

Review



Remarks on Some Results Related to the Thermal Casimir Effect in Einstein and Closed Friedmann Universes with a Cosmic String

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Abstract: In this paper, we present a review of some recent results concerning the thermal corrections to the Casimir energy of massless scalar, electromagnetic, and massless spinor fields in the Einstein and closed Friedmann universes with a cosmic string. In the case of a massless scalar field, it is shown that the Casimir energy can be written as a simple sum of two terms; the first one corresponds to the Casimir energy for the massless scalar field in the Einstein and Friedmann universes without a cosmic string, whereas the second one is simply the Casimir energy of the electromagnetic field in these backgrounds, multiplied by a parameter $\lambda = (1/\alpha) - 1$, where α is a constant that codifies the presence of the cosmic string, and is related to its linear mass density, μ , by the expression $\alpha = 1 - G\mu$. The Casimir free energy and the internal energy at a temperature different from zero, as well as the Casimir entropy, are given by similar sums. In the cases of the electromagnetic and massless spinor fields, the Casimir energy, free energy, internal energy, and Casimir entropy are also given by the sum of two terms, similarly to the previous cases, but now with both terms related to the same field. Using the results obtained concerning the mentioned thermodynamic quantities, their behavior at high and low temperatures limits are studied. All these results are particularized to the scenario in which the cosmic string is absent. Some discussions concerning the validity of the Nernst heat theorem are included as well.

Keywords: Casimir effect; scalar field; electromagnetic field; massless spinor field; Einstein universe; Friedmann universe; thermal Casimir effect; cosmic string

1. Introduction

The Casimir effect is a phenomenon connected with a fundamental type of physical reality termed quantum vacuum. It was predicted by H. B. G. Casimir [1] in 1948 as the attractive force arising between two parallel, uncharged metallic plates in vacuum. This phenomenon, unlike what is predicted by classical electrodynamics, tell us that there is a force acting between neutral plates. In fact, this is due to the quantum nature of this effect whose origin arises from the modification of the vacuum fluctuations (zero-point oscillations) by the electromagnetic field, induced by the presence of the material boundaries, namely the metallic plates. A consequence of this phenomenon is the appearance of a finite vacuum energy. Thus, the Casimir effect can be considered as a direct manifestation of the most intriguing basic type of physical reality, namely the quantum vacuum. Later on the Casimir effect was generalized for quantum fields with different spins and for material boundaries with different geometries, made of different materials [2–4]. During the last years some experiments were performed on measuring the Casimir force in different scenarios [5–18] with important results and progress. The Casimir effect was



Citation: Bezerra, V.B.; Mota, H.F.S.; Muniz, C.R.; Filho, C.A.R. Remarks on Some Results Related to the Thermal Casimir Effect in Einstein and Closed Friedmann Universes with a Cosmic String. *Universe* **2021**, 7, 232. https://doi.org/10.3390/ universe7070232

Academic Editor: Galina L. Klimchitskaya

Received: 31 May 2021 Accepted: 1 July 2021 Published: 7 July 2021

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Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). investigated over the last 73 years not only from the theoretical point of view, but also from the experimental point of view as well. Despite the advances made, a lot of problems from both point of views, theoretical and experimental, remains to resolve as pointed out by Mostepanenko [19].

As was noticed in the middle 1970's [20,21], the Casimir effect also manifests in a space without material boundaries, since that its topology is non-Euclidean. In such spaces, the identification conditions arising from the topology and imposed to the quantum fields play a role equivalent to the ones imposed by the boundary conditions due to the material body. The spaces with non-Euclidean topology can have any geometry, they can be flat or curved. In both cases, they give rise to the Casimir effect, due to the fact that the identification conditions modify the spectrum of the vacuum fluctuations (zero-point oscillations) analogously to what is done by the material boundaries. When the Casimir effect is calculated in spaces with non-Euclidean topology, some nonzero contribution to the stress–energy tensor of a quantum field in the vacuum state appears. These depends on geometrical parameters associated to the manifold. Thus, to appropriately study the evolution of the universe, we have to solve Einstein's equations taking this contribution to the total stress–energy tensor into account [4]. In other words, it is necessary, as well as important, to take the contribution arising from the Casimir effect into account in order to give an adequate description of all details of this evolution. Keeping in mind that the Casimir effect is a quantum phenomenon, one should expect that it may play the most important role at the early stages of the universe's evolution.

The first studies concerning the Casimir energy density and pressure in the context of cosmological models were conducted in the static Einstein universe, which possesses a topology $R^1 \times S^3$. In these studies, a conformally coupled massless scalar field was considered in the pioneer paper by Ford [21]. It was found that the Casimir energy density and pressure are given by

$$\varepsilon_0 = \frac{\hbar c}{480\pi^2 a^4}, \qquad P_0 = \frac{\varepsilon_0}{3},\tag{1}$$

where *a* is the radius of the universe and \hbar and *c* are the Planck constant and the velocity of light in vacuum. The same result was shown to be valid for a closed Friedmann universe with an instantaneous value of the radius by Ford [21] and Dowker and Critchley [22]. Others pioneering papers concerning this subject include a paper by Mamaev, Mostepanenko, and Starobinsky [23], who calculated the Casimir energy density in a closed Friedmann model using the Abel–Plana formula for the first time, as well as another paper by Mamaev and Mostepanenko [24], who obtained, for the first time, a self-consistent solution of Einstein equations taking the Casimir contribution, whose scale factor is equivalent to the same corresponding to the inflation, into account.

The Casimir energy density and pressure for the electromagnetic and neutrino fields in the Einstein cosmological model were also investigated by Ford [25], for the first time. In the closed Friedmann universe, the vacuum polarization of Casimir origin was shown to be a specific contribution to the total vacuum stress–energy tensor in addition to the conformal anomaly and the contribution of particles created from the vacuum [26,27].

After the publication of the pioneering papers by Ford [21,25], Dowker and Critchley [22], Mamayev, Mostepanneko and Starobinsky [23], the Casimir effect was investigated in different cosmological scenarios. Among the papers devoted to this subject, we can mention the ones by Bellucci and Saharian [28], Saharian and Mkhitaryan [29], Pavlov [30], Altaie [31] and Bezerra de Mello, Saharian and Setare [32]. Also some discussions concerning the Casimir effect in multi-dimensional cosmology [33] and the problem of dark energy [34] were presented. Numerous works have been considered the thermal corrections to the Casimir effect in the context of gravity and cosmological models with non-Euclidean topology. Among the works related to this subject, we can mention the one concerning the study of the thermal properties of Green's function in Rindler, de Sitter, and Schwarzschild spaces [35]; the calculation of the thermal contributions to the Casimir stress–energy tensor of the scalar field in Einstein universe [36] and in the closed Friedmann universe [37]. The approach adopted in [37] were applied to fields with other spins, namely electromagnetic and neutrino fields to obtain the thermal contribution [38]. The same approach developed in [37] were extended to obtain the thermal corrections to the Casimir energy in the case of a scalar field [39], as well as for electromagnetic and neutrino fields [40], in the cosmological scenario corresponding to the Friedmann universe, but now with a static, infinitely straight and thin cosmic string.

The Casimir free energy and the internal energy at temperature different from zero, for massless and massive scalar fields in the closed Friedmann universe were also investigated in [41]. Analogous results in a universe with a 3-torus topology, $R^1 \times S^1 \times S^1 \times S^1$, were obtained in [42]. With respect to electromagnetic and neutrino fields, the effects of the temperature on the total stress–energy tensors in the Einstein cosmological model were considered in [43], whose results were used [43,44] to determine the backreaction of the total thermal stress–energy tensor $< T_{ik} >$. Taking this contribution into account, the Einstein equations are written as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = < T_{\mu\nu} > .$$
 (2)

It is worth mentioning that $\langle T_{\mu\nu} \rangle$ can be identified as the cosmological constant. Λ . The role played by the back reaction due to the thermal Casimir effect in the Einstein universe was investigated for fields of different spin [45]. In the case of the electromagnetic field it was shown that it is necessary to take a contribution to the thermal stress–energy tensor, which arises from the zeroth mode in the Poisson summation formula [46], into account. With respect to the self-consistent solutions of the Einstein equations, an interesting result was obtained by Mamaev and Mostepanenko [24], where for the first time the effects of the quantum vacuum polarization associated to Casimir energy were taken into account.

In this review we will consider the presence of a topological defect, namely a cosmic string, in the Einstein and Friedmann universes. It was predicted that this linear topological defect was formed in the early stages of the universe [47,48] as a consequence of spontaneous symmetry breaking [49]. It is characterized by its tension which is proportional to its linear mass density, whose magnitude is the order of the energy scale of the symmetry breaking, This tension determines how strong is the the intensity of the gravitational interaction. Taking the data related to the Cosmic Microwave Background (CMB) [50–52] into account, the estimated upper bound of the string tension of the order of 10^{-6} .

We will consider a static, straight and infinite cosmic string. The spatial section of the geometry of the spacetime associated to the cosmic string has an azimuthal deficit angle given by $\Delta \varphi = 8\pi G \mu$. Thus, the space-time is locally analogous to the Minkowski spacetime, but globally it is different [53–55]. Despite of being locally flat, the cosmic string produces very interesting gravitational effects due to the global or topological features of its spacetime, as for example, it can act as a gravitational lens [56], induces a Casimir effect [57] and many others.

In the primordial universe, with extremely high temperatures, it is predicted that the cosmic strings were formed [49]. Thus, it seems to us that the investigations of different cosmological models with a cosmic string has some interest. Motivated by this conclusion, we will investigate the role played by a cosmic string in the Einstein and Friedmann universes on the thermal Casimir energies, vacuum, free, internal, and Casimir entropy, by taking different fields, namely massless scalar, electromagnetic, and massless spinor fields, into account.

It is worth pointing out that the stress-energy tensor of a quantum field depends on the global features of the background spacetime. In the case of the spacetime of a cosmic string, whose geometry of the spatial section is conical, and therefore, it is globally different from the Minkowski spacetime, the vacuum expectation value of the stress–energy tensor is affected by the quantity that gives the intensity of the gravitational interaction of the cosmic string, namely its tension, which is proportional to the angle of the cone that depends on the linear mass density of the cosmic string [58]. The Casimir effect is a manifestation of the properties of the quantum vacuum, and certainly, it has an important role in discussions related to the phenomena appeared in the the earlier stages of the universe, and also to what happens in the present stage of accelerated expansion of the universe [34]. In another cosmological scenario, namely related to cosmological braneworld models, the Casimir effect has also an important role [59]. If we assume that the cosmic strings were formed in the early stages of the universe, as predicted, they have been appeared at the end of inflationary era [60] and as a consequence, they certainly played an important role at that time.

This review is based on papers [36–39], but mainly on the papers [38,39], because these papers are generalizations of the papers [36,37] in the sense that the presence of a cosmic string is taken into account. It is organized as follows. In Section 2, we present the Einstein and Friedmann cosmological models with a cosmic string and some fundamental results concerning the eigenfrequencies of the quantum fields considered. The vacuum energy, thermodynamic quantities, and its thermal corrections are presented and discussed in Sections 3 and 4. The limits of high and low temperatures for the three different fields are considered in Section 5, for massless scalar field, and Section 6, for electromagnetic and massless spinor fields. Finally, the discussions and conclusions are presented in Section 7.

2. Spacetimes and Some Fundamental Results

In this section we present the metric associated to the closed Einstein universe with and without a cosmic string which can also be used to describe a instantaneously closed Friedmann universe. In the background spacetimes without a cosmic string, we present some results obtained in the middle 1970's by Ford [21,25] in a pioneer paper related to the Casimir energy at zero temperature, for scalar, electromagnetic and neutrino fields. These papers by Ford [21,25] were motivated by the desire to understand how important is the quantum vacuum energy in the context of the General Theory of Relativity.

2.1. Einstein and Friedmann Universes with a Cosmic String

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A cosmic string can be introduced in any spacetime with an axis of symmetry [61]. All that we have to do, from the geometrical point of view is to remove a wedge and then glue together the resulting edges. Now, let us proceed in this way to construct the Einstein universe with a cosmic string. In this case, from the mathematical point of view, we simply redefine the azimuthal coordinate, by redefining appropriately this coordinate in order to consider the presence of the cosmic string, as follow

$$ds^{2} = c^{2}d\tau^{2} - a_{0}^{2}d(\psi^{2} + \sin^{2}\psi(d\theta^{2} + \alpha^{2}\sin^{2}\theta d\varphi^{2})),$$
(3)

where $\alpha = 1 - 4G\mu$ is the parameter which codifies the presence of the cosmic string and $0 \le \psi \le \pi, 0 \le \theta \le \pi, 0 \le \varphi \le 2\pi$ are dimensionless coordinates, τ is the proper synchronous time and a_0 is the radius of the spatial section of the universe. The volume of the spatial section of this spacetime is finite and is calculated using the expression

$$V_{\alpha} = \int \sqrt{-g^{(3)}} d\psi d\theta d\varphi = a_0^3 \alpha \int_0^{\pi} \sin^2 \psi d\psi \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\varphi = 2\pi^2 \alpha a_0^3, \qquad (4)$$

with $g^{(3)}$ being the determinant of the spatial part of the metric (3). The line element of the Einstein spacetime with a topology $R^3 \times S^1$ is obtained by taking $\alpha = 1$ in Equation (3), and can be written as

$$ds^{2} = c^{2}d\tau^{2} - a_{0}^{2}(d\psi^{2} + \sin^{2}\psi(d\theta^{2} + \sin^{2}\theta d\varphi^{2})),$$
(5)

whose volume of the spatial section is given by $V = 2\pi^2 a_0^3$. As to the Friedmann line element, it can be obtained from Equation (5) through the following conformal transformation

$$d\bar{s}^2 = \Omega^{-2} ds^2, \tag{6}$$

where $\Omega = a_0/a(t)$, with a(t) being the scale factor. It is important to call attention to the fact that in the case of the Friedmann universe [38], which depends on time, there is a contribution to the energy-momentum tensor arising from the creation of particles and as well as from the conformal anomaly [26,27]. But, fortunately, when $a_0 \rightarrow a(t)$, for instantaneous values of a(t), all physical quantities calculated in the Einstein model continue valid in the Friedmann model. Therefore, all results concerning the Casimir thermodynamic quantities remain valid [25,37,38].

2.2. Eigenfrequencies: Massless Scalar Field

Firstly, let us consider the conformally massless scalar field ϕ in the Einstein universe with a cosmic string. Its behavior is described by the covariant Klein-Gordon equation, which is given as follows

$$\Box \phi + \frac{1}{6} R \phi = 0, \tag{7}$$

with $\Box = \partial_{\nu} \partial^{\nu}$ being the D'Alembertian operator and *R* is the scalar curvature, which for the spacetime under consideration, given by the line element (5) is $R = 6a_0^{-2}$. Let us assume that The solution of Equation (7) can be written, in the spacetime under under consideration, described in Equation (5), as follows

$$\phi(t,\psi,\theta,\varphi) = A_{l_{\alpha}m_{\alpha}}R(\psi)P_{l_{\alpha}}^{m_{\alpha}}(\cos\theta)e^{im\varphi}e^{-i\omega t},$$
(8)

where $A_{l_{\alpha}m_{\alpha}}$ are constants, while $P_{l_{\alpha}}^{m_{\alpha}}$ is the associated Legendre function. It is worth calling attention to the fact that the presence of the cosmic strings changes the quantum numbers *l* and *m* to l_{α} and m_{α} , defined as follows

$$l_{\alpha} = n + m_{\alpha}, \qquad n = 0, 1, 2, \dots,$$

$$m_{\alpha} = \frac{m}{\alpha}, \qquad -l \le m \le l. \qquad (9)$$

The quantum number *n* is independent of the parameter α and thus from Equation (9), we obtain

$$l_{\alpha} = l + |m| \left(\frac{1}{\alpha} - 1\right), \qquad l = 0, 1, 2, \dots,$$
 (10)

with l = n + m. Substituting Equation (8) into Equation (7), we obtain the equation for $R(\psi)$

$$\frac{d}{d\psi}\left(\sin^2\psi\frac{dR}{d\psi}\right) + \left(\frac{\omega^2}{c^2} - \frac{1}{6}R\right)a_0^2\sin^2\psi R - l_\alpha(l_\alpha + 1)R = 0,\tag{11}$$

whose solution is given in terms of the Gegenbauer functions C_{n-l}^{l+1} as

$$R(\psi) = B_{l_{\alpha}m_{\alpha}}(\sin\psi)^{l_{\alpha}}C_{n-l_{\alpha}}^{l_{\alpha}+1}(\cos\psi), \qquad (12)$$

with $B_{l_{\alpha}m_{\alpha}}$ being constants.

The eigenfrequencies of the massless scalar field in the background spacetime under consideration, namely the Einstein universe with a cosmic string are given by

$$\omega_{n,\lambda} = \frac{cn}{a_0} + \frac{c|m|\lambda}{a_0}, \qquad n = 1, 2, 3 \dots,$$
(13)

where $\lambda = \left(\frac{1}{\alpha} - 1\right)$. Taking $\lambda = 0$, which means that there is no cosmic string at all, we recover the eigenfrequencies of the massless scalar field

$$\omega_n = \frac{cn}{a_0}, \qquad n = 1, 2, 3 \dots, \tag{14}$$

Thus, we conclude that the presence of a cosmic string changes the eigenfrequencies, by adding a term proportional to λ , which codifies the presence of the cosmic string, according to what is shown explicitly in Equation (13).

2.3. Eigenfrequencies of the Electromagnetic and Massless Spinor Fields

The eigenfrequencies of the electromagnetic and massless spinor fields in the Einstein and closed Friedmann universes were calculated long time ago [25,62]. For the electromagnetic field they are given by [62]

$$\omega^{(1)} = \frac{c}{a_0} (n' + l + 1), \qquad n' = 0, 1, 2, \dots,$$
(15)

with l = 1, 2, 3, ... and $-l \le m \le l$. Taking the presence of the cosmic string into account, the quantum numbers l_{α} change in according to Equation (10).

Then, replacing *l* in Equation (15) by l_{α} in Equation (10), we get

$$\omega_{n,\lambda}^{(1)} = \frac{c}{a_0} (n + |m|\lambda), \qquad n = 2, 3, 4, \dots,$$
(16)

where n = n' + l + 1 and $\lambda = (1/\alpha) - 1$. If there is no cosmic string, then, $\alpha = 1$, and Equation (15) is recovered. The eigenfrequencies of the massless spinor field in the closed Einstein universe, were obtained in the middle 1970s [25]. They are given by the following expression

$$\omega^{(1/2)} = \frac{c}{a_0} \left(n' + l + \frac{1}{2} \right), \qquad n' = 0, 1, 2, 3, \dots,$$
(17)

Changing *l* by l_{α} , as we did previously, we get

$$\omega_{n,\lambda}^{(1/2)} = \frac{c}{a_0} \left[\left(n + \frac{1}{2} \right) + |m|\lambda \right], \qquad n = 1, 2, 3, \dots,$$
(18)

where n = n' + l. The results when the cosmic string is absent are reobtained by taking $\alpha = 1$ as should be expected.

3. Vacuum Energy

Now that we know the eigenfrequencies for all quantum fields under consideration, we can obtain the vacuum energy for these fields. Let us start with the massless scalar field. In the sequence we will consider the electromagnetic and massless spinor fields.

3.1. Massless Scalar Field

Firstly, we will consider a real quantum massless scalar field conformally coupled to the background spacetimes under consideration. The vacuum energy at zero temperature can be obtained from the following expression

$$E_0^{(s)} = \frac{\hbar}{2} \sum_j \omega_j^{(s)},$$
(19)

where *j* and *s* indicate the eigenmodes and spin of the fields, respectively.

For the massless scalar field in the Einstein universe with a cosmic string, Equation (19) yields

$$E_{0,\lambda} = \frac{\hbar}{2} \sum_{n=1}^{\infty} \sum_{l=0}^{n} \sum_{m=-l}^{l} \omega_{n,\lambda}^{(0)},$$

$$= \frac{\hbar}{2} \sum_{n=1}^{\infty} n^{2} \omega_{n}^{(0)} + \frac{\lambda}{6} \hbar \sum_{n=2}^{\infty} (n^{2} - 1) \omega_{n}^{(1)},$$
 (20)

The eigenfrequencies for the scalar, $\omega_n^{(0)}$ and electromagnetic fields, $\omega_n^{(1)}$, are both are given by the Equation (14).

The first term in Equation (20) is the vacuum energy of the massless scalar field in the Einstein universe and the the second term corresponds to the vacuum energy of the electromagnetic field multiplying the factor $\lambda/6$. which codifies the presence of the cosmic string in the Einstein universe [39].

In other words, Equation (20) states that

$$E_{0,\lambda} = E_0^{(0)} + \frac{\lambda}{6} E_0^{(1)}.$$
(21)

where

$$E_0^{(0)} = \frac{\hbar}{2} \sum_{n=1}^{\infty} n^2 \omega_n^{(0)},$$

$$E_0^{(1)} = \hbar \sum_{n=2}^{\infty} (n^2 - 1) \omega_n^{(1)}.$$
(22)

Therefore, the modification induced by the cosmic string in the Einstein universe is to add to the usual vacuum energy of the massless scalar field $E_0^{(0)}$, a term which can be seen as the usual vacuum energy of the electromagnetic field multiplied by $\lambda/6$.

In order to renormalize the vacuum energy $E_{0,\lambda}^{(0)}$ given by Equation (21), we have to do this separately for $E_0^{(0)}$ and $E_0^{(1)}$. This is done by adopting the method based on the Abel-Plana formula, which was applied for the first time in the pioneering paper by Mamaev and Mostepanenko [24]. Thus, using the Abel-Plana formula given by [2,4,63,64],

$$\sum_{n=1}^{\infty} \Phi(n) - \int_0^{\infty} \Phi(t) dt = -\frac{1}{2} \Phi(0) + i \int_0^{\infty} \frac{\Phi(it) - \Phi(-it)}{e^{2\pi t} - 1} dt,$$
(23)

where in this case, the function $\Phi(t)$ in Equation (23) has the same form as $E_0^{(s)}$, in such a way that one should just make the changing $n \to t$ in order to calculate the integral by Equation (23). Thus, the renormalized energies for the massless scalar and electromagnetic fields are given by [37,38]

$$E_{0,\text{ren}}^{(0)} = \frac{\hbar c}{240a_0}, \qquad E_{0,\text{ren}}^{(1)} = \frac{11\hbar c}{120a_0},$$
(24)

and therefore, Equation (21) is written as

$$E_{0,\lambda,\text{ren}} = E_{0,\text{ren}}^{(0)} + \frac{\lambda}{6} E_{0,\text{ren}}^{(1)}.$$
(25)

Taking into account the CMB data, which constrain the cosmic string linear mass density such that $G\mu \leq 10^{-6}$, this implies that parameter $\lambda \sim 10^{-6}$. Taking into account that the vacuum energies in Equation (25) are approximately of the same order of magnitude, we are forced to conclude that the role played by the cosmic string is negligible, in principle, differs from the conclusion in the literature [39].

3.2. Electromagnetic and Massless Spinor Fields

Now let us obtain the vacuum energies of the electromagnetic and massless spinor fields and then proceed to the renormalization, once again, using the Abel-Plana formula. Let us start by considering the equation

$$E_0^{(s)} = \pm \frac{\hbar}{2} \sum_j \omega_j^{(s)},$$
 (26)

which is similar to Equation (19), except that there an additional minus sign in Equation (26). Thus, For the electromagnetic and massless spinor fields, the vacuum energies are given by

$$E_{0,\lambda}^{(1)} = \frac{\hbar}{2} \sum_{\sigma=1}^{2} \sum_{n=2}^{\infty} \sum_{l=1}^{n-1} \sum_{m=-l}^{l} \omega_{n,\lambda}^{(1)},$$

= $\left(1 + \frac{\lambda}{3}\right) E_{0}^{(1)}$ (27)

$$E_{0,\lambda}^{(1/2)} = -4\frac{\hbar}{2} \sum_{n=1}^{\infty} \sum_{l=1}^{n} \sum_{m=-(l-1/2)}^{(l-1/2)} \omega_{n,\lambda}^{(1/2)},$$

= $\left(1 + \frac{\lambda}{3}\right) E_0^{(1/2)},$ (28)

where $\sigma = 1,2$ identify the two polarization states of the electromagnetic field and the factor 4 in the first line of Equation (28) takes care of the four components of the massless spinor field. The vacuum energies when the cosmic string is absent are

$$E_0^{(1)} = \hbar \sum_{n=1}^{\infty} (n^2 - 1)\omega_n^{(1)},$$
(29)

$$E_0^{(1/2)} = -2\hbar \sum_{n=1}^{\infty} n(n+1)\omega_n^{(1/2)},$$
(30)

where $\omega_n^{(1)}$ and $\omega_n^{(1/2)}$ are given by Equations (15) and (17), respectively. As we can see, the modifications due to the presence of a cosmic string in the Einstein universe (and consequently in the closed Friedmann universe) are contained in the terms which are multiplied by $\lambda/3$, as showed in Equations (27) and (28).

In the renormalization procedure, the Abel-Plana formula [2,4,63,64] is used, and the following results are achieved

$$\sum_{n=1}^{\infty} \Phi(n) - \int_0^{\infty} \Phi(t) dt = -\frac{1}{2} \Phi(0) + i \int_0^{\infty} \frac{\Phi(it) - \Phi(-it)}{e^{2\pi t} - 1} dt,$$
(31)

and

$$\sum_{n=1}^{\infty} \Phi\left(n + \frac{1}{2}\right) - \int_{0}^{\infty} \Phi(t)dt = -i \int_{0}^{\infty} \frac{\Phi(it) - \Phi(-it)}{e^{2\pi t} + 1} dt,$$
(32)

valid for bosons and fermions, respectively. In both cases, the function $\Phi(t)$ in Equations (31) and (32) have the same form as $E_0^{(s)}$. Therefore, we can calculate the integrals by making the identifications $n \to t$ and $(n + 1/2) \to t$, for bosons and fermions, respectively. The renormalized energies of the electromagnetic and massless spinor fields, are [38]

$$E_{0,\text{ren}}^{(1)} = \frac{11\hbar c}{120a_0}, \qquad E_{0,\text{ren}}^{(1/2)} = \frac{17\hbar c}{480a_0}.$$
 (33)

Therefore, we can write Equations (27) and (28) in the following compact form

$$E_{0,\lambda,\text{ren}}^{(s)} = \left(1 + \frac{\lambda}{3}\right) E_{0,\text{ren}}^{(s)}.$$
(34)

where $E_{0,\text{ren}}^{(1)}$ and $E_{0,\text{ren}}^{(1/2)}$ are given in Equation (33). Again, we conclude that, taking the contribution to these renormalized energies into account, arising due to the cosmic string is negligible.

4. Thermal Correction

From now on, we will consider the nonzero temperature contributions (thermal corrections) to the zero-temperature Casimir stress–energy tensor of the massless scalar, electromagnetic and massless spinor fields in the Einstein and closed Friedmann universes with a cosmic string, and also mention the particular results when the cosmic string is absent.

4.1. Massless Scalar Field

Firstly, we will consider the massless scalar field and generalize previous results [37], to take the presence of the cosmic string into account. In this case, the free energy can be written as

$$F_{\lambda}(T) = E_{0,\lambda} + \Delta F_{\lambda}(T), \qquad (35)$$

where $E_{0,\lambda}$ is the vacuum energy given by Equation (21). The renormalized vacuum energy $E_{0,\lambda,\text{ren}}$ in Equation (25), permit us to write Equation (35) as

$$F_{\text{tot},\lambda}(T) = E_{0,\lambda,\text{ren}} + \Delta F_{\lambda}(T).$$
(36)

According to [37,38], the thermal correction $\Delta F(T)$ can be renormalized by subtracting the contribution of the black body radiation and also a contribution of pure quantum nature. Thus, in any finite volume, the asymptotic expression for the free energy $F_{\text{tot}}(T)$ contains the following terms:

$$\alpha_0 \frac{(k_B T)^4}{(\hbar c)^3}, \qquad \alpha_1 \frac{(k_B T)^3}{(\hbar c)^2}, \qquad \alpha_2 \frac{(k_B T)^2}{\hbar c},$$
(37)

where the coefficients α_0 , α_1 , α_2 depend on the spin of the field. For example, for the massless scalar and electromagnetic fields, only the following coefficients are different from zero

$$\begin{aligned}
\alpha_0^{(0)} &= -\frac{\pi^2}{90}V \\
\alpha_0^{(1)} &= -\frac{\pi^2}{45}V, \qquad \alpha_2^{(1)} = \frac{\pi^2 a_0}{3}.
\end{aligned}$$
(38)

The thermal correction $\Delta F^{(s)}(T)$ is renormalized by using again the Abel-Plana Formula (23) [37,38]. Doing this procedure, we find the following result

$$F_{C,\lambda}(T) = E_{0,\lambda,\text{ren}} + \Delta F_{C,\lambda}(T),$$
(39)

where $\Delta F_{C,\lambda}(T)$ is the renormalized thermal correction corresponding to $\Delta F_{\lambda}^{(s)}(T)$, which should be obtained using the Abel-Plana formula as follows

$$\Delta F_{C,\lambda}(T) = \Delta F_{\lambda}(T) - \int_0^\infty \Phi_{\lambda}(t) dt.$$
(40)

Thus, the thermal correction $\Delta F_{\lambda}(T)$ is given by

$$\Delta F_{\lambda}(T) = \Delta F^{(0)}(T) + \frac{\lambda}{6} \Delta F^{(1)}(T), \qquad (41)$$

where

$$\Delta F^{(0)}(T) = k_B T \sum_{n=1}^{\infty} n^2 \ln \left[1 - e^{-(\hbar \omega_n^{(0)} / k_B T)} \right], \tag{42}$$

$$\Delta F^{(1)}(T) = 2k_B T \sum_{n=1}^{\infty} (n^2 - 1) \ln \left[1 - e^{-(\hbar \omega_n^{(1)}/k_B T)} \right],$$
(43)

for the massless scalar and electromagnetic fields [37,38], respectively. In order to calculate the Casimir free energy, we have to perform the calculation of the integral on the right hand side of Equation (40), whose result is

$$\int_0^\infty \Phi_\lambda(t)dt = \alpha_0^{(0)} \frac{(k_B T)^4}{(\hbar c)^3} + \frac{\lambda}{6} \left(\alpha_0^{(1)} \frac{(k_B T)^4}{(\hbar c)^3} + \alpha_2^{(1)} \frac{(k_B T)^2}{\hbar c} \right).$$
(44)

The integral (44) corresponds to the quantum contributions given in Equation (38). Therefore, for the massless scalar field in the background under consideration, the integral (44) has a term proportional to $\alpha_0^{(0)}$ in Equation (38), and also the terms proportional to $\alpha_0^{(1)}$ and $\alpha_2^{(1)}$, related to the electromagnetic field. In this case, the Casimir free energy can be obtained from Equation (39) and is written as

$$F_{C,\lambda}(T) = F_{\text{tot},\lambda}(T) - \alpha_0^{(0)} \frac{(k_B T)^4}{(\hbar c)^3} - \frac{\lambda}{6} \left(\alpha_0^{(1)} \frac{(k_B T)^4}{(\hbar c)^3} + \alpha_2^{(1)} \frac{(k_B T)^2}{\hbar c} \right).$$
(45)

Observing the approximated value of $\lambda \sim 10^{-6}$, we conclude that the contribution arising from the presence of the cosmic string is negligible.

We can obtain other thermodynamic quantities by using the results for the total and Casimir free energies through the relations

$$U(T) = -T^2 \frac{\partial}{\partial T} \left[\frac{F(T)}{T} \right], \tag{46}$$

for the internal energy and

$$P(T) = -\frac{\partial F(T)}{\partial V},\tag{47}$$

$$S(T) = -\frac{\partial F(T)}{\partial T}.$$
(48)

for the pressure and entropy, respectively.

4.2. Electromagnetic and Massless Spinor Fields

Now, let us consider the vacuum energies of the electromagnetic and massless spinor fields by generalizing some results in the literature [38], using the following general expression for the free energy

$$F_{\lambda}^{(s)}(T) = E_{0,\lambda}^{(s)} + \Delta F_{\lambda}^{(s)}(T),$$
(49)

where $E_{0,\lambda}^{(s)}$ is the vacuum energy at zero temperature given by Equations (27) and (28). Taking into account the renormalized vacuum energy $E_{0,\lambda,\text{ren}}$ in Equation (33), Equation (49) turns into

$$F_{\text{tot},\lambda}^{(s)}(T) = E_{0,\lambda,\text{ren}}^{(s)} + \Delta F_{\lambda}^{(s)}(T).$$
(50)

Again, the thermal correction $\Delta F(T)$ is renormalized by subtract not only the contribution of the black body radiation, as well as contribution of pure quantum nature [38], Considering a finite volume, the total free energy $F_{tot}(T)$ contains the following terms

$$\alpha_0^{(s)} \frac{(k_B T)^4}{(\hbar c)^3}, \qquad \alpha_1^{(s)} \frac{(k_B T)^3}{(\hbar c)^2}, \qquad \alpha_2^{(s)} \frac{(k_B T)^2}{\hbar c},$$
(51)

where the the coefficients $\alpha_0^{(s)}$, $\alpha_1^{(s)}$, $\alpha_2^{(s)}$ for the electromagnetic and massless spinor fields are given by

$$\begin{aligned}
\alpha_0^{(1)} &= -\frac{\pi^2}{45}V, \qquad \alpha_2^{(1)} = \frac{\pi^2 a_0}{3}, \\
\alpha_0^{(1/2)} &= -\frac{7\pi^2}{180}V, \qquad \alpha_2^{(1/2)} = \frac{\pi^2 a_0}{12}
\end{aligned}$$
(52)

The renormalization of $\Delta F_{\lambda}(T)$ in Equation (50), give us the following expression for the Casimir free energy

$$F_{C,\lambda}^{(s)}(T) = E_{0,\lambda,\text{ren}}^{(s)} + \Delta F_{C,\lambda}^{(s)}(T),$$
(53)

where $\Delta F_{C,\lambda}^{(s)}(T)$ is the renormalized thermal correction obtained from $\Delta F_{\lambda}^{(s)}(T)$ by using the Abel-Plana formulas as follows:

$$\Delta F_{C,\lambda}^{(s)}(T) = \Delta F_{\lambda}^{(s)}(T) - \int_0^\infty \Phi_{\lambda}^{(s)}(t) dt.$$
(54)

The thermal correction $\Delta F_{\lambda}^{(s)}(T)$ of the vacuum energies (27) and (28) is given by the general expression

$$\Delta F_{\lambda}^{(s)}(T) = \left(1 + \frac{\lambda}{3}\right) \Delta F^{(s)}(T), \tag{55}$$

where

$$\Delta F^{(1)}(T) = 2k_B T \sum_{n=1}^{\infty} (n^2 - 1) \ln\left[1 - e^{-(\hbar \omega_n^{(1)}/k_B T)}\right],$$
(56)

$$\Delta F^{(1/2)}(T) = -4k_B T \sum_{n=1}^{\infty} n(n+1) \ln \left[1 + e^{-(\hbar \omega_n^{(1/2)}/k_B T)} \right],$$
(57)

for the electromagnetic and massless spinor fields [38], respectively. To obtain the Casimir free energy, we need to calculate the integral in the right hand side of Equation (54), whose result is

$$\int_0^\infty \Phi_{\lambda}^{(1)}(t)dt = \left(1 + \frac{\lambda}{3}\right) \left(\alpha_0^{(1)} \frac{(k_B T)^4}{(\hbar c)^3} + \alpha_2^{(1)} \frac{(k_B T)^2}{\hbar c}\right),\tag{58}$$

for the electromagnetic field and

$$\int_{0}^{\infty} \Phi_{\lambda}^{(1/2)}(t) dt = \left(1 + \frac{\lambda}{3}\right) \left(\alpha_{0}^{(1/2)} \frac{(k_{B}T)^{4}}{(\hbar c)^{3}} + \alpha_{2}^{(1/2)} \frac{(k_{B}T)^{2}}{\hbar c}\right),\tag{59}$$

for the massless spinor fields. The integrals (58) and (59) provide the quantum contributions given by Equation (52). Thus, the Casimir free energies for electromagnetic and massless spinor fields can be obtained from Equation (54) and these are given by

$$F_{C,\lambda}^{(1)}(T) = F_{\text{tot},\lambda}^{(1)}(T) - \left(1 + \frac{\lambda}{3}\right) \left(\alpha_0^{(1)} \frac{(k_B T)^4}{(\hbar c)^3} + \alpha_2^{(1)} \frac{(k_B T)^2}{\hbar c}\right),\tag{60}$$

$$F_{C,\lambda}^{(1/2)}(T) = F_{\text{tot},\lambda}^{(1/2)}(T) - \left(1 + \frac{\lambda}{3}\right) \left(\alpha_0^{(1/2)} \frac{(k_B T)^4}{(\hbar c)^3} + \alpha_2^{(1/2)} \frac{(k_B T)^2}{\hbar c}\right).$$
(61)

It is worth pointing out once again that considering the approximated value of $\lambda \sim 10^{-6}$ into Equations (55), (60) and (61), we arise to the same previous conclusion, namely that the role played by the cosmic string is negligible.

5. High and Low Limits of Temperature: Massless Scalar Field

Now, let us investigate the behavior of the thermodynamic quantities obtained previously in the Einstein universe with a cosmic string, but at this time with focus on the limits of low and high temperatures.

5.1. Low Temperature Limit

The low temperature limit $k_B T << \hbar c / a_0$ of the Casimir free energy can be obtained from Equation (39) written in the following form:

$$F_{C,\lambda}(T) = F_C^{(0)}(T) + \frac{\lambda}{6} F_C^{(1)}(T),$$
(62)

where

$$F_{C}^{(0)}(T) = \frac{\hbar c}{240a_{0}} + \frac{\pi^{2}}{90}V\frac{(k_{B}T)^{4}}{(\hbar c)^{3}} - k_{B}Te^{-(\hbar c/a_{0}k_{B}T)},$$
(63)

is the expression for the Casimir free energy of the massless scalar field at low temperature limit [37] and

$$F_{C}^{(1)}(T) = \frac{11\hbar c}{120a_{0}} + \frac{\pi^{2}}{45}V\frac{(k_{B}T)^{4}}{(\hbar c)^{3}} - \frac{\pi^{2}a_{0}}{3}\frac{(k_{B}T)^{2}}{\hbar c} - 6k_{B}Te^{-(2\hbar c/a_{0}k_{B}T)}.$$
(64)

is the expression for the Casimir free energy of the electromagnetic field [38], in this limit. From Equations (46) and (39), we get the Casimir internal energy, which is written as

$$U_{C,\lambda}(T) = U_C^{(0)}(T) + \frac{\lambda}{6} U_C^{(1)}(T),$$
(65)

where

$$U_{\rm C}^{(0)}(T) = \frac{\hbar c}{240a_0} - \frac{\pi^2}{30} V \frac{(k_B T)^4}{(\hbar c)^3} + \frac{\hbar c}{a_0} e^{-(\hbar c/a_0 k_B T)},\tag{66}$$

is the Casimir internal energy of the massless scalar field [37] and

$$U_{C}^{(1)}(T) = \frac{11\hbar c}{120a_{0}} - \frac{\pi^{2}}{15}V\frac{(k_{B}T)^{4}}{(\hbar c)^{3}} + \frac{\pi^{2}a_{0}}{3}\frac{(k_{B}T)^{2}}{(\hbar c)} + \frac{12\hbar c}{a_{0}}e^{-(2\hbar c/a_{0}k_{B}T)},$$
(67)

is the Casimir internal energy of the electromagnetic field [38].

Analogously, we can obtain the Casimir entropy using Equations (48) and (39). Thus the following result

$$S_{C,\lambda}(T) = S_C^{(0)}(T) + \frac{\lambda}{6} S_C^{(1)}(T),$$
(68)

where

$$S_{C}^{(0)}(T) = -\frac{2\pi^{2}}{45}k_{B}\left(\frac{k_{B}T}{\hbar c}\right)^{3}V + \frac{\hbar c}{a_{0}T}e^{-(\hbar c/a_{0}k_{B}T)}.$$
(69)

is the Casimir entropy of the massless scalar field [37], while

$$S_{C}^{(1)}(T) = 2\pi^{2}k_{B}\frac{ak_{B}T}{3\hbar c} \left[1 - \frac{4\pi^{2}}{15}\frac{(a_{0}k_{B}T)^{2}}{(\hbar c)^{2}}\right] + \frac{12\hbar c}{aT}e^{-(2\hbar c/a_{0}k_{B}T)}.$$
(70)

is the Casimir entropy of the electromagnetic field [38]. It is worth calling attention to the fact that the Casimir entropy satisfies the third law of thermodynamics (the Nernst heat theorem) when T goes to zero, as should be expected due to the fact that the Casimir entropy for both the massless scalar field and the electromagnetic field, separately, satisfy the third law of thermodynamic [37,38,65,66].

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5.2. High Temperature Limit

In high temperature limit, which means that $k_B T >> \hbar c/a_0$, the Casimir free energy given by Equation (62) is the sum of the following contributions

$$F_{C}^{(0)}(T) = \frac{k_{B}T}{4\pi^{2}}\zeta(3) + 4\pi^{2} \left(\frac{a_{0}k_{B}T}{\hbar c}\right)^{3} \frac{\hbar c}{a_{0}} e^{-4\pi^{2}(a_{0}k_{B}T/\hbar c)},$$
(71)

arising from massless scalar field [37] and

$$F_{C}^{(1)}(T) = -k_{B}T\ln\frac{a_{0}k_{B}T}{\hbar c} - Rk_{B}T + 8\pi^{2}a_{0}^{2}\frac{(k_{B}T)^{3}}{(\hbar c)^{2}}e^{-(4\pi^{2}a_{0}k_{B}T/\hbar c)},$$
(72)

from the electromagnetic field, respectively [38]. The parameter *R* is a constant equal to 1.77698 see Ref. [38].

The contributions to the Casimir internal energy, corresponding to the massless scalar and to the electromagnetic fields, given by Equation (65), in this limit, result in the following

$$U_C^{(0)}(T) = 16\pi^4 \left(\frac{a_0 k_B T}{\hbar c}\right)^4 \frac{\hbar c}{a_0} e^{-4\pi^2 (a_0 k_B T/\hbar c)},\tag{73}$$

for massless scalar field contribution [37] and

$$U_C^{(1)}(T) = k_B T + \frac{32\pi^4 a_0^3 (k_B T)^4}{(\hbar c)^3} e^{-(4\pi^2 a_0 k_B T/\hbar c)}.$$
(74)

for the internal energy of the electromagnetic field [38]. The total free energy and its limits of low and high temperatures can be obtained from Equation (45).

6. High and Low Limits of Temperature: Electromagnetic and Massless Spinor Fields

In this section, we will proceed a similar analysis did previously, but now taking into consideration the electromagnetic and massless spinor fields. The procedure follows, in a straightforward manner, what was done in relation to the massless scalar field.

6.1. Low Temperature Limit

The low temperature limit means that $k_BT \ll \hbar c/a_0$. Applying this limit to the Casimir free energy for the electromagnetic and massless spinor fields considering Equation (55), and taking into account the general form

$$F_{C,\lambda}^{(s)}(T) = \left(1 + \frac{\lambda}{3}\right) F_C^{(s)}(T),\tag{75}$$

we obtain the following results

$$F_{\rm C}^{(1)}(T) = \frac{11\hbar c}{120a_0} + \frac{\pi^2}{45} V \frac{(k_B T)^4}{(\hbar c)^3} - \frac{\pi^2 a_0}{3} \frac{(k_B T)^2}{\hbar c} - 6k_B T e^{-(2\hbar c/a_0 k_B T)}.$$
(76)

in the case of the electromagnetic field [38] and

$$F_{C}^{(1/2)}(T) = \frac{17\hbar c}{480a_{0}} + \frac{7\pi^{2}}{180}V\frac{(k_{B}T)^{4}}{(\hbar c)^{3}} - \frac{\pi^{2}a_{0}}{12}\frac{(k_{B}T)^{2}}{\hbar c} - 8k_{B}Te^{-(3\hbar c/2a_{0}k_{B}T)},$$
(77)

in the case of the massless spinor field [38].

Using Equations (46) and (53), we get

$$U_{C,\lambda}^{(s)}(T) = \left(1 + \frac{\lambda}{3}\right) U_C^{(s)}(T),$$
(78)

where

$$U_{C}^{(1)}(T) = \frac{11\hbar c}{120a_{0}} - \frac{\pi^{2}}{15}V\frac{(k_{B}T)^{4}}{(\hbar c)^{3}} + \frac{\pi^{2}a_{0}}{3}\frac{(k_{B}T)^{2}}{(\hbar c)} + \frac{12\hbar c}{a_{0}}e^{-(2\hbar c/a_{0}k_{B}T)},$$
(79)

is the Casimir internal energy of the electromagnetic field [38] and

$$U_{C}^{(1/2)}(T) = \frac{17\hbar c}{480a_{0}} - \frac{7\pi^{2}}{60}V\frac{(k_{B}T)^{4}}{(\hbar c)^{3}} + \frac{\pi^{2}a_{0}}{12}\frac{(k_{B}T)^{2}}{(\hbar c)} + \frac{12\hbar c}{a_{0}}e^{-(3\hbar c/2a_{0}k_{B}T)},$$
 (80)

is the Casimir internal energy of the massless spinor field [38].

Concerning the Casimir entropy, it can be obtained using Equations (48) and (53). Thus, we get

$$S_{C,\lambda}^{(s)}(T) = \left(1 + \frac{\lambda}{3}\right) S_C^{(s)}(T).$$
 (81)

The Casimir entropy for the electromagnetic [38] and the massless spinor fields [38] are given, respectively, by

$$S_{C}^{(1)}(T) = 2\pi^{2}k_{B}\frac{ak_{B}T}{3\hbar c} \left[1 - \frac{4\pi^{2}}{15} \frac{(a_{0}k_{B}T)^{2}}{(\hbar c)^{2}} \right] + \frac{12\hbar c}{aT} e^{-(2\hbar c/a_{0}k_{B}T)},$$
(82)

and

$$S_{C}^{(1/2)}(T) = \pi^{2}k_{B}\frac{ak_{B}T}{3\hbar c} \left[\frac{1}{2} - \frac{14\pi^{2}}{15}\frac{(a_{0}k_{B}T)^{2}}{(\hbar c)^{2}}\right] + \frac{12\hbar c}{aT}e^{-(3\hbar c/2a_{0}k_{B}T)}.$$
(83)

The Casimir entropy (81) satisfies the third law of thermodynamics (the Nernst heat theorem). This is expected due to the fact that both entropies satisfy the third law of thermodynamics.

6.2. High Temperature Limit

The high temperature limit is obtained under the condition $k_B T >> \hbar c/a_0$. In this limit, the Casimir free energy (75) for the electromagnetic and massless spinor fields are given, respectively, by

$$F_{C}^{(1)}(T) = -k_{B}T\ln\frac{a_{0}k_{B}T}{\hbar c} - Rk_{B}T + 8\pi^{2}a_{0}^{2}\frac{(k_{B}T)^{3}}{(\hbar c)^{2}}e^{-(4\pi^{2}a_{0}k_{B}T/\hbar c)},$$
(84)

and

$$F_{C}^{(1/2)}(T) = 4\pi^{2}a_{0}^{2}\frac{(k_{B}T)^{3}}{(\hbar c)^{2}}e^{-(2\pi^{2}a_{0}k_{B}T/\hbar c)}.$$
(85)

The Casimir internal energy given by Equation (78), in the high temperature limit, turns into

$$U_{C}^{(1)}(T) = k_{B}T + \frac{32\pi^{4}a_{0}^{3}(k_{B}T)^{4}}{(\hbar c)^{3}}e^{-(4\pi^{2}a_{0}k_{B}T/\hbar c)}.$$
(86)

for the internal energy of the electromagnetic field [38] and

$$U_C^{(1/2)}(T) = \frac{8\pi^4 a_0^3 (k_B T)^4}{(\hbar c)^3} e^{-(2pi^2 a_0 k_B T/\hbar c)},$$
(87)

for the internal energy of the neutrino field [38].

7. Discussions and Conclusions

The Einstein and closed Friedmann universes with a cosmic string were considered as a background spacetime in which the Casimir vacuum energy, Casimir free energy, internal energy, and entropy were calculated for fields with different spin, namely the massless scalar, electromagnetic, and massless spinor fields.

Particularly in the massless scalar field case, we showed that the regularized quantum vacuum energy is the sum of two terms; the first is associated with the scalar field itself in the Einstein universe without the cosmic string, while the second is an electromagnetic-like term multiplied by a factor given by $\lambda = (1/\alpha) - 1$, which codifies the presence of the cosmic string. Therefore, all contribution which arises due to the to the presence of the cosmic string is contained, exclusively, in the second term. It is worth stressing that the contribution of this term is negligible as compared with the contribution given by the first term, which is exclusively due to the massless scalar field. With respect to the regularized free energy, internal energy and entropy, their expression also separates into two terms analogously to the regularized quantum vacuum energy.

These thermodynamic quantities were studied in the low and high temperature limits. One of the main results obtained is the fact that in the low temperature limit, the Casimir entropy is in accordance with the third law of thermodynamics. Particularly, in the context of the Friedmann universe. With respect to the the internal energy of the massless scalar field we observed that, at a very high temperature limit, it tends to $(\lambda/6)k_BT$ and in the low temperature limit, it tends to a constant value, according to result given by Equation (25).

The Casimir vacuum energy, free energy, internal energy and entropy for the electromagnetic and massless spinor fields were also calculated. The results obtained for these thermodynamic quantities tell us that the expressions for these fields, in the spacetimes considered, also splits up into two terms. but differently from the massless scalar field, both terms are of the same nature, or in other words, the two terms for the electromagnetic have origin in the electromagnetic field itself, and similarly for the massless spinor field. However, similarly to the massless scalar field case, the second term is multiplied by the factor $\lambda = (1/\alpha) - 1$ which codifies the presence of a cosmic string. It is worth calling attention to the fact this second term, in both cases, are of six orders of magnitude less than the first one, and can be neglected, in principle. Other thermodynamics quantities such as internal energy, entropy have the same structure.

The low and high temperature limits of the thermodynamics quantities were analyzed, with an interesting result in the low temperature limit, namely that the Casimir entropies of the electromagnetic and massless spinor fields also obey the third law of thermodynamics.

It is worth emphasizing that the contributions to the Casimir vacuum energy and to all thermodynamics quantities considered, for the three fields, which arise due to the presence of the cosmic string are very small as compared with the contribution arising from the first term, which does not depend on the presence of the cosmic string. Therefore, from the physical point of view, the presence of the comic string only changes slightly the results when compared to the ones obtained when the cosmic string is absent [36,37]. Mathematically, the results obtained in [38,39] just reduce to the results obtained in [36,37] by taking $\alpha = 1$, or, equivalently, $\lambda = 0$, which corresponds to the situation where the cosmic is absent.

Author Contributions: Conceptualization, V.B.B., H.F.S.M., C.R.M. and C.A.R.F.; methodology, V.B.B., H.F.S.M., C.R.M. and C.A.R.F.; software, V.B.B., H.F.S.M., C.R.M. and C.A.R.F.; validation, V.B.B., H.F.S.M., C.R.M. and C.A.R.F.; formal analysis, V.B.B., H.F.S.M., C.R.M. and C.A.R.F.; investigation, V.B.B., H.F.S.M., C.R.M. and C.A.R.F.; supervision, V.B.B., H.F.S.M., C.R.M. and C.A.R.F.; visualization, V.B.B., H.F.S.M., C.R.M. and C.A.R.F.; supervision, V.B.B., H.F.S.M., C.R.M. and C.A.R.F.; visualization, V.B.B., H.F.S.M., C.R.M. and C.A.R.F.; supervision, V.B.B., H.F.S.M., C.R.M. and C.A.R.F.; visualization, V.B.B., H.F.S.M., C.R.M. and C.A.R.F.; visualization, V.B.B., H.F.S.M., C.R.M. and C.A.R.F. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported partially by Conselho nacional de Desenvolvimento Cientifico e Tecnológico (CNPQ), from Brazil, grant numbers 307211/2020-7 (VBB), 311031/2020-0 (HFM), 308979/2018-4 (CRM) and 306401/2018-5 (CR).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: All necessary data are contained in this article.

Acknowledgments: The authors acknowledge the partial financial support from the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) (Brazilian Agency). The authors are grateful to V. M. Mostepanenko and G. I. Klimchitskaya for collaboration on our join papers cited in this review.

Conflicts of Interest: The authors declare no conflict of interest.

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