-metricity (through the connection field equations). Note that these energy-related tensors are not quite independent and are subject to the conservation law

$$\sqrt{-g}(2\tilde{\nabla}_\mu T^\mu{}_\alpha - \Delta^{\lambda\mu\nu}R_{\lambda\mu\nu\alpha}) + \hat{\nabla}_\mu \hat{\nabla}_\nu(\sqrt{-g}\Delta_\alpha{}^{\mu\nu}) + 2S_{\mu\alpha}{}^\lambda \hat{\nabla}_\nu(\sqrt{-g}\Delta_\lambda{}^{\mu\nu}) = 0 \quad (20)$$

$$\hat{\nabla}_\mu := 2S_\mu - \nabla_\mu \quad (21)$$

which comes from the diffeomorphism invariance of the matter sector of the action (see [32]). In the above discussion, we have briefly developed the geometric and physical setup needed for the rest of our study. Let us focus on the cosmological aspects of theories with torsion and non-metricity (i.e., non-Riemannian extensions).

3. Cosmology with Torsion and Non-Metricity

Let us consider a homogeneous flat FLRW cosmology, with the usual Robertson–Walker line element

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \quad (22)$$

where $i, j = 1, 2, 3$ and $a(t)$ are as usual the scale factor of the universe. As usual, the Hubble parameter is defined as $H := \dot{a}/a$. Now, let u^μ be the normalized 4-velocity field and

$$h_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu \quad (23)$$

be the projection tensor projecting objects on the space orthogonal to u^μ . The affine connection of the non-Riemannian FLRW spacetime reads [32]

$$\Gamma^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu} + X(t)u^\lambda h_{\mu\nu} + Y(t)u_\mu h^\lambda_\nu + Z(t)u_\nu h^\lambda_\mu + V(t)u^\lambda u_\mu u_\nu + \epsilon^\lambda_{\mu\nu\rho} u^\rho W(t)\delta_{n,4} \quad (24)$$

where the non-vanishing components of the Levi-Civita connection are, in this case,

$$\tilde{\Gamma}^0_{ij} = \tilde{\Gamma}^0_{ji} = \dot{a}a\delta_{ij} = Hg_{ij}, \quad \tilde{\Gamma}^i_{j0} = \tilde{\Gamma}^i_{0j} = \frac{\dot{a}}{a}\delta^i_j = H\delta^i_j \quad (25)$$

Continuing with the rest of the geometric objects, in this highly symmetric spacetime, the torsion and non-metricity tensors take the forms [32]

$$S^{(n)}_{\mu\nu\alpha} = 2u_{[\mu}h_{\nu]\alpha}\Phi(t) + \epsilon_{\mu\nu\alpha\rho}u^\rho P(t) \quad (26)$$

$$Q_{\alpha\mu\nu} = A(t)u_\alpha h_{\mu\nu} + B(t)h_{\alpha(\mu}u_{\nu)} + C(t)u_\alpha u_\mu u_\nu, \quad (27)$$

respectively. The five functions Φ, P, A, B, C describe the non-Riemannian cosmological effects. These, along with the scale factor, give the cosmic evolution of non-Riemannian geometries. Let us note that, using the relations of the torsion and non-metricity tensors with the distortion tensor, it is trivial to show that the functions $X(t), Y(t), Z(t), V(t), W(t)$ are linearly related to $\Phi(t), P(t), A(t), B(t), C(t)$ as [32]

$$2(X + Y) = B, \quad 2Z = A, \quad 2V = C, \quad 2\Phi = Y - Z, \quad P = W \quad (28)$$

or inverting them

$$W = P, \quad V = C/2, \quad Z = A/2 \quad (29)$$

$$Y = 2\Phi + \frac{A}{2}, \quad X = \frac{B}{2} - 2\Phi - \frac{A}{2}. \quad (30)$$

Now, using the Equations (11) and (12) for the torsion and non-metricity scalars and the above cosmological forms for torsion and non-metricity, we find for the former

$$T = 24\Phi^2 - 6P^2 \quad (31)$$

$$Q = \frac{3}{4}[2A^2 + B(C - A)], \quad (32)$$

respectively. These are the expressions for the torsion and non-metricity scalars in a homogeneous cosmological setup when no teleparallelism is imposed.

Finally, using the post-Riemannian decomposition of the Ricci scalar and the above forms of the torsion and non-metricity scalars, we find

$$R = \tilde{R} + 6\left[\frac{1}{4}A^2 + 4\Phi^2 + \Phi(2A - B)\right] + \frac{3}{4}B(C - A) - 6P^2 + 3\left(\frac{\dot{B}}{2} - \dot{A} - 4\Phi\right) + 9H\left(\frac{B}{2} - A - 4\Phi\right) \quad (33)$$

where

$$\tilde{R} = 6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right] \quad (34)$$

is the usual Riemannian part. The last decomposition will be very useful in our subsequent discussion.

4. MG-VIII Model and Extension: The $F(R, T, Q, \mathcal{T}, \mathcal{D})$ Theories

In this paper, we study the Myrzakulov gravity [46] VIII (MG-VIII) ⁴. Its action is given by [46]

$$S[g, \Gamma, \phi] = S_g + S_m = \frac{1}{2\kappa} \int \sqrt{-g} d^4x [F(R, T, Q, \mathcal{T}) + 2\kappa \mathcal{L}_m], \quad (35)$$

where R stands for the Ricci scalar (curvature scalar), T is the torsion scalar, Q is the non-metricity scalar and $\mathcal{T}$ is trace of the energy-momentum tensor of matter Lagrangian L_m . The MG-VIII can be seen as some kind of unification of $F(R)$, $F(T)$, $F(Q)$ or $F(R, \mathcal{T})$, $F(T, \mathcal{T})$, $F(Q, \mathcal{T})$ theories (see [51–53], respectively). For instance, if one imposes flatness (i.e., $R^\lambda_{\alpha\mu\nu} \equiv 0$) and metric compatibility ($Q_{\alpha\mu\nu} \equiv 0$), one arrives at the $f(T)$ gravity [7,54]. Demanding flatness and a torsionless connection, we get symmetric teleparallel $f(Q)$ gravity [8,9]. More generally, imposing only teleparallelism, we arrive at the recently developed generalized teleparallel scheme of $f(G)$ [48,55] theories. If no restriction on the connection is assumed, then (35) serves as a specific generalization of metric-affine $f(R)$ gravity where the energy-momentum trace $\mathcal{T}$ and certain quadratic combinations of torsion and non-metricity are added as well. In fact, in this generalized metric-affine setup, one could also consider the presence of the hypermomentum analogue of the (metrical) energy-momentum trace. Giving it a little thought, we observe that the divergence of the dilation current is similar to the trace $\mathcal{T}$, as they appear in the trace of the canonical ⁵ energy-momentum tensor (see, for instance, [32])

$$t = \mathcal{T} + \frac{1}{2\sqrt{-g}} \partial_\nu (\sqrt{-g} \Delta^\nu), \quad \Delta^\nu := \Delta_\mu^{\mu\nu}. \quad (36)$$

In this sense, $\mathcal{T}$ and the divergence of Δ^ν are placed on equal footing as is obvious from the above equation. Therefore, the scalar obtained by the divergence of the dilation current

$$\mathcal{D} = \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} \Delta^\nu) \quad (37)$$

would be a trace analogue for the hypermomentum. With this inclusion, we may generalize the class of theories (35) to

$$S[g, \Gamma, \phi] = S_g + S_m = \frac{1}{2\kappa} \int \sqrt{-g} d^4x [F(R, T, Q, \mathcal{T}, \mathcal{D}) + 2\kappa \mathcal{L}_m], \quad (38)$$

The field equations of the family of theories given by the the above action read as follows:

g-Variation:

$$\begin{aligned} & -\frac{1}{2} g_{\mu\nu} F + F_R R_{(\mu\nu)} + F_T (2S_{\nu\alpha\beta} S_\mu^{\alpha\beta} - S_{\alpha\beta\mu} S^{\alpha\beta}_\nu + 2S_{\nu\alpha\beta} S_\mu^{\beta\alpha} - 4S_\mu S_\nu) + F_Q L_{(\mu\nu)} \\ & + \hat{\nabla}_\lambda (F_Q J^\lambda_{(\mu\nu)}) + g_{\mu\nu} \hat{\nabla}_\lambda (F_Q \zeta^\lambda) + F_{\mathcal{T}} (\Theta_{\mu\nu} + T_{\mu\nu}) + F_{\mathcal{D}} M_{\mu\nu} = \kappa T_{\mu\nu} \end{aligned} \quad (39)$$

where

$$\hat{\nabla}_\lambda := \frac{1}{\sqrt{-g}} (2S_\lambda - \nabla_\lambda) \quad (40)$$

$$\Omega^{\alpha\mu\nu} = \frac{1}{4}Q^{\alpha\mu\nu} - \frac{1}{2}Q^{\mu\nu\alpha} - \frac{1}{4}g^{\mu\nu}Q^{\alpha} + \frac{1}{2}g^{\alpha\mu}Q^{\nu} \quad (41)$$

$$4L_{\mu\nu} = (Q_{\mu\alpha\beta} - 2Q_{\alpha\beta\mu})Q_{\nu}{}^{\alpha\beta} + (Q_{\mu} + 2q_{\mu})Q_{\nu} + (2Q_{\mu\nu\alpha} - Q_{\alpha\mu\nu})Q^{\alpha} - 4\Omega^{\alpha\beta}{}_{\nu}Q_{\alpha\beta\mu} - 4\Omega_{\alpha\mu\beta}Q^{\alpha\beta}{}_{\nu} \quad (42)$$

$$\Theta_{\mu\nu} := g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} \quad (43)$$

$$M_{\mu\nu} := \frac{\delta D}{\delta g^{\mu\nu}} \quad (44)$$

and we also define the densities

$$J^{\lambda}{}_{\mu\nu} := \sqrt{-g} \left(\frac{1}{4}Q^{\lambda}{}_{\mu\nu} - \frac{1}{2}Q_{\mu\nu}{}^{\lambda} + \Omega^{\lambda}{}_{\mu\nu} \right) \quad (45)$$

$$\zeta^{\lambda} = \sqrt{-g} \left(-\frac{1}{4}Q^{\lambda} + \frac{1}{2}q^{\lambda} \right) \quad (46)$$

Γ -Variation:

$$P_{\lambda}{}^{\mu\nu}(F_R) + 2F_T \left(S^{\mu\nu}{}_{\lambda} - 2S_{\lambda}{}^{[\mu\nu]} - 4S^{[\mu}{}^{\nu]\lambda} \right) - M_{\lambda}{}^{\mu\nu\alpha} \partial_{\alpha} F_D + F_Q \left(2Q^{[\nu\mu]}{}_{\lambda} - Q_{\lambda}{}^{\mu\nu} + (q^{\nu} - Q^{\nu})\delta_{\lambda}^{\mu} + Q_{\lambda}g^{\mu\nu} + \frac{1}{2}Q^{\mu}{}^{\nu}{}_{\lambda} \right) = F_T \Theta_{\lambda}{}^{\mu\nu} + \kappa \Delta_{\lambda}{}^{\mu\nu} \quad (47)$$

where

$$P_{\lambda}{}^{\mu\nu}(F_R) = -\frac{\nabla_{\lambda}(\sqrt{-g}F_R g^{\mu\nu})}{\sqrt{-g}} + \frac{\nabla_{\alpha}(\sqrt{-g}F_R g^{\mu\alpha} \delta_{\lambda}^{\nu})}{\sqrt{-g}} + 2F_R(S_{\lambda}g^{\mu\nu} - S^{\mu}{}^{\nu}{}_{\lambda} - S_{\lambda}{}^{\mu\nu}) \quad (48)$$

is the modified Palatini tensor and

$$\Theta_{\lambda}{}^{\mu\nu} := -\frac{\delta \mathcal{T}}{\delta \Gamma^{\lambda}{}_{\mu\nu}}, \quad M_{\lambda}{}^{\mu\nu\alpha} := \frac{\delta \Delta^{\alpha}}{\delta \Gamma^{\lambda}{}_{\mu\nu}}. \quad (49)$$

Note: if matter does not couple to the connection (e.g., classical perfect fluid with no inner structure) we have that $\Theta_{\lambda}{}^{\mu\nu} = 0$ as well as $\Delta_{\lambda}{}^{\mu\nu} = 0$ and $M_{\lambda}{}^{\mu\nu\alpha}$. The above set of field equations constitutes an extended (with the divergence of dilation included) metric-affine version of the Myrzakulov gravities [46]. Here, we derive the field equations with no restriction on the connection and also for the extended case $F(R, T, Q, \mathcal{T}, D)$. In the next section, we further analyze the linear case $F = R + \beta T + \gamma Q + \mu \mathcal{T} + \nu D$ and also touch upon cosmological applications.

5. Cosmological Applications

5.1. The Cosmology of $F = R + \beta T + \gamma Q + \mu \mathcal{T}$ Theory

Let us now analyze in more detail the linear case $F = R + \beta T + \gamma Q + \mu \mathcal{T}$ and also obtain the associated cosmological equations. To start with, let us note that even if we consider the theory $F = R + \beta T + \gamma Q + \mu \mathcal{T} + \nu D$, since $\sqrt{-g}D$ is a total divergence, the dilation current would not contribute to the field equations when included linearly⁶.

Therefore we can safely set $\nu = 0$ for the rest of our discussion. In addition, in this linear case, the metric field equations take the form

$$-\frac{1}{2}g_{\mu\nu}F + R_{(\mu\nu)} + \beta\left(2S_{\nu\alpha\beta}S_{\mu}^{\alpha\beta} - S_{\alpha\beta\mu}S^{\alpha\beta}_{\nu} + 2S_{\nu\alpha\beta}S_{\mu}^{\beta\alpha} - 4S_{\mu}S_{\nu}\right) + \gamma L_{(\mu\nu)} + \hat{\nabla}_{\lambda}(\gamma J^{\lambda}_{(\mu\nu)}) + g_{\mu\nu}\hat{\nabla}_{\lambda}(\gamma \tilde{\zeta}^{\lambda}) + \mu(\Theta_{\mu\nu} + T_{\mu\nu}) = \kappa T_{\mu\nu} \quad (50)$$

Taking the trace of the last equation, using the post-Riemannian expansion (33) and also employing (26) along with (27) and after some long calculations, we finally arrive at

$$\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + (1 + \beta)(4\Phi^2 - P^2) + \frac{1}{8}(2A^2 + B(C - A)) + \Phi(2A - B) + \dot{f} + 3Hf = -\mu(\Theta + \mathcal{T}) + \kappa\mathcal{T} \quad (51)$$

where

$$f := \frac{1}{2}\left[(1 - \gamma)\left(\frac{B}{2} - A\right) - 4\Phi\right], \quad \Theta := \Theta_{\mu\nu}g^{\mu\nu} \quad (52)$$

which is a variant of the modified Friedmann equation. As for the second Friedmann (acceleration) equation, its general form was derived in [31] for general non-Riemannian cosmological setups. It reads

$$\frac{\ddot{a}}{a} = -\frac{1}{3}R_{\mu\nu}u^{\mu}u^{\nu} + 2\left(\frac{\dot{a}}{a}\right)\Phi + 2\dot{\Phi} + \left(\frac{\dot{a}}{a}\right)\left(A + \frac{C}{2}\right) + \frac{\dot{A}}{2} - \frac{A^2}{2} - \frac{1}{2}AC - 2A\Phi - 2C\Phi \quad (53)$$

One could then proceed by contracting (50) with $u^{\mu}u^{\nu}$ in order to eliminate the first term ($R_{\mu\nu}u^{\mu}u^{\nu}$) and express everything in terms of the scale factor and the torsion and non-metricity variables. This results in a fairly complicated expression which we refrain from presenting here since it goes beyond the scope of the present study. As a final note, let us mention that in order to analyze the above cosmological model in depth, one should consider an appropriate form of matter for which both the metrical energy-momentum and hypermomentum tensors respect the cosmological principle. The fluid with such characteristics was constructed in [32] (also see, for a generalized version, [45]) and goes by the name perfect cosmological hyperfluid. The hypermomentum part of this fluid will then source the torsion and non-metricity variables $\Phi, P, A, \dots$, etc. by virtue of the connection field equations. We note that scalar fields coupled to the connection belong (are certain subcases) to the aforementioned fluid description. For the sake of illustration, below we present such an example with a scalar field non-minimally coupled to the connection in the case of vanishing non-metricity and also study some of the cosmological implications of this theory.

5.2. Scalar Field Coupled to Torsion

We now focus on the vanishing non-metricity sector and also set $\gamma = 0$, that is, we concentrate on the case $F = R + \beta T$. As for the matter part, let us consider a scalar field. In the usual (i.e., purely Riemannian) case, one would have the usual Lagrangian

$$\mathcal{L}_m^{(0)} = -\frac{1}{2}g^{\mu\nu}\nabla_{\mu}\phi\nabla_{\nu}\phi - V(\phi), \quad (54)$$

for the scalar field ϕ . However, in the presence of torsion, nothing prevents us from considering direct couplings of the scalar field with torsion. The most straightforward form of such a coupling is a torsion vector-scalar field derivative interaction of the form $\lambda_0 S^{\mu}\nabla_{\mu}\phi$, where λ_0 is the coupling constant measuring the strength of the interaction. Including this term, our full matter Lagrangian now reads

$$\mathcal{L}_m = -\frac{1}{2}g^{\mu\nu}\nabla_{\mu}\phi\nabla_{\nu}\phi - V(\phi) + \lambda_0 S^{\mu}\nabla_{\mu}\phi \quad (55)$$

Then, substituting this into (35) and varying the latter with respect to the scalar field, we obtain

$$\frac{1}{\sqrt{-g}}\partial_\mu \left[\sqrt{-g}(\partial^\mu \phi - \lambda_0 S^\mu) \right] = \frac{\partial V}{\partial \phi} \quad (56)$$

which is the evolution equation for the scalar field under the influence of torsion. In addition, the very presence of the interaction term $\lambda_0 S^\mu \nabla_\mu \phi$ produces a non-vanishing hypermomentum which is trivially computed to be

$$\Delta_\lambda^{\mu\nu} = 2\lambda_0 \delta_\lambda^{[\mu} \nabla^{\nu]} \phi \quad (57)$$

With this result, starting from the connection field Equation (47) which, in our case, reads

$$P_\lambda^{\mu\nu} + 2\beta \left(S^{\mu\nu}{}_\lambda - 2S_\lambda^{[\mu\nu]} - 4S^{[\mu} \delta_\lambda^{\nu]} \right) = \kappa \Delta_\lambda^{\mu\nu} \quad (58)$$

and contracting in $\mu = \lambda$, we find

$$S^\mu = \frac{3\kappa\lambda_0}{8\beta} \partial^\mu \phi \quad (59)$$

that in the presence of a scalar field produces spacetime torsion⁷. In addition, contracting (58) with $\varepsilon^\lambda_{\mu\nu\alpha}$, it follows that

$$t_\alpha = 0 \quad (60)$$

Note that we can now plug back into (55) the above form of the torsion tensor to end up with

$$\mathcal{L}_m = -\frac{1}{2} \left(1 - \frac{3\kappa\lambda_0^2}{4\beta} \right) g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \quad (61)$$

Interestingly, from the last equation, we conclude that the scalar–torsion interaction changes the factor of the kinetic term for the scalar field. We also see that this is a crucial value for the coupling $|\lambda_0| = 2\sqrt{\frac{\beta}{3\kappa}}$ above which the kinetic term changes sign and, for exactly this value, vanishes identically. Since this last case would require severe fine tuning, we disregard it and we also assume that λ_0 is under this bound so that the kinetic term keeps its original sign.

Up to this point, the above considerations have been general. Let us now focus on the homogeneous FLRW cosmology of this theory. In this case, Equation (60) implies that $p = 0$ and, as a result, upon using (59), the full torsion tensor is given by

$$S_{\mu\nu\alpha} = 2u_{[\mu} h_{\nu]\alpha} \Phi(t), \quad (62a)$$

$$\Phi = -\frac{\kappa\lambda_0}{8\beta} \dot{\phi} \quad (62b)$$

In the case of a free scalar field (i.e., $V(\phi) = 0$)⁸ inserting (59) into (56), we obtain

$$\left(1 - \frac{3\kappa\lambda_0^2}{8\beta} \right) \partial_\mu \left[\sqrt{-g} \partial^\mu \phi \right] = 0 \quad (63)$$

which for $|\lambda_0| \neq 2\sqrt{\frac{\beta}{3\kappa}}$ implies that

$$\dot{\phi} = \frac{c_0}{a^3} \quad (64)$$

On the other hand, the metric field equations in this case read

$$-\frac{1}{2}g_{\mu\nu}F + R_{(\mu\nu)} + \beta\left(2S_{\nu\alpha\beta}S_{\mu}^{\alpha\beta} - S_{\alpha\beta\mu}S^{\alpha\beta}_{\nu} + 2S_{\nu\alpha\beta}S_{\mu}^{\beta\alpha} - 4S_{\mu}S_{\nu}\right) = \kappa T_{\mu\nu} \quad (65)$$

and by taking the trace, using the same procedure we outlined previously, we finally obtain

$$\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = \left[-\frac{\kappa}{6} + (1 - \beta)\left(\frac{\kappa\lambda_0}{4\beta}\right)^2\right]\dot{\phi}^2 \quad (66)$$

which is again a variant of the modified Friedmann equation. Let us now derive the acceleration equation for this case. First, we contract the above field equations with $u^{\mu}u^{\nu}$ to obtain

$$R_{\mu\nu}u^{\mu}u^{\nu} = 24\beta\Phi^2 + \frac{\kappa}{2}(\rho + 3p) \quad (67)$$

which when substituted in (53) for vanishing non-metricity and the given scalar matter results in the acceleration equation

$$\frac{\ddot{a}}{a} = -8\beta\Phi^2 - \frac{\kappa}{6}(\rho + 3p) + 2H\Phi + 2\dot{\Phi} \quad (68)$$

where ρ and p are the density and pressure associated with the scalar field Lagrangian (61). It is interesting to note that the first term on the right-hand side of the acceleration equation has a fixed sign depending on the value of β . Intriguingly, for $\beta < 0$, the contribution from this term always has a fixed positive sign producing an accelerated expansion regardless of the sign of Φ (or equivalently $\dot{\phi}$). As for the last two terms, combining (62b) and (64), we observe that $\dot{\Phi} = -3H\Phi$ which, when substituted into the above acceleration equation, yields

$$\frac{\ddot{a}}{a} = -8\beta\Phi^2 - \frac{\kappa}{6}(\rho + 3p) + \frac{4}{3}\dot{\Phi} \quad (69)$$

We can conclude, therefore, that the last term aids acceleration when $\dot{\Phi} > 0$ and slows it down whenever $\dot{\Phi} < 0$. From the above analysis, we see that the non-Riemannian degrees of freedom play a crucial role in the cosmological evolution, providing new interesting phenomena. Now, using the latter form of the acceleration equation, we can obtain the first Friedmann equation from (66) by eliminating the double derivative of the scale factor. For the simple case $V(\phi) = 0$, we find

$$\left(\frac{\dot{a}}{a}\right)^2 = \left[\frac{\kappa}{6} + (1 + \beta)\left(\frac{\kappa\lambda_0}{4\beta}\right)^2\right]\dot{\phi}^2 - \frac{4}{3}\dot{\Phi} \quad (70)$$

as the modified first Friedmann equation. Note that on substituting (62b) in the above and completing the square in the resulting expression, we easily find the power-law solution

$$a(t) \propto t^{1/3} \quad (71)$$

which is the stiff matter solution. We see that in the simplified case of a zero potential for the scalar, we arrive at a known solution. However, we should remark that the situation changes drastically when one considers a non-vanishing potential. Note also that the torsion tensor in this case goes like $1/t$ and therefore its effect diminishes with time.

Needless to say, when non-metricity is also included, one obtains more complicated expressions with a much richer phenomenology. It would be quite interesting to see exactly to what degree the simultaneous presence of torsion and non-metricity alters the cosmological evolution in such models. This will be the theme of a separate work.

6. Conclusions

By working in a metric-affine approach (i.e., considering the metric and the connection as independent variables) we have considered a generalized version of the theory proposed in [46]. In particular, we derived the full set of field equations of the class of theories whose gravitational part of the Lagrangian is given by $F(R, T, Q, \mathcal{T}, \mathcal{D})$, where T , Q are the torsion and non-metricity scalars, $\mathcal{T}$ is the trace of the energy-momentum tensor and $\mathcal{D}$ is the divergence of the dilation current (one of the hypermomentum sources). The family of theories contained in our Lagrangian is fairly large since all metric and Palatini $f(R)$ theories, teleparallel $f(T)$, symmetric teleparallel $f(Q)$ or even generalized teleparallel $f(G)$ and generalizations of them such as $f(R, \mathcal{T})$, $f(T, \mathcal{T})$, $f(Q, \mathcal{T})$ can be seen as special cases of our theory.

Our contribution was two-fold. Firstly, we generalized the family of theories to those also including the divergence of the dilation current (which is the analogue of the energy-momentum trace for hypermomentum). Furthermore, as already mentioned above, we worked in a metric-affine framework, considering an independent affine connection as a fundamental variable along with the metric. This allows one not only to study the aforementioned theories (by restricting the connection one way or another), but also to analyze them in this general metric-affine scheme. Having derived the complete set of metric-affine $F(R, T, Q, \mathcal{T}, \mathcal{D})$ theories, we then concentrated our attention on the linear case $F = R + \beta T + \gamma Q + \mu \mathcal{T} + \nu \mathcal{D}$ and obtained a variant version of the modified Friedmann equation. Finally, we focused on the vanishing non-metricity sector and also considered a scalar field coupled to torsion as our matter sector. In this case, we derived both the first and second (acceleration) Friedmann equations and examined under what circumstances the presence of torsion can have an accelerating affect on the cosmological evolution. For this simple case, we were also able to provide an exact power-law solution for the scale factor.

In closing, let us note some further applications and additional developments of our study here. Firstly, it would be interesting to study in more detail the linear case, especially in regard to its cosmological implications in the presence of the cosmological hyperfluid [32,45]. In addition, as we have already mentioned, it would be worth elaborating more on the coupled scalar field we presented when both torsion and non-metricity are allowed and direct couplings of the latter with the scalar field occur. Finally, it would be quite interesting to go beyond linear functions F of the new dilation current term we considered. In this way, we will be able to investigate what exactly is the effect of this new addition/extension as well as its phenomenology, especially with regard to its energy-momentum trace counterpart.

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Notes

- ¹ From here onwards, we shall use the tilde notation in order to denote Riemannian objects, that is, objects computed with respect to the Levi-Civita connection $\tilde{\Gamma}^\lambda_{\mu\nu}$.

- Note that we define the combination of the torsion scalar in the usual way so as to obtain the usual teleparallel equivalent of GR. As a generalization, one could consider an arbitrary linear combination of the three independent torsion scalars connected with the three irreducible components of the torsion (see, for instance, [47]).
- Here, we are using the conventions of [16].
- See also [49,50] for some observational implications of this theory.
- Here $t = t^{\mu\nu} g_{\mu\nu}$ is the trace of the canonical energy-momentum tensor $t^{\mu\nu}$.
- If we considered a quadratic contribution νD^2 , we would have the additional terms $-\frac{\nu}{2} D^2 g_{\mu\nu} + 2\nu \frac{g_{\mu\nu}}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} D \Delta^\alpha)$ on the right-hand side of the metric field equations. These terms would then have an interesting impact in the cosmological setup we consider below, however, a detailed discussion goes beyond the purpose of this study and will be pursued elsewhere.
- Of course, this is so because of the connection coupling which yields a non-vanishing hypermomentum. If no such coupling is included, the scalar field can neither feel nor produce torsion.
- One can investigate the case of a non-vanishing potential by making use of reconstruction techniques developed in [56] (see also [57]).

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