

Article

Ellis–Bronnikov Wormholes in Asymptotically Safe Gravity

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Abstract: In this paper, we investigate the simplest wormhole solution—the Ellis–Bronnikov one—in the context of the asymptotically safe gravity (ASG) at the Planck scale. We work with three models, which employ the Ricci scalar, Kretschmann scalar, and squared Ricci tensor to improve the field equations by turning the Newton constant into a running coupling constant. For all the cases, we check the radial energy conditions of the wormhole solution and compare them with those that are valid in general relativity (GR). We verified that asymptotic safety guarantees that the Ellis–Bronnikov wormhole can satisfy the radial energy conditions at the throat radius, r_0 , within an interval of values of the latter, which is quite different from the result found in GR. Following this, we evaluate the effective radial state parameter, $\omega(r)$, at r_0 , showing that the quantum gravitational effects modify Einstein’s field equations in such a way that it is necessary to have a very exotic source of matter to generate the wormhole spacetime—phantom or quintessence-like matter. This occurs within some ranges of the throat radii, even though the energy conditions are or are not violated there. Finally, we find that, although at r_0 we have a quintessence-like matter, upon growing r , we inevitably came across phantom-like regions. We speculate whether such a phantom fluid must always be present in wormholes in the ASG context or even in more general quantum gravity scenarios.

Keywords: asymptotically safe gravity; general relativity; Ellis–Bronnikov wormhole



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1. Introduction

Wormholes and their traversability are an object of intense discussion in the communities which study general relativity and its extensions. By means of Einstein’s theory, one shows [1–4] that the wormhole is traversable only with the presence of exotic matter, including Casimir energy [5–9], even being capable of mimicking the behavior of black holes [10–12]. Classically modified gravity theories involving wormholes (e.g., Einstein–Gauss–Bonnet gravity [13], Lovelock gravity [14], Einstein–Born–Infeld gravity [15], and others [16–19]), as well as some quantum corrected ones [20], are also often discussed in the literature.

Nobody knows if quantum effects can change the energy conditions of wormholes and avoid the necessity of having non-exotic matter, but a definitive answer to these issues requires the formulation of the ultimate quantum theory of gravity, which is actually intensely researched. Notwithstanding, quantum effects in gravity can be described through an asymptotically safe quantum field theory, which is UV complete [21–24]. The existence of such a fixed point for the gravity re-normalization group flow is verified from several methods and in various scenarios [25–42]. Its physical applications are explored in [43–51]. However, solving the exact re-normalization group equation to derive the effective average

action is very hard, if not impossible. Therefore, the effects of this quantization method are usually considered (i.e., semi-classically) as a correction to the classical theory and studied by means of an effective theory obtained by turning the classical coupling constant into a running one, which is derived from the solution for the β -function [25,52,53].

An example of the re-normalization group improvement of the field equations can be shown via the action modification presented in [54,55]. In this method, the action functional is covariantly improved, leading to the modification of Einstein's equations:

$$G_{\mu\nu} = 8\pi G(\chi)T_{\mu\nu} + G(\chi)X_{\mu\nu}(\chi), \quad (1)$$

where $G(\chi)$ is the improved coupling constant introduced as a function of the curvature invariants χ , with the covariant tensor $X_{\mu\nu}$ being defined as:

$$X_{\mu\nu}(\chi) = \left(\nabla_\mu \nabla_\nu - g_{\mu\nu} \square \right) G(\chi)^{-1} - \frac{1}{2} \left(R\mathcal{K}(\chi) \frac{\delta\chi}{\delta g^{\mu\nu}} + \partial_\kappa \left(R\mathcal{K}(\chi) \frac{\partial\chi}{\partial(\partial_\kappa g^{\mu\nu})} \right) + \partial_\kappa \partial_\lambda \left(R\mathcal{K}(\chi) \frac{\partial\chi}{\partial(\partial_\lambda \partial_\kappa g^{\mu\nu})} \right) \right), \quad (2)$$

with $\mathcal{K}(\chi) \equiv \frac{2\partial G(\chi)/\partial\chi}{G(\chi)^2}$ [56].

The functional re-normalization group methods as well as other assumptions [25,57] lead to the anti-screening running gravitational coupling given by:

$$G(k) = \frac{G_0}{1 + \omega k^2 G_0}, \quad (3)$$

where k is the Euclidean 4-momentum, $\omega = (4/\pi)(1 - \pi^2/144)$, and G_0 is the Newton gravity constant [58]. Hence, one can introduce the RG improvement, in the form $\omega k^2 \rightarrow \xi f(\chi)$, where χ is a function of curvature invariants with the dimension of length square.

The function $f(\chi) \equiv \xi/\chi$ is called anti-screening running coupling since it goes to zero at the scale of very high energies ($\chi \rightarrow 0$). This behavior mimics that of the quarks and gluons, which are subject to the asymptotic freedom described by quantum chromodynamics. The scaling constant ξ can be written as $G_0\omega\xi_0$, and $\xi_0\omega$ is a constant of the order of unity [58].

The quantum correction term $X_{\mu\nu}$ depends on the scaling factor, and up to the first order one gets [55,56]:

$$X_{\mu\nu} \simeq \nabla_\mu \nabla_\nu G(\chi)^{-1} - g_{\mu\nu} \square G(\chi)^{-1}, \quad (4)$$

with χ depending on a well-defined function of all independent curvature invariants, such as $R, R_{\alpha\beta}R^{\alpha\beta}, R_{\alpha\beta\kappa\lambda}R^{\alpha\beta\kappa\lambda}, \dots$ [55]. However, one of the drawbacks of the theory is that there is no unique way to fix the form of χ [59–62], although for non-vacuum solutions, one can restrict the possible choices [56,59–61].

In this direction, the authors of [63] study the effects of the above modifications due to asymptotically safe gravity in wormholes. For this, they consider some simplifications, such as a linear Equation of State (EoS). They also study only the region very close to the throat. With this, they find that for some ranges of the parameters, a traversable Morris wormhole is possible. However, the supposition of a linear EoS excludes a lot of possible models. Additionally, the study of the model with an analytical solution valid for all r is important to analyze the behavior of the system throughout all of space. Research considering these features is lacking in the literature.

In this work, we will investigate a system that provides us with both the above possibilities: a non-trivial EoS with an analytical solution for all regions of spacetime. For this, we consider the quantum effects on the Ellis–Bronnikov wormhole at the Planck scale, employing the re-normalization group improved theory. Thus, in Section 2, we will study models using anti-screening functions based on Ricci scalar, squared Ricci tensor, and

Kretschmann scalar. We will verify that, besides the flare-out and anti-screening conditions, the principal radial energy conditions are satisfied at the Planck scale, which does not happen in general relativity, in certain intervals of the throat radius. In addition, we will show that the quantum effects associated with asymptotically safe gravity are such that the zero tidal Ellis–Bronnikov spacetime has to be supported by a highly exotic matter, such as phantom or quintessence-like energies, in a range of the wormhole throat radii and at certain regions away from the throat. In Section 3, we conclude the paper.

2. The Ellis–Bronnikov Wormhole Solution in ASG

We consider the spherical Morris–Thorne wormhole metric given by:

$$ds^2 = e^{2\Phi(r)} dt^2 - \frac{dr^2}{1 - b(r)/r} - r^2 d\Omega_2. \quad (5)$$

For an anisotropic matter $T^\mu_\nu = \text{Diag}[\rho(r), -p_r(r), -p_l(r), -p_l(r)]$, the improved field Equation (1) led to [63]:

$$\kappa\rho = (1+f)\frac{b'}{r^2} - (1-\frac{b}{r})(f'' + \frac{2}{r}f') + \frac{b'r-b}{2r^2}f', \quad (6)$$

$$\kappa p_r = -(1+f)\left(\frac{b}{r^3} - \frac{2\Phi'}{r}(1-\frac{b}{r})\right) + (1-\frac{b}{r})\left(\Phi' + \frac{2}{r}\right)f', \quad (7)$$

$$\begin{aligned} \kappa p_l = & -(1+f)\left(\frac{b'r-b}{2r^2}(\Phi' + \frac{1}{r}) - (1-\frac{b}{r})(\Phi'' + \Phi'^2 + \frac{\Phi'}{r})\right) \\ & + (1-\frac{b}{r})\left((\Phi' + \frac{1}{r})f' + f''\right) - \frac{b'r-b}{2r^2}f', \end{aligned} \quad (8)$$

with $\kappa = 8\pi G_0$.

For an asymptotically flat Morris wormhole, shape and redshift functions are given by:

$$b(r) = r_0^2/r, \quad (9)$$

$$\Phi(r) = 0. \quad (10)$$

characterizing a zero tidal wormhole, with r_0 being the throat radius. The flare-out condition $b'(r_0) < 1$ is readily satisfied. With this, our equations simplify to:

$$\kappa\rho = -(1+f)\frac{r_0^2}{r^4} - \frac{2}{r}\left(1 - \frac{1}{2}\frac{r_0^2}{r^2}\right)f' - \left(1 - \frac{r_0^2}{r^2}\right)f'', \quad (11)$$

$$\kappa p_r = -(1+f)\frac{r_0^2}{r^4} + \frac{2}{r}\left(1 - \frac{r_0^2}{r^2}\right)f', \quad (12)$$

$$\kappa p_l = (1+f)\frac{r_0^2}{r^4}f' + \left(1 - \frac{r_0^2}{r^2}\right)\left(\frac{1}{r}f' + f''\right) + \frac{r_0^2}{r^3}f', \quad (13)$$

Now, we must choose the function $f = \xi/\chi$. For the above case, we have:

$$R = -2\frac{r_0^2}{r^4}, R^{\mu\nu}R_{\mu\nu} = 4\frac{r_0^4}{r^8}, R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} = 12r_0^4/r^8$$

therefore, in order for the condition $f > 0$ to be obeyed and to obtain the correct dimensions, we must choose:

$$f_1 = -\xi R = 2\xi\frac{r_0^2}{r^4}, f_2 = \xi(R^{\mu\nu}R_{\mu\nu})^{1/2} = 2\xi\frac{r_0^2}{r^4}, f_3 = \xi(R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda})^{1/2} = \xi\sqrt{12}r_0^2/r^4. \quad (14)$$

With the above expressions, we see that for the Ellis–Bronnikov wormhole, we have $f_1 = f_2 = f_3/\sqrt{3}$. Therefore, for all cases, we must have very similar behaviors. We will first consider the cases f_1, f_2 , which are identical, and obtain analytical conclusions. At the end, we plot figures for the case f_3 .

Now, we will analyze the radial energy conditions of the Ellis–Bronnikov wormhole. Such conditions are verified from the substitution of Equation (14) into Equations (11) and (12). We get:

$$\kappa\rho_r = -\frac{r_0^2}{r^8}(-30r_0^2\zeta + 24\zeta r^2 + r^4), \quad (15)$$

$$\kappa p_r = -\frac{r_0^2}{r^8}(-14r_0^2\zeta + 16\zeta r^2 + r^4), \quad (16)$$

$$\kappa(\rho + p_r) = -\frac{2r_0^2}{r^8}(-22r_0^2\zeta + 20\zeta r^2 + r^4). \quad (17)$$

The first thing we can note is that the state equation is not linear for any value of ζ . Therefore, as mentioned in the introduction, the model considered in [63] does not take into account the Ellis–Bronnikov wormhole, which is considered the most simple wormhole. In fact, the above $b(r)$ is never a solution to their equations. It can be seen that the Ellis–Bronnikov wormhole satisfies the radial energy conditions in the ASG scenario. For this, we need to analyze the quantities ρ , p_r , and $\rho + p_r$, in order to verify if the null ($\rho + p_r \geq 0$), weak ($\rho \geq 0, \rho + p_r \geq 0$), and dominant ($\rho \geq 0, \rho \geq |p_r|$) energy conditions are satisfied near the throat.

What will determine the sign of the Equations (15)–(17) are the terms between parentheses. These terms are all bi-quadratic equations with only two symmetric roots, respectively given by:

$$r = \pm\sqrt{\sqrt{6}\sqrt{\zeta(5r_0^2 + 24\zeta)} - 12\zeta}, \quad (18)$$

$$r = \pm\sqrt{\sqrt{2}\sqrt{\zeta(7r_0^2 + 32\zeta)} - 8\zeta}, \quad (19)$$

$$r = \pm\sqrt{\sqrt{2}\sqrt{\zeta(11r_0^2 + 50\zeta)} - 10\zeta}. \quad (20)$$

This shows that the three quantities are positive in $r = 0$ and are monotonically decreasing, changing sign in the above roots. Therefore, close to $r = 0$, it can be seen that they are all positive, satisfying all the energy conditions. However, we also need the conditions close to $r = r_0$. In $r = r_0$, the above equations reduce to:

$$\kappa\rho = -\frac{1}{r_0^4}(-6\zeta + r_0^2), \quad (21)$$

$$\kappa p_r = -\frac{1}{r_0^4}(2\zeta + r_0^2), \quad (22)$$

$$\kappa(\rho + p_r) = -\frac{2}{r_0^4}(-2\zeta + r_0^2). \quad (23)$$

Therefore, we arrive at some general conclusions. We see that p_r must always be negative, since p_r is always decreasing and at $r = r_0$ it is negative. For $\rho > 0$, we see that we must add the relation $r_0^2 < 6\zeta$. For $\rho + p_r > 0$, we must have $r_0^2 < 2\zeta$. Therefore, we can obtain that the energy conditions depend on the relations between r_0 and ζ , as below:

$$\begin{cases} \text{Null:} & \rho + p_r > 0 & \text{if } r_0^2 < 2\zeta, \\ \text{Weak:} & \rho \geq 0, \rho + p_r \geq 0 & \text{if } r_0^2 < 2\zeta, \\ \text{Dominant:} & \rho \geq 0, \rho \geq |p_r| & \text{if } r_0^2 < 2\zeta. \end{cases}$$

We see therefore that for $r_0^2 < 2\zeta$, the null, weak, and dominant energy conditions are satisfied near the wormhole throat at the Planck scale. Therefore, a radical difference with general relativity is that now we have the possibility that the energy conditions (2)

are satisfied over the throat if $r_0^2 < 2\tilde{\xi}$. This also shows that the result of general relativity can be recovered in the limit $\tilde{\xi} \rightarrow 0$, as expected. From another viewpoint, in the limit $\tilde{\xi} \rightarrow 0$, none of the conditions can be satisfied, as expected. From now on, we will study the consequences of imposing that $r_0^2 < 2\tilde{\xi}$, and therefore that the energy conditions are satisfied over the throat.

First, we will investigate the presence of cosmological exotic matter in the Ellis–Bronnikov wormhole in the ASG scenario by evaluating the state parameter $\omega(r) = p_r/\rho$. Let us first see what kind of matter is allowed at our wormhole throat. For this, we have:

$$\omega = \frac{r_0^2 + 2\tilde{\xi}}{r_0^2 - 6\tilde{\xi}}$$

and we can analyze this as a function of r_0 . We easily obtain:

$$\begin{cases} \text{Quintessence: } -1 < \omega < -1/3 & \text{if } r_0 < \sqrt{2\tilde{\xi}}, \\ \text{Phantom: } \omega < -1 & \text{if } \sqrt{2\tilde{\xi}} < r_0 < \sqrt{6\tilde{\xi}}, \\ \text{Other Exotic Matter: } \omega > 1 & \text{if } r_0 > \sqrt{6\tilde{\xi}}. \end{cases} \quad (24)$$

An interesting point about the above result is that general relativity demands that $\omega = 1$, but this is forbidden here since $\omega = 1$ only if $\tilde{\xi} = 0$. This again shows that the result of general relativity can be recovered in the limit $\tilde{\xi} \rightarrow 0$, as expected. Now, we can see the consequences of imposing $r_0^2 < 2\tilde{\xi}$, and therefore, that over the throat, the energy conditions are satisfied. From Equation (24), we see that this implies a source with $-1 < \omega < -1/3$. Therefore, at $r = r_0$, our improved wormhole can satisfy the energy conditions, but must be sourced by quintessential fluid. In this case ($r_0^2 < 2\tilde{\xi}$), we can also study, region by region, what are the sources that surround our wormhole. The regions where the EoS is phantom-like are given by:

$$r^4 + 20\tilde{\xi}r^2 - 22r_0^2\tilde{\xi} > 0.$$

Again, this is a bi-quadratic equation with two symmetric solutions, given by:

$$r = \pm \sqrt{\sqrt{2}\sqrt{\tilde{\xi}(11r_0^2 + 50\tilde{\xi})} - 10\tilde{\xi}},$$

and therefore, we must have:

$$r > \sqrt{\sqrt{2}\sqrt{\tilde{\xi}(11r_0^2 + 50\tilde{\xi})} - 10\tilde{\xi}}$$

Since we are considering $r_0 < \sqrt{2\tilde{\xi}}$, the above expression implies that $r > \sqrt{2\tilde{\xi}}$. Therefore, this reinforces our result that at $r = r_0$, we can never have a phantom. Now, let us determine precisely the region that is Phantom-like. For this, we remember that ρ has a real root where it changes sign, and p_r is always negative. With this, we conclude that, beyond the root (18), we must have exotic matter with ($\omega > 1$), for instance, Casimir energy ($\omega = 3$). We also conclude that the region between $\sqrt{2\tilde{\xi}}$ and our singularity is phantom-like. Therefore, in asymptotically safe gravity, the wormhole requires very exotic matter as a source for $\sqrt{2\tilde{\xi}} < r < \sqrt{2\tilde{\xi}}\sqrt{\sqrt{51}-3}$, while in general relativity, it needs exotic matter with $\omega = 1$ (stiff matter [64]) at any r . Below, we provide the solution for all regions:

$$\begin{cases} \text{Quintessence: } -1 < \omega < -1/3 & \text{if } r_0 < r < \sqrt{2\tilde{\xi}}, \\ \text{Phantom: } \omega < -1 & \text{if } \sqrt{2\tilde{\xi}} < r < \sqrt{2\tilde{\xi}}\sqrt{\sqrt{51}-3}, \\ \text{Other Exotic Matter: } \omega > 1 & \text{if } r > \sqrt{2\tilde{\xi}}\sqrt{\sqrt{51}-3}. \end{cases} \quad (25)$$

Finally, we consider the Kretschmann scalar. As stated above, the behavior must be the same as for the other cases. Due to the similarity, we only provide the plot of the expressions. In Figures 1 and 2, we plot the quantities ρ , p_r , and $\rho + p_r$, in order to visualize that the null ($\rho + p_r \geq 0$), weak ($\rho \geq 0, \rho + p_r \geq 0$), and dominant ($\rho \geq 0, \rho \geq |p_r|$) energy conditions are satisfied or not near the throat.

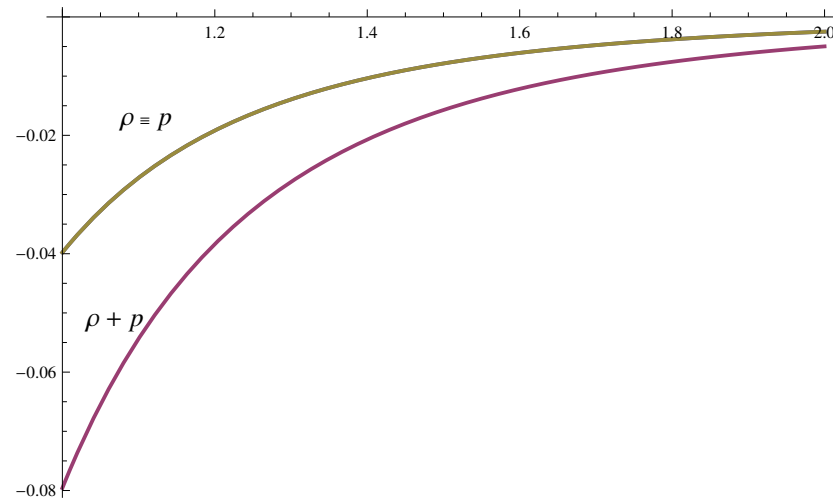


Figure 1. Plot of $\rho \equiv p_r$ and $\rho + p_r$, as functions of the coordinate r , in Planckian units, for $\zeta = 0$ and $r_0 = 1$.

Looking at Figure 2, we can notice that the null, weak, and dominant radial energy conditions are satisfied near the wormhole throat at the Planck scale, in the context of ASG.

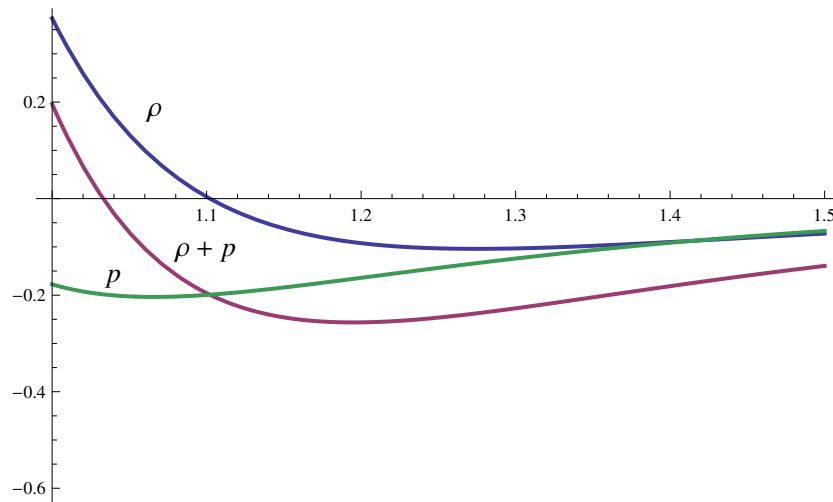


Figure 2. Plot of ρ , p_r , and $\rho + p_r$, as functions of the coordinate r , in Planckian units, for $\zeta = 1$ and $r_0 = 1$.

In Figure 1, we depict the same quantities, in the context of general relativity ($\zeta = 0$).

Hence, we conclude that, in general relativity, the radial null and weak energy conditions are not satisfied near the throat, and the dominant one is satisfied, and even so within the inferior limit, $\rho = |p|$.

In Figure 3, we depict the state parameter as a function of r_0 , considering quantum improved gravity.

The above figures have the same behaviors as the previous analytical expressions when we considered the functions f_1, f_2 . Therefore, the conclusions are basically the same. We should point out that the results for the Kretschmann scalar can be obtained just by doing $\zeta \rightarrow \sqrt{4}\zeta$.

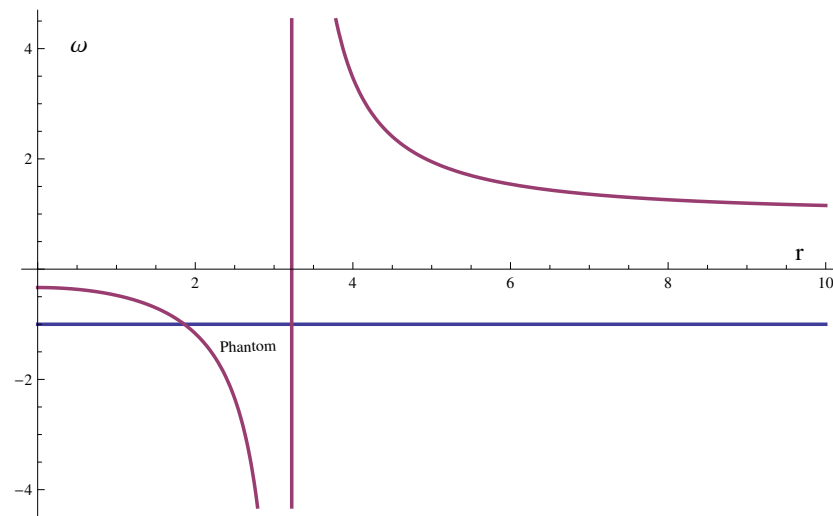


Figure 3. Plot of state parameter, $\omega = p_r/\rho$, as a function of the throat radius, r_0 , in Planckian units, for $\xi = 1$ and $r = r_0$.

3. Conclusions

In this paper, we have studied the presence of the simplest wormhole solution (the Ellis–Bronnikov one) in the context of asymptotic safety in quantum gravity, at the Planck scale. Thus, we have considered three models, which employ Ricci scalar, squared Ricci tensor, and Kretschmann scalar, to perform a re-normalization group improvement of the Ellis–Bronnikov solution of general relativity at that scale.

In this scenario, we checked the radial energy conditions for that wormhole solution and found that, besides both flare-out and anti-screening conditions, the null, weak, and dominant energy conditions were also satisfied near the throat if $r_0 < \sqrt{2}\xi$. On the other hand, the wormhole only obeyed the dominant condition in general relativity, and only at the inferior limit, $\rho = |p_r|$.

First, we considered the Ricci scalar and squared Ricci tensor since they provide identical improvements. We analyzed the improved Ellis–Bronnikov wormhole in the region very close to the throat by using $r = r_0$. We found that the effective EoS in this region forbids $\omega = 1$. Therefore, the quantum gravity effects cannot match with a perfect fluid, as it is considered in [63], to generate the wormhole under consideration. In fact, we considered ω as a function of r_0 . With this, in Equation (24), we showed that the quantum gravity effects on the region near the wormhole throat are such that there must be a phantom, quintessence, or exotic matter with $\omega > 1$ (including Casimir energy), thus generating the modified wormhole spacetime, regardless of whether the radial energy conditions are satisfied or not. The phantom-like matter, for example, was obtained for the throat radii $\sqrt{2}\xi < r_0 < \sqrt{6}\xi$. However, when we imposed that the wormhole does not violate the radial energy conditions over the throat, we found that $r_0 < \sqrt{2}\xi$, and the only possibility in the region was given by $-1 < \omega < -1/3$. Thus, the quantum gravity effects require the presence of the quintessence-like matter near the throat of the wormhole.

Next, we considered the other regions of the wormhole and what kind of sources are necessary for the quantum modified gravity. Here, we imposed that the wormhole satisfies the radial energy conditions from the beginning. With this, in Equation (25), we showed that the wormhole can be divided into three regions when r grows from the throat. The first, around the throat, must be sourced by quintessence-like matter, the second must be sourced by a phantom-like fluid, and the third by exotic matter with $\omega > 1$, including Casimir energy ($\omega = 3$).

Third, and since the Kretschmann scalar is proportional to the Ricci scalar, we only considered this case graphically. From Figure 1, it can be seen that the null, weak, and dominant energy conditions were satisfied near the wormhole throat at the Planck scale. In Figure 2, we depicted the same quantities, also in the context of general relativity ($\xi = 0$).

In Figure 3, we depicted the state parameter as a function of r_0 , considering quantum improved gravity. With these plots, we can also visualize all the results of the previous cases. For example, we can see from Figure 3 that the kind of fluid that must be present at the throat due to the quantum gravity effects can only be quintessence-like matter, $-1 < \omega < -1/3$.

Finally, we can conclude that the hypothesis of a perfect fluid is not possible in order to obtain traversable Ellis–Bronnikov wormholes in the context of ASG. However, at least for this case, the very exotic phantom-like matter must be present in some regions of the modified spacetime, and even at the throat for some of them, notwithstanding the non-violation of the radial energy conditions in the analyzed scenario. To study other examples with analytical solutions beyond the throat would be very important in order to verify whether the phantom-like matter is a necessity for wormholes in the asymptotically safe gravity or even in more general quantum gravity scenarios. These will be the topics of our next studies.

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