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Anisotropic Cosmological Model in a Modified Theory of Gravitation

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Abstract: Current observations indicate that, on a large enough scale, the universe is homogeneous and isotropic. However, this does not preclude the possibility of some anisotropy having occurred during the early stages of the evolution of the universe, which could then have been damped out later. This idea has aroused interest in the Bianchi models, which are homogeneous but anisotropic. Secondly, there is much interest in modified gravity these days due to the problems that the usual Λ CDM model faces in general relativity. Hence, in this paper, a study was conducted on the Bianchi type-I cosmological model in $f(R,T)$ -modified gravity. Following some ideas from cosmography, a specific form of the deceleration parameter was assumed, leading to a model that exhibited a transition from early deceleration to late-time acceleration. The derived model approached isotropy at late times. The physical properties of the model were discussed, and expressions for the various parameters of the model were derived. It is also possible to make progress towards solving the cosmological constant problem, since in this model in $f(R,T)$ gravity, a variable cosmological-type parameter arose, which was large early on but decreased to a constant value in later times.

Keywords: Bianchi type-I universe; $f(R,T)$ theory; deceleration parameter; variable cosmological parameter

1. Introduction

Recent theoretical and experimental studies have revealed that our universe is currently in an accelerating stage of expansion [1–5] and that an unknown form of matter called “dark energy” plays a significant role in driving this acceleration [6,7]. The most interesting characteristics of this dark energy are positive energy density but negative pressure. The results of the Wilkinson microwave anisotropy probe (WMAP) [8] and Planck indicate that the universe is composed of roughly 68.5% dark energy, 26.5% dark matter, and 5% baryonic matter. Dark energy can be expressed in either of two ways. The first is by means of so-called exotic matter—i.e., either by using the equation of state parameter (EOS) $\omega = \frac{p}{\rho}$, where p is the pressure and ρ is the energy density, or with respect to the cosmological constant. The second approach used to picture the expansion of the universe is a modified version of the Einstein–Hilbert action principle—i.e., alternative theories to Einstein’s theory of gravitation. In this process, an arbitrary function replaces the matter Lagrangian in the action. Hence, the acceleration in the expansion of the universe, together with effective causes linked to dark energy, is most attractively explained in these modified theories.

The cosmological constant Λ , introduced by Albert Einstein in his field equations to obtain a static universe, is now treated as a suitable representative for dark energy for

explaining the increase in the expansion of the universe. However, cosmological puzzles such as the fine tuning and cosmic coincidence problems currently surround it [9].

In the last few years, to explain the mechanism of the late-time acceleration, dark matter, and dark energy, many modified theories of gravity have been studied—e.g., $f(R)$, $f(T)$, $f(G)$ and $f(R, T)$ gravity. These models were put forward to explain dark energy and other problems of cosmology. Noteworthy amongst them is $f(R)$ gravity, which has been broadly investigated by several authors [5,10,11]. In $f(R)$ gravity, an arbitrary function of R replaces R in the Einstein–Hilbert action. Another recommendation to explain late-time acceleration is $f(T)$ gravity, which has recently been developed. This theory is a generalized version of tele-parallel gravity in which the Weitzenböck connection is used instead of the Levi-Civita connection. The fascinating attribute of this theory is that it can explain the current acceleration without invoking dark energy.

Another modified theory that has attracted a lot of attention in recent years is $f(R, T)$ gravity, which was introduced by Harko et al. [12]. In this theory, the gravitational Lagrangian is defined by an arbitrary function of the Ricci scalar R and the trace T of the energy momentum tensor. It is to be observed that the dependency upon T may be attributed to quantum effects or to an imperfect fluid. In their paper [12], Harko and his collaborators studied some specific forms of the function $f(R, T)$. This theory can be considered as a more convenient theory to depict the accelerating stage of the universe. Some other authors who have investigated various aspects of the Bianchi type-I model in $f(R, T)$ gravity will now be briefly mentioned.

The locally rotationally symmetric (LRS) Bianchi type-I models were found by Adhav [13]. Sharif and Zubair [14] studied exponential and power law solutions for the Bianchi type-I model with perfect fluid. Models with a constant deceleration parameter were found by Shamir [15]. Ram and Kumari [16] obtained Bianchi types I and V bulk viscous solutions by choosing a nonlinear form of the deceleration parameter. A model with a cosmological constant and quadratic equation of state was investigated by Singh and Bishi [17]. Sahoo and Sivakumar [18] found LRS models with a dynamic cosmological parameter by assuming a linearly varying deceleration parameter. By assuming an expansion scalar that is proportional to the shear scalar, Shamir [19] was able to derive exact solutions for an LRS Bianchi type-I model. Singh and Bishi [20] found transit solutions by assuming a quadratic equation of state as well as a scale factor which is a product of power-law and exponential. Zubair and Ali Hassan [21] studied the Bianchi types I and III and Kantowski–Sachs spacetimes in a unified way by assuming that the expansion scalar is proportional to the shear scalar and also that the derivative of $f(R, T)$ with respect to R is proportional to the scale factor. In Singh et al. [22], an LRS model with a scalar field was considered, as well as power-law and exponential forms for the scale factor and scalar field. A string cosmological model was derived by Sahoo [23] utilizing a constant deceleration parameter and the ansatz that the scalar expansion is proportional to the shear scalar. Shukla and Jayadev [24] found solutions for particle creation in an LRS model with gamma-law equation of state, and expansion proportional to shear. A string solution in the LRS case was solved by Kanakavalli and Rao [25] by taking the equation of state for strings. Zubair et al. [26] found Bianchi types I and V solutions by using a linear deceleration parameter and taking the expansion proportional to the expansion.

Solutions for Bianchi types I and V with magnetized strange quark matter and cosmological constant were found by Aktas [27], who used the usual equation of state for quark matter and a constant deceleration parameter. Caglar and Aygun [28] investigated a model with quark matter and a cosmological constant by assuming that the expansion was proportional to the shear, and by using the usual equation of state for quark matter. Tiwari et al. [29] looked at varying gravitational and cosmological parameters with a particular non-linear equation of state and expansion proportional to the shear. In general, energy-momentum is not conserved in $f(R, T)$ theory, but the requirement of energy-momentum conservation may be imposed to obtain solutions. An LRS model with a constant deceleration parameter was found by Bishi et al. [30]. Electromagnetic and scalar

fields were introduced by Solanke and Karade [31] and solutions were found by assuming the interaction between these fields to be linear. Gudekli and Caliskan [32] considered the LRS model and found perfect fluid solutions by assuming the two scale factors to be proportional to each other. In a very interesting paper, Yadav and Ali [33] applied Lie point symmetry analysis to find two solutions, one with a big-bang singularity and the other without. Bulk viscous LRS models were found by Sahoo and Reddy [34] using a specific time-dependent deceleration parameter. Yadav [35] found a transitioning solution by using a hybrid expansion law for the scale factor. The stability of the LRS models was discussed by Sharma et al. [36]. Pradhan et al. [37] found solutions that exhibited a transition from early deceleration to late-time acceleration by choosing two suitable forms of the scale factor. By choosing a hybrid form for the scale factor, Yadav et al. [38] were able to study bulk viscous LRS models. Cyclic LRS models with periodic varying deceleration parameters were derived by Bhardwaj and Rana [39], and Singh and Beesham [40] studied LRS models with a constant expansion rate. A reconstruction of the LRS model was made by Tiwari et al. [41] by choosing a non-linear form for the deceleration parameter. Bhardwaj and Dixit [42] constructed bouncing solutions by choosing a suitable form for the scale factor.

In this paper, we discussed the Bianchi type-I cosmological model by assuming a particular form for the deceleration parameter as a function of the Hubble parameter. The reason for choosing the Bianchi type-I model is that, although the universe is currently observed to be homogeneous and isotropic, it could have started off with some anisotropy. Then, as the universe evolved, the anisotropy became damped out, leading to the currently observable universe. The field equations are presented in Section 2. The solution of the field equations is derived and discussed in Section 3. The observational parameters such as cosmological red-shift, luminosity distance, and state-finder parameters for the model are also discussed in Section 3. Section 4 contains the conclusion.

This paper is based on results presented at the 1st International Electronic Conference on Universe and is a much extended version of the abbreviated paper that appeared in the conference proceedings [43].

2. Modified $f(R, T)$ Gravity

The action of $f(R, T)$ gravity is given by:

$$S = \int \sqrt{-g} \left(\frac{-1}{16\pi G} f(R, T) + L_m \right) d^4x, \quad (1)$$

where g is the determinant of the metric tensor g_{ij} ; $f(R, T)$ is an arbitrary function of the Ricci scalar R and the trace T of the energy-momentum tensor T_{ij} —i.e., $(T = g^{ij}T_{ij})$; and L_m is the matter Lagrangian density. It is worth mentioning here that the $f(R, T)$ theory of gravity is a modification of general relativity and can be considered as an extension of the $f(R)$ theory. As in $f(R)$ gravity models, the field equations are obtained by varying the total action of both the field and matter and equating this variation to zero.

Now, using gravitational units ($8\pi G = 1, c = 1$) and varying the action S in (1) with respect to the metric tensor g_{ij} , we obtain the field equations in $f(R, T)$ gravity as:

$$f_R(R, T)R_{ij} - \frac{1}{2}f(R, T)g_{ij} + (g_{ij}\square - \nabla_i\nabla_j)f(R, T) = -T_{ij} - f_T(R, T)T_{ij} - f_T(R, T)\Theta_{ij}, \quad (2)$$

where $f_R(R, T) = \frac{\partial f(R, T)}{\partial R}$, $f_T(R, T) = \frac{\partial f(R, T)}{\partial T}$, R_{ij} is the Ricci tensor, and T_{ij} is the energy-momentum tensor given by:

$$T_{ij} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_m)}{\delta g^{ij}}. \quad (3)$$

In Equation (2), $\square = \nabla^i\nabla_i$ is the D'Alembertian operator, where ∇_i represents the covariant derivative and:

$$\Theta_{ij} = -2T_{ij} + g_{ij}L_m - 2g^{\mu\nu} \frac{\partial^2 L_m}{\partial g^{ij} \partial g^{\mu\nu}}. \quad (4)$$

On contraction, Equation (2) yields an important relation connecting the Ricci scalar R and the trace T of the energy-momentum tensor:

$$f_R(R, T)R + 3 \square f_R(R, T) - 2f(R, T) = -T - f_T(R, T)(T + \Theta), \quad (5)$$

where $\Theta = \Theta_i^i$. If we assume that the matter Lagrangian density L_m depends only on the metric tensor component g_{ij} rather than its derivatives, then Equation (3) is reduced to the form:

$$T_{ij} = g_{ij}L_m - 2 \frac{\partial L_m}{\partial g^{ij}}. \quad (6)$$

For a perfect fluid distribution, the energy-momentum tensor of the matter has the form:

$$T_{ij} = (\rho + p)u_i u_j + p g_{ij}, \quad (7)$$

where ρ and p are the energy density and pressure of the fluid, respectively. Here u^i is the four-velocity vector satisfying $u^i u_i = -1$ and $u^i \nabla_j u_i = 0$. Now, using the fact that $L_m = -p$, Equation (4) can be rewritten as:

$$\Theta_{ij} = -p g_{ij} - 2T_{ij}. \quad (8)$$

On account of this, the field Equation (2) takes the form:

$$f_R(R, T)R_{ij} - \frac{1}{2}f(R, T)g_{ij} - (\nabla_i \nabla_j - \square g_{ij})f_R(R, T) = -T_{ij} + f_T(R, T)(T_{ij} + p g_{ij}). \quad (9)$$

Harko et al. [12] have considered three possible forms of the function $f(R, T)$:

$$f(R, T) = \begin{cases} R + 2f_1(T) \\ f_1(R) + f_2(T) \\ f_1(R) + f_2(R)f_3(T) \end{cases}. \quad (10)$$

In the present study, we shall concentrate on the first form of $f(R, T)$ —i.e., $f(R, T) = R + 2f_1(T)$ —and choose $f_1(T) = -\lambda T$, where λ is an arbitrary constant. For this consideration and energy-momentum tensor (7), Equation (9) is reduced to the form:

$$R_{ij} - \frac{1}{2}R g_{ij} = -(1 + 2\lambda)T_{ij} + \lambda(T + 2p)g_{ij}. \quad (11)$$

Now, Einstein's field equations with the cosmological term can be written as:

$$R_{ij} - \frac{1}{2}R g_{ij} = -T_{ij} + \Lambda g_{ij}. \quad (12)$$

By comparing Equations (11) and (12), and by taking the parameter λ to be small, we can make the identification $\Lambda = \Lambda(T) = \lambda(T + 2p)$. Therefore, in the $f(R, T)$ theory of gravity, the field equations with a variable cosmological parameter $\Lambda(T)$ can be expressed as:

$$R_{ij} - \frac{1}{2}R g_{ij} = -(1 + 2\lambda)T_{ij} + \Lambda g_{ij}. \quad (13)$$

In the case of a perfect fluid, the trace T of the energy-momentum tensor can be written as $T = \rho - 3p$. The cosmological parameter can be written as:

$$\Lambda = \lambda(\rho - p). \quad (14)$$

It can clearly be seen from Equation (13), which follows from Equation (11), that the usual energy conservation law does not hold in general in the $f(R, T)$ theory. It has been pointed out by Shabani and Ziaie [44] that the non-conservation of energy from the thermodynamic point of view implies an irreversible matter creation process. It is expected that this process could be justified by fundamental particle physics. Such particle creation corresponds to energy flow from the gravitational field to the created matter particles. The same authors in another paper [45] investigated the consequences of the energy conservation. They found that, in general, if there is energy conservation in $f(R, T)$ gravity then late-time stable accelerating solutions are not a general feature. However, with energy non-conservation, it is possible to find a large class of solutions with a dynamic $\Lambda(T)$ which have late-time acceleration and are stable.

In our case, let us examine, first, if we have energy conservation or not. The LHS of Equation (13) has zero divergence by virtue of the Bianchi identities. This implies that the RHS must also have zero divergence. From this, it can be seen that there is only one situation in which the usual energy conservation law holds. This is when $\frac{dp}{dt} = \frac{dp}{dt}$. Otherwise, in general, as is the case in this paper, there is a non-conservation of energy.

3. Model and Field Equations

The gravitational field for a spatially homogeneous and anisotropic Bianchi type-I space-time is given by the line element:

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 dy^2 + C^2 dz^2. \quad (15)$$

where A , B , C are metric functions of the cosmic time t . For the Bianchi type-I space-time (14), the field Equation (13) in $f(R, T)$ gravity yields the following dynamical equations:

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = \Lambda - (1 + 2\lambda)p, \quad (16)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} = \Lambda - (1 + 2\lambda)p, \quad (17)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = \Lambda - (1 + 2\lambda)p, \quad (18)$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} = \Lambda + (1 + 2\lambda)\rho, \quad (19)$$

where an over-dot denotes the ordinary derivative with respect to cosmic time t . We assume that the matter content obeys the usual equation of state:

$$p = \omega\rho, \quad -1 \leq \omega \leq 1. \quad (20)$$

The spatial volume (V) and average scale factor (a) for the Bianchi type-I space-time are given by, respectively:

$$V = ABC, \quad (21)$$

$$a = (ABC)^{\frac{1}{3}} = V^{\frac{1}{3}}. \quad (22)$$

An average Hubble parameter (H) for the Bianchi type-I is defined by:

$$H = \frac{1}{3}(H_1 + H_2 + H_3), \quad (23)$$

where $H_1 = \frac{\dot{A}}{A}$, $H_2 = \frac{\dot{B}}{B}$, and $H_3 = \frac{\dot{C}}{C}$ are the directional Hubble parameters along the X , Y , and Z axes, respectively.

Equations (22) and (23) can also be written in the form:

$$H = \frac{\dot{a}}{a} = \frac{1}{3} \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right). \quad (24)$$

The expansion scalar (θ), shear scalar (σ), and anisotropy parameter A_m are defined as, respectively:

$$\theta = 3H = 3\frac{\dot{a}}{a}, \quad (25)$$

$$\sigma^2 = \frac{1}{2} \left(\sum_{i=1}^3 H_i^2 - \frac{1}{3} \theta^2 \right), \quad (26)$$

$$A_m = \frac{2\sigma^2}{3H^2}. \quad (27)$$

From Equations (16)–(18), we can obtain the following equations:

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{\dot{C}}{C} \left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) = 0, \quad (28)$$

$$\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} + \frac{\dot{A}}{A} \left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) = 0, \quad (29)$$

$$\frac{\ddot{A}}{A} - \frac{\ddot{C}}{C} + \frac{\dot{B}}{B} \left(\frac{\dot{A}}{A} - \frac{\dot{C}}{C} \right) = 0. \quad (30)$$

These equations imply that:

$$\frac{A}{B} = c_1 \exp \left(d_1 \int \frac{dt}{a^3} \right), \quad (31)$$

$$\frac{B}{C} = c_2 \exp \left(d_2 \int \frac{dt}{a^3} \right), \quad (32)$$

$$\frac{A}{C} = c_3 \exp \left(d_3 \int \frac{dt}{a^3} \right), \quad (33)$$

where c_1, c_2, c_3 and d_1, d_2, d_3 are the constants of integration. From Equations (31)–(33), we can easily obtain the metric potentials A , B , and C as:

$$A = m_1 a \exp \left[\frac{2k_1 + k_2}{3} \int \frac{dt}{a^3} \right], \quad (34)$$

$$B = m_2 a \exp \left[\frac{k_2 - k_1}{3} \int \frac{dt}{a^3} \right], \quad (35)$$

$$C = m_3 a \exp \left[-\frac{k_1 + 2k_2}{3} \int \frac{dt}{a^3} \right], \quad (36)$$

where m_1, m_2, m_3 and k_1, k_2 are arbitrary constants of integration satisfying $m_1 m_2 m_3 = 1$. The deceleration parameter (q) is defined as:

$$q = -\frac{a\ddot{a}}{\dot{a}^2}. \quad (37)$$

Then, Equations (13)–(16) can be expressed in terms of H , q , and σ as:

$$3H^2 - \sigma^2 = \Lambda + (1 + 2\lambda)\rho, \quad (38)$$

$$H^2(2q - 1) - \sigma^2 = (1 + 2\lambda)p - \Lambda. \quad (39)$$

4. Solution to Field Equations

Equations (14) and (16)–(20), which are obtained from the field Equation (13), represent six equations in the six unknown quantities—i.e., A , B , C , ρ , p , and Λ , respectively. Hence, one can try to solve for the system directly. However, this is very difficult. Additionally, we care about seeking suitable cosmological solutions that exhibit a transition from deceleration early on, to acceleration at late times, in keeping with recent observations [1–5]. In this investigation, we assume that the deceleration parameter q can be expanded as a function of the Hubble parameter H [41]. There are many different assumptions that can be adopted to solve this system. The motivation for considering the time-dependent deceleration parameter q is due to the fact that the universe exhibits a phase transition from the past decelerating expansion to the recent accelerating one, as evidenced by observations [1–5]. The deceleration parameter is a geometric parameter which describes the acceleration or deceleration of the universe depending on its sign. In this context, it is known that if $q < 0$, then the universe has accelerating expansion; if $q > 0$, then the universe has decelerating expansion; if $q = 0$, then the universe has a constant rate of expansion; and if $q < -1$, then the accelerating expansion is dubbed super-exponential expansion.

Motivated by the above, in order to explain the behavior of the universe, we chose the deceleration parameter as a function of the Hubble parameter H as proposed by Tiwari et al. [41]:

$$q = \alpha - \frac{\beta}{H}. \quad (40)$$

Here, α and β are constants and $\beta > 0$. This form of the deceleration parameter yields the required transition from positive to negative as we desire. Equation (40) leads to the following solution for the scale factor:

$$a = k_1 \left(e^{\beta t} - 1 \right)^{\frac{1}{1+\alpha}}, \quad (41)$$

where k_1 is a constant. The directional Hubble parameters H_1 , H_2 , and H_3 are given by, respectively:

$$H_1 = \frac{\beta e^{\beta t}}{(1 + \alpha)(e^{\beta t} - 1)} + \frac{(2k_1 + k_2)}{3k_1^3(e^{\beta t} - 1)^{3/(1+\alpha)}}, \quad (42)$$

$$H_2 = \frac{\beta e^{\beta t}}{(1 + \alpha)(e^{\beta t} - 1)} + \frac{(k_2 - k_1)}{3k_1^3(e^{\beta t} - 1)^{3/(1+\alpha)}}, \quad (43)$$

$$H_3 = \frac{\beta e^{\beta t}}{(1 + \alpha)(e^{\beta t} - 1)} - \frac{(k_1 + 2k_2)}{3k_1^3(e^{\beta t} - 1)^{3/(1+\alpha)}}. \quad (44)$$

The spatial volume V , Hubble parameter H , expansion scalar θ , shear scalar σ^2 , anisotropy parameter A_m , and deceleration parameter q take the following forms, respectively:

$$V = k_1^3 \left(e^{\beta t} - 1 \right)^{\frac{3}{1+\alpha}}, \quad (45)$$

$$H = \frac{\beta e^{\beta t}}{(1 + \alpha)(e^{\beta t} - 1)}, \quad (46)$$

$$\theta = \frac{3\beta}{(1 + \alpha)(1 - e^{-\beta t})}, \quad (47)$$

$$\sigma^2 = \frac{k_1^2 + k_2^2 + k_1 k_2}{3k_1^6(e^{\beta t} - 1)^{\frac{6}{1+\alpha}}}, \quad (48)$$

$$A_m = \frac{2(k_1^2 + k_2^2 + k_1 k_2)(1 + \alpha)^2}{9\beta^2 k_1^6 e^{2\beta t} (e^{\beta t} - 1)^{\frac{4-2\alpha}{1+\alpha}}}, \quad (49)$$

$$q = -1 + (1 + \alpha)e^{-\beta t}. \quad (50)$$

Equations (41)–(50) are determined essentially from (40), and are the kinematic quantities. The field Equation (13), on the other hand is basically used to determine the dynamical quantities, viz., the energy density ρ , pressure p , and cosmological parameter Λ .

From Equations (17)–(19), we obtain energy density ρ and pressure p :

$$\rho = \frac{1}{(1 + \omega)(1 + 2\lambda)} \left[\frac{2\beta^2 e^{\beta t}}{(1 + \alpha)(e^{\beta t} - 1)^2} - \frac{2(k_1^2 + k_2^2 + k_1 k_2)}{3k_1^6 (e^{\beta t} - 1)^{\frac{6}{1+\alpha}}} \right], \quad (51)$$

$$p = \frac{\omega}{(1 + \omega)(1 + 2\lambda)} \left[\frac{2\beta^2 e^{\beta t}}{(1 + \alpha)(e^{\beta t} - 1)^2} - \frac{2(k_1^2 + k_2^2 + k_1 k_2)}{3k_1^6 (e^{\beta t} - 1)^{\frac{6}{1+\alpha}}} \right], \quad (52)$$

The cosmological parameter $\Lambda = \lambda(\rho - p)$ is given by:

$$\Lambda = \lambda \left[\frac{2(1 - \omega)\beta^2 e^{\beta t}}{(1 + \omega)(1 + 2\lambda)(1 + \alpha)(e^{\beta t} - 1)^2} - \frac{(1 - \omega)}{(1 + \omega)(1 + 2\lambda)} \frac{2(k_1^2 + k_2^2 + k_1 k_2)}{3k_1^6 (e^{\beta t} - 1)^{\frac{6}{1+\alpha}}} \right]. \quad (53)$$

For our Bianchi model (14), we observe that the spatial volume V is zero and expansion scalar θ are infinite at $t = 0$. Thus, the universe starts evolving with zero volume and an infinite rate of expansion at $t = 0$. Equations (34)–(36) and (41) show that the scale factors also vanish at $t = 0$, hence the model has a “point type” singularity at the initial epoch. Initially, at $t = 0$ the Hubble parameter H and shear scalar σ^2 are infinite. The energy density ρ , pressure p and cosmological constant Λ are also infinite. As t tends to infinity, V becomes infinitely large, whereas σ^2 approaches zero. Later, the energy density ρ and pressure p converge to zero. The cosmological parameter Λ also approaches a constant later. The deceleration parameter q for the model is a constant α at $t = 0$, and as t increases—i.e., when it is $(1/\beta) \log(1 + \alpha)$ — q is zero, which shows that there will be a transition to acceleration. It is equal to -1 when t tends to infinity, which shows that the model describes the accelerating phase of the universe. The anisotropy parameter A_m gives a measure of the anisotropy of the model, and is given by Equation (49), which is large early on as $t \rightarrow 0$ but decreases very rapidly [46]. Depending upon the choice of parameters of the model (α and β), we can make the anisotropy less than one part in 10^5 at the time of decoupling, in keeping with the observations of the cosmic microwave background radiation. In other words, the anisotropy is effectively erased at the time of decoupling, and the universe is effectively isotropic thereafter.

As a matter of interest, the solution for $\Lambda = 0$, which also means $\lambda = 0$ from Equation (14), can now be easily given. All the kinematic quantities are the same as before, viz., Equations (41)–(50). The density and pressure are given by:

$$\rho = p = \frac{1}{(1 + \omega)} \left[\frac{2\beta^2 e^{\beta t}}{(1 + \alpha)(e^{\beta t} - 1)^2} - \frac{2(k_1^2 + k_2^2 + k_1 k_2)}{3k_1^6 (e^{\beta t} - 1)^{\frac{6}{1+\alpha}}} \right].$$

This corresponds to a stiff matter solution.

4.1. Some Cosmological Distance Parameters: Cosmological Red-Shift

The age and size of the universe is defined by the Hubble parameter. From Equation (46), the Hubble parameter is:

$$H = \frac{\beta e^{\beta t}}{(1 + \alpha)(e^{\beta t} - 1)}.$$

From this, we obtain:

$$\frac{H}{H_0} = \frac{e^{\beta t}(e^{\beta t_0} - 1)}{e^{\beta t_0}(e^{\beta t} - 1)}, \quad (54)$$

where H_0 is the present value of the Hubble parameter and t_0 is the present time.

The following equation explains the relationship between the scale factor a and red-shift z :

$$a = \frac{a_0}{1+z}, \quad (55)$$

where a_0 is the present value of the scale factor. Here, we assume $a_0 = 1$. Using Equation (41), we can also write:

$$a = \frac{1}{1+z} = k_1(e^{\beta t} - 1)^{\frac{1}{1+\alpha}}. \quad (56)$$

This enables us to write the Hubble parameter as:

$$H = H_0(1 - e^{-\beta t_0}) \left[\{k_1(1+z)\}^{1+\alpha} + 1 \right]. \quad (57)$$

Equation (57) represents the value of the Hubble parameter in terms of the red shift parameter.

The distance modulus (μ) is given by:

$$\mu(z) = 5 \log d_L + 25, \quad (58)$$

where d_L stands for the luminosity distance, which is defined by:

$$d_L = r_1(1+z)a_0. \quad (59)$$

A source emits a photon at $r = r_1$ at time $t = t_0$, and an observer receives it at time t , located at $r = 0$. Then, we can calculate r_1 from the following equation:

$$r_1 = \int_t^{t_0} \frac{dt}{a} = \int_t^{t_0} \frac{dt}{k_1(e^{\beta t} - 1)^{\frac{1}{1+\alpha}}}. \quad (60)$$

To solve this integral, we take $\alpha = 0$ without any loss of generality. We obtain the value of r_1 as:

$$r_1 = \frac{1}{\beta k_1} \log \left(\frac{1 - e^{-\beta t_0}}{1 - e^{-\beta t}} \right). \quad (61)$$

Hence, from Equations (59) and (60), we obtain the expression for the luminosity distance as:

$$d_L = \frac{1}{\beta k_1} \log \left[\left(1 - e^{-\beta t_0} \right) \left(\{k_1(1+z)\}^{1+\alpha} + 1 \right) \right] (1+z). \quad (62)$$

From Equations (58) and (62), we obtain the expression for the distance modulus.

4.2. State-Finder Parameters

The state-finder parameters are a cosmological diagnostic pair $\{r, s\}$ which permit us to examine the characteristics of dark energy independent of a model. Moreover, like the dependence of the Hubble and deceleration parameters on the first and second derivatives, respectively, the state-finder parameters depend on the third derivative of the scale factor, $a(t)$. These parameters were introduced by Sahni et al. [47] and Alam et al. [48]. They are defined as:

$$r = \frac{\ddot{a}}{aH^3}, \quad (63)$$

$$s = \frac{r-1}{3\left(q - \frac{1}{2}\right)}. \quad (64)$$

The values of the state-finder parameters for our model are, respectively:

$$r = 1 - \frac{3\beta + (1 + \alpha)^2(1 - e^{-2\beta t})}{(e^{\beta t} - 1)}, \quad (65)$$

$$s = \frac{2(1 + \alpha)^2(1 - e^{-2\beta t}) - 6\beta}{(e^{\beta t} - 1)[6(1 + \alpha)e^{-\beta t} - 9]}. \quad (66)$$

When $(r, s) = (1, 0)$, we have the Λ CDM model, while for $(r, s) = (1, 1)$, we have the cold dark matter (CDM) limit [49]. Additionally, when $r < 1$ we have a quintessence region and for $s > 0$ the phantom region. We observe that for our model, when $t \rightarrow 0$, $\{r, s\} \rightarrow \{\infty, -\infty\}$, and as $t \rightarrow \infty$, $\{r, s\} \rightarrow \{1, 0\}$. This shows that our model starts from an Einstein static era and asymptotically approaches the Λ CDM model as $t \rightarrow \infty$.

5. Conclusions

In this paper, we discussed a spatially homogeneous and anisotropic Bianchi type-I space-time in the framework of $f(R, T)$ gravity. A specific choice of $f(R, T) = R + 2f_1(T)$, where $f_1(T) = -\lambda T$, has been considered to explore some exact solutions of an anisotropic and homogeneous Bianchi type-I space-time. One can ask the question regarding what constraints are placed on the coupling parameter λ . It is most interesting to note that solar system tests do not place any restrictions on the value of λ , since such tests are based on the vacuum field equations—i.e., the energy momentum tensor is zero, which also implies for the trace that $T = 0$ [50]. Nagpal et al. [51] have shown that the value $\lambda = 65$ allows for structure formation, which is consistent with a wide variety of observational data and a transition from deceleration to acceleration. Bhattacharjee and Sahoo [52] have studied bounds from big bang nucleosynthesis in $f(R, T)$ gravity, and found the following stronger bound: $-0.42 \leq \lambda \leq 0.07$ from the abundances of helium and deuterium. The Lithium problem persists as in the standard model.

For obtaining deterministic solutions of the field equations, we employed a variation law in which the deceleration parameter q is assumed to be a function of the Hubble parameter H —i.e., $q = \alpha - \frac{\beta}{H}$, which gives the scale factor $a = k_1(e^{\beta t} - 1)^{\frac{1}{1+\alpha}}$ (where α, β , and k_1 are constants and $\beta > 0$). Since we also assumed a barotropic equation of state, Equation (20), we can have the usual radiation and matter-dominated eras. We find that the universe expands exponentially until later times and that it also becomes more or less isotropic by the time of decoupling. The cosmological parameter Λ is very large at initial times and approaches a constant as t tends to infinity. This is in agreement with the work of Amirhashchi [53] and Yadav [54], and can help in finding a solution for the cosmological constant problem. The cosmological constant problem [55] has yet to be solved, despite much research in the area. One possibility is a dynamic cosmological parameter [56] which can fit the observations [57] or fit the observations even better than the standard Λ CDM model [58]. For $t \rightarrow 0$, the deceleration parameter q tends to be constant. We also discussed some cosmological distance parameters and state-finder parameters. Finally, we noticed from the state-finder parameters $\{r, s\}$ that the evolution of the universe begins from an Einstein static era ($r \rightarrow \infty, s \rightarrow -\infty$) and approaches the Λ CDM model ($r \rightarrow 1, s \rightarrow 0$) at later times.

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