

Article

Unifying Theory for Casimir Forces: Bulk and Surface Formulations

Giuseppe Bimonte ^{1,2,*} and Thorsten Emig ³

¹ Dipartimento di Fisica E. Pancini, Università di Napoli Federico II, Complesso Universitario di Monte S. Angelo, Via Cintia, I-80126 Napoli, Italy

² INFN Sezione di Napoli, I-80126 Napoli, Italy

³ Laboratoire de Physique Théorique et Modèles Statistiques, CNRS UMR 8626, Université Paris-Saclay, 91405 Orsay CEDEX, France; thorsten.emig@universite-paris-saclay.fr

* Correspondence: giuseppe.bimonte@na.infn.it

Abstract: The principles of the electromagnetic fluctuation-induced phenomena such as Casimir forces are well understood. However, recent experimental advances require universal and efficient methods to compute these forces. While several approaches have been proposed in the literature, their connection is often not entirely clear, and some of them have been introduced as purely numerical techniques. Here we present a unifying approach for the Casimir force and free energy that builds on both the Maxwell stress tensor and path integral quantization. The result is presented in terms of either bulk or surface operators that describe corresponding current fluctuations. Our surface approach yields a novel formula for the Casimir free energy. The path integral is presented both within a Lagrange and Hamiltonian formulation yielding different surface operators and expressions for the free energy that are equivalent. We compare our approaches to previously developed numerical methods and the scattering approach. The practical application of our methods is exemplified by the derivation of the Lifshitz formula.



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1. Introduction

The interaction induced by quantum and thermal fluctuation of the electromagnetic field is an everyday phenomenon that acts between all neutral objects, both on atomic and macroscopic scales [1–5]. For the Casimir interaction between macroscopic bodies, the last two decades have witnessed unparalleled progress in experimental observations and the development of novel theoretical approaches [6,7]. In most of the recent theoretical approaches, the computation of Casimir forces between multiple objects of different shapes and material composition has been achieved by the use of scattering methods or the so-called TGTG formula [8–16]. These approaches have the advantage of relatively low numerical effort; they are rapidly converging and can achieve in principle any desired precision [17–22]. Another merit of these methods is the exclusion of UV divergencies by performing the subtraction analytically before any numerical computation. Other efficient approaches that have been developed before the scattering approaches include path integral quantizations where the boundary conditions at the surfaces are implemented by delta functions [23]. These approaches are limited to scalar fields with Dirichlet or Neumann boundary conditions [24], or the electromagnetic field with perfectly conducting boundary conditions [25], with the exception of a similar approach for dielectric boundaries [26]. However, such analytical (and semi-analytical) methods have been restricted to symmetric and simple shapes, like spheres, cylinders or ellipsoids [27–31]. Geometries where parts of the bodies interpenetrate, such as those shown in Figure 1a, cannot be studied with scattering approaches. For general shapes and arbitrary geometries, new methods are needed. Purely numerical methods based on surface current fluctuations have been developed [32], but they rely on a full-scale numerical evaluation of matrices and their determinants, which complicates these approaches when high precision of the force is required.

Hence, there is a need to develop methods that predict Casimir interactions between objects of arbitrary geometries composed of materials with arbitrary frequency-dependent electromagnetic properties. The Casimir force can be viewed as arising from the interaction of fluctuating currents distributions. In fact, these effective fluctuating electric and magnetic currents can be considered to be localized either in the bulk of the bodies or just on their surfaces. The surface approach relies on the observation that the electromagnetic response of bodies can be represented entirely in terms of their surfaces, known as the “equivalence principle”, which is based on the observation that many source distributions outside a given region can produce the same field inside the region [33]. The surface approach has been introduced in the literature as a method for a purely numerical computation of Casimir interactions [32]. There are two different methods to implement the idea of computing Casimir forces from fluctuating currents. One can either integrate the Maxwell stress tensor over a closed surface enclosing the body, directly yielding the Casimir force, or integrate over all electromagnetic gauge field fluctuations in a path integral, yielding the Casimir free energy. We shall consider both approaches here.

Compared to scattering theory-based approaches, the surface formulation has the advantage that it does not require the use of eigenfunctions of the vector wave equation that are specific to the shape of the bodies. Hence, our approach is applicable to general geometries and shapes, including interpenetrating structures. In fact, the power of the surface approach has been demonstrated by numerical implementations in Reference [32], where it was used to compute the Casimir force in complicated geometries.

In this paper, we present both the Maxwell stress tensor and path integral-based approaches for the Casimir force and free energy in terms of bulk or surface operators. Our main advancements are

- A new, compact and elegant derivation of the Casimir force from the Maxwell stress tensor within both a T-operator approach and a surface operator approach;
- A new surface formula for the Casimir free energy expressed in terms of a surface operator;
- A new path integral-based derivation of a Lagrange and Hamiltonian formulation for the Casimir free energy.

We also compare the approaches presented here to methods existing in the literature. For the special case of bodies that can be separated by non-overlapping enclosing surfaces, along which one of the coordinates in which the wave equation is separable is constant, our approach is shown to be equivalent to the scattering approach. Our approaches also show the general equivalence of the use of the Maxwell stress tensor in combination with the fluctuation-dissipation theorem on one side and the path integral representation of the Casimir force on the other side. As the most simple application of our approaches, we re-derive the Lifshitz formula for the Casimir free energy of two dielectric slabs. Other analytical applications of our approach will be presented elsewhere.

The geometries and shapes to which our approaches can be applied are shown in Figure 1a. For comparison, in Figure 1b, we display non-penetrating bodies to which scattering theory-based approaches are limited. In general, we assume a configuration composed of N bodies with dielectric functions $\epsilon_r(\omega)$ and magnetic permeabilities $\mu_r(\omega)$, $r = 1, \dots, N$. The bodies occupy the volumes V_r with surfaces Σ_r and outward pointing surface normal vectors $\hat{\mathbf{n}}_r$. The space with volume V_0 in between the bodies is filled by matter with dielectric function $\epsilon_0(\omega)$ and magnetic permeability $\mu_0(\omega)$.

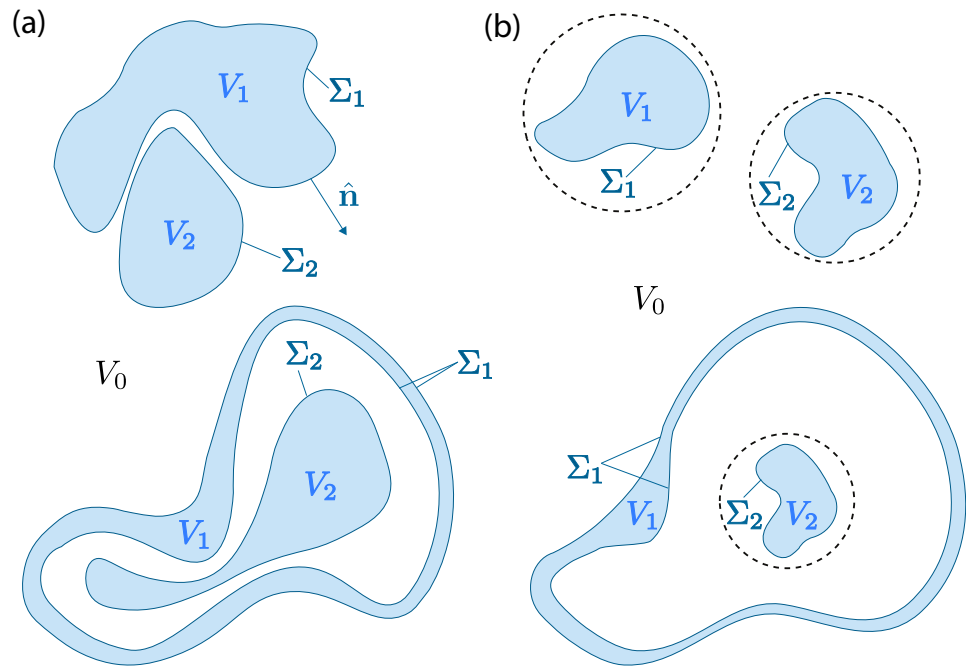


Figure 1. Configuration of bodies: (a) general shapes and positions that can be studied with the approaches presented in this work, (b) non-penetrating configurations that can be studied within the scattering approach.

2. Stress-Tensor Approach

2.1. Bulk and Surface Expressions for the Force

Consider a collection of N magneto-dielectric bodies in vacuum. In the stress-tensor approach, the (bare) Casimir force $F_i^{(\text{bare}|r)}$ on body r is obtained by integrating the expectation value $\langle T_{ij} \rangle$ of the Maxwell stress tensor

$$\langle T_{ij}(\mathbf{x}) \rangle = \frac{1}{4\pi} \left\{ \langle E_i(\mathbf{x})E_j(\mathbf{x}) \rangle + \langle H_i(\mathbf{x})H_j(\mathbf{x}) \rangle - \frac{1}{2}\delta_{ij}[\langle E_k(\mathbf{x})E_k(\mathbf{x}) \rangle + \langle H_k(\mathbf{x})H_k(\mathbf{x}) \rangle] \right\} \quad (1)$$

over any closed surface S_r drawn in the vacuum, which surrounds that body (but excludes all other bodies):

$$F_i^{(\text{bare}|r)} = \oint_{S_r} d^2\sigma \hat{n}_j(\mathbf{x}) \langle T_{ji}(\mathbf{x}) \rangle, \quad (2)$$

where $\hat{n}$ is the unit normal oriented outside S_r , and the angular brackets denote the expectation value taken with respect to quantum and thermal fluctuations. For a system in thermal equilibrium at temperature T , the (equal-time) expectation values of the products of field components (at points $\mathbf{x}$ and $\mathbf{x}'$ in the vacuum region) are provided by the fluctuation-dissipation theorem [34,35]:

$$\begin{aligned} \langle \hat{E}_i(\mathbf{x})\hat{E}_j(\mathbf{x}') \rangle &= 2k_B T \sum_{n=0}' \mathcal{G}_{ij}^{(EE)}(\mathbf{x}, \mathbf{x}'; i\xi_n), \\ \langle \hat{H}_i(\mathbf{x})\hat{H}_j(\mathbf{x}') \rangle &= 2k_B T \sum_{n=0}' \mathcal{G}_{ij}^{(HH)}(\mathbf{x}, \mathbf{x}'; i\xi_n), \end{aligned} \quad (3)$$

where $\xi_n = 2\pi n k_B T / \hbar$ are the Matsubara imaginary frequencies, and the prime in the summations mean that the $n = 0$ term is taken with a weight of one half. When the r.h.s of the above equations are plugged into Equation (1), one obtains for $\langle T_{ij}(\mathbf{x}) \rangle$ a formally divergent expression, since the Green functions $\mathcal{G}_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}'; i\xi_n)$ are singular in the

coincidence limit $\mathbf{x} = \mathbf{x}'$. This divergence can however be easily disposed of by noticing that the Green's functions admit the decomposition:

$$\mathcal{G}_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}'; i\zeta_n) = \mathcal{G}_{ij}^{(\alpha\beta;0)}(\mathbf{x} - \mathbf{x}'; i\zeta_n) + \Gamma_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}'; i\zeta_n), \quad (4)$$

where $\mathcal{G}_{ij}^{(\alpha\beta;0)}(\mathbf{x} - \mathbf{x}'; i\zeta_n)$ is the Green's function of free space, while $\Gamma_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}'; i\zeta_n)$ describes the effect of scattering of electromagnetic fields by the bodies. In the coincidence limit, only $\mathcal{G}_{ij}^{(\alpha\beta;0)}(\mathbf{x} - \mathbf{x}'; i\zeta_n)$ diverges, while $\Gamma_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}'; i\zeta_n)$ attains a finite limit. When the decomposition in Equation (4) is used to evaluate the r.h.s. of Equation (1), one finds that the expectation value of the stress tensor is decomposed in a way analogous to Equation (4):

$$\langle T_{ij}(\mathbf{x}) \rangle = \langle T_{ij}^{(0)}(\mathbf{x}) \rangle + \Theta_{ij}(\mathbf{x}), \quad (5)$$

where $\langle T_{ij}^{(0)}(\mathbf{x}) \rangle$ is the divergent expectation value of the stress tensor in empty space, and $\Theta_{ij}(\mathbf{x})$ is the *finite* expression

$$\begin{aligned} \Theta_{ij}(\mathbf{x}) = & \frac{k_B T}{2\pi} \sum_{n=0}^{\infty} \left[\Gamma_{ij}^{(EE)}(\mathbf{x}, \mathbf{x}; i\zeta_n) + \Gamma_{ij}^{(HH)}(\mathbf{x}, \mathbf{x}; i\zeta_n) \right. \\ & \left. - \frac{1}{2} \delta_{ij} \left(\Gamma_{kk}^{(EE)}(\mathbf{x}, \mathbf{x}; i\zeta_n) + \Gamma_{kk}^{(HH)}(\mathbf{x}, \mathbf{x}; i\zeta_n) \right) \right]. \end{aligned} \quad (6)$$

Since the divergent contribution $\langle T_{ij}^{(0)}(\mathbf{x}) \rangle$ is independent of the presence of the bodies, one can just neglect it and then one obtains the following finite expression for the Casimir force on body r due to the presence of the other bodies,

$$F_i^{(r)} = \oint_{S_r} d^2\sigma \hat{n}_j(\mathbf{x}) \Theta_{ji}(\mathbf{x}). \quad (7)$$

The further development of the theory starts from the observation that the dyadic Green's functions $\Gamma_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}')$ (for brevity, from now on we shall not display the dependence of the Green's functions on the Matsubara frequencies ζ_n) can be expressed in two distinct possible forms.

The first representation is general, since it is valid for arbitrary constitutive equations of the magneto-dielectric materials constituting the bodies, which can possibly be non-homogeneous, anisotropic, and non-local. For all points $\mathbf{x}$ and $\mathbf{x}'$, it expresses $\Gamma_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}')$ in the form of an integral of the T -operator $\hat{T}$ (for its definition, see Appendix B.1) over the *volume* V occupied by all bodies:

$$\Gamma_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}') = \sum_{r,r'=1}^N \int_{V_r} d^3\mathbf{y} \int_{V_{r'}} d^3\mathbf{y}' \mathcal{G}_{ik}^{(\alpha\rho;0)}(\mathbf{x} - \mathbf{y}) T_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}') \mathcal{G}_{lj}^{(\sigma\beta;0)}(\mathbf{y}' - \mathbf{x}'). \quad (8)$$

The above formula has a simple intuitive interpretation, if one recalls that according to its definition, the T -operator provides the polarization induced in the volume of the bodies when they are immersed in the electromagnetic field generated by a certain distribution of external sources.

The second representation is less general than Equation (8) because it applies only to magneto-dielectric bodies that are (piecewise) homogeneous and isotropic¹. It expresses $\Gamma_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}')$ in the form of an integral of the *surface* operator $\hat{M}^{-1}$ defined in Equation (A44),

over the union $\Sigma = \bigcup \Sigma_r$ of the surfaces Σ_r of the bodies. For two points $\mathbf{x}$ and $\mathbf{x}'$ both lying in the vacuum region outside the bodies², the surface representation of $\Gamma_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}')$ reads:

$$\begin{aligned} \Gamma_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}') &= - \sum_{r,r'=1}^N \int_{V_r} d^3\mathbf{y} \int_{V_{r'}} d^3\mathbf{y}' \delta(F_r(\mathbf{y})) \delta(F_{r'}(\mathbf{y}')) \\ &\times \mathcal{G}_{ik}^{(\alpha\rho;0)}(\mathbf{x} - \mathbf{y}) \left(M^{-1}\right)_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}') \mathcal{G}_{lj}^{(\sigma\beta;0)}(\mathbf{y}' - \mathbf{x}') , \end{aligned} \quad (9)$$

where $F_r(\mathbf{y}) = 0$ is the equation of the surface Σ_r . This representation also has a simple intuitive meaning, if one considers that $-\hat{M}^{-1}$ (see Appendix B.2 for details) is defined as the operator that provides the *fictitious* surface polarizations that radiate outside the bodies the same scattered electromagnetic field as the one radiated by the physically induced volumic polarization, in response to an external field. The derivations of Equations (8) and (9) are presented in Appendix B. It is apparent that both representations have the same mathematical structure, consisting of a two-sided convolution of a certain kernel $\mathcal{K}_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}')$ with the free-space Green's functions $\mathcal{G}_{ij}^{(\alpha\beta;0)}(\mathbf{x} - \mathbf{x}')$:

$$\Gamma_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}') = \int_V d^3\mathbf{y} \int_V d^3\mathbf{y}' \mathcal{G}_{ik}^{(\alpha\rho;0)}(\mathbf{x} - \mathbf{y}) \mathcal{K}_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}') \mathcal{G}_{lj}^{(\sigma\beta;0)}(\mathbf{y}' - \mathbf{x}') . \quad (10)$$

The only difference between the two representations consists in the expression of $\mathcal{K}$, which in the case of Equation (8) is the three-dimensional kernel $T_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}')$ supported in the volume V occupied by the bodies:

$$\mathcal{K}_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}') = T_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}') , \quad (11)$$

while in Equation (9) $\mathcal{K}$ is the two-dimensional kernel $-(M^{-1})_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}')$ supported on the union Σ of their surfaces:

$$\mathcal{K}_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}') = - \sum_{r,r'=1}^N \delta(F_r(\mathbf{y})) \delta(F_{r'}(\mathbf{y}')) \left(M^{-1}\right)_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}') \quad (12)$$

In both cases, the above equation can be concisely written using the operator notation described in Appendix A:

$$\hat{\Gamma} = \hat{\mathcal{G}}^{(0)} \hat{\mathcal{K}} \hat{\mathcal{G}}^{(0)} . \quad (13)$$

In Appendix C, we prove that the structure of the representation of $\Gamma_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}')$ given in Equation (10), allows to re-express the Casimir force Equation (7) in the following remarkably simple form:

$$F_i^{(r)} = 2k_B T \sum_{n=0}^{\infty} \int_{V_r} d^3\mathbf{y} \int_V d^3\mathbf{y}' \left(\mathcal{K}_{lj}^{(\alpha\beta)}(\mathbf{y}', \mathbf{y}; i\xi_n) \frac{\partial}{\partial y_i} \mathcal{G}_{jl}^{(\beta\alpha;0)}(\mathbf{y} - \mathbf{y}'; i\xi_n) \right) . \quad (14)$$

A crucial role in the derivation of Equation (14) is played by the fact that the kernel $\mathcal{K}$ satisfies a set of reciprocity relations analogous to those satisfied by the Green's functions:

$$\mathcal{K}_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}') = (-1)^{s(\rho)+s(\sigma)} \mathcal{K}_{lk}^{(\sigma\rho)}(\mathbf{y}', \mathbf{y}) , \quad (15)$$

where $s(E) = 0$ and $s(H) = 1$. It is possible to verify that the reciprocity relations satisfied by $\mathcal{G}_{ij}^{(\alpha\beta;0)}(\mathbf{y} - \mathbf{y}')$ and $\mathcal{K}_{ij}^{(\alpha\beta)}(\mathbf{y}, \mathbf{y}')$ ensure vanishing of the “self-force” $F_i^{(\text{self}|r)}$:

$$F_i^{(\text{self}|r)} = 2k_B T \sum_{n=0}^{\infty} \int_{V_r} d^3\mathbf{y} \int_{V_r} d^3\mathbf{y}' \left(\mathcal{K}_{ij}^{(\alpha\beta)}(\mathbf{y}', \mathbf{y}; i\xi_n) \frac{\partial}{\partial y_i} \mathcal{G}_{ji}^{(\beta\alpha;0)}(\mathbf{y} - \mathbf{y}'; i\xi_n) \right) = 0 . \quad (16)$$

This implies that in Equation (14) the $\mathbf{y}'$ integral is in fact restricted to $V - V_r$, in accord with one's intuition that the force on body r is due to the interaction with the other bodies. It is possible to present Equation (14) in a more compact and symmetric form, by defining the derivative $\partial/\partial\mathbf{x}_r$ of any kernel $\mathcal{A}(\mathbf{y}, \mathbf{y}')$, with respect to rigid translations of the r -th body:

$$\frac{\partial}{\partial\mathbf{x}_r}\mathcal{A}(\mathbf{y}, \mathbf{y}') \equiv \psi_r(\mathbf{y})\frac{\partial}{\partial\mathbf{y}}\mathcal{A}(\mathbf{y}, \mathbf{y}') + \psi_r(\mathbf{y}')\frac{\partial}{\partial\mathbf{y}'}\mathcal{A}(\mathbf{y}, \mathbf{y}') , \quad (17)$$

where $\psi_r(\mathbf{y})$ is the characteristic function of V_r : $\psi_r(\mathbf{y}) = 1$ if $\mathbf{y} \in V_r$, $\psi_r(\mathbf{y}) = 0$ if $\mathbf{y} \notin V_r$. Using $\partial/\partial\mathbf{x}_r$, we can rewrite Equation (14) as

$$\mathbf{F}^{(r)} = k_B T \sum_{n=0}^{\infty} \int_V d^3\mathbf{y} \int_V d^3\mathbf{y}' \left(\mathcal{K}_{ij}^{(\alpha\beta)}(\mathbf{y}', \mathbf{y}; i\zeta_n) \frac{\partial}{\partial\mathbf{x}_r} \mathcal{G}_{ji}^{(\beta\alpha;0)}(\mathbf{y} - \mathbf{y}'; i\zeta_n) \right) . \quad (18)$$

The expression on the r.h.s. of the above formula can be compactly expressed using the operator notation and the trace operation described in Appendix A:

$$\mathbf{F}^{(r)} = k_B T \sum_{n=0}^{\infty} \text{Tr} \left[\hat{\mathcal{K}}(i\zeta_n) \frac{\partial}{\partial\mathbf{x}_r} \hat{\mathcal{G}}^{(0)}(i\zeta_n) \right] . \quad (19)$$

Depending on whether we use for the kernel $\hat{\mathcal{K}}$ the T -operator of Equation (8) or rather the surface operator $-\hat{\mathcal{M}}$ of Equation (9), Equation (19) provides us with two distinct but formally similar representations of the Casimir force, which is expressed either as an integral over the volume V occupied by the bodies or as an integral over their surfaces Σ . One feature of Equation (19) is worth stressing. Since the force is expressed as a trace, Equation (19) can be evaluated in an *arbitrary* basis, leaving one with complete freedom in the choice of the most convenient basis in a concrete situation. A representation of the Casimir force in the form of a volume integral equivalent to Equation (19) was derived in [11], while the surface-integral representation was obtained in [32]. Equation (19) can be computed numerically for any shapes and dispositions of the bodies, by using discrete meshes covering the bodies. An efficient numerical scheme based on surface-elements methods is described in [32], where it was used to compute the Casimir force in complex geometries, not amenable to analytical techniques.

2.2. Casimir Free Energy

In this section, we compute the Casimir free energy $\mathcal{F}$ of the system of bodies, starting from the force formula Equation (19). We shall see that the T -operator and the surface-operator approaches lead to two distinct but equivalent representations of the Casimir energy.

2.2.1. T-Operator Approach

Plugging into Equation (19) the expression of the T -operator Equation (A36), we find that the Casimir force can be expressed in the form:

$$\mathbf{F}^{(r)} = k_B T \sum_{n=0}^{\infty} \text{Tr} \left(\hat{T}(i\zeta_n) \frac{\partial}{\partial\mathbf{x}_r} \hat{\mathcal{G}}^{(0)}(i\zeta_n) \right) = k_B T \sum_{n=0}^{\infty} \text{Tr} \left[\frac{1}{1 - \hat{\chi} \hat{\mathcal{G}}^{(0)}} \frac{\partial}{\partial\mathbf{x}_r} (\hat{\chi} \hat{\mathcal{G}}^{(0)}) \right] , \quad (20)$$

where in the last passage, we made use of the fact that the polarization operator $\hat{\chi}$ defined in Equation (A24) is invariant under a rigid displacement of the body. The r.h.s. of the above equation can be formally expressed as a gradient:

$$F^{(r)} = - \frac{\partial}{\partial\mathbf{x}_r} \mathcal{F}_{\text{bare}} , \quad (21)$$

where $\mathcal{F}_{\text{bare}}$ is the *bare* free energy:

$$\mathcal{F}_{\text{bare}} = k_B T \sum_{n=0}^{\infty} \text{Tr} \log[1 - \hat{\chi} \hat{\mathcal{G}}^{(0)}] \quad (22)$$

Unfortunately, $\mathcal{F}_{\text{bare}}$ is formally divergent. To obtain the finite Casimir free energy $\mathcal{F}$, one has to subtract from $\mathcal{F}_{\text{bare}}$ the divergent self-energies $\mathcal{F}_{\text{self}}^{(r)}$ of the individual bodies:

$$\mathcal{F}_{\text{self}}^{(r)} = \left\{ k_B T \sum_{n=0}^{\infty} \text{Tr} \log[1 - \hat{\chi}_r \hat{\mathcal{G}}^{(0)}] \right\}, \quad (23)$$

where $\hat{\chi}_r = \hat{\psi}_r \hat{\chi} \hat{\psi}_r$ is the polarizability operator of body r in isolation. In the case of a system composed by two bodies, the renormalized Casimir free energy can be recast in the following TGTG form:

$$\mathcal{F} = \mathcal{F}_{\text{bare}} - \mathcal{F}_{\text{self}}^{(1)} - \mathcal{F}_{\text{self}}^{(2)} = k_B T \sum_{n=0}^{\infty} \text{Tr} \log[1 - \hat{T}_1 \hat{\mathcal{G}}^{(0)} \hat{T}_2 \hat{\mathcal{G}}^{(0)}], \quad (24)$$

where

$$\hat{T}_r = \frac{1}{1 - \hat{\chi}_r \hat{\mathcal{G}}^{(0)}} \hat{\chi}_r, \quad (25)$$

is the T-operator of body r in isolation. To prove Equation (24), one notes that for each Matsubara mode the operator identity holds:

$$(1 - \hat{\mathcal{G}}^{(0)} \hat{\chi}_1) \hat{\mathcal{G}}^{(0)} \hat{T}_1 \hat{\mathcal{G}}^{(0)} \hat{T}_2 (1 - \hat{\mathcal{G}}^{(0)} \hat{\chi}_2) = \hat{\mathcal{G}}^{(0)} \hat{\chi}_1 \hat{\mathcal{G}}^{(0)} \hat{\chi}_2. \quad (26)$$

The above identity in turn allows to prove the following chain of identities:

$$\begin{aligned} & \text{Tr} \log[1 - \hat{\mathcal{G}}^{(0)} \hat{\chi}_1] + \text{Tr} \log[1 - \hat{\mathcal{G}}^{(0)} \hat{T}_1 \hat{\mathcal{G}}^{(0)} \hat{T}_2] + \text{Tr} \log[1 - \hat{\mathcal{G}}^{(0)} \hat{\chi}_2] \\ &= \text{Tr} \log[(1 - \hat{\mathcal{G}}^{(0)} \hat{\chi}_1)(1 - \hat{\mathcal{G}}^{(0)} \hat{T}_1 \hat{\mathcal{G}}^{(0)} \hat{T}_2)(1 - \hat{\mathcal{G}}^{(0)} \hat{\chi}_2)] \\ &= \text{Tr} \log[(1 - \hat{\mathcal{G}}^{(0)} \hat{\chi}_1)(1 - \hat{\mathcal{G}}^{(0)} \hat{\chi}_2) - (1 - \hat{\mathcal{G}}^{(0)} \hat{\chi}_1) \hat{\mathcal{G}}^{(0)} \hat{T}_1 \hat{\mathcal{G}}^{(0)} \hat{T}_2 (1 - \hat{\mathcal{G}}^{(0)} \hat{\chi}_2)] \\ &= \text{Tr} \log[(1 - \hat{\mathcal{G}}^{(0)} \hat{\chi}_1)(1 - \hat{\mathcal{G}}^{(0)} \hat{\chi}_2) - \hat{\mathcal{G}}^{(0)} \hat{\chi}_1 \hat{\mathcal{G}}^{(0)} \hat{\chi}_2] = \text{Tr} \log[(1 - \hat{\mathcal{G}}^{(0)})(\hat{\chi}_1 + \hat{\chi}_2)] \\ &= \text{Tr} \log[1 - \hat{\mathcal{G}}^{(0)} \hat{\chi}]. \end{aligned} \quad (27)$$

Equating the first line with the last line, we obtain the identity:

$$\begin{aligned} & \text{Tr} \log[1 - \hat{\mathcal{G}}^{(0)} \hat{\chi}] \\ &= \text{Tr} \log[1 - \hat{\mathcal{G}}^{(0)} \hat{\chi}_1] + \text{Tr} \log[1 - \hat{\mathcal{G}}^{(0)} \hat{T}_1 \hat{\mathcal{G}}^{(0)} \hat{T}_2] + \text{Tr} \log[1 - \hat{\mathcal{G}}^{(0)} \hat{\chi}_2]. \end{aligned} \quad (28)$$

Upon summing the above identity over all Matsubara modes (with weight one half for the $n = 0$ term), and then multiplying it by $k_B T$, we find:

$$\mathcal{F}_{\text{self}}^{(1)} + \mathcal{F}_{\text{self}}^{(2)} + \mathcal{F} = \mathcal{F}_{\text{bare}}, \quad (29)$$

which is equivalent to Equation (24). The energy formula Equation (24) was derived in [13] using the path-integral method and in [11], using Rytov's fluctuational electrodynamics [36].

2.2.2. Surface Operator Approach

Now we derive the surface-operator representation of the Casimir energy. To do that, we start from the surface-operator representation of the force, which is obtained by

replacing $\hat{K}$ in Equation (19) with minus the inverse of the surface operator $\hat{M}$ defined in Equation (A44):

$$\mathbf{F}^{(r)} = -k_B T \sum_{n=0}^{\infty} \text{Tr} \left[\hat{M}^{-1}(\mathbf{i} \zeta_n) \frac{\partial}{\partial \mathbf{x}_r} \hat{\mathcal{G}}^{(0)}(\mathbf{i} \zeta_n) \right]. \quad (30)$$

This can also be written as:

$$\mathbf{F}^{(r)} = -k_B T \sum_{n=0}^{\infty} \text{Tr} \left[\hat{M}^{-1}(\mathbf{i} \zeta_n) \frac{\partial}{\partial \mathbf{x}_r} \left(\hat{\Pi} \hat{\mathcal{G}}^{(0)}(\mathbf{i} \zeta_n) \hat{\Pi} \right) \right], \quad (31)$$

where $\hat{\Pi}$ is the tangential projection operator defined in Appendix B.2. Now, one notes the identity:

$$\frac{\partial}{\partial \mathbf{x}_r} \left(\hat{\Pi} \hat{\mathcal{G}}^{(0)}(\mathbf{i} \zeta_n) \hat{\Pi} \right) = \frac{\partial \hat{M}}{\partial \mathbf{x}_r}, \quad (32)$$

which is a direct consequence of Equation (A44) since

$$\frac{\partial}{\partial \mathbf{x}_r} \sum_{s=1}^N \hat{\Pi}_s \hat{\mathcal{G}}^{(s)} \hat{\Pi}_s = 0. \quad (33)$$

Plugging Equation (32) into Equation (31), we obtain:

$$F^{(r)} = -k_B T \sum_{n=0}^{\infty} \text{Tr} \left[\hat{M}^{-1}(\mathbf{i} \zeta_n) \frac{\partial}{\partial \mathbf{x}_r} \hat{M}(\mathbf{i} \zeta_n) \right], \quad (34)$$

The r.h.s. of the above equation can be formally expressed as a gradient:

$$F^{(r)} = -\frac{\partial}{\partial \mathbf{x}_r} \tilde{\mathcal{F}}_{\text{bare}}, \quad (35)$$

where $\tilde{\mathcal{F}}_{\text{bare}}$ is the *bare* free energy:

$$\tilde{\mathcal{F}}_{\text{bare}} = k_B T \sum_{n=0}^{\infty} \text{Tr} \log \hat{M}(\mathbf{i} \zeta_n). \quad (36)$$

Similarly to what we found in the T-operator approach, the surface formula of the bare-energy $\tilde{\mathcal{F}}_{\text{bare}}$ is formally divergent. The finite Casimir free energy is obtained by subtracting from $\tilde{\mathcal{F}}_{\text{bare}}$ the limit $\tilde{\mathcal{F}}_{\text{bare}}^{(\infty)}$ of the bare energy when the bodies are taken infinitely apart from each other. From Equation (A44), one sees that in the limit of infinite separations, the operator $\hat{M}$ approaches the limit $\hat{M}_{\infty}$

$$\hat{M}_{\infty} = \sum_{r=1}^N \hat{M}_r, \quad (37)$$

where

$$\hat{M}_r = \hat{\Pi}_r (\hat{\mathcal{G}}^{(r)} + \hat{\mathcal{G}}^{(0)}) \hat{\Pi}_r. \quad (38)$$

Notice that the surface operator $\hat{M}_r$ is localized onto the surface Σ_r of the r -th body. This implies that:

$$\hat{M}_r \hat{M}_s = 0, \quad \text{for } r \neq s. \quad (39)$$

Using Equation (37), we find that $\tilde{\mathcal{F}}_{\text{bare}}^{(\infty)}$ is the formally divergent quantity:

$$\tilde{\mathcal{F}}_{\text{bare}}^{(\infty)} = k_B T \sum_{n=0}^{\infty} \text{Tr} \log \hat{M}_{\infty}(\mathbf{i} \zeta_n) = \sum_{r=1}^N k_B T \sum_{n=0}^{\infty} \text{Tr} \log \hat{M}_r(\mathbf{i} \zeta_n) \equiv \sum_{r=1}^N \tilde{\mathcal{F}}_{\text{self}}^{(r)}. \quad (40)$$

The additive character of $\hat{\mathcal{F}}_{\text{bare}}^{(\infty)}$ allows to interpret $\hat{\mathcal{F}}_{\text{self}}^{(r)}$ as representing the (infinite) the self-energy of the bodies in the surface approach. Upon subtracting Equation (40) from Equation (36), we arrive at the following formula for the Casimir energy:

$$\mathcal{F} = k_B T \sum_{n=0}^{\infty} \log \det \frac{\hat{M}(\mathbf{i} \xi_n)}{\hat{M}_{\infty}(\mathbf{i} \xi_n)}. \quad (41)$$

An easy computation shows that

$$\frac{1}{\hat{M}_{\infty}} \hat{M} = 1 + \sum_{r \neq s} \frac{1}{\hat{M}_r} \hat{\mathcal{G}}_{rs}^{(0)}. \quad (42)$$

where

$$\hat{\mathcal{G}}_{rs}^{(0)} = \hat{\Pi}_r \hat{\mathcal{G}}^{(0)} \hat{\Pi}_s. \quad (43)$$

Substitution of the above formula into Equation (41) results in the following surface formula for the Casimir energy:

$$\mathcal{F} = k_B T \sum_{n=0}^{\infty} \log \det \left[1 + \sum_{r \neq s} \frac{1}{\hat{M}_r} \hat{\mathcal{G}}_{rs}^{(0)} \right]. \quad (44)$$

In the simple case of two bodies, the above formula reduces to:

$$\mathcal{F} = k_B T \sum_{n=0}^{\infty} \text{Tr} \log \left[1 - \frac{1}{\hat{M}_1} \hat{\mathcal{G}}_{12}^{(0)} \frac{1}{\hat{M}_2} \hat{\mathcal{G}}_{21}^{(0)} \right]. \quad (45)$$

The surface formulas for the Casimir energy given in Equations (44) and (45) were not known before and are presented here for the first time. Comparison of Equation (45) with Equation (24) reveals the striking similarity of the T-operator and surface-approach representations of the Casimir energy. Indeed we see that both formulas can be written in the form:

$$\mathcal{F} = k_B T \sum_{n=0}^{\infty} \text{Tr} \log \left[1 - \hat{\mathcal{K}}_1 \hat{\mathcal{G}}^{(0)} \hat{\mathcal{K}}_2 \hat{\mathcal{G}}^{(0)} \right], \quad (46)$$

where $\hat{\mathcal{K}}_r$ is the kernel, which gives the scattering Green's function of body r in isolation:

$$\hat{\Gamma}_r = \hat{\mathcal{G}}^{(0)} \hat{\mathcal{K}}_r \hat{\mathcal{G}}^{(0)}. \quad (47)$$

3. Equivalence of the Surface-Formula with the Scattering Formula for the Casimir Energy

In the previous sections, we have shown that, both in the T -operator and in the surface approaches, the Casimir energy $\mathcal{F}$ of two bodies can be expressed by the general Equation (46). This formula is valid for any shape and relative dispositions of the two bodies, and in particular for two interleaved bodies (see Figure 1a). Now we show that when the two bodies can be enclosed within two non-overlapping spheres (see Figure 1b), Equation (46) is the same as the well-known scattering formula [8–12]:

$$\mathcal{F} = k_B T \sum_{n=0}^{\infty} \text{tr} \log \left[1 - \mathcal{T}^{(1)} \mathcal{U}^{(12)} \mathcal{T}^{(2)} \mathcal{U}^{(21)} \right]. \quad (48)$$

where $\mathcal{T}^{(r)}$ is the scattering matrix of body r (see Equation (A76) for the definition of $\mathcal{T}^{(r)}$), $\mathcal{U}^{(rs)}$ are the translation matrices defined in Equation (A82) and tr denotes a trace over multipole indices.

To prove equivalence of Equation (46) with Equation (48), one starts from the observation that the trace operation in Equation (46) involves evaluating the Green functions $\hat{\mathcal{G}}^{(\alpha\beta;0)}(\mathbf{y}, \mathbf{y}')$ at points $\mathbf{y}$ and $\mathbf{y}'$, one of which (call it $\mathbf{y}_1$) belongs to body 1, while the other

(call it $\mathbf{y}_2$) belongs to body 2. For two bodies that can be separated by non-overlapping spheres, it is warranted that $|\mathbf{y}_1 - \mathbf{X}_1| < |\mathbf{y}_2 - \mathbf{X}_1|$ and $|\mathbf{y}_2 - \mathbf{X}_2| < |\mathbf{y}_1 - \mathbf{X}_2|$, where $\mathbf{X}_1$ and $\mathbf{X}_2$ are the positions of the centers of the spheres $S^{(1)}$ and $S^{(2)}$, respectively, and $d = |\mathbf{X}_2 - \mathbf{X}_1|$ is their distance. This condition satisfied by $\mathbf{y}_1$ and $\mathbf{y}_2$ ensures that it is legitimate to express $\mathcal{G}^{\leftrightarrow(\alpha\beta;0)}(\mathbf{y}_r, \mathbf{y}_s)$ (with $r \neq s = 1, 2$) by the partial-wave expansion (see Appendix D):

$$\begin{aligned} \mathcal{G}^{\leftrightarrow(\alpha\beta;0)}(\mathbf{y}_r, \mathbf{y}_s) &= \lambda(-1)^{s(\beta)} \sum_{plm} \Phi_{plm}^{(\alpha|\text{reg})}(\mathbf{y}_r - \mathbf{X}_r) \otimes \Phi_{pl-m}^{(\beta|\text{out})}(\mathbf{y}_s - \mathbf{X}_r) \\ &= \lambda(-1)^{s(\beta)} \sum_{plm} \sum_{p'l'm'} \Phi_{plm}^{(\alpha|\text{reg})}(\mathbf{y}_r - \mathbf{X}_r) \otimes \mathcal{U}_{pl;p'l'}^{(rs)}(d) \Phi_{p'l'-m}^{(\beta|\text{reg})}(\mathbf{y}_s - \mathbf{X}_s), \quad (49) \end{aligned}$$

where $\Phi_{plm}^{(\text{reg}/\text{out})}(\mathbf{y}_r - \mathbf{X}_r)$ are a basis of regular and outgoing spherical waves with origin at $\mathbf{X}_r$. When the above expansion is substituted into Equation (46) and the trace is evaluated, one finds that $\mathcal{F}$ can be recast in the form:

$$\mathcal{F} = k_B T \sum_{n=0}^{\infty} \text{tr} \log[1 - \mathcal{N}]. \quad (50)$$

where $\mathcal{N}$ is the matrix of elements:

$$\begin{aligned} \mathcal{N}_{plm;p'l'm'} &\equiv \sum_{p''l''} \sum_{p'''l'''} \sum_{p''''l''''} \mathcal{U}_{pl;p''l''}^{(21)}(d) \mathcal{U}_{p'''l''';p''''l''''}^{(12)}(d) \\ &\times \lambda \sum_{\alpha,\mu} (-1)^{s(\alpha)} \int_{V_1} d^3\mathbf{y}_1 \int_{V_1} d^3\mathbf{y}'_1 \Phi_{p''l''-m}^{(\alpha|\text{reg})}(\mathbf{y}_1 - \mathbf{X}_1) \cdot \mathcal{K}_1^{\leftrightarrow(\alpha\mu)}(\mathbf{y}_1, \mathbf{y}'_1) \cdot \Phi_{p'''l'''-m'}^{(\mu|\text{reg})}(\mathbf{y}'_1 - \mathbf{X}_1) \\ &\times \lambda \sum_{\beta,\nu} (-1)^{s(\nu)} \int_{V_2} d^3\mathbf{y}_2 \int_{V_2} d^3\mathbf{y}'_2 \Phi_{p'''l'''-m'''}^{(\nu|\text{reg})}(\mathbf{y}_2 - \mathbf{X}_2) \cdot \mathcal{K}_2^{\leftrightarrow(\nu\beta)}(\mathbf{y}_2, \mathbf{y}'_2) \cdot \Phi_{p''l''-m'}^{(\beta|\text{reg})}(\mathbf{y}'_2 - \mathbf{X}_2). \end{aligned}$$

Recalling the formula Equation (A79) for the scattering matrices $\mathcal{T}^{(r)}$ of the two bodies, we see that $\mathcal{N}$ is the matrix:

$$\mathcal{N} = \mathcal{U}^{(21)}(d) \mathcal{T}^{(1)} \mathcal{U}^{(12)}(d) \mathcal{T}^{(2)}. \quad (51)$$

Upon substituting the above expression into the r.h.s. of Equation (50), and using cyclicity of the trace, we see that Equation (50) indeed coincides with the scattering formula Equation (48).

4. Path Integral Approach

As in the previous sections, we consider again N dielectric bodies occupying the volumes V_r , $r = 1, \dots, N$, bounded by surfaces Σ_r . Their electromagnetic properties are described by the dielectric functions $\epsilon^{(r)}$ and magnetic permeability $\mu^{(r)}$. The bodies are embedded in a homogeneous medium occupying the outside volume of the bodies, V_0 , with dielectric function $\epsilon^{(0)}$ and magnetic permeability $\mu^{(0)}$.

In the Euclidean path integral quantization of the electromagnetic field, the Casimir free energy at finite temperature T can be obtained as

$$\mathcal{F} = -k_B T \sum_{n=0}^{\infty} \log \frac{\mathcal{Z}(\kappa_n)}{\mathcal{Z}_{\infty}(\kappa_n)}, \quad (52)$$

where the sum runs over the Matsubara momenta $\kappa_n = 2\pi n k_B T / \hbar c$, with a weight of $1/2$ for $n = 0$. The partition function $\mathcal{Z}$ is given by a path integral that we shall derive now. The partition function $\mathcal{Z}_{\infty}$ describes the configuration of infinitely separated bodies and subtracts the self-energies of the bodies from the bare free energy. In the following two

sections, we shall derive both a Lagrangian and a Hamiltonian path integral expression of the partition function. In both cases, we employ a fluctuating surface current approach. A path integral approach that is based on bulk currents can be found, e.g., in Reference [10].

4.1. Lagrange Formulation

The action of the electromagnetic field coupled to bound sources $\mathbf{P}_{\text{ind}}$ in the absence of free sources is in general given by

$$S_{\text{EM}} = \int d^3\mathbf{x} \left[\frac{1}{2} \left(\epsilon_{\mathbf{x}} \mathbf{E}^2 - \frac{1}{\mu_{\mathbf{x}}} \mathbf{B}^2 \right) + \mathbf{P}_{\text{ind}} \cdot \mathbf{E} \right]. \quad (53)$$

In the following, we express the action in terms of the gauge field $\mathbf{A}$ choosing the transverse or temporal gauge with $A_0 = 0$. The functional integral will then run over $\mathbf{A}$ only. The electric field is given by $\mathbf{E} = ik\mathbf{A} \rightarrow -\kappa\mathbf{A}$ and the magnetic field by $\mathbf{B} = \nabla \times \mathbf{A}$. Then the action in terms of the induced sources at fixed frequency κ is given by

$$\hat{S}[\mathbf{A}] = -\frac{1}{2} \int_{\mathbb{R}^3} d^3\mathbf{x} \left[\mathbf{A}^2 \epsilon_{\mathbf{x}} \kappa^2 + \frac{1}{\mu_{\mathbf{x}}} (\nabla \times \mathbf{A})^2 \right] - \kappa \sum_{r=1}^N \int_{V_r} d^3\mathbf{x} \mathbf{A} \cdot \mathbf{P}_r, \quad (54)$$

for fluctuations $\mathbf{A}$ of the gauge field, and induced bulk currents $\mathbf{P}_r$ inside the objects. The inverse of the kernel of the quadratic part of this action is given by the Green tensor $G(\mathbf{x}, \mathbf{x}')$, which is defined by

$$\nabla \times \frac{1}{\mu_{\mathbf{x}}} \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA)}(\mathbf{x}, \mathbf{x}') + \epsilon_{\mathbf{x}} \kappa^2 \overset{\leftrightarrow}{\mathcal{G}}^{(AA)}(\mathbf{x}, \mathbf{x}') = 4\pi \mathbf{1} \delta(\mathbf{x} - \mathbf{x}'). \quad (55)$$

For spatially constant ϵ and μ with body r , this yields the free Green's tensor

$$\overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') = \mu_r \left(\mathbf{1} - \frac{1}{\epsilon_r \mu_r \kappa^2} \nabla \otimes \nabla \right) \frac{e^{-\sqrt{\epsilon_r \mu_r \kappa} |\mathbf{x} - \mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|}, \quad (56)$$

which is symmetric, reflecting reciprocity. From the relation between the gauge field $\mathbf{A}$ and the electric field $\mathbf{E}$ follows the relation $-\kappa^2 \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') = \overset{\leftrightarrow}{\mathcal{G}}^{(EE;r)}(\mathbf{x}, \mathbf{x}')$, which allows to compare the results below to those of the stress-tensor-based derivation.

Next, we define the classical solutions $\mathcal{A}_r$ of the vector wave equation in each region V_r , obeying

$$\nabla \times \nabla \times \mathcal{A}_r + \epsilon_r \mu_r \kappa^2 \mathcal{A}_r = -\kappa \mu_r \mathbf{P}_r \quad (57)$$

We use this definition together with the fact that $\mathbf{A}$ has no sources inside V_r , i.e., obeys above wave equation with vanishing right-hand side, to rewrite the source terms of Equation (54) as

$$\begin{aligned} -\kappa \int_{V_r} d^3\mathbf{x} \mathbf{A} \cdot \mathbf{P}_r &= \int_{V_r} d^3\mathbf{x} \mathbf{A} \cdot \left[\frac{1}{\mu_r} \nabla \times \nabla \times \mathcal{A}_r + \epsilon_r \kappa^2 \mathcal{A}_r \right] \\ &= \frac{1}{\mu_r} \int_{V_r} d^3\mathbf{x} [\mathbf{A} \cdot (\nabla \times \nabla \times \mathcal{A}_r) - (\nabla \times \nabla \times \mathbf{A}) \cdot \mathcal{A}_r] \\ &= \frac{1}{\mu_r} \int_{V_r} d^3\mathbf{x} [\nabla \cdot ((\nabla \times \mathcal{A}_r) \times \mathbf{A}) - \nabla \cdot ((\nabla \times \mathbf{A}) \times \mathcal{A}_r)] \\ &= \frac{1}{\mu_r} \int_{\Sigma_r} d^3\mathbf{x} [\mathbf{n}_r \cdot ((\nabla \times \mathcal{A}_r) \times \mathbf{A}) - \mathbf{n}_r \cdot ((\nabla \times \mathbf{A}) \times \mathcal{A}_r)] \\ &= \frac{1}{\mu_r} \int_{\Sigma_r} d^3\mathbf{x} [\mathbf{A} \cdot (\mathbf{n}_r \times (\nabla \times \mathcal{A}_r)) + (\nabla \times \mathbf{A}) \cdot (\mathbf{n}_r \times \mathcal{A}_r)]. \end{aligned} \quad (58)$$

Now we have to consider the electric field $\mathbf{E} = -\kappa\mathbf{A}$ only on the surfaces Σ_r . However, the values of the electric field $\mathbf{E}$ and its curl $\nabla \times \mathbf{E}$ are those when the surface is approached

from the inside, denoted by $\mathbf{E}_-$ and $(\nabla \times \mathbf{E})_-$. It is important to realize that in the above surface integral, $\mathbf{A}$ and $\nabla \times \mathbf{A}$ multiply vectors that are tangential to the surface, and hence only the tangential components of $\mathbf{A}$ and $\nabla \times \mathbf{A}$ contribute to the integral. Hence, we can use the continuity conditions of the tangential components of $\mathbf{E}$ and $\mathbf{H}$,

$$\mathbf{n}_r \times \mathbf{E}_- = \mathbf{n}_r \times \mathbf{E}_+, \quad \frac{1}{\mu_r} \mathbf{n}_r \times (\nabla \times \mathbf{E})_- = \frac{1}{\mu_0} \mathbf{n}_r \times (\nabla \times \mathbf{E})_+, \quad (59)$$

to write the source terms as

$$\begin{aligned} -\kappa \int_{V_r} d^3\mathbf{x} \mathbf{A} \cdot \mathbf{P}_r &= \frac{1}{\mu_r} \int_{\Sigma_r} d^3\mathbf{x} [\mathbf{A}_- \cdot (\mathbf{n}_r \times (\nabla \times \mathcal{A}_r)) + (\nabla \times \mathbf{A})_- \cdot (\mathbf{n}_r \times \mathcal{A}_r)] \\ &= \int_{\Sigma_r} d^3\mathbf{x} \left[\frac{1}{\mu_r} \mathbf{A}_+ \cdot (\mathbf{n}_r \times (\nabla \times \mathcal{A}_r)) + \frac{1}{\mu_0} (\nabla \times \mathbf{A})_+ \cdot (\mathbf{n}_r \times \mathcal{A}_r) \right], \quad (60) \end{aligned}$$

where the first form applies to $\mathbf{A}$ inside the objects and the second form to $\mathbf{A}$ outside the objects. There is another advantage of having expressed the latter integrals in terms of the values of $\mathbf{A}$ and $\nabla \times \mathbf{A}$ when the surfaces are approached from either the outside or the inside of the objects. In the region V_0 , the field $\mathbf{A} \equiv \mathbf{A}_0$ is fully determined by its values on the surfaces Σ_r and the dielectric function ϵ_0 and permeability μ_0 , which are constant across V_0 . When integrating out $\mathbf{A}_0$, in fact, one computes the two-point correlation function of $\mathbf{A}_+$ and $(\nabla \times \mathbf{A})_+$ on the surfaces Σ_r , and hence the behavior of $\mathbf{A}_0$ inside the regions V_r with $r > 0$ is irrelevant. Following the same arguments for $\mathbf{A} \equiv \mathbf{A}_\alpha$ inside the objects, the behavior of $\mathbf{A}_r$ outside of region V_r is irrelevant for computing the correlations of $\mathbf{A}_-$ and $(\nabla \times \mathbf{A})_-$ on the surfaces Σ_r . Hence, we can replace in the action $\hat{S}[\mathbf{A}, \{\mathcal{A}_r\}]$ the spatially dependent ϵ_x by ϵ_0 when the coupling of $\mathbf{A}_0$ to the surface fields $\mathcal{A}_r$ is represented by the second line of Equation (60), and similarly replace ϵ_x by ϵ_r when the coupling of $\mathbf{A}_r$ to the surface fields $\mathcal{A}_r$ is represented by the first line of Equation (60).

That this is justified can also be understood as follows. The field $\mathbf{A}_0$ in region V_0 can be expanded in a basis of functions that obey the wave equation with ϵ_0 . The same can be done for $\mathbf{A}_r$ in the interior of each object, i.e., $\mathbf{A}_r$ can be expanded in a basis of functions that obey the wave equation with ϵ_r in V_r . For each given set of expansion coefficients in V_0 there are corresponding coefficients within each region V_r that are determined by the continuity conditions at the surfaces Σ_r . The functional integral over $\mathbf{A}$ then corresponds to integrating over consistent sets of expansion coefficients that are related by the continuity conditions. The two-point correlations of $\mathbf{A}_+$ and $(\nabla \times \mathbf{A})_+$ on the surfaces Σ_r are then fully determined by the integral over the expansion coefficients of $\mathbf{A}_0$ in V_0 only, and the interior expansion coefficients play no role. Equivalently, the two-point correlations of $\mathbf{A}_-$ and $(\nabla \times \mathbf{A})_-$ on the surfaces Σ_r are then fully determined by the integral over the expansion coefficients of $\mathbf{A}_r$ in V_r only, and now the exterior expansion coefficients are irrelevant. Hence, in the functional integral, the integration of $\mathbf{A}$ can be replaced by $N + 1$ integrations over the fields $\mathbf{A}_r$, $r = 0, \dots, N$, where each $\mathbf{A}_r$ is allowed to extend over unbounded space with the action for a free field in a homogeneous space with ϵ_r , μ_r . However, it is important that the correct of the two possible forms of the surface integral in Equation (60) is used. The multiple counting of degrees of freedom that results from $N + 1$ functional integrations poses no problem since the (formally infinite) factor in the partition function cancels when the Casimir energy is computed from Equation (52).

With this representation, we can write the partition function as a functional integral over $\mathbf{A}$, separately in each region V_r , and the surface fields $\mathcal{A}_r$ on body r , leading to the partition function

$$\mathcal{Z}(\kappa) = \prod_{r=0}^N \int \mathcal{D}\mathbf{A}_r \prod_{r=1}^N \int \mathcal{D}\mathcal{A}_r \exp[-\beta \hat{S}[\{\mathbf{A}_r\}, \{\mathcal{A}_r\}]]. \quad (61)$$

with the action

$$\begin{aligned}\hat{S}[\{\mathbf{A}_r\}, \{\mathcal{A}_r\}] &= -\frac{1}{2} \sum_{r=0}^N \int_{\mathbb{R}^3} d^3\mathbf{x} \left[\mathbf{A}_r^2 \epsilon_r \kappa^2 + \frac{1}{\mu_r} (\nabla \times \mathbf{A}_r)^2 \right] \\ &+ \sum_{r=1}^N \int_{\Sigma_r} d^3\mathbf{x} \left[\frac{1}{\mu_r} \mathbf{A}_0(\mathbf{n}_r \times (\nabla \times \mathcal{A}_r)) + \frac{1}{\mu_0} (\nabla \times \mathbf{A}_0)(\mathbf{n}_r \times \mathcal{A}_r) \right] \\ &+ \sum_{r=1}^N \int_{\Sigma_r} d^3\mathbf{x} \left[\frac{1}{\mu_r} \mathbf{A}_r(\mathbf{n}_r \times (\nabla \times \mathcal{A}_r)) + \frac{1}{\mu_r} (\nabla \times \mathbf{A}_r)(\mathbf{n}_r \times \mathcal{A}_r) \right]\end{aligned}\quad (62)$$

Now, the fluctuations $\mathbf{A}$ can be integrated out easily, noting that the two point correlation function $\langle \mathbf{A}_r(\mathbf{x}) \mathbf{A}_{r'}(\mathbf{x}') \rangle = 0$ for all $r, r' = 0, \dots, N$ with $r \neq r'$, and for equal-region correlations

$$\langle A_{r,j}(\mathbf{x}) A_{r,k}(\mathbf{x}') \rangle = \mathcal{G}_{jk}^{\leftrightarrow(AA;r)}(\mathbf{x}, \mathbf{x}') \quad (63)$$

$$\langle (\nabla \times \mathbf{A})_{r,j}(\mathbf{x}) A_{r,k}(\mathbf{x}') \rangle = \left[\nabla \times \mathcal{G}^{\leftrightarrow(AA;r)} \right]_{jk}(\mathbf{x}, \mathbf{x}') \quad (64)$$

$$\langle A_{r,j}(\mathbf{x}) (\nabla \times \mathbf{A})_{r,k}(\mathbf{x}') \rangle = - \left[\mathcal{G}^{\leftrightarrow(AA;r)} \times \nabla \right]_{jk}(\mathbf{x}, \mathbf{x}') = \left[\nabla \times \mathcal{G}^{\leftrightarrow(AA;r)} \right]_{jk}(\mathbf{x}, \mathbf{x}') \quad (65)$$

$$\begin{aligned}\langle (\nabla \times \mathbf{A})_{r,j}(\mathbf{x}) (\nabla \times \mathbf{A})_{r,k}(\mathbf{x}') \rangle &= - \left[\nabla \times \mathcal{G}^{\leftrightarrow(AA;r)} \times \nabla \right]_{jk}(\mathbf{x}, \mathbf{x}') \\ &= \left[\nabla \times \nabla \times \mathcal{G}^{\leftrightarrow(AA;r)} \right]_{jk}(\mathbf{x}, \mathbf{x}')\end{aligned}\quad (66)$$

where ∇ always acts on the argument $\mathbf{x}$ of $\mathcal{G}^{\leftrightarrow(AA;r)}$ and the notation $\nabla \times \mathcal{G}^{\leftrightarrow(AA;r)}$ means that ∇ acts column-wise on the tensor $\mathcal{G}^{\leftrightarrow(AA;r)}$ whereas $\mathcal{G}^{\leftrightarrow(AA;r)} \times \nabla$ means that ∇ acts row-wise on the tensor $\mathcal{G}^{\leftrightarrow(AA;r)}$. We obtain for the partition function

$$\begin{aligned}\mathcal{Z}(\kappa) &= \prod_{r=1}^N \int \mathcal{D}\mathcal{A}_r \exp \left[-\frac{\beta}{2} \left(\sum_{r=1}^N \int_{\Sigma_r} d^3\mathbf{x} \int_{\Sigma_r} d^3\mathbf{x}' \mathcal{A}_r(\mathbf{x}) L_r(\mathbf{x}, \mathbf{x}') \mathcal{A}_r(\mathbf{x}') \right. \right. \\ &\quad \left. \left. + \sum_{r,r'=1}^N \int_{\Sigma_r} d^3\mathbf{x} \int_{\Sigma_{r'}} d^3\mathbf{x}' \mathcal{A}_r(\mathbf{x}) M_{rr'}(\mathbf{x}, \mathbf{x}') \mathcal{A}_{r'}(\mathbf{x}') \right) \right].\end{aligned}\quad (67)$$

with the kernels

$$\begin{aligned}
 L_r(\mathbf{x}, \mathbf{x}') &= \frac{1}{\mu_r^2} \left[\nabla \times \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}')(\mathbf{n}_r \times \vec{\cdot})(\mathbf{n}'_r \times \vec{\cdot}) \right. \\
 &\quad + \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}')(\mathbf{n}_r \times (\nabla \times \vec{\cdot}))(\mathbf{n}'_r \times \vec{\cdot}) \\
 &\quad + \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}')(\mathbf{n}_r \times \vec{\cdot})(\mathbf{n}'_r \times (\nabla' \times \vec{\cdot})) \\
 &\quad \left. + \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}')(\mathbf{n}_r \times (\nabla \times \vec{\cdot}))(\mathbf{n}'_r \times (\nabla' \times \vec{\cdot})) \right] \\
 M_{rr'}(\mathbf{x}, \mathbf{x}') &= \frac{1}{\mu_0^2} \nabla \times \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;0)}(\mathbf{x}, \mathbf{x}')(\mathbf{n}_r \times \vec{\cdot})(\mathbf{n}'_{r'} \times \vec{\cdot}) \\
 &\quad + \frac{1}{\mu_0 \mu_r} \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;0)}(\mathbf{x}, \mathbf{x}')(\mathbf{n}_r \times (\nabla \times \vec{\cdot}))(\mathbf{n}'_{r'} \times \vec{\cdot}) \\
 &\quad + \frac{1}{\mu_0 \mu_{r'}} \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;0)}(\mathbf{x}, \mathbf{x}')(\mathbf{n}_r \times \vec{\cdot})(\mathbf{n}'_{r'} \times (\nabla' \times \vec{\cdot})) \\
 &\quad + \frac{1}{\mu_r \mu_{r'}} \overset{\leftrightarrow}{\mathcal{G}}^{(AA;0)}(\mathbf{x}, \mathbf{x}')(\mathbf{n}_r \times (\nabla \times \vec{\cdot}))(\mathbf{n}'_{r'} \times (\nabla' \times \vec{\cdot})) \quad (68)
 \end{aligned}$$

where $\overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}')$ is the free Green function of Equation (56), and the arrow over the placeholder $\cdot$ indicates to which side of the kernel M acts. This notation implies that the derivatives are taken before the kernel is evaluated with $\mathbf{x}$ and $\mathbf{x}'$ on the surfaces Σ_r .

4.2. Hamiltonian Formulation

The representation of the partition function in the previous subsection sums over all configurations of the surface fields $\mathcal{A}_r$, and the action depends both on $\mathcal{A}_r$ and the tangential part of its curl, which is functionally dependent on $\mathcal{A}_r$. Hence, the situation is similar to classical mechanics where the Lagrangian depends on the trajectory $q(t)$ and its velocity $\dot{q}(t)$. The Lagrangian path integral runs then over all of path $q(t)$ with $\dot{q}(t)$ determined by the path automatically. To obtain a representation in terms of a space of functions that are defined strictly on the surfaces Σ_r only, it would be useful to be able to integrate over $\mathcal{A}_r$ and its derivatives *independently*. In classical mechanics, this is achieved by Lagrange multipliers that lead to a Legendre transformation of the action to its Hamiltonian form. Here the situation is similar. To see this, it is important to realize that the bilinear form described by L_r is degenerate on the space of functions over which the functional integral runs, i.e., $\int_{\Sigma_r} d^3\mathbf{x} \int_{\Sigma_r} d^3\mathbf{x}' \mathcal{A}(\mathbf{x}) L_r(\mathbf{x}, \mathbf{x}') \mathcal{A}(\mathbf{x}') = 0$ for all $\mathcal{A}(\mathbf{x})$ that are regular solutions of the vector wave equation $\nabla \times \nabla \times \mathcal{A} + \epsilon_r \mu_r \kappa^2 \mathcal{A} = 0$ inside region V_r . With a basis $\{\mathcal{A}_v^{(\text{reg},r)}(\mathbf{x})\}$ for this functional space, the elements of L_r can be expressed as

$$\begin{aligned}
 L_r(v, v') &= \int_{\Sigma_r} d^3\mathbf{x} \int_{\Sigma_r} d^3\mathbf{x}' \mathcal{A}_v^{(\text{reg},r)}(\mathbf{x}) L_r(\mathbf{x}, \mathbf{x}') \mathcal{A}_{v'}^{(\text{reg},r)}(\mathbf{x}') \\
 &= \frac{1}{\mu_r^2} \int_{\Sigma_r} d^3\mathbf{x} \left[(\nabla \times \mathcal{A}_{v'}(\mathbf{x})) (\mathbf{n}_r \times \mathcal{A}_v^{(\text{reg},r)}(\mathbf{x})) + \mathcal{A}_{v'}(\mathbf{x}) (\mathbf{n}_r \times (\nabla \times \mathcal{A}_v^{(\text{reg},r)}(\mathbf{x}))) \right] \\
 &= \frac{1}{\mu_r^2} \int_{V_r} d^3\mathbf{x} \left[\mathcal{A}_{v'}(\mathbf{x}) (\nabla \times \nabla \times \mathcal{A}_v^{(\text{reg},r)}(\mathbf{x})) - \mathcal{A}_v^{(\text{reg},r)}(\mathbf{x}) (\nabla \times \nabla \times \mathcal{A}_{v'}(\mathbf{x})) \right] \\
 &= 0 \quad (69)
 \end{aligned}$$

where we used the relations of Equation (58), and defined

$$\begin{aligned} \mathcal{A}_{v'}(\mathbf{x}) = \int_{\Sigma_r} d^3\mathbf{x}' \left[\nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \left(\mathbf{n}'_r \times \mathcal{A}_{v'}^{(\text{reg},r)}(\mathbf{x}') \right) \right. \\ \left. + \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \left(\mathbf{n}'_r \times \left(\nabla' \times \mathcal{A}_{v'}^{(\text{reg},r)}(\mathbf{x}') \right) \right) \right], \end{aligned} \quad (70)$$

and made use of the fact that $\mathcal{A}_{v'}(\mathbf{x})$ is also a solution of the vector wave equation inside V_r . This implies that the kernel L_r can be ignored in the above functional integral over regular waves $\mathcal{A}_r$ inside the objects.

However, the appearance of the kernel L_r is important in what follows. Let us consider the part of the action $\hat{S}[\{\mathbf{A}_r\}, \{\mathcal{A}_r\}]$ which, after functional integration over $\mathbf{A}_r$, generates the kernel L_r . It is given by

$$\begin{aligned} S_r = -\frac{1}{2} \int_{\mathbb{R}^3} d^3\mathbf{x} \left[\mathbf{A}_r^2 \epsilon_r \kappa^2 + \frac{1}{\mu_r} (\nabla \times \mathbf{A}_r)^2 \right] \\ + \frac{1}{\mu_r} \int_{\Sigma_r} d^3\mathbf{x} \left[\mathbf{A}_r (\mathbf{n}_r \times (\nabla \times \mathcal{A}_r)) + (\nabla \times \mathbf{A}_r) (\mathbf{n}_r \times \mathcal{A}_r) \right]. \end{aligned} \quad (71)$$

The exponential of this action can be written as a functional integral over two new vector fields $\mathbf{K}_r$ and $\mathbf{K}'_r$ that are defined on the surfaces Σ_r and are *tangential* to the surfaces,

$$\begin{aligned} \exp(-\beta S_r) \\ = \mathcal{Z}_r \oint \mathcal{D}\mathbf{K}_r \mathcal{D}\mathbf{K}'_r \exp \left\{ -\frac{\beta}{2} \frac{1}{\mu_r^2} \int_{\Sigma_r} d^3\mathbf{x} \int_{\Sigma_r} d^3\mathbf{x}' \left[\mathbf{K}_r(\mathbf{x}) \cdot \nabla \times \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \cdot \mathbf{K}_r(\mathbf{x}') \right. \right. \\ + \mathbf{K}_r(\mathbf{x}) \cdot \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \cdot \mathbf{K}'_r(\mathbf{x}') + \mathbf{K}'_r(\mathbf{x}) \cdot \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \cdot \mathbf{K}_r(\mathbf{x}') \\ + \left. \left. \mathbf{K}'_r(\mathbf{x}) \cdot \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \cdot \mathbf{K}'_r(\mathbf{x}') \right] \right\} \\ + \frac{1}{\mu_r} \int_{\Sigma_r} d^3\mathbf{x} \left[\mathbf{A}_r \cdot ((\mathbf{n}_r \times (\nabla \times \mathcal{A}_r)) - \mathbf{K}'_r) + (\nabla \times \mathbf{A}_r) \cdot ((\mathbf{n}_r \times \mathcal{A}_r - \mathbf{K}_r)) \right] \end{aligned} \quad (72)$$

where $\mathcal{Z}_r$ is some normalization coefficient, and we have used $\oint \mathcal{D}\mathbf{K}_r \mathcal{D}\mathbf{K}'_r$ to indicate that the functional integral extends only over vector fields that are tangential to the surface Σ_r . This representation shows that $\mathbf{A}_r$ acts as a Lagrange multiplier. Integration over this field removes the imposed constraints between the dependent tangential fields $\mathbf{n}_r \times \mathcal{A}_r$, $\mathbf{n}_r \times (\nabla \times \mathcal{A}_r)$ by replacing them with the independent tangential fields $\mathbf{K}_r$ and $\mathbf{K}'_r$, respectively.

Substituting Equation (72) for each object into the expression for the partition in Equation (61), we obtain with $\underline{\mathbf{K}}_r = (\mathbf{K}_r, \mathbf{K}'_r)$ the partition function

$$\begin{aligned} \mathcal{Z}(\kappa) = \int \mathcal{D}\mathbf{A}_0 \prod_{r=1}^N \oint \mathcal{D}\underline{\mathbf{K}}_r \\ \times \exp[-\beta S_{\text{eff}}[\mathbf{A}_0, \{\underline{\mathbf{K}}_r\}]] \exp \left[-\frac{\beta}{2} \sum_{r=1}^N \int_{\Sigma_r} d^3\mathbf{x} \int_{\Sigma_r} d^3\mathbf{x}' \underline{\mathbf{K}}_r(\mathbf{x}) \hat{L}_r(\mathbf{x}, \mathbf{x}') \underline{\mathbf{K}}_r(\mathbf{x}') \right] \end{aligned} \quad (73)$$

with the kernel from Equation (72),

$$\hat{L}_r(\mathbf{x}, \mathbf{x}') = \frac{1}{\mu_r^2} \begin{pmatrix} \nabla \times \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') & \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \\ \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') & \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \end{pmatrix}, \quad (74)$$

and with the effective action

$$S_{\text{eff}}[\mathbf{A}_0, \{\mathbf{K}_r\}] = \frac{1}{2} \int_{\mathbb{R}^3} d^3\mathbf{x} \left[\mathbf{A}_0^2 \epsilon_0 \kappa^2 + \frac{1}{\mu_0} (\nabla \times \mathbf{A}_0)^2 \right] + \sum_{r=1}^N \int_{\Sigma_r} d^3\mathbf{x} \left[\frac{1}{\mu_r} \mathbf{A}_0 \mathbf{K}_r' + \frac{1}{\mu_0} (\nabla \times \mathbf{A}_0) \mathbf{K}_r \right], \quad (75)$$

where we have integrated out $\mathbf{A}_r$ for $r = 1, \dots, N$, constraining the functional integral over $\mathcal{A}_r$ to be replaced by the substitutions $\mathbf{n}_r \times \mathcal{A}_r \rightarrow \mathbf{K}_r$ and $\mathbf{n}_r \times (\nabla \times \mathcal{A}_r) \rightarrow \mathbf{K}_r'$. Integrating out $\mathbf{A}_0$, finally yields

$$\mathcal{Z}(\kappa) = \prod_{r=1}^N \oint \mathcal{D}\mathbf{K}_r \exp \left[-\frac{\beta}{2} \left(\sum_{r=1}^N \int_{\Sigma_r} d^3\mathbf{x} \int_{\Sigma_r} d^3\mathbf{x}' \mathbf{K}_r(\mathbf{x}) \hat{L}_r(\mathbf{x}, \mathbf{x}') \mathbf{K}_r(\mathbf{x}') + \sum_{r,r'=1}^N \int_{\Sigma_r} d^3\mathbf{x} \int_{\Sigma_{r'}} d^3\mathbf{x}' \mathbf{K}_r(\mathbf{x}) \hat{M}_{rr'}(\mathbf{x}, \mathbf{x}') \mathbf{K}_{r'}(\mathbf{x}') \right) \right], \quad (76)$$

with the additional kernel

$$\hat{M}_{rr'}(\mathbf{x}, \mathbf{x}') = \begin{pmatrix} \frac{1}{\mu_0^2} \nabla \times \nabla \times \hat{\mathcal{G}}^{(AA;0)}(\mathbf{x}, \mathbf{x}') & \frac{1}{\mu_0 \mu_{r'}} \nabla \times \hat{\mathcal{G}}^{(AA;0)}(\mathbf{x}, \mathbf{x}') \\ \frac{1}{\mu_0 \mu_r} \nabla \times \hat{\mathcal{G}}^{(AA;0)}(\mathbf{x}, \mathbf{x}') & \frac{1}{\mu_r \mu_{r'}} \hat{\mathcal{G}}^{(AA;0)}(\mathbf{x}, \mathbf{x}') \end{pmatrix}. \quad (77)$$

It should be noted again that the functional integral in Equation (76) runs over tangential vector fields $\mathbf{K}_r, \mathbf{K}_r'$ defined on the surfaces Σ_r only. The kernels $\hat{L}$ and $\hat{M}$ can be combined into the joint kernel

$$\hat{N}_{rr'} = \hat{L}_r \delta_{rr'} + \hat{M}_{rr'}. \quad (78)$$

Since $\hat{N}$ acts in the path integral only on tangential vectors, the projections of $\hat{N}$ on the tangent space of the surfaces Σ_r have to be taken. Let $\mathbf{t}_{r,1}(\mathbf{x}), \mathbf{t}_{r,2}(\mathbf{x})$ be two tangent vector fields that span the tangent space of Σ_r at $\mathbf{x}$. The 4×4 matrix kernels then become

$$\begin{aligned} \tilde{L}_{r,mn}(\mathbf{x}, \mathbf{x}') &= \mathbf{t}_{r,m}(\mathbf{x}) \hat{L}_r(\mathbf{x}, \mathbf{x}') \mathbf{t}_{r,n}(\mathbf{x}') \\ &= \frac{1}{\mu_r} \begin{pmatrix} (\mathbf{t}_{r,m}(\mathbf{x}) \cdot \nabla)(\mathbf{t}_{r,n}(\mathbf{x}') \cdot \nabla) g_r - \mathbf{t}_{r,m}(\mathbf{x}) \cdot \mathbf{t}_{r,n}(\mathbf{x}') \nabla^2 g_r & -(\mathbf{t}_{r,m}(\mathbf{x}) \times \mathbf{t}_{r,n}(\mathbf{x}')) \cdot \nabla g_r \\ -(\mathbf{t}_{r,m}(\mathbf{x}) \times \mathbf{t}_{r,n}(\mathbf{x}')) \cdot \nabla g_r & -\frac{1}{\epsilon_r \mu_r \kappa^2} (\mathbf{t}_{r,m}(\mathbf{x}) \cdot \nabla)(\mathbf{t}_{r,n}(\mathbf{x}') \cdot \nabla) g_r + \mathbf{t}_{r,m}(\mathbf{x}) \cdot \mathbf{t}_{r,n}(\mathbf{x}') g_r \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \tilde{M}_{rr',mn}(\mathbf{x}, \mathbf{x}') &= \mathbf{t}_{r,m}(\mathbf{x}) \hat{M}_{rr'}(\mathbf{x}, \mathbf{x}') \mathbf{t}_{r',n}(\mathbf{x}') \\ &= \frac{1}{\mu_0} \begin{pmatrix} (\mathbf{t}_{r,m}(\mathbf{x}) \cdot \nabla)(\mathbf{t}_{r,n}(\mathbf{x}') \cdot \nabla) g_0 - \mathbf{t}_{r,m}(\mathbf{x}) \cdot \mathbf{t}_{r,n}(\mathbf{x}') \nabla^2 g_0 & -\frac{\mu_0}{\mu_{r'}} (\mathbf{t}_{r,m}(\mathbf{x}) \times \mathbf{t}_{r,n}(\mathbf{x}')) \cdot \nabla g_0 \\ -\frac{\mu_0}{\mu_r} (\mathbf{t}_{r,m}(\mathbf{x}) \times \mathbf{t}_{r,n}(\mathbf{x}')) \cdot \nabla g_0 & -\frac{\mu_0}{\epsilon_0 \mu_r \mu_{r'} \kappa^2} (\mathbf{t}_{r,m}(\mathbf{x}) \cdot \nabla)(\mathbf{t}_{r,n}(\mathbf{x}') \cdot \nabla) g_0 + \frac{\mu_0^2}{\mu_r \mu_{r'}} \mathbf{t}_{r,m}(\mathbf{x}) \cdot \mathbf{t}_{r,n}(\mathbf{x}') g_0 \end{pmatrix} \end{aligned}$$

which expresses all kernels in terms of tangential and normal derivatives of the scalar Green function $g_r(|\mathbf{x} - \mathbf{x}'|) = e^{-\sqrt{\epsilon_r \mu_r \kappa} |\mathbf{x} - \mathbf{x}'|} / |\mathbf{x} - \mathbf{x}'|$. These expressions simplify when an orthonormal basis $\mathbf{t}_{r,1}(\mathbf{x}), \mathbf{t}_{r,2}(\mathbf{x}), \mathbf{n}_r(\mathbf{x}) = \mathbf{t}_{r,1}(\mathbf{x}) \times \mathbf{t}_{r,2}(\mathbf{x})$ is used. The Casimir free energy is then given by

$$\mathcal{F} = -k_B T \sum_{n=0}^{\infty} \log \det [\hat{N}(\kappa_n) \hat{N}_{\infty}^{-1}(\kappa_n)], \quad (79)$$

where the determinant runs over all indices, i.e., $\mathbf{x}, \mathbf{x}'$ located on the surfaces Σ_r , and $r, r' = 1, \dots, N$. The kernel $\hat{N}_{\infty}$ is obtained from the kernel $\hat{N}$ by taking the distance between all bodies to infinity, i.e., by setting $\hat{M}_{rr'} = 0$ for all $r \neq r'$. In the following we shall again denote the form of the partition function in Equation (67) as Lagrange representation, and

the one of Equation (76) as a Hamiltonian representation. By a simple computation, one can verify that the Hamiltonian representation of the Casimir free energy in Equation (79) is indeed equivalent to the surface formula Equation (44).

5. Application: Derivation of the Lifshitz Theory

As a simple example to demonstrate the practical application of the surface formulations, we consider two dielectric half-spaces, one covering the region $z \leq z_1 = 0$, with the surface Σ_1 and dielectric function ϵ_1 and magnetic permeability μ_1 , and the other covering the region $z \geq z_2 = H$, with the surface Σ_2 and dielectric function ϵ_2 magnetic permeability μ_2 . We shall consider both the Lagrange and Hamiltonian representation in the following.

5.1. Lagrange Representation

We compute the matrix elements of the kernels L and M of Equation (68) in the basis of transverse vector plane waves, given by

$$\mathbf{M}_{1,\mathbf{k}_{\parallel}} = \nabla \times \left(e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel} + p_1 z} \hat{\mathbf{z}} \right) = (-ik_y, ik_x, 0) e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel} + p_1 z} \quad (80)$$

$$\mathbf{N}_{1,\mathbf{k}_{\parallel}} = \frac{1}{\kappa} \nabla \times \nabla \times \left(e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel} + p_1 z} \hat{\mathbf{z}} \right) = \frac{1}{\kappa} (-ik_x p_1, -ik_y p_1, k_{\parallel}^2) e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel} + p_1 z} \quad (81)$$

$$\mathbf{M}_{2,\mathbf{k}_{\parallel}} = \nabla \times \left(e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel} - p_2(z-H)} \hat{\mathbf{z}} \right) = (-ik_y, ik_x, 0) e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel} - p_2(z-H)} \quad (82)$$

$$\mathbf{N}_{2,\mathbf{k}_{\parallel}} = \frac{1}{\kappa} \nabla \times \nabla \times \left(e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel} - p_2(z-H)} \hat{\mathbf{z}} \right) = \frac{1}{\kappa} (ik_x p_2, ik_y p_2, k_{\parallel}^2) e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel} - p_2(z-H)} \quad (83)$$

with $p_r = \sqrt{\epsilon_r \mu_r \kappa^2 + \mathbf{k}_{\parallel}^2}$ and the sign of z is fixed so that the waves are regular inside the half-spaces. Note that we include here a z dependence to be able to compute the curl on the surfaces. For the Green tensor, we use the representation

$$\begin{aligned} \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') &= \int_{\mathbf{q}} e^{i\mathbf{q}(\mathbf{x}-\mathbf{x}')} \frac{1/(\epsilon_r \kappa^2)}{\epsilon_r \mu_r \kappa^2 + \mathbf{q}^2} \begin{pmatrix} \epsilon_r \mu_r \kappa^2 + q_x^2 & q_x q_y & q_x q_z \\ q_y q_x & \epsilon_r \mu_r \kappa^2 + q_y^2 & q_y q_z \\ q_z q_x & q_z q_y & \epsilon_r \mu_r \kappa^2 + q_z^2 \end{pmatrix} \\ &\equiv \int_{\mathbf{q}} e^{i\mathbf{q}(\mathbf{x}-\mathbf{x}')} \frac{\mu_r \tilde{G}_r(\kappa, \mathbf{q})}{\epsilon_r \mu_r \kappa^2 + \mathbf{q}^2}, \end{aligned} \quad (84)$$

which yields after the curl operations

$$\begin{aligned} \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') &= \int_{\mathbf{q}} e^{i\mathbf{q}(\mathbf{x}-\mathbf{x}')} \frac{\mu_r}{\epsilon_r \mu_r \kappa^2 + \mathbf{q}^2} \begin{pmatrix} 0 & -iq_z & iq_y \\ iq_z & 0 & -iq_x \\ -iq_y & iq_x & 0 \end{pmatrix} \\ &\equiv \int_{\mathbf{q}} e^{i\mathbf{q}(\mathbf{x}-\mathbf{x}')} \frac{\mu_r \tilde{G}_r'(\mathbf{q})}{\epsilon_r \mu_r \kappa^2 + \mathbf{q}^2} \end{aligned} \quad (85)$$

$$\begin{aligned} \nabla \times \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') &= \int_{\mathbf{q}} e^{i\mathbf{q}(\mathbf{x}-\mathbf{x}')} \frac{\mu_r}{\epsilon_r \mu_r \kappa^2 + \mathbf{q}^2} \begin{pmatrix} q_y^2 + q_z^2 & -q_x q_y & -q_x q_z \\ -q_x q_y & q_x^2 + q_z^2 & -q_y q_z \\ -q_x q_z & -q_y q_z & q_x^2 + q_y^2 \end{pmatrix} \\ &\equiv \int_{\mathbf{q}} e^{i\mathbf{q}(\mathbf{x}-\mathbf{x}')} \frac{\mu_r \tilde{G}_r''(\mathbf{q})}{\epsilon_r \mu_r \kappa^2 + \mathbf{q}^2}. \end{aligned} \quad (86)$$

We also need the following expressions for the operators that appear in the kernels, acting on the basis functions, which are tangential to the surfaces. On surface Σ_1 we have

$$\hat{\mathbf{z}} \times \mathbf{M}_{1,\mathbf{k}_{\parallel}} = (-ik_x, -ik_y, 0)e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel}} \equiv \mathbf{u}_{m1}e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel}} \quad (87)$$

$$\hat{\mathbf{z}} \times \mathbf{N}_{1,\mathbf{k}_{\parallel}} = \frac{1}{\kappa}(ik_y p_1, -ik_x p_1, 0)e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel}} \equiv \mathbf{u}_{n1}e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel}} \quad (88)$$

$$\hat{\mathbf{z}} \times \nabla \times \mathbf{M}_{1,\mathbf{k}_{\parallel}} = (ik_y p_1, -ik_x p_1, 0)e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel}} \equiv \mathbf{v}_{m1}e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel}} \quad (89)$$

$$\hat{\mathbf{z}} \times \nabla \times \mathbf{N}_{1,\mathbf{k}_{\parallel}} = \frac{1}{\kappa}(ik_x \epsilon_1 \mu_1 \kappa^2, ik_y \epsilon_1 \mu_1 \kappa^2, 0)e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel}} \equiv \mathbf{v}_{n1}e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel}} \quad (90)$$

and similarly on surface Σ_2 ,

$$\hat{\mathbf{z}} \times \mathbf{M}_{2,\mathbf{k}_{\parallel}} = (-ik_x, -ik_y, 0)e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel}} \equiv \mathbf{u}_{m2}e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel}} \quad (91)$$

$$\hat{\mathbf{z}} \times \mathbf{N}_{2,\mathbf{k}_{\parallel}} = \frac{1}{\kappa}(-ik_y p_2, ik_x p_2, 0)e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel}} \equiv \mathbf{u}_{n2}e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel}} \quad (92)$$

$$\hat{\mathbf{z}} \times \nabla \times \mathbf{M}_{2,\mathbf{k}_{\parallel}} = (-ik_y p_2, ik_x p_2, 0)e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel}} \equiv \mathbf{v}_{m2}e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel}} \quad (93)$$

$$\hat{\mathbf{z}} \times \nabla \times \mathbf{N}_{2,\mathbf{k}_{\parallel}} = \frac{1}{\kappa}(ik_x \epsilon_2 \mu_2 \kappa^2, ik_y \epsilon_2 \mu_2 \kappa^2, 0)e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel}} \equiv \mathbf{v}_{n2}e^{-i\mathbf{k}_{\parallel}\mathbf{x}_{\parallel}}. \quad (94)$$

It is straightforward to show that the matrix elements of L_r in the above basis all vanish, as the basis functions are regular solutions of the vector wave equation. This observation is in agreement with the above finding that the kernel L_r is degenerate on the space of those solutions. We proceed with the computation of the elements of kernel M . We find for the case $r = r'$,

$$\begin{aligned} M_{rr}(\mathbf{k}_{\parallel}, \mathbf{k}'_{\parallel}) &= \int_{\Sigma_r} d^3\mathbf{x} \int_{\Sigma_r} d^3\mathbf{x}' \begin{pmatrix} \mathbf{M} \\ \mathbf{N} \end{pmatrix}_{r,\mathbf{k}_{\parallel}}(\mathbf{x}) M_{rr}(\mathbf{x}, \mathbf{x}') \begin{pmatrix} \mathbf{M} \\ \mathbf{N} \end{pmatrix}_{r,\mathbf{k}'_{\parallel}}(\mathbf{x}') \\ &= \delta(\mathbf{k}_{\parallel} + \mathbf{k}'_{\parallel}) \int_{-\infty}^{\infty} \frac{dq_z}{2\pi} \left[\frac{1}{\mu_0^2} \begin{pmatrix} \mathbf{u}_m \\ \mathbf{u}_n \end{pmatrix}_{r,\mathbf{k}_{\parallel}} \tilde{G}_0''(\mathbf{k}_{\parallel}, q_z) \begin{pmatrix} \mathbf{u}_m \\ \mathbf{u}_n \end{pmatrix}_{r,-\mathbf{k}_{\parallel}} \right. \\ &\quad + \frac{1}{\mu_0 \mu_r} \begin{pmatrix} \mathbf{v}_m \\ \mathbf{v}_n \end{pmatrix}_{r,\mathbf{k}_{\parallel}} \tilde{G}_0'(\mathbf{k}_{\parallel}, q_z) \begin{pmatrix} \mathbf{u}_m \\ \mathbf{u}_n \end{pmatrix}_{r,-\mathbf{k}_{\parallel}} + \frac{1}{\mu_0 \mu_r} \begin{pmatrix} \mathbf{u}_m \\ \mathbf{u}_n \end{pmatrix}_{r,\mathbf{k}_{\parallel}} \tilde{G}_0'(\mathbf{k}_{\parallel}, q_z) \begin{pmatrix} \mathbf{v}_m \\ \mathbf{v}_n \end{pmatrix}_{r,-\mathbf{k}_{\parallel}} \\ &\quad \left. + \frac{1}{\mu_r^2} \begin{pmatrix} \mathbf{v}_m \\ \mathbf{v}_n \end{pmatrix}_{r,\mathbf{k}_{\parallel}} \tilde{G}_0(\kappa, \mathbf{k}_{\parallel}, q_z) \begin{pmatrix} \mathbf{v}_m \\ \mathbf{v}_n \end{pmatrix}_{r,-\mathbf{k}_{\parallel}} \right] \frac{\mu_0}{\epsilon_0 \mu_0 \kappa^2 + \mathbf{k}_{\parallel}^2 + q_z^2} e^{iq_z(z-z')} \Big|_{z,z' \rightarrow z_r} \\ &= \delta(\mathbf{k}_{\parallel} + \mathbf{k}'_{\parallel}) \frac{\mu_0 \mathbf{k}_{\parallel}^2}{2p_0} \begin{pmatrix} \frac{\mu_0^2 p_r^2 - \mu_r^2 p_0^2}{(\mu_0 \mu_r)^2} & 0 \\ 0 & -\frac{\epsilon_0^2 p_r^2 - \epsilon_r^2 p_0^2}{\epsilon_0 \mu_0} \end{pmatrix} \\ &\equiv \delta(\mathbf{k}_{\parallel} + \mathbf{k}'_{\parallel}) M_{rr}(\mathbf{k}_{\parallel}) \end{aligned} \quad (95)$$

and for the case $r \neq r'$ we get

$$\begin{aligned}
 M_{rr'}(\mathbf{k}_{\parallel}, \mathbf{k}'_{\parallel}) &= \int_{\Omega_r} d^3\mathbf{x} \int_{\Omega_{r'}} d^3\mathbf{x}' \begin{pmatrix} \mathbf{M} \\ \mathbf{N} \end{pmatrix}_{r, \mathbf{k}_{\parallel}}(\mathbf{x}) M_{rr'}(\mathbf{x}, \mathbf{x}') \begin{pmatrix} \mathbf{M} \\ \mathbf{N} \end{pmatrix}_{r', \mathbf{k}'_{\parallel}}(\mathbf{x}') \\
 &= \delta(\mathbf{k}_{\parallel} + \mathbf{k}'_{\parallel}) \int_{-\infty}^{\infty} \frac{dq_z}{2\pi} \left[\frac{1}{\mu_0^2} \begin{pmatrix} \mathbf{u}_m \\ \mathbf{u}_n \end{pmatrix}_{r, \mathbf{k}_{\parallel}} \tilde{G}_0''(\mathbf{k}_{\parallel}, q_z) \begin{pmatrix} \mathbf{u}_m \\ \mathbf{u}_n \end{pmatrix}_{r', -\mathbf{k}_{\parallel}} \right. \\
 &\quad + \frac{1}{\mu_0 \mu_r} \begin{pmatrix} \mathbf{v}_m \\ \mathbf{v}_n \end{pmatrix}_{r, \mathbf{k}_{\parallel}} \tilde{G}_0'(\mathbf{k}_{\parallel}, q_z) \begin{pmatrix} \mathbf{u}_m \\ \mathbf{u}_n \end{pmatrix}_{r', -\mathbf{k}_{\parallel}} + \frac{1}{\mu_0 \mu_{r'}} \begin{pmatrix} \mathbf{u}_m \\ \mathbf{u}_n \end{pmatrix}_{r, \mathbf{k}_{\parallel}} \tilde{G}_0'(\mathbf{k}_{\parallel}, q_z) \begin{pmatrix} \mathbf{v}_m \\ \mathbf{v}_n \end{pmatrix}_{r', -\mathbf{k}_{\parallel}} \\
 &\quad \left. + \frac{1}{\mu_r \mu_{r'}} \begin{pmatrix} \mathbf{v}_m \\ \mathbf{v}_n \end{pmatrix}_{r, \mathbf{k}_{\parallel}} \tilde{G}_0(\mathbf{k}_{\parallel}, q_z) \begin{pmatrix} \mathbf{v}_m \\ \mathbf{v}_n \end{pmatrix}_{r', -\mathbf{k}_{\parallel}} \right] \frac{\mu_0}{\epsilon_0 \mu_0 \kappa^2 + \mathbf{k}_{\parallel}^2 + q_z^2} e^{iq_z(-1)^r H} \\
 &= \delta(\mathbf{k}_{\parallel} + \mathbf{k}'_{\parallel}) \frac{\mathbf{k}_{\parallel}^2}{2\mu_0 p_0} \begin{pmatrix} \frac{(\mu_0 p_1 - \mu_1 p_0)(\mu_0 p_2 - \mu_2 p_0)}{\mu_1 \mu_2} & 0 \\ 0 & -(\epsilon_0 p_1 - \epsilon_1 p_0)(\epsilon_0 p_2 - \epsilon_2 p_0) \frac{\mu_0}{\epsilon_0} \end{pmatrix} e^{-p_0 H}, \\
 &\equiv \delta(\mathbf{k}_{\parallel} + \mathbf{k}'_{\parallel}) M_{rr'}(\mathbf{k}_{\parallel}),
 \end{aligned} \tag{96}$$

where the sign in $e^{iq_z(-1)^r H}$ determines upon integration over q_z the sign of the terms $\sim iq_z$. The vanishing of the off-diagonal elements reflects the fact that the two polarizations described by the basis functions $\mathbf{M}$ and $\mathbf{N}$ do not couple for planar surfaces. The total kernel M can be written as

$$M(\mathbf{k}_{\parallel}) = \frac{\mathbf{k}_{\parallel}^2}{2p_0} \begin{pmatrix} \frac{\mu_0^2 p_1^2 - \mu_1^2 p_0^2}{\mu_0 \mu_1^2} & 0 & \frac{(\mu_0 p_1 - \mu_1 p_0)(\mu_0 p_2 - \mu_2 p_0)}{\mu_0 \mu_1 \mu_2} e^{-p_0 H} & 0 \\ 0 & -\frac{\epsilon_0^2 p_1^2 - \epsilon_1^2 p_0^2}{\epsilon_0} & 0 & -\frac{(\epsilon_0 p_1 - \epsilon_1 p_0)(\epsilon_0 p_2 - \epsilon_2 p_0)}{\epsilon_0} e^{-p_0 H} \\ \frac{(\mu_0 p_1 - \mu_1 p_0)(\mu_0 p_2 - \mu_2 p_0)}{\mu_0 \mu_1 \mu_2} e^{-p_0 H} & 0 & \frac{\mu_0^2 p_2^2 - \mu_2^2 p_0^2}{(\mu_0 \mu_2^2)} & 0 \\ 0 & -\frac{(\epsilon_0 p_1 - \epsilon_1 p_0)(\epsilon_0 p_2 - \epsilon_2 p_0)}{\epsilon_0} e^{-p_0 H} & 0 & -\frac{\epsilon_0^2 p_2^2 - \epsilon_2^2 p_0^2}{\epsilon_0} \end{pmatrix}$$

The Casimir free energy is given by

$$\mathcal{F} = k_B T \sum_{n=0}^{\infty} \int \frac{d^2 \mathbf{k}_{\parallel}}{(2\pi)^2} \log \det [MM^{-1}(\mathbf{k}_{\parallel})]_{\kappa=\kappa_n} \tag{97}$$

in terms of the determinant of the matrix

$$MM^{-1}(\mathbf{k}_{\parallel}) = \begin{pmatrix} 1 & 0 & \frac{\mu_2}{\mu_1} \frac{\mu_0 p_1 - \mu_1 p_0}{\mu_0 p_2 + \mu_2 p_0} e^{-p_0 H} & 0 \\ 0 & 1 & 0 & \frac{\epsilon_0 p_1 - \epsilon_1 p_0}{\epsilon_0 p_2 + \epsilon_2 p_0} e^{-p_0 H} \\ \frac{\mu_1}{\mu_2} \frac{\mu_0 p_2 - \mu_2 p_0}{\mu_0 p_1 + \mu_1 p_0} e^{-p_0 H} & 0 & 1 & 0 \\ 0 & \frac{\epsilon_0 p_2 - \epsilon_2 p_0}{\epsilon_0 p_1 + \epsilon_1 p_0} e^{-p_0 H} & 0 & 1 \end{pmatrix}$$

which has four dimensions due to two sets of basis functions (polarisations) $\mathbf{M}$ and $\mathbf{N}$ per surface. This yields the final result

$$\begin{aligned}
 \mathcal{F} &= k_B T \sum_{n=0}^{\infty} \int \frac{d\mathbf{k}_{\parallel}}{(2\pi)^2} \log \left[\left(1 - \frac{(\epsilon_0 p_1 - \epsilon_1 p_0)(\epsilon_0 p_2 - \epsilon_2 p_0)}{(\epsilon_0 p_1 + \epsilon_1 p_0)(\epsilon_0 p_2 + \epsilon_2 p_0)} e^{-2p_0 H} \right) \right. \\
 &\quad \times \left. \left(1 - \frac{(\mu_0 p_1 - \mu_1 p_0)(\mu_0 p_2 - \mu_2 p_0)}{(\mu_0 p_1 + \mu_1 p_0)(\mu_0 p_2 + \mu_2 p_0)} e^{-2p_0 H} \right) \right]_{\kappa=\kappa_n}.
 \end{aligned} \tag{98}$$

This result is in agreement with the Lifshitz formula [37].

5.2. Hamiltonian Representation

Now we derive the Lifshitz expression for the free energy of two dielectric half-spaces in the Hamiltonian representation. Since the kernels $\hat{L}$ and $\hat{M}$ of Equations (74) and (77) act

on vector fields that are tangential to the surfaces, we need to compute the matrix elements in a basis of tangential vectors $\mathbf{t}_{r,1}(\mathbf{x})$ and $\mathbf{t}_{r,2}(\mathbf{x})$ that span the tangent space of surface Σ_r at position $\mathbf{x}$. For a planar surface, one can simply set $\mathbf{t}_{r,1}(\mathbf{x}) = \hat{\mathbf{x}}_1$ and $\mathbf{t}_{r,2}(\mathbf{x}) = \hat{\mathbf{x}}_2$. For a given pair of positions $\mathbf{x}, \mathbf{x}'$ on the surface and fixed surface indices r, r' we obtain the following 4×4 dimensional matrices

$$\begin{aligned} \hat{L}_r(\mathbf{x}, \mathbf{x}') &= \frac{1}{\mu_r^2} \begin{pmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \end{pmatrix} \begin{pmatrix} \nabla \times \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') & \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \\ \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') & \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \end{pmatrix} \\ &\equiv \frac{1}{\mu_r^2} \begin{pmatrix} \hat{\mathbf{x}}_1 \cdot \nabla \times \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \cdot \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_1 \cdot \nabla \times \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \cdot \hat{\mathbf{x}}_2 & \hat{\mathbf{x}}_1 \cdot \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \cdot \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_1 \cdot \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \cdot \hat{\mathbf{x}}_2 \\ \hat{\mathbf{x}}_2 \cdot \nabla \times \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \cdot \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 \cdot \nabla \times \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \cdot \hat{\mathbf{x}}_2 & \hat{\mathbf{x}}_2 \cdot \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \cdot \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 \cdot \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \cdot \hat{\mathbf{x}}_2 \\ \hat{\mathbf{x}}_1 \cdot \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \cdot \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_1 \cdot \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \cdot \hat{\mathbf{x}}_2 & \hat{\mathbf{x}}_1 \cdot \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \cdot \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_1 \cdot \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \cdot \hat{\mathbf{x}}_2 \\ \hat{\mathbf{x}}_2 \cdot \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \cdot \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 \cdot \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \cdot \hat{\mathbf{x}}_2 & \hat{\mathbf{x}}_2 \cdot \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \cdot \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 \cdot \overset{\leftrightarrow}{\mathcal{G}}^{(AA;r)}(\mathbf{x}, \mathbf{x}') \cdot \hat{\mathbf{x}}_2 \end{pmatrix} \\ &= \frac{1}{\mu_r} \int_{\mathbf{q}} \frac{e^{i\mathbf{q}_{\parallel}(\mathbf{x}_{\parallel} - \mathbf{x}'_{\parallel})}}{p_r^2 + q_z^2} \begin{pmatrix} q_y^2 + q_z^2 & -q_x q_y & 0 & -iq_z \\ -q_x q_y & q_x^2 + q_z^2 & iq_z & 0 \\ 0 & -iq_z & 1 + \frac{q_x^2}{\epsilon_r \mu_r \kappa^2} & \frac{q_x q_y}{\epsilon_r \mu_r \kappa^2} \\ iq_z & 0 & \frac{q_x q_y}{\epsilon_r \mu_r \kappa^2} & 1 + \frac{q_y^2}{\epsilon_r \mu_r \kappa^2} \end{pmatrix} \\ &= \frac{1}{\mu_r} \int_{\mathbf{q}_{\parallel}} \frac{e^{i\mathbf{q}_{\parallel}(\mathbf{x}_{\parallel} - \mathbf{x}'_{\parallel})}}{2p_r} \begin{pmatrix} q_y^2 - p_r^2 & -q_x q_y & 0 & \mp p_r \\ -q_x q_y & q_x^2 - p_r^2 & \pm p_r & 0 \\ 0 & \mp p_r & 1 + \frac{q_x^2}{\epsilon_r \mu_r \kappa^2} & \frac{q_x q_y}{\epsilon_r \mu_r \kappa^2} \\ \pm p_r & 0 & \frac{q_x q_y}{\epsilon_r \mu_r \kappa^2} & 1 + \frac{q_y^2}{\epsilon_r \mu_r \kappa^2} \end{pmatrix}, \end{aligned}$$

where we set $\mathbf{x} = (\mathbf{x}_{\parallel}, 0)$ and $\mathbf{x}' = (\mathbf{x}_{\parallel}, H)$ for surfaces 1 and 2, respectively. We determined the sign of the terms $\sim iq_z$ from the q_z -integration by the observation that z, z' have to be taken to the surface with $z - z'$ staying inside the object. The upper (lower) sign of p_r refers to $r = 1$ ($r = 2$).

Analogously, for kernel M we get for the case $r = r'$

$$\begin{aligned} \hat{M}_{rr}(\mathbf{x}, \mathbf{x}') &= \begin{pmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \end{pmatrix} \begin{pmatrix} \frac{1}{\mu_0^2} \nabla \times \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;0)}(\mathbf{x}, \mathbf{x}') & \frac{1}{\mu_0 \mu_r} \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;0)}(\mathbf{x}, \mathbf{x}') \\ \frac{1}{\mu_0 \mu_r} \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;0)}(\mathbf{x}, \mathbf{x}') & \frac{1}{\mu_r^2} \overset{\leftrightarrow}{\mathcal{G}}^{(AA;0)}(\mathbf{x}, \mathbf{x}') \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \end{pmatrix} \\ &= \int_{\mathbf{q}} \frac{\mu_0 e^{i\mathbf{q}_{\parallel}(\mathbf{x}_{\parallel} - \mathbf{x}'_{\parallel})}}{p_0^2 + q_z^2} \begin{pmatrix} \frac{q_y^2 + q_z^2}{\mu_0^2} & -\frac{q_x q_y}{\mu_0^2} & 0 & -\frac{iq_z}{\mu_0 \mu_r} \\ -\frac{q_x q_y}{\mu_0^2} & \frac{q_x^2 + q_z^2}{\mu_0^2} & \frac{iq_z}{\mu_0 \mu_r} & 0 \\ 0 & -\frac{iq_z}{\mu_0 \mu_r} & \frac{1}{\mu_r^2} \left(1 + \frac{q_x^2}{\epsilon_0 \mu_0 \kappa^2}\right) & \frac{1}{\mu_r^2} \frac{q_x q_y}{\epsilon_0 \mu_0 \kappa^2} \\ \frac{iq_z}{\mu_0 \mu_r} & 0 & \frac{1}{\mu_r^2} \frac{q_x q_y}{\epsilon_0 \mu_0 \kappa^2} & \frac{1}{\mu_r^2} \left(1 + \frac{q_y^2}{\epsilon_0 \mu_0 \kappa^2}\right) \end{pmatrix} \\ &= \int_{\mathbf{q}_{\parallel}} \frac{\mu_0 e^{i\mathbf{q}_{\parallel}(\mathbf{x}_{\parallel} - \mathbf{x}'_{\parallel})}}{2p_0} \begin{pmatrix} \frac{q_y^2 - p_0^2}{\mu_0^2} & -\frac{q_x q_y}{\mu_0^2} & 0 & \frac{\pm p_0}{\mu_0 \mu_r} \\ -\frac{q_x q_y}{\mu_0^2} & \frac{q_x^2 - p_0^2}{\mu_0^2} & \frac{\mp p_0}{\mu_0 \mu_r} & 0 \\ 0 & \frac{\pm p_0}{\mu_0 \mu_r} & \frac{1}{\mu_r^2} \left(1 + \frac{q_x^2}{\epsilon_0 \mu_0 \kappa^2}\right) & \frac{1}{\mu_r^2} \frac{q_x q_y}{\epsilon_0 \mu_0 \kappa^2} \\ \frac{\mp p_0}{\mu_0 \mu_r} & 0 & \frac{1}{\mu_r^2} \frac{q_x q_y}{\epsilon_0 \mu_0 \kappa^2} & \frac{1}{\mu_r^2} \left(1 + \frac{q_y^2}{\epsilon_0 \mu_0 \kappa^2}\right) \end{pmatrix} \quad (99) \end{aligned}$$

and for $r \neq r'$,

$$\begin{aligned}
 \hat{M}_{rr'}(\mathbf{x}, \mathbf{x}') &= \begin{pmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \end{pmatrix} \begin{pmatrix} \frac{1}{\mu_0^2} \nabla \times \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;0)}(\mathbf{x}, \mathbf{x}') & \frac{1}{\mu_0 \mu_{r'}} \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;0)}(\mathbf{x}, \mathbf{x}') \\ \frac{1}{\mu_0 \mu_r} \nabla \times \overset{\leftrightarrow}{\mathcal{G}}^{(AA;0)}(\mathbf{x}, \mathbf{x}') & \frac{1}{\mu_r \mu_{r'}} \overset{\leftrightarrow}{\mathcal{G}}^{(AA;0)}(\mathbf{x}, \mathbf{x}') \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \end{pmatrix} \\
 &= \int_{\mathbf{q}} \frac{\mu_0 e^{i\mathbf{q}_{\parallel}(\mathbf{x}_{\parallel} - \mathbf{x}'_{\parallel}) \mp i q_z H}}{p_0^2 + q_z^2} \begin{pmatrix} \frac{q_y^2 + q_z^2}{\mu_0^2} & -\frac{q_x q_y}{\mu_0^2} & 0 & -\frac{i q_z}{\mu_0 \mu_{r'}} \\ -\frac{q_x q_y}{\mu_0^2} & \frac{q_x^2 + q_z^2}{\mu_0^2} & \frac{i q_z}{\mu_0 \mu_{r'}} & 0 \\ 0 & -\frac{i q_z}{\mu_0 \mu_r} & \frac{1}{\mu_r \mu_{r'}} \left(1 + \frac{q_x^2}{\epsilon_0 \mu_0 \kappa^2}\right) & \frac{1}{\mu_r \mu_{r'}} \frac{q_x q_y}{\epsilon_0 \mu_0 \kappa^2} \\ \frac{i q_z}{\mu_0 \mu_r} & 0 & \frac{1}{\mu_r \mu_{r'}} \frac{q_x q_y}{\epsilon_0 \mu_0 \kappa^2} & \frac{1}{\mu_r \mu_{r'}} \left(1 + \frac{q_y^2}{\epsilon_0 \mu_0 \kappa^2}\right) \end{pmatrix} \\
 &= \int_{\mathbf{q}_{\parallel}} \frac{\mu_0 e^{i\mathbf{q}_{\parallel}(\mathbf{x}_{\parallel} - \mathbf{x}'_{\parallel})}}{2p_0} \begin{pmatrix} \frac{q_y^2 - p_0^2}{\mu_0^2} & -\frac{q_x q_y}{\mu_0^2} & 0 & \frac{\mp p_0}{\mu_0 \mu_{r'}} \\ -\frac{q_x q_y}{\mu_0^2} & \frac{q_x^2 - p_0^2}{\mu_0^2} & \frac{\pm p_0}{\mu_0 \mu_{r'}} & 0 \\ 0 & \frac{\mp p_0}{\mu_0 \mu_r} & \frac{1}{\mu_r \mu_{r'}} \left(1 + \frac{q_x^2}{\epsilon_0 \mu_0 \kappa^2}\right) & \frac{1}{\mu_r \mu_{r'}} \frac{q_x q_y}{\epsilon_0 \mu_0 \kappa^2} \\ \frac{\pm p_0}{\mu_0 \mu_r} & 0 & \frac{1}{\mu_r \mu_{r'}} \frac{q_x q_y}{\epsilon_0 \mu_0 \kappa^2} & \frac{1}{\mu_r \mu_{r'}} \left(1 + \frac{q_y^2}{\epsilon_0 \mu_0 \kappa^2}\right) \end{pmatrix} e^{-p_0 H} \quad (100)
 \end{aligned}$$

where again the upper (lower) sign everywhere refers to $r = 1$ ($r = 2$). For the kernel $\hat{M}$ we determined the sign of the terms $\sim i q_z$ from the q_z -integration by the observation that z, z' have to be taken to the surface with $z - z'$ staying outside the object. When combining the kernels $\hat{L}$ and $\hat{M}$ into the joint kernel $N_{rr'} = \hat{L}_r \delta_{rr'} + \hat{M}_{rr'}$, it is diagonal in $\mathbf{q}_{\parallel}$ -space with the blocks $N(\mathbf{q}_{\parallel})$ on the diagonal given by the 8×8 matrix shown in Figure 2.

The Casimir free energy is given by

$$\mathcal{F} = k_B T \sum_{n=0}^{\infty} \int \frac{d^2 \mathbf{q}_{\parallel}}{(2\pi)^2} \log \det [N N_{\infty}^{-1}(\mathbf{q}_{\parallel})]_{\kappa=\kappa_n} \quad (101)$$

in terms of the determinant of the above matrix where N_{∞} is the matrix with $H \rightarrow \infty$, i.e., the matrix N with the off-diagonal 4×4 blocks $\sim e^{-p_0 H}$ vanishing. This yields the final result

$$\begin{aligned}
 \mathcal{F} &= k_B T \sum_{n=0}^{\infty} \int \frac{d^2 \mathbf{q}_{\parallel}}{(2\pi)^2} \log \left[\left(1 - \frac{(\epsilon_0 p_1 - \epsilon_1 p_0)(\epsilon_0 p_2 - \epsilon_2 p_0)}{(\epsilon_0 p_1 + \epsilon_1 p_0)(\epsilon_0 p_2 + \epsilon_2 p_0)} e^{-2p_0 H} \right) \right. \\
 &\quad \times \left. \left(1 - \frac{(\mu_0 p_1 - \mu_1 p_0)(\mu_0 p_2 - \mu_2 p_0)}{(\mu_0 p_1 + \mu_1 p_0)(\mu_0 p_2 + \mu_2 p_0)} e^{-2p_0 H} \right) \right], \quad (102)
 \end{aligned}$$

which is again identical to the Lifshitz formula. Note that in the Hamiltonian approach there is no need to express the kernels in a basis for the space of functions that are regular solutions of the wave equation inside the objects. However, the number of fields per surface is now doubled compared to the Lagrangian approach, resembling the situation in quantum mechanics where the Hamiltonian path integrals run over the canonical coordinates q and p independently.

$$\begin{aligned}
N(\mathbf{q}_{||}) = & \begin{pmatrix} \frac{q_y^2 - p_1^2}{2p_1\mu_1} & -\frac{q_x q_y}{2p_1\mu_1} & 0 & -\frac{1}{2\mu_1} \\ -\frac{q_x q_y}{2p_1\mu_1} & \frac{q_x^2 - p_1^2}{2p_1\mu_1} & \frac{1}{2\mu_1} & 0 \\ 0 & -\frac{1}{2\mu_1} & \frac{1}{2p_1\mu_1} \left(1 + \frac{q_x^2}{\epsilon_1\mu_1\kappa^2}\right) & \frac{1}{2p_1\mu_1} \frac{q_x q_y}{\epsilon_1\mu_1\kappa^2} \\ \frac{1}{2\mu_1} & 0 & \frac{1}{2p_1\mu_1} \frac{q_x q_y}{\epsilon_1\mu_1\kappa^2} & \frac{1}{2p_1\mu_1} \left(1 + \frac{q_y^2}{\epsilon_1\mu_1\kappa^2}\right) \end{pmatrix} \\
& + \frac{\mu_0}{2p_0} \begin{pmatrix} \frac{q_y^2 - p_0^2}{\mu_0^2} & -\frac{q_x q_y}{\mu_0^2} & 0 & \frac{p_0}{\mu_0\mu_1} & \frac{q_y^2 - p_0^2}{\mu_0^2} e^{-p_0 H} & -\frac{q_x q_y}{\mu_0^2} e^{-p_0 H} & 0 & -\frac{p_0}{\mu_0\mu_2} e^{-p_0 H} \\ -\frac{q_x q_y}{\mu_0^2} & \frac{q_x^2 - p_0^2}{\mu_0^2} & -\frac{p_0}{\mu_0\mu_1} & 0 & -\frac{q_x q_y}{\mu_0^2} e^{-p_0 H} & \frac{q_x^2 - p_0^2}{\mu_0^2} e^{-p_0 H} & \frac{p_0}{\mu_0\mu_2} e^{-p_0 H} & 0 \\ 0 & \frac{p_0}{\mu_0\mu_1} & \frac{1}{\mu_1^2} \left(1 + \frac{q_x^2}{\epsilon_0\mu_0\kappa^2}\right) & \frac{1}{\mu_1^2} \frac{q_x q_y}{\epsilon_0\mu_0\kappa^2} & 0 & -\frac{p_0}{\mu_0\mu_1} e^{-p_0 H} & \frac{1}{\mu_1\mu_2} \left(1 + \frac{q_x^2}{\epsilon_0\mu_0\kappa^2}\right) e^{-p_0 H} & \frac{1}{\mu_1\mu_2} \frac{q_x q_y}{\epsilon_0\mu_0\kappa^2} e^{-p_0 H} \\ -\frac{p_0}{\mu_0\mu_1} & 0 & \frac{1}{\mu_1^2} \frac{q_x q_y}{\epsilon_0\mu_0\kappa^2} & \frac{1}{\mu_1^2} \left(1 + \frac{q_y^2}{\epsilon_0\mu_0\kappa^2}\right) & \frac{p_0}{\mu_0\mu_1} e^{-p_0 H} & 0 & \frac{1}{\mu_1\mu_2} \frac{q_x q_y}{\epsilon_0\mu_0\kappa^2} e^{-p_0 H} & \frac{1}{\mu_1\mu_2} \left(1 + \frac{q_y^2}{\epsilon_0\mu_0\kappa^2}\right) e^{-p_0 H} \\ \frac{q_y^2 - p_0^2}{\mu_0^2} e^{-p_0 H} & -\frac{q_x q_y}{\mu_0^2} e^{-p_0 H} & 0 & \frac{1}{\mu_1^2} \frac{q_x q_y}{\epsilon_0\mu_0\kappa^2} & \frac{q_y^2 - p_0^2}{\mu_0^2} & -\frac{q_x q_y}{\mu_0^2} & 0 & -\frac{p_0}{\mu_0\mu_2} \\ -\frac{q_x q_y}{\mu_0^2} e^{-p_0 H} & \frac{q_x^2 - p_0^2}{\mu_0^2} e^{-p_0 H} & -\frac{p_0}{\mu_0\mu_1} e^{-p_0 H} & 0 & -\frac{q_x q_y}{\mu_0^2} & \frac{q_x^2 - p_0^2}{\mu_0^2} & \frac{p_0}{\mu_0\mu_2} & 0 \\ 0 & \frac{p_0}{\mu_0\mu_2} e^{-p_0 H} & \frac{1}{\mu_1\mu_2} \left(1 + \frac{q_x^2}{\epsilon_0\mu_0\kappa^2}\right) e^{-p_0 H} & \frac{1}{\mu_1\mu_2} \frac{q_x q_y}{\epsilon_0\mu_0\kappa^2} e^{-p_0 H} & 0 & -\frac{p_0}{\mu_0\mu_2} & \frac{1}{\mu_2^2} \left(1 + \frac{q_x^2}{\epsilon_0\mu_0\kappa^2}\right) & \frac{1}{\mu_2^2} \frac{q_x q_y}{\epsilon_0\mu_0\kappa^2} \\ -\frac{p_0}{\mu_0\mu_2} e^{-p_0 H} & 0 & \frac{1}{\mu_1\mu_2} \frac{q_x q_y}{\epsilon_0\mu_0\kappa^2} e^{-p_0 H} & \frac{1}{\mu_1\mu_2} \left(1 + \frac{q_y^2}{\epsilon_0\mu_0\kappa^2}\right) e^{-p_0 H} & \frac{p_0}{\mu_0\mu_2} & 0 & \frac{1}{\mu_2^2} \frac{q_x q_y}{\epsilon_0\mu_0\kappa^2} & \frac{1}{\mu_2^2} \left(1 + \frac{q_y^2}{\epsilon_0\mu_0\kappa^2}\right) \end{pmatrix} \\
= & \frac{1}{2} \begin{pmatrix} \frac{q_y^2 - p_0^2}{\mu_0 p_0} + \frac{q_y^2 - p_1^2}{\mu_1 p_1} & -\frac{q_x q_y}{\mu_1 p_1} - \frac{q_x q_y}{\mu_0 p_0} & 0 & 0 & \frac{e^{-H p_0} (q_y^2 - p_0^2)}{\mu_0 p_0} & -\frac{e^{-H p_0} q_x q_y}{\mu_0 p_0} & 0 & -\frac{e^{-H p_0}}{\mu_2} \\ -\frac{q_x q_y}{\mu_1 p_1} - \frac{q_x q_y}{\mu_0 p_0} & \frac{q_x^2 - p_0^2}{\mu_0 p_0} + \frac{q_x^2 - p_1^2}{\mu_1 p_1} & 0 & 0 & -\frac{e^{-H p_0} q_x q_y}{\mu_0 p_0} & \frac{e^{-H p_0} (q_x^2 - p_0^2)}{\mu_0 p_0} & 0 & 0 \\ 0 & 0 & \frac{(\epsilon_0 p_0 + \epsilon_1 p_1) q_x^2}{\kappa^2 \mu_1^2 p_0 p_1 \epsilon_0 \epsilon_1} + \frac{\mu_0 p_1 + \mu_1 p_0}{\mu_1^2 p_0 p_1} & \frac{(\epsilon_0 p_0 + \epsilon_1 p_1) q_x q_y}{\kappa^2 \mu_1^2 p_0 p_1 \epsilon_0 \epsilon_1} & 0 & -\frac{e^{-H p_0}}{\mu_1} & \frac{e^{-H p_0} (q_y^2 - p_0^2)}{\kappa^2 \mu_1 \mu_2 p_0 \epsilon_0} & \frac{e^{-H p_0} q_x q_y}{\kappa^2 \mu_1 \mu_2 p_0 \epsilon_0} \\ 0 & 0 & \frac{(\epsilon_0 p_0 + \epsilon_1 p_1) q_y^2}{\kappa^2 \mu_1^2 p_0 p_1 \epsilon_0 \epsilon_1} + \frac{\mu_0 p_1 + \mu_1 p_0}{\mu_1^2 p_0 p_1} & \frac{(\epsilon_0 p_0 + \epsilon_1 p_1) q_x q_y}{\kappa^2 \mu_1^2 p_0 p_1 \epsilon_0 \epsilon_1} & \frac{e^{-H p_0}}{\mu_1} & 0 & \frac{e^{-H p_0} q_x q_y}{\kappa^2 \mu_1 \mu_2 p_0 \epsilon_0} & \frac{e^{-H p_0} (q_y^2 - p_0^2)}{\kappa^2 \mu_1 \mu_2 p_0 \epsilon_0} \\ \frac{e^{-H p_0} (q_y^2 - p_0^2)}{\mu_0 p_0} & -\frac{e^{-H p_0} q_x q_y}{\mu_0 p_0} & 0 & \frac{e^{-H p_0}}{\mu_1} & \frac{q_y^2 - p_0^2}{\mu_0 p_0} + \frac{q_y^2 - p_2^2}{\mu_2 p_2} & -\frac{q_x q_y}{\mu_2 p_2} - \frac{q_x q_y}{\mu_0 p_0} & 0 & 0 \\ -\frac{e^{-H p_0} q_x q_y}{\mu_0 p_0} & \frac{e^{-H p_0} (q_x^2 - p_0^2)}{\mu_0 p_0} & -\frac{e^{-H p_0}}{\mu_1} & 0 & -\frac{q_x q_y}{\mu_2 p_2} - \frac{q_x q_y}{\mu_0 p_0} & \frac{q_x^2 - p_0^2}{\mu_0 p_0} + \frac{q_x^2 - p_2^2}{\mu_2 p_2} & 0 & 0 \\ 0 & \frac{e^{-H p_0}}{\mu_2} & \frac{e^{-H p_0} (q_y^2 - p_0^2)}{\kappa^2 \mu_1 \mu_2 p_0 \epsilon_0} & \frac{e^{-H p_0} q_x q_y}{\kappa^2 \mu_1 \mu_2 p_0 \epsilon_0} & 0 & 0 & \frac{(\epsilon_0 p_0 + \epsilon_2 p_2) q_x^2}{\kappa^2 \mu_2^2 p_0 p_2 \epsilon_0 \epsilon_2} + \frac{\mu_0 p_2 + \mu_2 p_0}{\mu_2^2 p_0 p_2} & \frac{(\epsilon_0 p_0 + \epsilon_2 p_2) q_x q_y}{\kappa^2 \mu_2^2 p_0 p_2 \epsilon_0 \epsilon_2} \\ -\frac{e^{-H p_0}}{\mu_2} & 0 & \frac{e^{-H p_0} q_x q_y}{\kappa^2 \mu_1 \mu_2 p_0 \epsilon_0} & \frac{e^{-H p_0} (q_y^2 - p_0^2)}{\kappa^2 \mu_1 \mu_2 p_0 \epsilon_0} & 0 & 0 & \frac{(\epsilon_0 p_0 + \epsilon_2 p_2) q_y^2}{\kappa^2 \mu_2^2 p_0 p_2 \epsilon_0 \epsilon_2} + \frac{\mu_0 p_2 + \mu_2 p_0}{\mu_2^2 p_0 p_2} & \frac{(\epsilon_0 p_0 + \epsilon_2 p_2) q_y q_x}{\kappa^2 \mu_2^2 p_0 p_2 \epsilon_0 \epsilon_2} \end{pmatrix}
\end{aligned}$$

Figure 2. Matrix $N(\mathbf{q}_{||})$ forming the diagonal blocks of the matrix N .

6. Conclusions

To date, analytical and purely numerical approaches to compute Casimir interactions have been developed independently, and it remained an open question if and how they are related. Analytical methods build on ideas from scattering theory and hence require an expansion of the Green function and bulk or surface operators in terms of special functions that are solutions of the wave equation. Hence, the very existence of such functions and the convergence of the expansion limit these approaches to sufficiently symmetric problems. Purely numerical approaches, such as that developed in [32], can be applied to basically arbitrary geometries but the numerical effort can be extremely high. Hence, it appeared useful to us to study the relation between these approaches in order to develop methods that can serve as semi-numerical approaches that combine the versatility of the purely numerical approaches with the smaller numerical effort of analytical methods. Hence, we have presented in this work a new compact derivation of formulas for the Casimir force, which is based both on bulk and surface operators that also enable analytical evaluations. This we have demonstrated for the simplest case of two dielectric slabs. Further semi-analytical implementations of our approaches are underway.

Our Hamiltonian path integral representation is equivalent to the one derived by Johnson et al. as a purely numerical approach using Lagrange multipliers to enforce the boundary conditions in the path integral. Interestingly, the here-presented derivation of this representation from a Lagrangian path integral demonstrates the relation of this approach to the scattering approach when the T-matrix is defined, as originally by Waterman, by surface integrals of regular solutions of the wave equation over the bodies' surfaces [38]. This shows the close connection of these approaches, motivating further research in the direction of new semi-analytical methods to compute Casimir forces.

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Appendix A. Dyadic Green's Functions and Notations

It is well known [34] that the expectation values of quantum fields in systems in thermal equilibrium can be expressed, via the Matsubara formalism, in terms of the analytic continuation to the positive imaginary axis of the appropriate Green's functions. Along the imaginary axis, response functions have a simpler behavior than along the real frequency axis. For example, it can be shown that the electric and the magnetic permittivities of a magneto-dielectric medium are real-valued and positive definite for imaginary frequencies [39]. In the same manner, retarded Green's functions are real-valued for positive imaginary frequencies. In this Appendix, we briefly review their definitions and main properties.

To be specific, consider a system consisting of a collection of possibly spatially dispersive magneto-dielectric bodies in vacuum, occupying the (non-overlapping) regions of space $V_1, \dots, V_N$. We let V_0 be the vacuum region that separates the bodies. The electro-

magnetic response of the system is described by the following real-valued and positive definite frequency-dependent electric and magnetic permittivities

$$\begin{aligned}\epsilon_{ij}(\mathbf{x}, \mathbf{x}'; i\zeta) &= \sum_{r=0}^N \psi_r(\mathbf{x}) \epsilon_{ij}^{(r)}(\mathbf{x}, \mathbf{x}'; i\zeta) \psi_r(\mathbf{x}') , \\ \mu_{ij}(\mathbf{x}, \mathbf{x}'; i\zeta) &= \sum_{r=0}^N \psi_r(\mathbf{x}) \mu_{ij}^{(r)}(\mathbf{x}, \mathbf{x}'; i\zeta) \psi_r(\mathbf{x}') ,\end{aligned}\quad (\text{A1})$$

where $\epsilon_{ij}^{(0)}(\mathbf{x}, \mathbf{x}'; i\zeta) = \mu_{ij}^{(0)}(\mathbf{x}, \mathbf{x}'; i\zeta) = \delta_{ij} \delta^{(3)}(\mathbf{x} - \mathbf{x}')$ is the permittivity of the vacuum. For local media, $\epsilon_{ij}(\mathbf{x}, \mathbf{x}'; i\zeta) = \epsilon_{ij}(\mathbf{x}) \delta^{(3)}(\mathbf{x} - \mathbf{x}')$, and $\mu_{ij}(\mathbf{x}, \mathbf{x}'; i\zeta) = \mu_{ij}(\mathbf{x}) \delta^{(3)}(\mathbf{x} - \mathbf{x}')$. The principle of microscopic reversibility [40] requires that the permittivities satisfy the symmetry relations:

$$\begin{aligned}\epsilon_{ij}(\mathbf{x}, \mathbf{x}'; i\zeta) &= \epsilon_{ji}(\mathbf{x}', \mathbf{x}; i\zeta) , \\ \mu_{ij}(\mathbf{x}, \mathbf{x}'; i\zeta) &= \mu_{ji}(\mathbf{x}', \mathbf{x}; i\zeta) .\end{aligned}\quad (\text{A2})$$

The imaginary-frequency Maxwell's Equations are:

$$\begin{aligned}-\nabla \times \mathbf{E} &= \kappa(\mathbf{B} + 4\pi \mathbf{M}_{\text{ext}}) , \\ \nabla \times \mathbf{H} &= \kappa(\mathbf{D} + 4\pi \mathbf{P}_{\text{ext}}) ,\end{aligned}\quad (\text{A3})$$

where $\kappa = \zeta/c$, $\mathbf{P}_{\text{ext}}$ and $\mathbf{M}_{\text{ext}}$ are the external polarization and magnetization, and

$$\begin{aligned}D_i(\mathbf{x}) &= \int_{R^3} d^3\mathbf{x}' \epsilon_{ij}(\mathbf{x}, \mathbf{x}'; i\zeta) E_j(\mathbf{x}') , \\ B_i(\mathbf{x}) &= \int_{R^3} d^3\mathbf{x}' \mu_{ij}(\mathbf{x}, \mathbf{x}'; i\zeta) H_j(\mathbf{x}') .\end{aligned}\quad (\text{A4})$$

At the boundary separating media r and s , the tangential components of $\mathbf{E}$ and $\mathbf{H}$ are continuous:

$$\hat{\mathbf{n}} \times [\mathbf{E}^{(r)} - \mathbf{E}^{(s)}] = \hat{\mathbf{n}} \times [\mathbf{H}^{(r)} - \mathbf{H}^{(s)}] = 0 , \quad (\text{A5})$$

where $\hat{\mathbf{n}}$ is the unit normal to the boundary. The dyadic Green's functions $\mathcal{G}_{ij}^{(\alpha\beta)}(\mathbf{x} - \mathbf{x}'; i\zeta_n)$ are defined such that:

$$\begin{aligned}\mathbf{E}(\mathbf{x}) &= \int_{R^3} d^3\mathbf{x}' \left[\overset{\leftrightarrow}{\mathcal{G}}^{(EE)}(\mathbf{x}, \mathbf{x}'; i\zeta) \cdot \mathbf{P}_{\text{ext}}(\mathbf{x}') + \overset{\leftrightarrow}{\mathcal{G}}^{(EH)}(\mathbf{x}, \mathbf{x}'; i\zeta) \cdot \mathbf{M}_{\text{ext}}(\mathbf{x}') \right] , \\ \mathbf{H}(\mathbf{x}) &= \int_{R^3} d^3\mathbf{x}' \left[\overset{\leftrightarrow}{\mathcal{G}}^{(HE)}(\mathbf{x}, \mathbf{x}'; i\zeta) \cdot \mathbf{P}_{\text{ext}}(\mathbf{x}') + \overset{\leftrightarrow}{\mathcal{G}}^{(HH)}(\mathbf{x}, \mathbf{x}'; i\zeta) \cdot \mathbf{M}_{\text{ext}}(\mathbf{x}') \right] .\end{aligned}\quad (\text{A6})$$

It can be shown [33] that the Green's function satisfies the reciprocity relations:

$$\mathcal{G}_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}'; i\zeta) = (-1)^{s(\alpha)+s(\beta)} \mathcal{G}_{ji}^{(\beta\alpha)}(\mathbf{x}', \mathbf{x}; i\zeta) , \quad (\text{A7})$$

where $s(E) = 0$ and $s(H) = 1$.

In a local medium:

$$\begin{aligned}\epsilon_{ij}(\mathbf{x}, \mathbf{x}'; i\zeta) &= \delta^{(3)}(\mathbf{x} - \mathbf{x}') \epsilon_{ij}(\mathbf{x}; i\zeta) , \\ \mu_{ij}(\mathbf{x}, \mathbf{x}'; i\zeta) &= \delta^{(3)}(\mathbf{x} - \mathbf{x}') \mu_{ij}(\mathbf{x}; i\zeta) .\end{aligned}\quad (\text{A8})$$

In a local medium, it is possible to express the three Green's function $\overset{\leftrightarrow}{\mathcal{G}}^{(EH)}(\mathbf{x}, \mathbf{x}'; i\zeta)$, $\overset{\leftrightarrow}{\mathcal{G}}^{(HE)}(\mathbf{x}, \mathbf{x}'; i\zeta)$ and $\overset{\leftrightarrow}{\mathcal{G}}^{(HH)}(\mathbf{x}, \mathbf{x}'; i\zeta)$ in terms of the electric Green's function $\overset{\leftrightarrow}{\mathcal{G}}^{(EE)}(\mathbf{x}, \mathbf{x}'; i\zeta)$, as:

$$\overset{\leftrightarrow}{\mathcal{G}}^{(HE)}(\mathbf{x}, \mathbf{x}'; i\zeta) = -\frac{1}{\kappa} \overset{\leftrightarrow}{\mu}^{-1}(\mathbf{x}, i\zeta) \vec{\nabla} \times \overset{\leftrightarrow}{\mathcal{G}}^{(EE)}(\mathbf{x}, \mathbf{x}'; i\zeta), \quad (\text{A9})$$

$$\overset{\leftrightarrow}{\mathcal{G}}^{(EH)}(\mathbf{x}, \mathbf{x}'; i\zeta) = -\frac{1}{\kappa} \overset{\leftrightarrow}{\mathcal{G}}^{(EE)}(\mathbf{x}, \mathbf{x}'; i\zeta) \times \overset{\leftarrow}{\nabla}' \overset{\leftrightarrow}{\mu}^{-1}(\mathbf{x}', i\zeta), \quad (\text{A10})$$

$$\begin{aligned} \overset{\leftrightarrow}{\mathcal{G}}^{(HH)}(\mathbf{x}, \mathbf{x}'; i\zeta) &= \frac{1}{\kappa^2} \overset{\leftrightarrow}{\mu}^{-1}(\mathbf{x}, i\zeta) \vec{\nabla} \times \overset{\leftrightarrow}{\mathcal{G}}^{(EE)}(\mathbf{x}, \mathbf{x}'; i\zeta) \times \overset{\leftarrow}{\nabla}' \overset{\leftrightarrow}{\mu}^{-1}(\mathbf{x}', i\zeta) \\ &\quad - 4\pi \overset{\leftrightarrow}{\mu}^{-1}(\mathbf{x}, i\zeta) \delta^{(3)}(\mathbf{x} - \mathbf{x}'), \end{aligned} \quad (\text{A11})$$

where $\overset{\leftarrow}{\nabla}'$ denotes derivation w.r.t. $\mathbf{x}'$, acting from right. The electric Green function solves the differential Equation:

$$\vec{\nabla} \times \left[\overset{\leftrightarrow}{\mu}^{-1}(\mathbf{x}, i\zeta) \vec{\nabla} \times \overset{\leftrightarrow}{\mathcal{G}}^{(EE)}(\mathbf{x}, \mathbf{x}'; i\zeta) \right] + \kappa^2 \overset{\leftrightarrow}{\epsilon}(\mathbf{x}, i\zeta) \overset{\leftrightarrow}{\mathcal{G}}^{(EE)}(\mathbf{x}, \mathbf{x}'; i\zeta) = -4\pi \kappa^2 \overset{\leftrightarrow}{1} \delta^{(3)}(\mathbf{x} - \mathbf{x}').$$

In the case of a homogeneous isotropic medium, this Equation can be solved explicitly:

$$\mathcal{G}_{ij}^{(EE)}(\mathbf{x} - \mathbf{x}'; i\zeta) = -\left(\frac{1}{\epsilon} \frac{\partial^2}{\partial x_i \partial x_j} + \mu \kappa^2 \delta_{ij} \right) g_0(\mathbf{x} - \mathbf{x}'), \quad (\text{A12})$$

From Equations (A9)–(A11), one then gets:

$$\mathcal{G}_{ij}^{(HH)}(\mathbf{x} - \mathbf{x}'; i\zeta) = -\left(\frac{1}{\mu} \frac{\partial^2}{\partial x_i \partial x_j} + \epsilon \kappa^2 \delta_{ij} \right) g_0(\mathbf{x} - \mathbf{x}'), \quad (\text{A13})$$

$$\mathcal{G}_{ij}^{(HE)}(\mathbf{x} - \mathbf{x}'; i\zeta) = -\kappa \epsilon_{ijk} \frac{\partial}{\partial x_k} g_0(\mathbf{x} - \mathbf{x}'), \quad (\text{A14})$$

$$\mathcal{G}_{ij}^{(EH)}(\mathbf{x} - \mathbf{x}'; i\zeta) = -\kappa \epsilon_{ijk} \frac{\partial}{\partial x'_k} g_0(\mathbf{x} - \mathbf{x}'), \quad (\text{A15})$$

where

$$g_0(\mathbf{x} - \mathbf{x}') = \frac{e^{-\kappa \sqrt{\epsilon \mu} |\mathbf{x} - \mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|}. \quad (\text{A16})$$

is the Green's function of the scalar Helmholtz Equation in free space:

$$(\nabla^2 - \epsilon \mu \kappa^2) g_0(\mathbf{x} - \mathbf{x}') = -4\pi \delta^{(3)}(\mathbf{x} - \mathbf{x}'), \quad (\text{A17})$$

It is convenient to introduce a compact notation for the fields and the Green's functions. We collect the fields and the sources into six-rows column-vectors:

$$\begin{aligned} \Phi &\equiv \begin{pmatrix} \Phi^{(E)}(\mathbf{x}) \\ \Phi^{(H)}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \mathbf{E}(\mathbf{x}) \\ \mathbf{H}(\mathbf{x}) \end{pmatrix}, \\ \mathcal{D} &\equiv \begin{pmatrix} \mathcal{D}^{(E)}(\mathbf{x}) \\ \mathcal{D}^{(H)}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \mathbf{D}(\mathbf{x}) \\ \mathbf{B}(\mathbf{x}) \end{pmatrix}, \\ K_{\text{ext}} &\equiv \begin{pmatrix} \mathbf{K}_{\text{ext}}^{(E)}(\mathbf{x}) \\ \mathbf{K}_{\text{ext}}^{(H)}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \mathbf{P}_{\text{ext}}(\mathbf{x}) \\ \mathbf{M}_{\text{ext}}(\mathbf{x}) \end{pmatrix}. \end{aligned} \quad (\text{A18})$$

The defining Equations of the Green's functions Equation (A6) can then be interpreted as defining the action of the linear operator $\hat{\mathcal{G}}$ onto the vector K_{ext} :

$$\Phi = \hat{\mathcal{G}} K_{\text{ext}} . \quad (\text{A19})$$

With this notation, the reciprocity relations Equation (A7) are expressed by the operator Equation:

$$\hat{\mathcal{G}}^T = \hat{S} \hat{\mathcal{G}} \hat{S} , \quad (\text{A20})$$

where $\hat{S}$ is the 6×6 diagonal matrix of elements $S_{ij}^{(\alpha\beta)} = (-1)^{s(\alpha)} \delta_{\alpha\beta} \delta_{ij}$. The electric and the magnetic permittivities in Equation (A1) can be both collected into the permittivity operator $\hat{\epsilon}$. Equation (A1) becomes:

$$\hat{\epsilon} = \sum_{r=0}^N \hat{\psi}_r \hat{\epsilon}^{(r)} \hat{\psi}_r , \quad (\text{A21})$$

where the operator $\hat{\psi}_r$ is

$$\hat{\psi}_r \equiv \psi_{r|ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}') = \delta_{\alpha\beta} \delta_{ij} \delta^{(3)}(\mathbf{x} - \mathbf{x}') \psi_r(\mathbf{x}) . \quad (\text{A22})$$

The constitutive Equations (A4) are expressed as:

$$\mathcal{D} = \hat{\epsilon} \Phi . \quad (\text{A23})$$

For later use, it is convenient to define the polarization operator $\hat{\chi}$:

$$\hat{\chi} \equiv \frac{1}{4\pi} (\hat{\epsilon} - \hat{1}) . \quad (\text{A24})$$

Equation (A1) implies that the polarizability χ of a collection of bodies is:

$$\hat{\chi} = \sum_{r=1}^N \hat{\chi}^{(r)} , \quad (\text{A25})$$

where $\hat{\chi}^{(r)}$ is the polarizability of body r . Clearly $\hat{\chi}^{(r)}$ is supported in the volume V_r . This implies:

$$\hat{\chi}^{(r)} \hat{\chi}^{(s)} = 0 , \quad \text{if } r \neq s . \quad (\text{A26})$$

Finally, we define the trace operation Tr of any operator $\hat{A}$:

$$\text{Tr} \hat{A} = \sum_i \sum_{\alpha} \int_V d^3 \mathbf{y} A_{ii}^{(\alpha\alpha)}(\mathbf{y}) . \quad (\text{A27})$$

Appendix B. Derivation of Equation (10)

In this Appendix, we derive the volumic and surface representations, Equation (8) and Equation (9), respectively, for the scattering Green function $\Gamma_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}')$.

Appendix B.1. Volumic Representation

Let $\Phi_{\text{ext}} = \hat{\mathcal{G}}^{(0)} K_{\text{ext}}$ be the external field generated by a certain distribution of external sources K_{ext} . If one or more magneto-dielectric bodies are exposed to the field Φ_{ext} , they get polarized, and we let K_{ind} be the induced polarization field. The polarization field K_{ind} depends linearly on the external field Φ_{ext} :

$$K_{\text{ind}} = \hat{T} \Phi_{\text{ext}} . \quad (\text{A28})$$

The above equation defines the T-operator of the system. The definition of the T-operator implies that the scattered field Φ_{sc} produced by the bodies is:

$$\Phi_{\text{sc}} = \hat{\mathcal{G}}^{(0)} K_{\text{ind}} = \hat{\mathcal{G}}^{(0)} \hat{T} \Phi_{\text{ext}} = \hat{\mathcal{G}}^{(0)} \hat{T} \hat{\mathcal{G}}^{(0)} K_{\text{ext}} . \quad (\text{A29})$$

However, by definition of $\hat{\Gamma}$:

$$\Phi_{\text{sc}} = \hat{\Gamma} K_{\text{ext}} . \quad (\text{A30})$$

A comparison of the above two equations gives:

$$\hat{\Gamma} = \hat{\mathcal{G}}^{(0)} \hat{T} \hat{\mathcal{G}}^{(0)} , \quad (\text{A31})$$

which coincides with Equation (8).

Now we show that the T-operator can be expressed in terms of the polarizability operator $\hat{\chi}$. Indeed, by definition of polarizability it holds:

$$K_{\text{ind}} = \hat{\chi} \Phi_{\text{tot}} , \quad (\text{A32})$$

where the *total* field $\Phi_{\text{tot}} = \Phi_{\text{ext}} + \Phi_{\text{ind}}$ is the sum of the external field and of the induced field $\Phi_{\text{ind}} = \hat{\mathcal{G}}_0 K_{\text{ind}}$ generated by K_{ind} . Therefore:

$$K_{\text{ind}} = \hat{\chi} (\Phi_{\text{ext}} + \Phi_{\text{ind}}) = \hat{\chi} (\Phi_{\text{ext}} + \hat{\mathcal{G}}^{(0)} K_{\text{ind}}) . \quad (\text{A33})$$

Using Equation (A28) to eliminate K_{ind} from the previous equation, we find:

$$\hat{T} \Phi_{\text{ext}} = \hat{\chi} (\hat{1} + \hat{\mathcal{G}}^{(0)} \hat{T}) \Phi_{\text{ext}} . \quad (\text{A34})$$

Since the above equation holds for an arbitrary distribution of sources K_{ext} , i.e., for an arbitrary external field Φ_{ext} , it implies the operator identity:

$$\hat{T} = \hat{\chi} (\hat{1} + \hat{\mathcal{G}}^{(0)} \hat{T}) . \quad (\text{A35})$$

The above relation constitutes an equation of Lippmann–Schwinger form, which determines the T-operator. Its formal solution is:

$$\hat{T} = \frac{1}{1 - \hat{\chi} \hat{\mathcal{G}}^{(0)}} \hat{\chi} = \hat{\chi} \frac{1}{1 - \hat{\mathcal{G}}^{(0)} \hat{\chi}} . \quad (\text{A36})$$

Appendix B.2. Surface Representation

In this section, we derive the surface formula Equation (9) of the scattering Green's function. The existence of this representation is a direct consequence of the equivalence principle [33] of classical electromagnetism. This principle is applicable to bodies constituted by homogeneous and isotropic magneto-dielectric materials whose electric and magnetic permeabilities are of the form³:

$$\begin{aligned} \epsilon_{ij}(\mathbf{x}, \mathbf{x}'; i\zeta) &= \delta_{ij} \delta^{(3)}(\mathbf{x} - \mathbf{x}') \sum_{r=0}^N \psi_r(\mathbf{x}) \epsilon^{(r)}(i\zeta) \psi_r(\mathbf{x}') , \\ \mu_{ij}(\mathbf{x}, \mathbf{x}'; i\zeta) &= \delta_{ij} \delta^{(3)}(\mathbf{x} - \mathbf{x}') \sum_{r=0}^N \psi_r(\mathbf{x}) \mu^{(r)}(i\zeta) \psi_r(\mathbf{x}') . \end{aligned} \quad (\text{A37})$$

The equivalence principle expresses the scattered field in terms of fictitious equivalent currents in a homogeneous medium replacing the scatterer. Let Φ be the electromagnetic field that solves the Maxwell Equations (A3), with the constitutive Equation (A4) and the permittivities ϵ and μ as in Equation (A37), subjected to the boundary conditions Equation (A5) on the surfaces of N bodies in a vacuum. According to the equivalence

principle, there exist *tangential surface polarizations* K_{surf} concentrated on the union Σ of the surfaces Σ_r of the bodies: such that the field Φ can be expressed as

$$\Phi = \hat{\psi}_0(\Phi^{(0+)}) + \hat{\mathcal{G}}^{(0)} K_{\text{surf}} + \sum_{r=1}^N \hat{\psi}_r(\Phi^{(r+)} - \hat{\mathcal{G}}^{(r)} K_{\text{surf}}^{(r)}), \quad (\text{A38})$$

where $K_{\text{surf}}^{(r)} = \hat{\psi}_r K_{\text{surf}}$ is the restriction of the surface current to the surface Σ_r of the r -th body, $\hat{\mathcal{G}}^{(r)}$, $r = 0, 1, \dots, N$ is the Green's functions of an infinite homogeneous medium having constant electric and magnetic permittivities equal to $\epsilon^{(r)}(i\xi)$ and $\mu^{(r)}(i\xi)$, respectively, and

$$\Phi^{(r+)} = \hat{\mathcal{G}}^{(r)} K_{\text{ext}}^{(r)}, \quad (\text{A39})$$

is the external field generated by $K_{\text{ext}}^{(r)}$. The Green's functions $\hat{\mathcal{G}}^{(r)}$ are obtained by replacing ϵ and μ by $\epsilon^{(r)}(i\xi)$ and $\mu^{(r)}(i\xi)$, respectively, in Equations (A12)–(A15). The equivalence principle shows [33] that the surface currents $K_{\text{surf}}^{(r)}$ coincide with the tangential values of the field Φ on the surface Σ_r :

$$K_{\text{surf}}^{(r)} \equiv \begin{pmatrix} \mathbf{K}_{\text{surf}}^{(r|E)}(\mathbf{x}) \\ \mathbf{K}_{\text{surf}}^{(r|H)}(\mathbf{x}) \end{pmatrix} = \delta(F_r(\mathbf{x})) \begin{pmatrix} \hat{\mathbf{n}} \times \hat{\mathbf{H}}(\mathbf{x}) \\ -\hat{\mathbf{n}} \times \hat{\mathbf{E}}(\mathbf{x}) \end{pmatrix}, \quad r = 1, \dots, N \quad (\text{A40})$$

In concrete applications of the surface current method to scattering problems, the surface current K_{surf} is unknown. However, it can be uniquely determined a posteriori by imposing the requirement that the field Φ in Equation (A38) has continuous tangential components across the boundaries of the N bodies. This requirement leads to the following set of Equations:

$$\hat{\Pi}_r(\Phi^{(0+)} + \hat{\mathcal{G}}^{(0)} K_{\text{surf}}) = \hat{\Pi}_r(\Phi^{(r+)} - \hat{\mathcal{G}}^{(r)} K_{\text{surf}}^{(r)}), \quad (\text{A41})$$

where $\hat{\Pi}_r$ is the surface operator, which acts on the field $\Phi(\mathbf{x})$ by projecting it onto the tangential plane to Σ_r at $\mathbf{x}$:

$$\hat{\Pi}_r^{\leftrightarrow}(\mathbf{x}, \mathbf{x}') = \delta^{(3)}(\mathbf{x} - \mathbf{x}') \delta(F_r(\mathbf{x})) \left[\hat{1} - \hat{\mathbf{n}}(\mathbf{x}) \otimes \hat{\mathbf{n}}(\mathbf{x}) \right], \quad r = 1, \dots, N \quad (\text{A42})$$

Equation (A41) can be recast in the form:

$$\hat{M} K_{\text{surf}} = \sum_{r=1}^N \hat{\Pi}_r(\Phi^{(r+)} - \Phi^{(0+)}), \quad (\text{A43})$$

where $\hat{M}$ is the surface operator

$$\hat{M} = \hat{\Pi} \hat{\mathcal{G}}^{(0)} \hat{\Pi} + \sum_{r=1}^N \hat{\Pi}_r \hat{\mathcal{G}}^{(r)} \hat{\Pi}_r, \quad (\text{A44})$$

where $\hat{\Pi} = \sum_{r=1}^N \hat{\Pi}_r$. The operator $\hat{M}$ is invertible [41]. Equation (A43) then determines the unique surface current K_{surf} that solves the boundary-value problem:

$$K_{\text{surf}} = \hat{M}^{-1} \sum_{r=1}^N \hat{\Pi}_r(\Phi^{(r+)} - \Phi^{(0+)}) \quad (\text{A45})$$

Note that according to its definition, $\hat{M}^{-1}$ acts on tangential fields defined on the union Σ of the surfaces Σ_r of the bodies and returns as a result a tangential polarization field K_{surf} defined on Σ . Suppose now that the external sources K_{ext} are localized in the vacuum region V_0 :

$$K_{\text{ext}} = K_{\text{ext}}^{(0)}. \quad (\text{A46})$$

Since $\Phi^{(r+)} = 0$, Equation (A45) now gives:

$$K_{\text{surf}} = -\hat{M}^{-1} \Phi^{(0+)} = -\hat{M}^{-1} \hat{\mathcal{G}}^{(0)} K_{\text{ext}}^{(0)}. \quad (\text{A47})$$

According to Equation (A38), the scattered field $\Phi_{(\text{sc})}^{(0)} = \hat{\psi}_0 \Phi_{(\text{sc})}$ in the vacuum region V_0 is:

$$\Phi_{(\text{sc})}^{(0)} = \hat{\mathcal{G}}^{(0)} K_{\text{surf}} = -\hat{\mathcal{G}}^{(0)} \hat{M}^{-1} \hat{\mathcal{G}}^{(0)} K_{\text{ext}}^{(0)}. \quad (\text{A48})$$

The above equation shows that when $\mathbf{x}$ and $\mathbf{x}'$ belong to V_0 , the scattering Green's function is:

$$\hat{\Gamma} = -\hat{\mathcal{G}}^{(0)} \hat{M}^{-1} \hat{\mathcal{G}}^{(0)}. \quad (\text{A49})$$

We thus see that in the vacuum outside the bodies $\hat{\Gamma}$ has precisely the form of Equation (13), with

$$\hat{\mathcal{K}} = -\hat{M}^{-1}. \quad (\text{A50})$$

Appendix C. Proof of the Force Formula Equation (14)

In this Appendix, we demonstrate the force formula Equation (14). The simple proof presented below holds for smooth kernels $\mathcal{K}_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}')$. Unfortunately, the kernels $\mathcal{K}_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}')$ involved in both the volume and the surface representations of the scattering Green's functions lack the necessary smoothness. Indeed the T -operator in Equation (8) is in general discontinuous on the surfaces of the bodies. As to the surface operator $\hat{M}$ in Equation (9), it is a singular kernel supported on the surfaces of the bodies. Fortunately, it is possible to remedy this difficulty by representing the kernel $\mathcal{K}_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}')$ as the limit of a one-parameter family of smooth real kernels $\tilde{\mathcal{K}}_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}'; \lambda)$ that are supported on an arbitrarily small open neighborhood $\mathcal{O}$ of the domain of the original kernel $\mathcal{K}_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}')$:

$$\mathcal{K}_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}') = \lim_{\lambda \rightarrow 0} \tilde{\mathcal{K}}_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}'; \lambda). \quad (\text{A51})$$

The approximating kernels $\tilde{\mathcal{K}}_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}'; \lambda)$ can be so defined as to satisfy the reciprocity relations Equation (15)⁴:

$$\tilde{\mathcal{K}}_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}'; \lambda) = (-1)^{s(\rho)+s(\sigma)} \tilde{\mathcal{K}}_{lk}^{(\sigma\rho)}(\mathbf{y}', \mathbf{y}; \lambda), \quad (\text{A52})$$

where we recall that $s(\alpha)$ is defined such that $s(E) = 0$ and $s(H) = 1$. It thus holds:

$$\tilde{\Gamma}_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}') = \lim_{\lambda \rightarrow 0} \tilde{\Gamma}_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}'; \lambda), \quad (\text{A53})$$

where we set

$$\tilde{\Gamma}_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}'; \lambda) \equiv \int_{\mathcal{O}} d^3\mathbf{y} \int_{\mathcal{O}} d^3\mathbf{y}' \mathcal{G}_{ik}^{(\alpha\rho;0)}(\mathbf{x} - \mathbf{y}) \tilde{\mathcal{K}}_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}'; \lambda) \mathcal{G}_{jl}^{(\sigma\beta;0)}(\mathbf{y}' - \mathbf{x}'). \quad (\text{A54})$$

For brevity, below we shall omit writing the parameter λ , and we shall consider the $\lambda \rightarrow 0$ limit only at the end. In the first step, we use reciprocity relations satisfied by the free-space Green function to rewrite Equation (A54) as:

$$\tilde{\Gamma}_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}') = \int_{\mathcal{O}} d^3\mathbf{y} \int_{\mathcal{O}} d^3\mathbf{y}' \mathcal{G}_{ik}^{(\alpha\rho;0)}(\mathbf{x} - \mathbf{y}) \tilde{\mathcal{K}}_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}') (-1)^{s(\sigma)+s(\beta)} \mathcal{G}_{jl}^{(\beta\sigma;0)}(\mathbf{x}' - \mathbf{y}'). \quad (\text{A55})$$

Next, one notes that, by virtue of the reciprocity relations Equation (A52), the real kernel $(-1)^{s(\sigma)} \tilde{\mathcal{K}}_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}')$ is symmetric, and therefore, it can be diagonalized:

$$(-1)^{s(\sigma)} \tilde{\mathcal{K}}_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}') = \sum_m w_m K_{m|k}^{(\rho)}(\mathbf{y}) K_{m|l}^{(\sigma)}(\mathbf{y}'), \quad (\text{A56})$$

where the eigenvalues w_m are non-vanishing real numbers, and the eigenvectors $K_{m|k}^{(\rho)}(\mathbf{y})$ are real smooth fields supported in $\mathcal{O}$. By plugging the above expansion into Equation (A55), we see that $\tilde{\Gamma}_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}')$ can be expressed as:

$$\tilde{\Gamma}_{ij}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}') = (-1)^{s(\beta)} \sum_m w_m \Phi_{m|i}^{(\alpha)}(\mathbf{x}) \Phi_{m|j}^{(\beta)}(\mathbf{x}') , \quad (\text{A57})$$

where $\Phi_{m|i}^{(\alpha)}(\mathbf{x})$ are the real fields:

$$\Phi_{m|i}^{(\alpha)}(\mathbf{x}) = \int_{\mathcal{O}} d^3\mathbf{y} \mathcal{G}_{ik}^{(\alpha\rho;0)}(\mathbf{x} - \mathbf{y}) K_{m|k}^{(\rho)}(\mathbf{y}) . \quad (\text{A58})$$

Now one notes that the fields $\Phi_{m|i}^{(\alpha)}(\mathbf{x})$ are well defined in all space, and by construction they satisfy the following euclidean-time Maxwell Equations:

$$-\nabla \times \mathbf{E}_m = \kappa(\mathbf{H}_m + 4\pi \mathbf{M}_m) , \quad (\text{A59})$$

$$\nabla \times \mathbf{H}_m = \kappa(\mathbf{E}_m + 4\pi \mathbf{P}_m) , \quad (\text{A60})$$

where we set $\Phi_m \equiv (\mathbf{E}_m, \mathbf{H}_m)$, $\mathbf{K}_m \equiv (\mathbf{P}_m, \mathbf{M}_m)$ and $\kappa = \xi_n/c$. When the expansion Equation (A57) is substituted into Equation (6), we find for the dyad $\overset{\leftrightarrow}{\Theta}$ the following expression:

$$\overset{\leftrightarrow}{\Theta} = 2k_B T \sum_{n=0}^{\infty} \sum_m w_m \left[\overset{\leftrightarrow}{\Theta}_m^{(EE)} - \overset{\leftrightarrow}{\Theta}_m^{(HH)} \right] , \quad (\text{A61})$$

where

$$\overset{\leftrightarrow}{\Theta}_m^{(EE)} = \frac{1}{4\pi} \left[\mathbf{E}_m \otimes \mathbf{E}_m - \frac{E_m^2}{2} \mathbf{1} \right] , \quad (\text{A62})$$

$$\overset{\leftrightarrow}{\Theta}_m^{(HH)} = \frac{1}{4\pi} \left[\mathbf{H}_m \otimes \mathbf{H}_m - \frac{H_m^2}{2} \mathbf{1} \right] . \quad (\text{A63})$$

It is worth noticing that the minus sign that multiplies $\overset{\leftrightarrow}{\Theta}_m^{(HH)}$ in Equation (A61) is a direct consequence of the factor $(-1)^{s(\beta)}$ in the r.h.s. of Equation (A57). At this point, we use the divergence theorem to convert the surface integral in Equation (7), giving the force $\mathbf{F}^{(r)}$, into a volume integral. That gives:

$$\mathbf{F}^{(r)} = \int_{\mathcal{O}_r} d^3\mathbf{y} \nabla \cdot \overset{\leftrightarrow}{\Theta} = 2k_B T \sum_{n=0}^{\infty} \sum_m w_m \int_{\mathcal{O}_r} d^3\mathbf{y} \nabla \cdot \left[\overset{\leftrightarrow}{\Theta}_m^{(EE)} - \overset{\leftrightarrow}{\Theta}_m^{(HH)} \right] . \quad (\text{A64})$$

where $\mathcal{O}_r$ is the portion of $\mathcal{O}$ included within the surface S_r . By using standard identities of vector calculus, and the Maxwell Equations satisfied by the fields $(\mathbf{E}_m, \mathbf{H}_m)$ one finds:

$$\begin{aligned} \nabla \cdot \left[\overset{\leftrightarrow}{\Theta}_m^{(EE)} - \overset{\leftrightarrow}{\Theta}_m^{(HH)} \right] &= \kappa \mathbf{E}_m \times \mathbf{M}_m - \mathbf{E}_m (\nabla \cdot \mathbf{P}_m) + \kappa \mathbf{H}_m \times \mathbf{P}_m + \mathbf{H}_m (\nabla \cdot \mathbf{M}_m) \\ &= (\nabla \times \mathbf{H}_m) \times \mathbf{M}_m - \mathbf{E}_m (\nabla \cdot \mathbf{P}_m) - (\nabla \times \mathbf{E}_m) \times \mathbf{P}_m + \mathbf{H}_m (\nabla \cdot \mathbf{M}_m) . \end{aligned} \quad (\text{A65})$$

Upon substituting the r.h.s. of the above equation into the r.h.s. of Equation (A64) we get

$$\mathbf{F}_i^{(r)} = 2k_B T \sum_{n=0}^{\infty} \sum_m w_m \int_{\mathcal{O}_r} d^3\mathbf{y} \left[P_{m|j} \partial_i E_{m|j} - M_{m|j} \partial_i H_{m|j} \right] . \quad (\text{A66})$$

If the fields $\mathbf{E}_m$ and $\mathbf{H}_m$ are expressed in terms of the sources $\mathbf{P}_m$ and $\mathbf{M}_m$, via Equation (A58) one can recast the r.h.s. of the above equation in the form:

$$\mathbf{F}_i^{(r)} = 2k_B T \sum_{n=0}^{\infty} \sum_m w_m \int_{\mathcal{O}_r} d^3\mathbf{y} \int_{\mathcal{O}} d^3\mathbf{y}' (-1)^{s(\alpha)} K_{m|j}^{(\alpha)}(\mathbf{y}) \frac{\partial}{\partial y_i} \mathcal{G}_{jk}^{(\alpha\beta;0)}(\mathbf{y} - \mathbf{y}') K_{m|k}^{(\beta)}(\mathbf{y}')$$

$$= 2k_B T \sum_{n=0}^{\infty} \int_{\mathcal{O}_r} d^3 \mathbf{y} \int_{\mathcal{O}} d^3 \mathbf{y}' \tilde{K}_{kj}^{(\beta\alpha)}(\mathbf{y}', \mathbf{y}; \lambda) \frac{\partial}{\partial y_i} \mathcal{G}_{jk}^{(\alpha\beta;0)}(\mathbf{y} - \mathbf{y}') , \quad (\text{A67})$$

where in the last passage, we made use of Equation (A56), and restored the dependence of $\tilde{K}$ on the parameter λ . By taking the limit $\lambda \rightarrow 0$ of the r.h.s., we see that Equation (A67) reproduces the formula for force in Equation (14).

Appendix D. Partial Wave Expansion and the Scattering Matrix

The Green's function $\mathcal{G}^{\leftrightarrow(\alpha\beta;0)}(\mathbf{x}, \mathbf{x}')$ admits an expansion in partial waves in any co-ordinate system in which the vector Helmholtz Equation is separable. When spherical coordinates are used, the partial-wave expansion takes the form of an infinite series over spherical multipoles:

$$\begin{aligned} \mathcal{G}^{\leftrightarrow(\alpha\beta;0)}(\mathbf{x}, \mathbf{x}') &= (-1)^{s(\beta)} \lambda \sum_{ilm} \left[\theta(|\mathbf{x}| - |\mathbf{x}'|) \Phi_{ilm}^{(\alpha|\text{out})}(\mathbf{x}) \otimes \Phi_{il-m}^{(\beta|\text{reg})}(\mathbf{x}') \right. \\ &\quad \left. + \theta(|\mathbf{x}'| - |\mathbf{x}|) \Phi_{ilm}^{(\alpha|\text{reg})}(\mathbf{x}) \otimes \Phi_{il-m}^{(\beta|\text{out})}(\mathbf{x}') \right] , \end{aligned} \quad (\text{A68})$$

where $\lambda = -4\pi \kappa^3$, $i = M, N$ and (lm) are, respectively, polarization and multipole indices. The partial waves $\Phi_{ilm}^{(\alpha|\text{out}/\text{reg})}(\mathbf{x})$ are defined as follows. For the electric field, they are: [10]:

$$\Phi_{Mlm}^{(E|\text{out}/\text{reg})}(\mathbf{x}) = \frac{1}{\sqrt{l(l+1)}} \nabla \times \mathbf{x} \phi_{lm}^{(\text{out}/\text{reg})}(\mathbf{x}, \kappa) , \quad (\text{A69})$$

$$\Phi_{Nlm}^{(E|\text{out}/\text{reg})}(\mathbf{x}) = \frac{i}{\kappa} \nabla \times \Phi_{Mlm}^{(E|\text{out}/\text{reg})}(\mathbf{x}, \kappa) , \quad (\text{A70})$$

where $\phi_{lm}^{(\text{reg}/\text{out})}(\mathbf{x}, \kappa)$ are the following regular and the outgoing spherical waves:

$$\phi_{lm}^{(\text{reg})}(\mathbf{x}, \kappa) = i_l(\kappa|\mathbf{x}|) Y_{lm}(\hat{\mathbf{x}}) , \quad (\text{A71})$$

$$\phi_{lm}^{(\text{out})}(\mathbf{x}, \kappa) = k_l(\kappa|\mathbf{x}|) Y_{lm}(\hat{\mathbf{x}}) . \quad (\text{A72})$$

Here, $i_l(z) = \sqrt{\pi/2z} I_{l+1/2}(z)$ is the modified spherical Bessel function of the first kind, and $k_l(z) = \sqrt{\pi/2z} K_{l+1/2}(z)$ is the modified spherical Bessel function of the third kind. Notice the relation:

$$\Phi_{Mlm}^{(E|\text{out}/\text{reg})}(\mathbf{x}) = \frac{i}{\kappa} \nabla \times \Phi_{Nlm}^{(E|\text{out}/\text{reg})}(\mathbf{x}) . \quad (\text{A73})$$

According to Maxwell Equations, the magnetic partial waves $\Phi_{ilm}^{(H|\text{reg}/\text{out})}(\mathbf{x})$ are obtained by taking a curl of the electric waves:

$$\Phi_{ilm}^{(H|\text{reg}/\text{out})}(\mathbf{x}) \equiv -\frac{1}{\kappa} \nabla \times \Phi_{ilm}^{(E|\text{reg}/\text{out})}(\mathbf{x}) . \quad (\text{A74})$$

However, using the two relations Equations (A70) and (A73), one finds:

$$\Phi_{ilm}^{(H|\text{reg}/\text{out})}(\mathbf{x}) = i \Phi_{\tau(i)lm}^{(E|\text{reg}/\text{out})}(\mathbf{x}) . \quad (\text{A75})$$

where $\tau(M) = N$, $\tau(N) = M$.

Next, we define the scattering matrix $\mathcal{T}^{(r)}$ of an isolated object r placed in vacuum. Let $\mathcal{G}_r^{\leftrightarrow(\alpha\beta)}(\mathbf{x}, \mathbf{x}')$ be the scattering part of the Green's function of body r in isolation. The scattering matrix $\mathcal{T}^{(r)}$ is defined such that at all points $(\mathbf{x}, \mathbf{x}')$ lying outside a sphere $S^{(r)}$

containing the body in its interior and centered at the point $\mathbf{X}_r$, $\overset{\leftrightarrow}{\Gamma}_r^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}')$ has the partial wave expansion:

$$\overset{\leftrightarrow}{\Gamma}_r^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}') = (-1)^{s(\beta)} \lambda \sum_{plm} \sum_{p'l'm'} \Phi_{plm}^{(\alpha|\text{out})}(\mathbf{x} - \mathbf{X}_r) \otimes \Phi_{p'l'm'}^{(\beta|\text{out})}(\mathbf{x}' - \mathbf{X}_r) \mathcal{T}_{plm,p'l'm'}^{(r)}, \quad (\text{A76})$$

The fact that all Green's functions $\overset{\leftrightarrow}{\Gamma}_r^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}')$ can be expressed in terms of a single scattering matrix $\mathcal{T}^{(r)}$ follows from Equation (A74), together with the following identities satisfied by $\overset{\leftrightarrow}{\Gamma}_r^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}')$ at all points in the vacuum outside the body, which in turn result from the identities of Equations (A9)–(A11) satisfied by the full Green's functions $\overset{\leftrightarrow}{G}^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}')$ in the presence of body r :

$$\begin{aligned} \overset{\leftrightarrow}{\Gamma}_r^{(HE)}(\mathbf{x}, \mathbf{x}'; i\zeta) &= -\frac{1}{\kappa} \vec{\nabla} \times \overset{\leftrightarrow}{\Gamma}_r^{(EE)}(\mathbf{x}, \mathbf{x}'; i\zeta), \\ \overset{\leftrightarrow}{\Gamma}_r^{(EH)}(\mathbf{x}, \mathbf{x}'; i\zeta) &= -\frac{1}{\kappa} \overset{\leftrightarrow}{\Gamma}_r^{(EE)}(\mathbf{x}, \mathbf{x}'; i\zeta) \times \vec{\nabla}', \\ \overset{\leftrightarrow}{\Gamma}_r^{(HH)}(\mathbf{x}, \mathbf{x}'; i\zeta) &= \frac{1}{\kappa^2} \vec{\nabla} \times \overset{\leftrightarrow}{\Gamma}_r^{(EE)}(\mathbf{x}, \mathbf{x}'; i\zeta) \times \vec{\nabla}'. \end{aligned} \quad (\text{A77})$$

Now we prove that the scattering matrix is given by the matrix elements of the operator $\hat{\mathcal{K}}_r$ of body r defined in Equation (47), taken between two regular partial waves. To see this, one notes that for any two points $(\mathbf{x}, \mathbf{x}')$ outside the sphere $S^{(r)}$, it is legitimate to replace $\hat{\mathcal{G}}^{(0)}$ in Equation (47) by its partial wave expansion Equation (A68). This substitution results in the expansion:

$$\begin{aligned} \overset{\leftrightarrow}{\Gamma}_r^{(\alpha\beta)}(\mathbf{x}, \mathbf{x}') &= -(-1)^{s(\beta)} \lambda^2 \sum_{plm} \sum_{p'l'm'} \Phi_{plm}^{(\alpha|\text{out})}(\mathbf{x} - \mathbf{X}_r) \otimes \Phi_{p'l'm'}^{(\beta|\text{out})}(\mathbf{x}' - \mathbf{X}_r) \\ &\times \sum_{\mu\nu} (-1)^{s(\mu)} \int_{V_r} d^3\mathbf{y} \int_{V_r} d^3\mathbf{y}' \Phi_{pl-m}^{(\mu|\text{reg})}(\mathbf{y} - \mathbf{X}_r) \cdot \overset{\leftrightarrow}{\mathcal{K}}_r^{(\mu\nu)}(\mathbf{y}, \mathbf{y}') \cdot \Phi_{p'l'm'}^{(\nu|\text{reg})}(\mathbf{y}' - \mathbf{X}_r). \end{aligned} \quad (\text{A78})$$

A comparison of Equation (A76) with Equation (A78) gives the desired formula:

$$\begin{aligned} \mathcal{T}_{plm,p'l'm'}^{(r)} &= \sum_{\mu\nu} (-1)^{s(\mu)} \lambda \int_{V_r} d^3\mathbf{y} \int_{V_r} d^3\mathbf{y}' \\ &\times \Phi_{pl-m}^{(\mu|\text{reg})}(\mathbf{y} - \mathbf{X}_r) \cdot \overset{\leftrightarrow}{\mathcal{K}}_r^{(\mu\nu)}(\mathbf{y}, \mathbf{y}') \cdot \Phi_{p'l'm'}^{(\nu|\text{reg})}(\mathbf{y}' - \mathbf{X}_r), \end{aligned} \quad (\text{A79})$$

which indeed shows that $\mathcal{T}_{plm,p'l'm'}^{(r)}$ is the matrix element of the operator $\hat{\mathcal{K}}_r$ between two partial waves. Note that in the T -operator approach, $\mathcal{T}_{plm,p'l'm'}^{(r)}$ is expressed by an integral of the body's T -operator $\hat{T}_r$ over the body's volume V_r , while in the surface approach it is an integral of the surface operator $-\hat{M}_r^{-1}$ over the boundary Σ_r of the body.

Now we define the translation matrices $\mathcal{U}^{(ij)}(d)$. Consider two points $\mathbf{X}_1$ and $\mathbf{X}_2$ in space that differ by a displacement of magnitude d along the z -axis:

$$\mathbf{X}_2 - \mathbf{X}_1 = d \hat{\mathbf{z}}. \quad (\text{A80})$$

The translation matrix $\mathcal{U}^{(21)}(d)$ is defined such that at all points $\mathbf{x}$ whose distances from $\mathbf{X}_1$ and $\mathbf{X}_2$ are both smaller than d (i.e., $|\mathbf{x} - \mathbf{X}_2| < d$ and $|\mathbf{x} - \mathbf{X}_1| < d$) it holds:

$$\Phi_{p'l'm'}^{(E|out)}(\mathbf{x} - \mathbf{X}_2) = \sum_{pl} \mathcal{U}_{p'l';pl}^{(21)}(d) \Phi_{plm}^{(E|reg)}(\mathbf{x} - \mathbf{X}_1), \quad (A81)$$

$$\Phi_{p'l'm'}^{(E|out)}(\mathbf{x} - \mathbf{X}_1) = \sum_{pl} \mathcal{U}_{p'l';pl}^{(12)}(d) \Phi_{plm}^{(E|reg)}(\mathbf{x} - \mathbf{X}_2), \quad (A82)$$

It can be shown that the relation holds:

$$\mathcal{U}^{(21)}(d) = \mathcal{U}^{(12)\dagger}(d). \quad (A83)$$

By applying the operator $\mathcal{L} = -(\mathbf{i}/\kappa) \nabla \times$ to both members of the Equation (A82), and recalling the definition of the magnetic partial waves Equation (A74) one finds:

$$\Phi_{p'l'm'}^{(H|out)}(\mathbf{x} - \mathbf{X}_2) = \sum_{pl} \mathcal{U}_{p'l';pl}^{(21)}(d) \Phi_{plm}^{(H|reg)}(\mathbf{x} - \mathbf{X}_1), \quad (A84)$$

$$\Phi_{p'l'm'}^{(H|out)}(\mathbf{x} - \mathbf{X}_1) = \sum_{pl} \mathcal{U}_{p'l';pl}^{(12)}(d) \Phi_{plm}^{(H|reg)}(\mathbf{x} - \mathbf{X}_2) \quad (A85)$$

which shows that the translation matrices of the magnetic partial waves coincide with those for the electric partial waves.

Notes

- ¹ This restriction is not so severe in practice, since the vast majority of Casimir experiments use test bodies that can be modelled in this way.
- ² A representation analogous to Equation (9) also exists when one or both points belong to the regions occupied by bodies, but we shall not display it since the surface integral expressing the force in Equation (7) involves only points $\mathbf{x}$ in the vacuum region.
- ³ Bodies that are only piecewise homogeneous can also be considered by a slight generalization of the homogeneous case.
- ⁴ For example, one can set $\tilde{\mathcal{K}}_{kl}^{(\rho\sigma)}(\mathbf{y}, \mathbf{y}'; \lambda) = \lambda^{-6} \int_V d^3\mathbf{x} \int_V d^3\mathbf{x}' f((\mathbf{y} - \mathbf{x})/\lambda) f((\mathbf{y}' - \mathbf{x}')/\lambda) \mathcal{K}_{kl}^{(\rho\sigma)}(\mathbf{x}, \mathbf{x}')$, where $f(\mathbf{x})$ is any rotationally invariant, smooth non-negative function, supported in a ball of unit radius centered in the origin, such that $\int d^3\mathbf{x} f(\mathbf{x}) = 1$.

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