



Article On the Coupling of Generalized Proca Fields to Degenerate Scalar-Tensor Theories †

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Abstract: We prove that vector fields described by the generalized Proca class of theories do not admit consistent coupling with a gravitational sector defined by a scalar–tensor theory of the degenerate type. Under the assumption that there exists a frame in which the Proca field interacts with gravity only through the metric tensor, our analysis shows that at least one of the constraints associated with the degeneracy of the scalar–tensor sector is inevitably lost whenever the vector theory includes coupling with the Christoffel connection.

Keywords: dark energy; alternative theories of gravity; scalar–tensor theories; generalized proca theories; Hamiltonian analysis



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1. Introduction

The extension of general relativity (GR) by additional light degrees of freedom is arguably the most natural way to provide a dynamical explanation of dark energy, thereby dispensing with the cosmological constant as the source of the observed late-time cosmic acceleration. Considering a single scalar field in addition to the metric tensor is, in this regard, particularly well motivated. These so-called scalar–tensor theories of gravity [1,2] thus provide the most minimal working changes to Einsteinian gravity in terms of local degrees of freedom, under some standard assumptions such as Poincaré invariance and locality. This is a virtue both from the theoretical and experimental perspectives, as its relative simplicity allows for strong analytical control while maintaining much of the phenomenology of GR. It is also not the least telling case for scalar–tensor theories that the related mechanism of inflation was likely to be at work during the pre-Big Bang epoch¹.

The complete classification of scalar-tensor theories thus seems to be an interesting and timely theoretical problem. In this effort, the assumption of having precisely three local degrees of freedom—two propagated by the metric and one by the scalar field—severely restricts the space for possible models. Although the physically meaningful question should make a distinction between light and heavy degrees of freedom, it has nevertheless proved fruitful to demand the strict absence of additional fields beyond the aforementioned three, since the resulting models often enjoy interesting properties that may have been difficult to discover through more agnostic constructions based on the rules of effective field theory.

This restriction on the number of degrees of freedom makes the classification problem mathematically well defined, although not easy as it turns out. Given the symmetries of the theory, it is sufficient to demand second-order field equations, and taking this as a premise, the problem has indeed been fully solved. The solution is given by Horndeski's scalar-tensor theory [7–10]. The remarkable observation is that this premise is, however, not a necessary one. That is, higher order equations of motion are not necessarily associated with extra unwanted degrees of freedom—unwanted indeed, as they are generically associated

with ghost-type instabilities according to the Ostrogradski theorem. This is so because the equations may happen to be degenerate, in the sense that a subset of them follows as a consequence of the others, implying in particular, a reduction of the number of pieces of initial data that one would have naively inferred. The development and classification of these so-called degenerate scalar-tensor theories has been an active research topic over the past decade [11–19]. New models have been discovered throughout the years and have been given different names. We will refer to all of them collectively as DHOST, an acronym that stands for "degenerate higher-order scalar-tensor" theories. See [20–22] for reviews.

DHOST theories provide, then, a very interesting solution to the classification problem of scalar–tensor gravity. They are consistent theories within the scope of that problem, at least according to the way we have formulated it, although it is clear that physical consistency will reduce the space of allowed models by the imposition of further constraints. Most of these constraints arise from experimental tests of gravity, although here we will not be concerned with them—not because they are not important, but because their importance is contingent on the physical context. For instance, constraints derived from cosmological observations [23–33] need not apply to the scales of compact astrophysical objects. Theoretical constraints on the other hand have the chance to be more generally applicable, even if experiments must have the last word.

One such theoretical constraint that has remained largely overlooked is the question on the consistency of matter coupling in DHOST theories. The fact that matter fields can be problematic is seen easily in the Hamiltonian language, in which the degeneracy of the field equations is manifested in the form of a constraint on the phase space variables. The mixing with matter fields can then obstruct this constraint, leading to the reappearance of the ghost degree of freedom and an inconsistent theory [34]. This may occur even if matter is minimally coupled with the metric tensor, for an indirect coupling with the DHOST scalar is still present. It is worth remarking that this issue is of course not specific to DHOST theories, and may happen whenever either one or both of two theories have constraints when considered separately, are coupled in some way [35]. It is thus a virtue of the Hamiltonian language to make it manifest that the degeneracy condition is in truth a constraint, on equal footing with other constraints.

Understanding the precise ways in which the DHOST constraint may be lost was the subject of the work [36]. Let us denote the constraint by $\Psi \approx 0$, where Ψ is a phase space function to be made explicit later, and the symbol " \approx " means weak equality. We can then distinguish two types of pathological matter theories:

(I) The constraint Ψ is lost, and no analogue of it exists.

This will be the case when the rank of the Hessian matrix

$$\mathcal{H}_{IJ} := \frac{\partial^2 \mathcal{L}}{\partial \dot{\psi}^I \partial \dot{\psi}^J} \,, \tag{1}$$

(where ψ^{l} stands for all the fields) is greater than the sum of the ranks of the DHOST and matter Hessians that one would have in the absence of coupling. This cannot occur when the full Hessian is block-diagonal in the DHOST and matter variables. As we are restricting our attention to minimal matter coupling, any matter Lagrangian that does not involve the Christoffel connection will lead to a block-diagonal Hessian and thus be safe according to this criterion. The converse of this is of course not true. Although a non-block-diagonal Hessian is at risk of failing this consistency check, it may still enjoy a (possibly modified) degeneracy constraint.

(II) The constraint Ψ (or some analogue of it) does exist, but it fails to Poisson-commute with one or more constraints present in the matter sector. In the absence of matter, the DHOST constraint Ψ is a primary, second-class constraint, and it Poisson-commutes with all the other primary constraints in the gravity sector. It therefore leads to a secondary constraint, which together with Ψ is responsible for removing the would-be ghost degree of freedom. If now the matter sector itself has

removing the would-be ghost degree of freedom. If now the matter sector itself has some constraints, there is the risk that they may not commute with Ψ , implying the

loss of the associated secondary constraint and the reappearance of the unwanted degree of freedom.

It is not difficult to find examples that fail either of these two criteria; some explicit pathological matter models were studied in [36]. The aim of the present article is to analyze these consistency criteria in detail for a more interesting model, namely, the generalization of the Proca theory of a massive spin-1 field [37,38]. This class of models, dubbed Generalized Proca (GP), has been subject to intense scrutiny for their potential role in cosmology for a dark energy fluid and also in the physics of compact astrophysical objects [39–46]. GP theory extends the linear Proca model by the inclusion of derivative interactions while maintaining the constraints that ensure that one of the components of the vector field is non-dynamical. The theory thus falls into the "dangerous" class of matter fields when coupled with DHOST: the non-trivial interactions produce coupling with the Christoffel connection upon covariantization, whereas the Proca constraint risks spoiling the Poisson algebra of the coupled DHOST-GP system.

Our main result is the proof that GP theory cannot be consistently coupled with DHOST gravity within the framework we consider. The main assumptions are the following: (i) We focus exclusively on the so-called quadratic DHOST class, i.e., scalar–tensor theories whose Lagrangians involve operators that are at most quadratic in $\nabla^2 \phi$ (where ϕ is the scalar field). (ii) We consider a truncated version of GP theory with at most cubic derivative self-interactions; (iii) the GP vector field couples with the DHOST sector only through the metric tensor. Assumptions (i) and (ii) are not essential, and we expect all our results to hold for more general DHOST models and for the complete GP Lagrangian. Assumption (iii) is, on the other hand, more restrictive, but is certainly reasonable and in line with our set-up of treating the Proca field as a matter field which couples with gravity in accordance with the equivalence principle. We will come back to this point in the final discussion.

2. ADM Decomposition of DHOST and GP Theories

In this section we review the definitions of the DHOST and GP theories that we focus on in this article. We then perform a 3 + 1 decomposition of the Lagrangians in terms of ADM variables.

2.1. DHOST Lagrangian

The gravitational sector of our framework is given by the quadratic DHOST Lagrangian:

$$S_{g}[\phi,g] = \int d^{4}x \sqrt{-g} \Big[F(\phi,X)R + P(\phi,X) + Q(\phi,X)\Box\phi + C^{\mu\nu\rho\sigma}[\phi]\nabla_{\mu}\nabla_{\nu}\phi\nabla_{\rho}\nabla_{\sigma}\phi \Big] .$$
(2)

where *R* is the curvature scalar constructed from the metric $g_{\mu\nu}$; *F*, *P* and *Q* are generic functions of the scalar field ϕ ; and

$$X := \nabla^{\mu} \phi \nabla_{\mu} \phi \,. \tag{3}$$

The tensor $C^{\mu\nu\rho\sigma}$ is defined as

$$C^{\mu\nu\rho\sigma} := A_{1}g^{\mu(\rho}g^{\sigma)\nu} + A_{2}g^{\mu\nu}g^{\rho\sigma} + \frac{A_{3}}{2}(\phi^{\mu}\phi^{\nu}g^{\rho\sigma} + \phi^{\rho}\phi^{\sigma}g^{\mu\nu}) + \frac{A_{4}}{2}(\phi^{\mu}\phi^{(\rho}g^{\sigma)\nu} + \phi^{\nu}\phi^{(\rho}g^{\sigma)\mu}) + A_{5}\phi^{\mu}\phi^{\nu}\phi^{\rho}\phi^{\sigma},$$
(4)

where $\phi_{\mu} := \nabla_{\mu} \phi$ and the *A*s are also functions of ϕ and *X*.

For the purpose of analyzing the constraints in the Hamiltonian language, we carry out a time-space split or 3 + 1 decomposition of the Lagrangian. The metric tensor is expanded in ADM variables [47], i.e., the lapse *N*, shift N^i and 3-metric γ_{ii} ,

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^k N_k & N_i \\ N_j & \gamma_{ij} \end{pmatrix}, \qquad g^{\mu\nu} = \frac{1}{N^2} \begin{pmatrix} -1 & N^i \\ N^j & N^2 \gamma^{ij} - N^i N^j \end{pmatrix}, \tag{5}$$

and the measure factor is $\sqrt{-g} = N\sqrt{\gamma}$. Spatial indices are raised and lowered with the 3-metric and its inverse—for example, $N_i = \gamma_{ij}N^j$ (the shift function is defined with an upper index). The extrinsic curvature of the constant-time hypersurfaces is

$$K_{ij} = \frac{1}{2N} \left(\dot{\gamma}_{ij} - 2D_{(i}N_{j)} \right), \tag{6}$$

where D_i is the covariant derivative compatible with the 3-metric and a dot denotes differentiation with respect to the time coordinate $x^0 = t$. We also introduce

$$n^{\mu} := \frac{1}{N} \left(1, -N^{i} \right), \qquad a^{\mu} := n^{\nu} \nabla_{\nu} n^{\mu} = \left(0, \frac{D^{i} N}{N} \right), \tag{7}$$

and note that $n^{\mu}n_{\mu} = -1$.

In the Hamiltonian language, one introduces a canonical momentum associated with each field velocity. The DHOST Lagrangian is a function of the second derivative of the scalar field; therefore, both ϕ and $\nabla_{\mu}\phi$ have conjugate momenta in phase space. It is convenient to introduce an auxiliary vector field A_{μ} which is constrained as $A_{\mu} = \nabla_{\mu}\phi$ by means of a Lagrange multiplier [15,48,49]. Thus the modified DHOST action we will inspect is

$$S_{g}[\phi, A, g] = \int d^{4}x \sqrt{-g} \Big[F(\phi, X)R + P(\phi, X) + Q(\phi, X)\nabla_{\mu}A^{\mu} + C^{\mu\nu\rho\sigma}[\phi, A]\nabla_{\mu}A_{\nu}\nabla_{\rho}A_{\sigma} + \lambda^{\mu}(A_{\mu} - \nabla_{\mu}\phi) \Big],$$
(8)

where it is understood that every instance of $\nabla_{\mu}\phi$ in $C^{\mu\nu\rho\sigma}$ has been replaced by A_{μ} , and similarly *X* now stands for $A^{\mu}A_{\mu}$. The Lagrangian is now purely first order in derivatives and the passage to the Hamiltonian proceeds as usual. Following the analysis of [21], we decompose the vector A_{μ} in its spatial components A_i and the redefined time component

$$A_* := n^{\mu} A_{\mu} = \frac{1}{N} \left(A_0 - N^i A_i \right).$$
(9)

Details of the 3 + 1 decomposition may be found in [21,36], so here we only quote the final result:

$$S_{g} = \int dt d^{3}x \left\{ N \sqrt{\gamma} \left[\mathcal{A} V_{*}^{2} + 2\mathcal{B}^{ij} V_{*} K_{ij} + \mathcal{K}^{ij,kl} K_{ij} K_{kl} + 2\mathcal{C}^{ij} K_{ij} + 2\mathcal{C}^{0} V_{*} - \mathcal{U} \right] + \lambda^{0} \left(N A_{*} + N^{i} A_{i} - \dot{\phi} \right) + \lambda^{i} (A_{i} - D_{i} \phi) \right\},$$

$$(10)$$

where

$$V_* := \frac{1}{N} (\dot{A}_* - \Xi_A) , \qquad \Xi_A := A^i D_i N + N^i D_i A_* .$$
(11)

The coefficients appearing in (10) are given explicitly as follows:

$$\mathcal{A} = A_{1} + A_{2} - (A_{3} + A_{4})A_{*}^{2} + A_{5}A_{*}^{4},$$

$$\mathcal{B}^{ij} = A_{*} \left(2F_{X} + A_{2} - \frac{A_{3}}{2}A_{*}^{2}\right)\gamma^{ij} - \frac{A_{*}}{2}\left(A_{3} + 2A_{4} - 2A_{5}A_{*}^{2}\right)A^{i}A^{j},$$

$$\mathcal{K}^{ij,kl} = \left(F + A_{1}A_{*}^{2}\right)\gamma^{i(k}\gamma^{l)j} - \left(F - A_{2}A_{*}^{2}\right)\gamma^{ij}\gamma^{kl} + \left(2F_{X} - \frac{A_{3}}{2}A_{*}^{2}\right)\left(\gamma^{ij}A^{k}A^{l} + \gamma^{kl}A^{i}A^{j}\right) - A_{2}\left(A^{i}A^{(k}\gamma^{l)j} + A^{j}A^{(k}\gamma^{l)i}\right) - \left(A_{4} - A_{5}A_{*}^{2}\right)A^{i}A^{j}A^{k}A^{l},$$
(12)

and while the expressions for C^{ij} , C^0 and U (which multiply terms that are at most linear in the velocities) will not be needed in our analysis; the interested reader may find them in Appendix A.

The degeneracy of the DHOST Lagrangian is manifested in the fact that the determinant of the Hessian matrix of second time derivatives vanishes identically²:

$$\mathcal{A} - \mathcal{K}_{ii\,kl}^{-1} \mathcal{B}^{ij} \mathcal{B}^{kl} = 0.$$
⁽¹³⁾

This relation translates into a set of algebraic equations for the coefficient functions A_I , I = 1, 2, 3, 4, 5, and the solutions have been classified in [15]. Note the implicit assumption that the gravitational kinetic matrix \mathcal{K} must be invertible, ensuring that DHOST can be connected smoothly, in theory space, to standard GR. The inversion of \mathcal{K} can be done explicitly, and the reader may find the result in Appendix B.

2.2. GP Lagrangian

GP is a vector-tensor theory that describes the coupled dynamics of a vector field B_{μ} and metric $g_{\mu\nu}$. In isolation, this theory is consistent in the sense that it describes 3 + 2 degrees of freedom, corresponding to massive spin-1 and massless spin-2 particles, at the complete non-linear level. The Lagrangian is given by [37,38]

$$S_{\rm GP} = \int d^4x \sqrt{-g} \sum_{I=2}^5 \mathcal{L}_I[g, B] \,, \tag{14}$$

and we have explicitly

$$\mathcal{L}_{2} = G_{2}(Y, \mathcal{F}, \mathcal{G}),$$

$$\mathcal{L}_{3} = G_{3}(Y)\nabla_{\mu}B^{\mu},$$

$$\mathcal{L}_{4} = G_{4}(Y)R - 2G'_{4}(Y)\left[(\nabla_{\mu}B^{\mu})^{2} - \nabla_{\mu}B^{\nu}\nabla_{\nu}B^{\mu}\right],$$

$$\mathcal{L}_{5} = G_{5}(Y)G^{\mu\nu}\nabla_{\mu}B_{\nu}$$

$$+ \frac{G_{5}(Y)}{3}\left[(\nabla_{\mu}B^{\mu})^{3} - 3(\nabla_{\rho}B^{\rho})\nabla_{\mu}B^{\nu}\nabla_{\nu}B^{\mu} + 2\nabla_{\mu}B^{\nu}\nabla_{\nu}B^{\rho}\nabla_{\rho}B^{\mu}\right],$$
(15)

with the definitions

$$Y := B^{\mu}B_{\mu}, \qquad \mathcal{F} := -\frac{1}{4} B^{\mu\nu}B_{\mu\nu}, \qquad \mathcal{G} := B^{\mu}B^{\nu}B_{\mu}^{\ \rho}B_{\nu\rho}, \qquad (16)$$

and $B_{\mu\nu} := \nabla_{\mu}B_{\nu} - \nabla_{\nu}B_{\mu}$; *R* and $G_{\mu\nu}$ are, respectively, the curvature scalar and Einstein tensor constructed from the metric $g_{\mu\nu}$. A prime on the coefficient functions denotes differentiation with respect to the argument *Y*, e.g., $G'_4 \equiv \frac{dG_4}{dY}$. The operators in (15) do not exhaust the whole GP class. We do not expect the additional terms to effect any of our conclusions, so the truncated model we consider is general enough to illustrate the message of this paper. See the final discussion section for further comments on this point.

Like DHOST, GP is a degenerate theory in the sense that not all among the components of B_{μ} are dynamical. As is well known, in the standard Proca theory there exists a (local) frame in which B_0 does not propagate, and GP theory is precisely constructed so as to generalize this property to include non-trivial derivative interactions. In the Hamiltonian language, this degeneracy will manifest itself in the fact that the kinetic part of the Lagrangian (i.e., the operators that are at least quadratic in the velocity variables) will be independent of the time component of the vector field velocity.

In the following subsections we detail the 3 + 1 decomposition of the operators entering in the GP terms defined above. The metric is again expanded in ADM variables while the Proca field, similarly to the DHOST auxiliary vector A_{μ} , is decomposed into its spatial part B_i and

$$B_* := n^{\mu} B_{\mu} = \frac{1}{N} \left(B_0 - N^i B_i \right).$$
(17)

A reader uninterested in the particulars may skip to the next section where we provide the relevant collected results.

2.2.1. \mathcal{L}_2 Term

The GP term \mathcal{L}_2 is a generic function of the scalars *Y*, \mathcal{F} and \mathcal{G} . By expanding ADM components, we find

$$Y = -B_*^2 + B^i B_i,$$

$$\mathcal{F} = \frac{1}{2} F^i F_i - \frac{1}{4} b^{ij} b_{ij},$$

$$\mathcal{G} = B_*^2 F^i F_i - B^i B^j F_i F_j - 2B_* B^i \left(F^j b_{ij} + \frac{N^j}{N} b_i^{\ k} b_{jk} \right) - \frac{N^i N^j}{N^2} B_*^2 b_i^{\ k} b_{jk},$$
(18)

where

$$F_{i} := \frac{1}{N} (\dot{B}_{i} - Y_{i}), \qquad Y_{i} := D_{i} (NB_{*}) + B_{j} D_{i} N^{j} + N^{j} D_{j} B_{i},$$

$$b_{ij} := D_{i} B_{j} - D_{j} B_{i}.$$
(19)

Therefore, \mathcal{L}_2 is manifestly degenerate as it is independent of \dot{B}_* .

2.2.2. \mathcal{L}_3 Term

For the GP term \mathcal{L}_3 we only need the expression

$$\nabla_{\mu}B^{\mu} = -W_* - B_*K + D_iB^i \,, \tag{20}$$

where

$$W_* := \frac{1}{N} (\dot{B}_* - \Xi_B) , \qquad \Xi_B := B^i D_i N + N^i D_i B_* .$$
(21)

We see that \mathcal{L}_3 gives a non-trivial contribution to the canonical momenta conjugate to B_* and γ_{ij} . However, the fact that \dot{B}_* appears only linearly still ensures the degeneracy.

2.2.3. \mathcal{L}_4 Term

We work out \mathcal{L}_4 in two steps. The non-minimal coupling to the curvature scalar is straightforward to expand, but it must be integrated by parts so as to remove second time derivatives. Thus we have

$$\begin{aligned} \mathcal{L}_{4}^{(1)} &:= G_{4}R \\ &= G_{4} \Big(K^{ij} K_{ij} - K^{2} + R^{(3)} + 2 \nabla_{\mu} (K n^{\mu} - a^{\mu}) \Big) \\ &= G_{4} \Big(K^{ij} K_{ij} - K^{2} + R^{(3)} \Big) - 2 G'_{4} \nabla_{\mu} Y (K n^{\mu} - a^{\mu}) + \text{t.d.} \\ &= G_{4} \Big(K^{ij} K_{ij} - K^{2} + R^{(3)} \Big) + 2 G'_{4} \Big[2 B^{i} B^{j} K K_{ij} + 2 B_{*} W_{*} K - 2 B^{i} F_{i} K \\ &- 2 B^{i} D_{i} B_{*} K + \frac{1}{N} D^{i} N D_{i} Y \Big] + \text{t.d.} , \end{aligned}$$

$$(22)$$

where $K := \gamma^{ij} K_{ij}$, $R^{(3)}$ is the curvature scalar built out of γ_{ij} , and "t.d." means total derivative. Next, the minimally covariantized GP term is

$$\mathcal{L}_{4}^{(2)} := \widetilde{G}_{4} \left[(\nabla_{\mu} B^{\mu})^{2} - \nabla_{\mu} B^{\nu} \nabla_{\nu} B^{\mu} \right] \\
= \widetilde{G}_{4} \left[-B_{*}^{2} (K^{ij} K_{ij} - K^{2}) + 2B^{i} B^{j} K_{i}^{k} K_{jk} - 2B^{i} F^{j} K_{ij} + 2B_{*} W_{*} K \right. \\
\left. - 2D_{i} B^{i} W_{*} - 2D_{i} B^{i} B_{*} K - 4B_{i} D_{j} B_{*} K^{ij} + 2B_{*} D_{i} B_{j} K^{ij} + 2D_{i} B_{*} F^{i} \\
\left. + 2D^{i} B_{*} D_{i} B_{*} + (D_{i} B^{i})^{2} - D_{i} B^{j} D_{j} B^{i} \right].$$
(23)

Note that we have "detuned" the relative coefficients multiplying $\mathcal{L}_4^{(1)}$ and $\mathcal{L}_4^{(2)}$ so that we may understand later the role it plays in the coupled DHOST-GP system. When taken in isolation, however, we see that \mathcal{L}_4 contains

$$\mathcal{L}_4 \supset 2\Big(\widetilde{G}_4 + 2G_4'\Big)B_*W_*K\,,\tag{24}$$

which mixes the Proca field and metric velocities, and thus spoils the degeneracy unless we choose $\tilde{G}_4 = -2G'_4$, in agreement with (15).

2.2.4. \mathcal{L}_5 Term

To expand \mathcal{L}_5 , we consider the two contributions separately, while again keeping the GP "tuning" of relative coefficients for later:

$$\mathcal{L}_{5}^{(1)} := G_{5}(Y) G^{\mu\nu} \nabla_{\mu} B_{\nu} ,$$

$$\mathcal{L}_{5}^{(2)} := \widetilde{G}_{5}(Y) \left[(\nabla_{\mu} B^{\mu})^{3} - 3(\nabla_{\rho} B^{\rho}) \nabla_{\mu} B^{\nu} \nabla_{\nu} B^{\mu} + 2 \nabla_{\mu} B^{\nu} \nabla_{\nu} B^{\rho} \nabla_{\rho} B^{\mu} \right].$$
(25)

For the sake of brevity we will focus here on the kinetic terms, i.e., the terms which are at least quadratic in the velocities, while delegating the full expressions to Appendix A.

For the first contribution we need the components of the Einstein tensor in ADM variables:

$$\begin{aligned}
G^{00} &= -\frac{1}{2N^2} \left(K^{kl} K_{kl} - K^2 - R^{(3)} \right), \\
G^{0i} &= \frac{N^i}{2N^2} \left(K^{kl} K_{kl} - K^2 - R^{(3)} \right) - \frac{1}{N} D_j \widetilde{K}^{ij}, \\
G^{ij} &= G^{(3)ij} - \frac{N^i N^j}{2N^2} \left(K^{kl} K_{kl} - K^2 - R^{(3)} \right) + \frac{1}{N\sqrt{\gamma}} \partial_t \left(\sqrt{\gamma} \, \widetilde{K}^{ij} \right) \\
&\quad - \frac{1}{N} \left(D_i D_j N - \gamma^{ij} D^2 N \right) + \frac{1}{N} D_k \left(2N^{(i} \widetilde{K}^{j)k} - N^k \widetilde{K}^{ij} \right) \\
&\quad + 2 \left(K^i_k K^{jk} - K K^{ij} \right) - \frac{1}{2} \gamma^{ij} \left(K^{kl} K_{kl} - K^2 \right),
\end{aligned} \tag{26}$$

where $\widetilde{K}_{ij} := K_{ij} - \gamma_{ij}K$ and $G_{ij}^{(3)}$ is the Einstein tensor built out of the 3-metric. Let us emphasize that the last result is only valid in three spatial dimensions. After collecting terms and integrating by parts, we obtain

$$\mathcal{L}_{5,\text{kin}}^{(1)} = \frac{1}{2N} G_5' \dot{Y} \Big[B_* (K^{ij} K_{ij} - K^2) - 2 \widetilde{K}^{ij} D_i B_j \Big] + \text{t.d.}$$
(27)

The result is proportional to $G'_5(Y)$ (the derivative of $G_5(Y)$ with respect to its argument), not surprisingly since $\mathcal{L}_5^{(1)}$ is a total derivative when G_5 is constant. Note that

$$\dot{Y} = 2\left(-B_*\dot{B}_* + B^k\dot{B}_k - NB^kB^lK_{kl} - B^kB^lD_kN_l\right).$$
(28)

By expanding $\mathcal{L}_5^{(2)}$ next, we can find

$$\mathcal{L}_{5,kin}^{(2)} = \widetilde{G}_{5} \left\{ \frac{3}{N} \dot{B}_{*} B_{*} \left[B_{*} (K^{ij} K_{ij} - K^{2}) - 2 \widetilde{K}^{ij} D_{i} B_{j} \right] \right. \\ \left. - B_{*}^{3} \left[K^{3} - 3KK^{ij} K_{ij} + 2K_{j}^{i} K_{k}^{j} K_{i}^{k} \right] + 6B_{*} B^{i} B^{j} \left[K^{kl} K_{ik} K_{jl} - KK_{i}^{k} K_{jk} \right] \right. \\ \left. - \frac{3}{N} (K^{ij} K_{ij} - K^{2}) B_{*}^{2} \left[D_{k} (NB^{k}) + N^{k} D_{k} B_{*} \right] \right. \\ \left. + 6(K_{i}^{k} K_{jk} - KK_{ij}) B_{*} \left[B_{*} D^{i} B^{j} - F^{i} B^{j} - 2B^{i} D^{j} B_{*} \right] \right. \\ \left. - 6B^{i} B^{j} \left[D^{k} B^{l} K_{ik} K_{jl} - D_{l} B^{l} K_{i}^{k} K_{jk} \right] \right.$$

$$\left. + 6F_{i} \left[\widetilde{K}^{ik} B_{*} D_{k} B_{*} + K^{kl} B_{k} D^{i} B_{l} - K^{ik} B_{k} D_{l} B^{l} \right] \right\}.$$

$$(29)$$

When comparing the two contributions, we see that the offending terms proportional to \dot{B}_* are indeed canceled upon choosing $\tilde{G}_5 = \frac{1}{3}G'_5$.

3. Constraint Analysis

In this section we collect the contributions to the GP terms in the Hamiltonian formalism and analyze the conditions for the Proca and DHOST constraints to be maintained once the two sectors are coupled through the metric tensor. We focus on each GP term independently, although in the end it will become clear that the results remain unchanged if one includes the whole Lagrangian. We consider the addition to the gravitational action (2) the following GP vector matter term:

$$S_{\rm m} = \int d^4x \sqrt{-g} \left[G_2(Y, \mathcal{F}, \mathcal{G}) + G_3(Y) \nabla_\mu B^\mu \right]$$

$$= \int dt d^3x N \sqrt{\gamma} \left[2\mathcal{C}_{\rm m}^{ij} K_{ij} + 2\overline{\mathcal{C}}^0 W_* - \mathcal{U}_{\rm m} \right],$$
(30)

where

$$C_{\rm m}^{ij} = -\frac{1}{2} G_3 B_* \gamma^{ij} ,$$

$$\overline{C}^0 = -\frac{1}{2} G_3 ,$$

$$U_{\rm m} = -G_2 - G_3 D_i B^i .$$
(31)

With some abuse of terminology we can think of U_m as a potential term because it is independent of \dot{B}_* and K_{ij} ; however, one should keep in mind that it does depend on \dot{B}_i .

The complete action $S_g + S_m$ is manifestly degenerate because the Hessian matrix is not affected by S_m as far as \dot{B}_* , \dot{A}_* and K_{ij} are concerned. Nevertheless, the primary constraints are still affected by the linear terms (in the velocities) brought in by \mathcal{L}_1 . In particular the Proca constraint is modified as follows:

$$q_* := \frac{\partial \mathcal{L}}{\partial \dot{B}_*} = \frac{1}{N} \frac{\partial \mathcal{L}}{\partial W_*} = -\sqrt{\gamma} G_3$$

$$\Rightarrow \quad \Lambda := q_* + \sqrt{\gamma} G_3 \approx 0.$$
(32)

To obtain the DHOST constraint, we first compute the momenta

$$\pi^{ij} := \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ij}} = \frac{1}{2N} \frac{\partial \mathcal{L}}{\partial K_{ij}} = \sqrt{\gamma} \Big[\mathcal{K}^{ij,kl} K_{kl} + \mathcal{B}^{ij} V_* + \mathcal{C}^{ij}_{\text{tot}} \Big],$$

$$p_* := \frac{\partial \mathcal{L}}{\partial \dot{A}_*} = \frac{1}{N} \frac{\partial \mathcal{L}}{\partial V_*} = 2\sqrt{\gamma} \Big[\mathcal{A} V_* + \mathcal{B}^{ij} K_{ij} + \mathcal{C}^0 \Big],$$
(33)

$$\Rightarrow \qquad \Psi' := p_* - 2\mathcal{K}_{ij,kl}^{-1} \pi^{ij} \mathcal{B}^{kl} + 2\sqrt{\gamma} \left(\mathcal{K}_{ij,kl}^{-1} \mathcal{C}_{tot}^{ij} \mathcal{B}^{kl} - \mathcal{C}^0 \right) \approx 0, \tag{34}$$

where $C_{tot}^{ij} = C^{ij} + C_m^{ij}$ includes the contribution from the matter action. This is to be compared with the "vacuum" constraint,

$$\Psi := p_* - 2\mathcal{K}_{ij,kl}^{-1} \pi^{ij} \mathcal{B}^{kl} + 2\sqrt{\gamma} \left(\mathcal{K}_{ij,kl}^{-1} \mathcal{C}^{ij} \mathcal{B}^{kl} - \mathcal{C}^0 \right) \approx 0, \qquad (35)$$

which one would have in the absence of matter. We conclude that there is no inconsistency at this stage: the \mathcal{L}_3 GP term maintains the primary constraints in the coupled GP–DHOST theory and is therefore safe with regard to the criterion (I) explained in the Introduction.

The inconsistency of the model is manifested in the failure to generate the secondary constraints that S_g and S_m possess when taken in isolation. That is, the model fails criterion (II). This is because the primary constraints Ψ' and Λ do not Poisson-commute:

$$\{\Psi',\Lambda\} = 2\sqrt{\gamma} \,\mathcal{K}_{ij,kl}^{-1} \left[\frac{\delta \mathcal{C}_{m}^{ij}}{\delta B_{*}} + \frac{1}{\sqrt{\gamma}} \,\frac{\delta\Lambda}{\delta\gamma_{ij}} \right] \mathcal{B}^{kl}$$

$$= 2\sqrt{\gamma} \,\mathcal{K}_{ij,kl}^{-1} G'_{3}(Y) \left(B_{*}^{2} \gamma^{ij} - B^{i} B^{j} \right) \mathcal{B}^{kl} \,.$$
(36)

Clearly, $G'_3 \neq 0$, since otherwise \mathcal{L}_3 is a total derivative and hence trivial. Thus the only way for the constraints to commute is that

$$\mathcal{K}_{ij,kl}^{-1} \left(B_*^2 \gamma^{ij} - B^i B^j \right) \mathcal{B}^{kl} = 0.$$
(37)

Recall that this condition should be understood as an identity valid for all field configurations. It implies a set of equations for the coefficients A_I . We can see that (37) has a unique solution when complemented with the DHOST constraint (13),

$$A_1 = -A_2 = 2F_X, \qquad A_3 = A_4 = A_5 = 0,$$
 (38)

which is in fact the same solution that yields $\mathcal{B}^{ij} = 0 = \mathcal{A}$. The vanishing of both \mathcal{B}^{ij} and \mathcal{A} is trivially a sufficient condition for both constraints to hold; what we have proved is that it is also a necessary condition.

With the result (38) for the functions A_I the covariant DHOST action reduces to

$$S_{\rm g} = \int d^4x \sqrt{-g} \Big[FR + 2F_X \Big(\nabla^\mu \nabla^\nu \phi \nabla_\mu \nabla_\nu \phi - (\Box \phi)^2 \Big) \Big] , \qquad (39)$$

which is nothing but the non-degenerate quadratic (in $\nabla^2 \phi$) Horndeski Lagrangian.

3.2. \mathcal{L}_4 Term

Next we consider adding to the DHOST action (2) the \mathcal{L}_4 GP term:

$$S_{\rm m} = \int d^4x \sqrt{-g} \bigg[G_2(Y, \mathcal{F}, \mathcal{G}) + G_4(Y)R + \widetilde{G}_4(Y) \Big((\nabla_\mu B^\mu)^2 - \nabla_\mu B^\nu \nabla_\nu B^\mu \Big) \bigg]$$

=
$$\int dt d^3x N \sqrt{\gamma} \bigg[2\overline{\mathcal{B}}^{ij} W_* K_{ij} + \mathcal{K}_{\rm m}^{ij,kl} K_{ij} K_{kl} + 2\mathcal{C}_{\rm m}^{ij} K_{ij} + 2\overline{\mathcal{C}}^0 W_* - \mathcal{U}_{\rm m} \bigg],$$
(40)

where

$$\begin{split} \overline{\mathcal{B}}^{ij} &= (2G'_4 + \widetilde{G}_4)B_*\gamma^{ij}, \\ \mathcal{K}^{ij,kl}_{m} &= \left(G_4 - \widetilde{G}_4B^2_*\right) \left(\gamma^{k(i}\gamma^{j)l} - \gamma^{ij}\gamma^{kl}\right) + 2G'_4\left(\gamma^{ij}B^kB^l + \gamma^{kl}B^iB^j\right) + 2\widetilde{G}_4B^{(i}\gamma^{j)(k}B^l), \\ \mathcal{C}^{ij}_{m} &= -2G'_4B^k(F_k + D_kB_*)\gamma^{ij} - \widetilde{G}_4\left(B^{(i}F^{j)} + B_*D_kB^k\gamma^{ij} - B_*D^{(i}B^{j)} + B^{(i}D^{j)}B_*\right), \\ \overline{\mathcal{C}}^0 &= -\widetilde{G}_4D_iB^i, \\ \mathcal{U}_{m} &= -G_2 - G_4R^{(3)} - \frac{2G'_4}{N}D^iND_iY \\ &\quad - \widetilde{G}_4\left[2D^iB_*(F_i + D_iB_*) + (D_iB^i)^2 - D_iB^jD_jB^i\right]. \end{split}$$

$$(41)$$

As before, we are abusing the notation by including the velocities \dot{B}_i (contained in the definition of F_i ; see Equation (19)) into the above coefficient tensors; note that such terms will not affect the degeneracy conditions within our set-up.

The critical question is whether we can find analogues of the DHOST and Proca primary constraints for this theory. To address this we compute the canonical momenta:

$$q_{*} = 2\sqrt{\gamma} \left[\overline{\mathcal{B}}^{ij} K_{ij} + \overline{\mathcal{C}}^{0} \right],$$

$$p_{*} = 2\sqrt{\gamma} \left[\mathcal{A}V_{*} + \mathcal{B}^{ij} K_{ij} + \mathcal{C}^{0} \right],$$

$$\pi^{ij} = \sqrt{\gamma} \left[\mathcal{K}_{\text{tot}}^{ij,kl} K_{kl} + \mathcal{B}^{ij} V_{*} + \overline{\mathcal{B}}^{ij} W_{*} + \mathcal{C}_{\text{tot}}^{ij} \right],$$
(42)

where $\mathcal{K}_{\text{tot}}^{ij,kl} = \mathcal{K}^{ij,kl} + \mathcal{K}_{\text{m}}^{ij,kl}$ and $\mathcal{C}_{\text{tot}}^{ij} = \mathcal{C}^{ij} + \mathcal{C}_{\text{m}}^{ij}$. In order for the two constraints to exist, the Hessian matrix must possess two independent null eigenvectors. We will demand that one of them be in \dot{B}_* direction—this is essentially what we mean by a GP theory, although it is in principle possible that the Proca constraint be realized in a more general way. This vector will be a null eigenvector if and only if $\overline{\mathcal{B}}^{ij} = 0$ identically, and so we recover the usual relation $\tilde{G}_4 = -2G'_4$ of GP theory.

Given this condition, the DHOST constraint will be present if and only if

$$\mathcal{A} - \mathcal{K}_{\text{tot}\,ij,kl}^{-1} \mathcal{B}^{ij} \mathcal{B}^{kl} = 0.$$
(43)

There are two ways for this relation to hold. First, we may choose to define the DHOST sector independently of the GP sector, so that we would have the usual constraint $\mathcal{A} = \mathcal{K}_{ij,kl}^{-1} \mathcal{B}^{ij} \mathcal{B}^{kl}$. This would be in line with the treatment of the GP vector as a matter field which couples with the gravitational sector described by DHOST only through the metric, in the same way as any other matter field. The second way is to include the GP vector field in the very definition of the DHOST Lagrangian and impose the condition $\mathcal{A} = \mathcal{K}_{\text{tot}\,ij,kl}^{-1} \mathcal{B}^{ij} \mathcal{B}^{kl}$ as a constraint on the coefficient functions. This option would be akin to constructing a particular type of scalar-vector-tensor model from the bottom-up, and is therefore beyond our current scope exposed in the Introduction.

Focusing then on the first possibility, we investigate whether the equation

$$\left(\mathcal{K}_{ij,kl}^{-1} - \mathcal{K}_{\text{tot}\,ij,kl}^{-1}\right)\mathcal{B}^{ij}\mathcal{B}^{kl} = 0\,,\tag{44}$$

could hold as an identity. We first note that the matrix on the left-hand side can be written as

$$\mathcal{K}_{ij,kl}^{-1} - \mathcal{K}_{\text{tot}\,ij,kl}^{-1} = \mathcal{K}_{\text{tot}\,ij,mn}^{-1} \mathcal{K}_{\text{m}}^{mn,pq} \mathcal{K}_{pq,kl}^{-1} \,.$$
(45)

Inverting the matrix \mathcal{K}_{tot} requires some formidable amount of algebra, so for convenience we will expand perturbatively in the Proca field B_{μ} , i.e.,

$$\mathcal{K}_{ij,kl}^{-1} - \mathcal{K}_{\text{tot}\,ij,kl}^{-1} = \mathcal{K}_{ij,mn}^{-1} \mathcal{K}_{\text{m}}^{mn,pq} \mathcal{K}_{pq,kl}^{-1} - \mathcal{K}_{ij,mn}^{-1} \mathcal{K}_{\text{m}}^{mn,pq} \mathcal{K}_{pq,rs}^{-1} \mathcal{K}_{\text{m}}^{rs,tu} \mathcal{K}_{tu,kl}^{-1} + \mathcal{O}(\mathcal{K}_{m}^{3}).$$
(46)

Note that it does not matter in which order in B_{μ} the tensor \mathcal{K}_{m} starts. Indeed we have

$$\mathcal{K}_{m}^{ij,kl} = \left(G_{4} + 2G_{4}'B_{*}^{2}\right)\left(\gamma^{k(i}\gamma^{j)l} - \gamma^{ij}\gamma^{kl}\right) + 2G_{4}'\left(\gamma^{ij}B^{k}B^{l} + \gamma^{kl}B^{i}B^{j}\right) - 4G_{4}'B^{(i}\gamma^{j)(k}B^{l)};$$
(47)

that is, regardless of the form of $G_4(Y)$, each tensor structure in \mathcal{K}_m starts in the same order in B_{μ} . This may seem to require that $G_4(Y)$ be an analytic function of Y; however, in reality all we demand is that there exists a field configuration for which an expansion in powers of \mathcal{K}_m is admissible, as in Equation (46). For instance, any $G_4(Y)$ admitting a Laurent series representation near Y = 0 would give such a consistent expansion.

Equation (44) together with the DHOST condition (13) gives two equations that must be satisfied identically. After replacing $A^i A_i = X + A_*^2$, $B^i B_i = Y + B_*^2$ and $A^i B_i = Z + A_* B_*$ (where $Z := A^{\mu} B_{\mu}$), the equations can be expanded in powers of A_* and B_* so that the coefficient of each monomial must separately vanish. This yields a system of equations which at leading order in \mathcal{K}_m , i.e., keeping only the first term on the RHS of (46), involves only the DHOST functions A_I and F. We found two independent solutions. The first is given by

$$A_1 = A_2 = A_5 = 0$$
, $A_3 = -A_4 = -\frac{4F_X}{X}$, (48)

and the second corresponds to $\mathcal{B}^{ij} = 0$. The latter of course solves the degeneracy conditions not just to leading order in B_{μ} but in general.

Among these two candidate solutions, the first yields $\mathcal{A} = \mathcal{K}_{ij,kl}^{-1} \mathcal{B}^{ij} \mathcal{B}^{kl} = 0$, so that the DHOST constraint is indeed satisfied. However, the modified degeneracy condition fails at the next order in \mathcal{K}_m in (46) (except in the trivial cases where $F_X = 0$ or $G'_4 = 0$). In conclusion, the unique consistent solution to the degeneracy conditions (44) and (13) is the trivial one with $\mathcal{B}^{ij} = 0$, which takes us again back to (39), i.e., the standard Horndeski scalar–tensor theory.

3.3. \mathcal{L}_5 Term

Focusing next on the \mathcal{L}_5 GP term, we envisage the matter action

$$S_{\rm m} = \int d^4x \sqrt{-g} \left[G_2(Y, \mathcal{F}, \mathcal{G}) + G_5(Y) G^{\mu\nu} \nabla_{\mu} B_{\nu} + \widetilde{G}_5(Y) \left((\nabla_{\mu} B^{\mu})^3 - 3(\nabla_{\rho} B^{\rho}) \nabla_{\mu} B^{\nu} \nabla_{\nu} B^{\mu} + 2 \nabla_{\mu} B^{\nu} \nabla_{\nu} B^{\rho} \nabla_{\rho} B^{\mu} \right) \right]$$

$$= \int dt d^3x N \sqrt{\gamma} \left[2 \overline{\mathcal{D}}^{ij,kl} W_* K_{ij} K_{kl} + 2 \overline{\mathcal{B}}^{ij} W_* K_{ij} + \frac{2}{3} \mathcal{J}_{\rm m}^{ij,kl,mn} K_{ij} K_{kl} K_{mn} + \mathcal{K}_{\rm m}^{ij,kl} K_{ij} K_{kl} + 2 \mathcal{C}_{\rm m}^{ij} K_{ij} + 2 \overline{\mathcal{C}}^0 W_* - \mathcal{U}_{\rm m} \right],$$

$$(49)$$

where

$$\begin{split} \overline{\mathcal{D}}^{ij,kl} &= -\frac{1}{2} (G'_{5} - 3\widetilde{G}_{5}) B_{*}^{2} \left(\gamma^{i(k} \gamma^{l)k} - \gamma^{ij} \gamma^{kl} \right), \\ \overline{\mathcal{B}}^{ij} &= (G'_{5} - 3\widetilde{G}_{5}) B_{*} \left(D^{(i}B^{j)} - \gamma^{ij} D_{k} B^{k} \right), \\ \mathcal{J}_{m}^{ij,kl,mn} &= -\frac{G'_{5}}{2} \left[B^{i}B^{j} \gamma^{k(m} \gamma^{n)l} - B^{i}B^{j} \gamma^{kl} \gamma^{mn} + (2 \text{ perm.}) \right] \\ &+ 3\widetilde{G}_{5} \left[B^{i}B^{k} \gamma^{j(m} \gamma^{n)l} - B^{i}B^{k} \gamma^{jl} \gamma^{mn} + (2 \text{ perm.}) \right] \\ &- \frac{3\widetilde{G}_{5}}{2} B_{*} \left[\gamma^{ij} \gamma^{kl} \gamma^{mn} - \left(\gamma^{ij} \gamma^{k(m} \gamma^{n)l} + (2 \text{ perm.}) \right) \right. \\ &+ \left(\gamma^{j(k} \gamma^{l)(m} \gamma^{n)i} + \gamma^{i(k} \gamma^{l)(m} \gamma^{n)j} \right) \right], \\ \mathcal{K}_{m}^{ij,kl} &= G'_{5} \left[B_{*} \left(B^{m}F_{m} + B^{m}D_{m}B_{*} + \frac{1}{2N} N^{m}D_{m}Y \right) \left(\gamma^{i(k} \gamma^{l)k} - \gamma^{ij} \gamma^{kl} \right) \\ &+ \left(B^{i}B^{j}D^{(k}B^{l)} + B^{k}B^{l}D^{(i}B^{j)} \right) - D_{m}B^{m} \left(B^{i}B^{j} \gamma^{kl} + B^{k}B^{l} \gamma^{ij} \right) \right] \\ &+ 6\widetilde{G}_{5} \left[-\frac{1}{2N} B_{*}^{2} (D_{m}(NB^{m}) + N^{m}D_{m}B_{*}) \left(\gamma^{i(k} \gamma^{l)k} - \gamma^{ij} \gamma^{kl} \right) \\ &+ B_{*} \left((B_{*}D^{i}B^{(k} - B^{i}F^{(k} - 2B^{i}D^{(k}B_{*}) \gamma^{l)j} + (1 \text{ perm.}) \right) \\ &- B_{*} \left((B_{*}D^{(i}B^{j)} - B^{(i}F^{j)} - 2B^{(i}D^{j)}B_{*}) \gamma^{kl} + (1 \text{ perm.}) \right) \\ &- \left(B^{(i}D^{j)}B^{(k}B^{l)} - D_{m}B^{m}B^{(i} \gamma^{j)(k}B^{l)} \right) \right]. \end{split}$$

Once again we abuse the notation to include terms involving \dot{B}_i in these coefficient tensors. The remaining coefficients are entered into (49); the ones most linear being the velocity variables and of no immediate interest in the constraint analysis, are provided in full in Appendix A.

The relevant set of canonical momenta is given by

$$q_{*} = 2\sqrt{\gamma} \left[\overline{D}^{ij,kl} K_{ij} K_{kl} + \overline{\mathcal{B}}^{ij} K_{ij} + \overline{\mathcal{C}}^{0} \right],$$

$$p_{*} = 2\sqrt{\gamma} \left[\mathcal{A}V_{*} + \mathcal{B}^{ij} K_{ij} + \mathcal{C}^{0} \right],$$

$$\pi^{ij} = \sqrt{\gamma} \left[\mathcal{J}_{m}^{ij,kl,mn} K_{kl} K_{mn} + 2\overline{D}^{ij,kl} W_{*} K_{kl} + \mathcal{K}_{tot}^{ij,kl} K_{kl} + \mathcal{B}^{ij} V_{*} + \overline{\mathcal{B}}^{ij} W_{*} + \mathcal{C}_{tot}^{ij} \right],$$
(51)

where $\mathcal{K}_{tot}^{ij,kl} = \mathcal{K}^{ij,kl} + \mathcal{K}_m^{ij,kl}$ and $\mathcal{C}_{tot}^{ij} = \mathcal{C}^{ij} + \mathcal{C}_m^{ij}$. It is clear that in the absence of the GP tuning, the standard Proca constraint fails to be realized; i.e., the Hessian matrix does not have a null eigenvector along the \dot{B}_* direction. As before, we will insist that this

eigenvector be present while keeping in mind that other options may be available in principle. Therefore, at this stage we set $\tilde{G}_5 = \frac{1}{3}G'_5$, so that in particular $\overline{D}^{ij,kl} = 0 = \overline{\mathcal{B}}^{ij}$.

The novelty brought in by the \mathcal{L}_5 GP term is that the relation between K_{ij} , V_* and the canonical momenta π^{ij} , p_* is now non-linear because of the presence of the tensor $\mathcal{J}_m^{ij,kl,mn}$ (which is non-zero since $G'_5 \neq 0$; otherwise \mathcal{L}_5 is a total derivative). Such a non-linear system can only be degenerate in a trivial manner, i.e., if the coefficients are such that one of the variables disappears from the system. In this case, for V_* to drop out, we must have $\mathcal{A} = 0 = \mathcal{B}^{ij}$. The conclusion is that the \mathcal{L}_5 term of GP does not admit a consistent coupling to DHOST except in the non-degenerate case of Horndeski theory.

4. Discussion

We have demonstrated that generalized Proca fields described by GP theory do not allow for a consistent coupling with a gravitational sector given by the DHOST class of models. Although our analysis considered the individual GP Lagrangians separately, it is clear in hindsight that none of the results would change if we were to envisage the complete model: the \mathcal{L}_5 GP term immediately spoils the DHOST degeneracy because of the cubic operators in the extrinsic curvature, and the \mathcal{L}_4 also fails the degeneracy test irrespective of \mathcal{L}_3 . The exceptions that bypass our no-go result are rather trivial, at least from the perspective of the constraint structure: either the DHOST sector must reduce to the standard, non-degenerate Horndeski theory, or the GP sector must reduce to the \mathcal{L}_2 term which is independent of the Christoffel connection.

It is important to emphasize the relation between having the correct number of constraints and the consistency of the theory. The appearance of an additional degree of freedom in the DHOST sector as a consequence of the coupling with GP theory is expected to be associated with a ghost instability. This follows from the Ostrogradsky theorem, since the DHOST equations of motion are higher than second order, and when the constraint is thwarted, there is no degeneracy responsible for reducing the number of pieces of initial data. Hence, the Hamiltonian in this situation is unbounded from below and an instability will be present. As usual, this instability may be non-linear; i.e., the ghost mode need not appear as a linear perturbation in every background field configuration, but it will necessarily manifest itself in some backgrounds or at the non-linear level, as it occurs with the Boulware–Deser ghost in massive gravity [50].

We stressed in the introduction that our set-up relies on various assumptions which we think worth reiterating. The GP–DHOST system we studied is not the most general one. The analysis of the full model including all known operators would be a straightforward extension of our work, and we expect our main conclusions to remain unchanged. Indeed, the additional terms of the GP class that we have omitted contain operators that are cubic and quartic in powers of $\nabla_{\mu}B_{\nu}$; hence, they are likely to lead to the same issues as the \mathcal{L}_5 GP term. More crucial was the assumed prescription for coupling the GP and DHOST sectors. The premise was that there exists a Jordan frame such that all matter fields experience gravity through the same metric tensor and that our Proca field follows suit. Relaxing this assumption would be tantamount to constructing a scalar–vector–tensor type of theory in which all three fields interact in a non-trivial way. It would be interesting to address this problem within the context of degenerate theories (see, e.g., [51–55] for some recent related work).

Finally, an additional assumption was made in the analysis of the primary constraints, where we demanded that the Proca constraint had to match that of GP theory—that is, with a Hessian null eigenvector that is such that the time component of the vector field is rendered non-dynamical (in some local frame). It would be intriguing to explore whether this hypothesis might be dropped in order for the Proca and DHOST constraints to be realized in a way that would mix the canonical momenta associated with the vector and scalar fields. We remark that a related generalization of the Proca constraint has been studied recently in [56] in the context of pure vector–tensor theories. We plan to revisit these questions in future work.

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Appendix A. Full Results of the ADM Decomposition

In the main text we omitted the explicit results for some of the coefficient tensors appearing in the 3 + 1-decomposed DHOST and \mathcal{L}_5 GP term Lagrangians. Those terms are not important for the constraint analysis, but we provide the full expressions in this Appendix for the sake of completeness.

The tensors C^{ij} , C^0 and \overline{U} that are entered into the DHOST action are

$$\begin{aligned} \mathcal{C}^{ij} &= -\left(2F_{X}A^{k}D_{k}A_{*} + F_{\phi}A_{*} + \frac{1}{2}QA_{*}\right)\gamma^{ij} + A_{*}\left[-A_{1}D^{(i}A^{j)} - A_{2}D_{K}A^{k}\gamma^{ij} \right. \\ &+ \frac{A_{3}}{2}\left(A^{i}A^{j}D_{k}A^{k} - A^{k}A^{l}D_{k}A_{l}\gamma^{ij}\right) + A_{5}A^{k}A^{l}D_{k}A_{l}A^{i}A^{j}\right], \\ \mathcal{C}^{0} &= -\frac{1}{2}Q - \left(A_{2} - \frac{A_{3}}{2}A_{*}^{2}\right)D_{i}A^{i} - \left(\frac{A_{3}}{2} - A_{5}A_{*}^{2}\right)A^{i}A^{j}D_{i}A_{j} \\ &+ \left(A_{3} + A_{4} - 2A_{5}A_{*}^{2}\right)A^{i}A_{*}D_{i}A_{*}, \end{aligned}$$
(A1)
$$\mathcal{U} &= -R^{(3)} + 2D_{i}\left(F_{X}D^{i}X + F_{\phi}A^{i}\right) - P - QD_{i}A^{i} - \left(A_{1}D^{i}A^{j}D_{i}A_{j} + A_{2}(D_{i}A^{i})^{2} \right. \\ &+ A_{3}A^{i}A^{j}D_{i}A_{j}D_{k}A^{k} + A_{4}A^{i}A^{j}D_{i}A^{k}D_{j}A_{k} + A_{5}A^{i}A^{j}A^{k}A^{l}D_{i}A_{j}D_{k}A_{l}\right) \\ &+ \left(2A_{1} - A_{4}A_{*}^{2}\right)D^{i}A_{*}D_{i}A_{*} + \left(A_{4} - 4A_{5}A_{*}^{2}\right)A^{i}A^{j}D_{i}A_{*}D_{j}A_{*} \\ &+ 2A_{*}D^{i}A_{*}\left(A_{3}A_{i}D\right)jA^{j} + A_{4}A^{j}D_{j}A_{i} + 2A_{5}A_{i}A^{j}A^{k}D_{j}A_{k}\right). \end{aligned}$$

The tensors $\mathcal{C}_m^{ij}, \overline{\mathcal{C}}^0$ and \mathcal{U}_m that are entered into the \mathcal{L}_5 GP term are

$$\begin{aligned} \mathcal{C}_{m}^{ij} &= \frac{3\widetilde{G}_{5}}{N} \left[\left(NF_{k} - N^{l}b_{kl} \right) \left(B_{*}D^{(i}B_{*}\gamma^{j)k} - B_{*}D^{k}B_{*}\gamma^{ij} + D^{k}B^{(i}B^{j)} - D_{m}B^{m}B^{(i}\gamma^{j)k} \right) \\ &- \frac{B_{*}}{2} \left(2ND^{k}B_{*}D_{k}B_{*} + N(D_{l}B^{k})^{2} + 2N^{k}D_{k}B_{*}D_{l}B^{l} + 2B^{k}D_{k}ND_{l}B^{l} \\ &- 2N^{k}D_{k}B^{l}D_{l}B_{*} + 2N^{k}D^{l}B_{k}D_{l}B_{*} - ND_{k}B^{l}D_{l}B^{k} \right) \gamma^{ij} \\ &+ NB_{*} \left(D^{i}B_{*}D^{j}B_{*} - D^{k}B^{(i}D^{j)}B_{k} + D^{(i}B^{j)}D_{k}B^{k} \right) + B_{*}N^{k} \left(D_{k}B_{*}D^{(i}B^{j)} \\ &- D_{k}B^{(i}D^{j)}B_{*} + D^{(i}B_{k}D^{j)}B_{*} \right) + N^{k}B^{(i} \left(D_{l}B^{j)}b_{k}^{l} - b_{j}^{j}{}_{k}D_{l}B^{l} \right) \\ &+ NB^{(i} \left(D^{j}B_{k}D^{k}B_{*} + D_{k}B^{j)}D^{k}B_{*} - 2D^{j)}B_{*}D_{l}B^{l} \right) + B_{*}B^{k}D_{k}ND^{(i}B^{j)} \right], \end{aligned} \tag{A2}$$

$$\begin{split} \mathcal{U}_{\rm m} &= -\frac{3G_5}{N} \left[2(NF_i - N^j b_{ij}) \left(D^i B_* D_k B^k - D_k B_* D^i B^k \right) + \frac{N}{3} \left((D_i B^i)^3 - 3D_i B^i D_j B^k D_k B^j + 2D_i B^j D_j B^k D_k B^i \right) + 2N \left(D_i B^i D^j B_* D_j B_* - D^i B^j D_i B_* D_j B_* \right) \\ &+ 2N^i \left(D_i B^j D_j B^k D_k B_* - D_j B^k D^j B_i D_k B_* - b_{ij} D^j B_* D_k B^k \right) \right]. \end{split}$$

Appendix B. Inverse of DHOST Kinetic Tensor

The metric kinetic tensor that appears in the Hamiltonian analysis of DHOST has the following structure (see Equation (10)):

$$\mathcal{K}^{ij,kl} = a\gamma^{i(k}\gamma^{l)j} + b\gamma^{ij}\gamma^{kl} + c\left(\gamma^{ij}A^{k}A^{l} + \gamma^{kl}A^{i}A^{j}\right) + d\left(A^{i}A^{(k}\gamma^{l)j} + A^{j}A^{(k}\gamma^{l)i}\right) + eA^{i}A^{j}A^{k}A^{l}.$$
(A3)

We wish to find the inverse tensor such that

$$\mathcal{K}^{ij,mn}\mathcal{K}_{mn,kl}^{-1} = \delta^i_{(k}\delta^j_{l)} \,. \tag{A4}$$

This is achieved by taking the ansatz

$$\mathcal{K}_{ij,kl}^{-1} = a' \gamma_{i(k} \gamma_{l)j} + b' \gamma_{ij} \gamma_{kl} + c' (\gamma_{ij} A_k A_l + \gamma_{kl} A_i A_j) + d' \Big(A_i A_{(k} \gamma_{l)j} + A_j A_{(k} \gamma_{l)i} \Big) + e' A_i A_j A_k A_l ,$$
(A5)

and a straightforward calculation yields

$$\begin{aligned} a' &= \frac{1}{a}, \\ b' &= \frac{1}{a} \frac{x(c^2x - b(2d + ex)) - ab}{a^2 + 3ab + ax(2(c + d) + ex) + 2x(2bd + bex - c^2x)}, \\ c' &= \frac{1}{a} \frac{-ac + 2bd + bex - c^2x}{a^2 + 3ab + ax(2(c + d) + ex) + 2x(2bd + bex - c^2x)}, \\ d' &= -\frac{1}{a} \frac{d}{a + dx}, \\ e' &= \frac{1}{a} \frac{-a^2e + a(-3be + 3c^2 + 4cd + 2d^2 + dex) + d(2bd + bex - c^2x)}{(a + dx)(a^2 + 3ab + ax(2(c + d) + ex) + 2x(2bd + bex - c^2x))}, \end{aligned}$$
(A6)

with $x := A^i A_i$.

Notes

- ¹ In the context of higher-derivative theories, it is worth remarking that the Starobinsky model of inflation provides a healthy and consistent example of this class [3]. See also [4–6] for related works on inflationary models based on generalized scalar–tensor theories.
- ² It is important to remark that the vanishing of the Hessian determinant implies the existence of a primary constraint, which is not by itself enough to remove a full Lagrangian degree of freedom. Nevertheless, the existence of an associated secondary constraint is guaranteed by the general covariance of the action, as shown explicitly in [15]. The same consideration holds for the situation wherein the DHOST constraint is modified in the presence of matter (as exemplified in Equation (43)), provided of course the latter admits consistent coupling according to the criteria explained in the introduction.

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